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AN ORLICZ-SOBOLEV SPACE SETTING FOR QUASILINEAR ELLIPTIC PROBLEMS

NIKOLAOS HALIDIAS

Abstract. In this paper we give two existence theorems for a class of elliptic problems in an Orlicz-Sobolev space setting concerning both the sublinear and the superlinear case with Neumann boundary conditions. We use the classical critical point theory with the Cerami (PS)-condition.

1. Introduction

In this paper we consider the following elliptic problem with Neumann boundary conditions,

$$\begin{gathered} -\operatorname{div}(\alpha(|\nabla u(x)|)\nabla u(x)) = g(x,u) \quad \text{a.e. on } \Omega \\ \frac{\partial u}{\partial v} = 0, \text{ a.e. on } \partial\Omega. \end{gathered} \tag{1.1}$$

We assume that Ω is a bounded domain with smooth boundary $\partial\Omega$. By $\frac{\partial}{\partial v}$ we denote the outward normal derivative. As in [2] we assume that the function α is such that $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(s) = \alpha(|s|)s$ if $s \neq 0$ and 0 otherwise, is an increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$.

In [2], the authors study a Dirichlet problem when the right-hand side is superlinear. They show the existence of a nontrivial solution and show that it is important to use an Orlicz-Sobolev space setting. Here, we consider a Neumann problem when the right-hand side is sublinear. Also we consider the superlinear case using the ideas in [4]. Assuming Landesman-Laser conditions for the sublinear case and using the interpolation inequality for the superlinear case, we prove the existence of a nontrivial solution.

Let us recall the Cerami (PS) condition [1]. Let X be a Banach space. We say that a functional $I : X \to \mathbb{R}$ satisfies the $(PS)_c$ condition if for any sequence such that $|I(u_n)| \leq M$ and $(1 + \|u_n\|)\langle I'(u_n), \phi\rangle \to 0$ for all $\phi \in X$ we can show that there exists a convergent subsequence.

Let

$$\Phi(s) = \int_0^s \phi(t)dt, \quad \Phi^*(s) = \int_0^s \phi^{-1}(t)dt, \quad s \in \mathbb{R},$$

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it is well-known that Φ and Φ^* are complementary N functions which define the Orlicz spaces L_Φ, L_{Φ^*} respectively. We use the well-known Luxenburg norm,

$$\|u\|_\Phi = \inf\{k > 0 : \int_\Omega \Phi(\frac{|u(x)|}{k})dx \leq 1\}.$$

As in [2] we denote by W^1L_Φ the corresponding Orlicz-Sobolev space with the norm $\|u\|_{1,\Phi} = \|u\|_\Phi + \||\nabla u|\|_\Phi$.

Now we introduce the Orlicz-Sobolev conjugate Φ_* of Φ, defined as

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}, d\tau,$$

and as in [2], we suppose that

$$\lim_{t\to 0}\int_t^1 \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}, d\tau < +\infty\,, \quad \lim_{t\to\infty}\int_1^t \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}, d\tau = +\infty.$$

To state our hypotheses on ϕ, g, we need the following three numbers,

$$p^1 = \inf_{t>0}\frac{t\phi(t)}{\Phi(t)}, \quad p_\Phi = \liminf_{t\to\infty}\frac{t\phi(t)}{\Phi(t)}, \quad p^0 = \sup_{t>0}\frac{t\phi(t)}{\Phi(t)}\,.$$

(H1) The function ϕ is such that
- (i) For every $\varepsilon > 0$, there is $k_\varepsilon > 1$ such that $\Phi'((1+\varepsilon)x) \geq k_\varepsilon \Phi'(x)$, $x \geq x_o(\varepsilon) \geq 0$ and that Φ is strictly convex.
- (ii) Both Φ, Φ^* satisfy a Δ_2 condition, namely

$$1 < \liminf_{s\to\infty}\frac{s\phi(s)}{\Phi(s)} \leq \limsup_{s\to\infty}\frac{s\phi(s)}{\Phi(s)} < +\infty.$$

Remark 1.1. Under hypotheses (H1), L_Φ is uniformly convex [8, p.288].

We assume the following conditions on g.

(H2) The function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous and satisfies the following hypotheses:
- (i) There exists nonnegative constants a_1, a_2 such that $|g(x,s)| \leq a_1 + a_2|s|^{a-1}$, for all $(x,s) \in \Omega \times \mathbb{R}$, with $p^0 \leq a < \frac{Np^1}{N-p^1}$.
- (ii) For all $x \in \Omega$,

$$\limsup_{u\to 0}\frac{G(x,u)}{\Phi(u)} \leq -\mu < 0, \quad \lim_{u\to\infty}\frac{G(x,u)}{|u|^{p^1}} = 0\,.$$

- (iii) There is a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ with the property $\liminf \frac{h(a_n b_n)}{h(b_n)} > 0$, $h(b_n) \to \infty$ when $a_n \to a > 0$ and $b_n \to +\infty$ such that

$$\liminf_{|u|\to\infty}\frac{p^1 G(x,u) - g(x,u)u}{h(|u|)} \geq k(x) > 0,$$

  with $k \in L^1(\Omega)$,

with $G(x,u) = \int_0^u g(x,r)dr$.

Remark 1.2. Using the definition of p^1 we can prove that $\Phi(t) \geq ct^{p^1}$ for $t \geq 1$. From this we obtain that $W^1L_\Phi \hookrightarrow L^{\frac{Np^1}{N-p^1}}$ (see [2]).

Our energy functional $I : W^1L_\Phi \to \mathbb{R}$ is defined as

$$I(u) = \int_\Omega \Phi(|\nabla u(x)|)dx - \int_\Omega G(x, u(x))dx.$$

From the arguments of [2, 5] we know that this functional is well defined and C^1.

Lemma 1.3. *If (H1), (H2) hold, then the energy functional satisfies the $(PS)_c$ condition.*

Proof. Let $X = W^1L_\Phi(\Omega)$. Suppose that there exists a sequence $\{u_n\} \subseteq X$ such that $|I(u_n)| \le M$ and

$$|\langle I'(u_n), \phi\rangle| \le \varepsilon_n \frac{\|\phi\|_{1,\Phi}}{1 + \|u_n\|_{1,\Phi}}. \tag{1.2}$$

Suppose that $\|u_n\|_{1,\Phi} \to \infty$. Let $y_n(x) = \frac{u_n(x)}{\|u_n\|_{1,\Phi}}$. It is easy to see that $y_n \to y$ weakly in X and $y_n \to y$ strongly in $L_\Phi(\Omega)$. From the first inequality we have

$$\Big|\int_\Omega \Phi(|\nabla u_n(x)|)dx - \int_\Omega G(x, u_n(x))dx\Big| \le M. \tag{1.3}$$

We can prove that $\Phi(t) \ge \rho^{p^1}\Phi(\frac{t}{\rho})$. Indeed, we have that $\Phi(t)p^1 \le t\phi(t)$ for $t > 0$. Then we obtain

$$\int_{t/\rho}^{t} \frac{p^1}{s} ds \le \int_{t/\rho}^{t} \frac{\phi(s)}{\Phi(s)} ds,$$

for all $t > 0$ and for $\rho > 1$. Calculating the above integrals we arrive at the fact that $\Phi(t) \ge \rho^{p^1}\Phi(\frac{t}{\rho})$ for all $t > 0$ and all $\rho > 1$. When we divide the above inequality by $\|u_n\|_{1,\Phi}^{p^1} > 1$, we obtain

$$\int_\Omega \Phi(|\nabla y_n(x)|dx \le \int_\Omega \frac{G(x, u_n(x))}{\|u_n\|_{1,\Phi}^{p^1}} dx\,.$$

Next, we prove that $\int_\Omega \frac{G(x,u_n(x))}{\|u_n\|_{1,\Phi}^{p^1}} dx \to 0$. Indeed, from $(H2)(ii)$ we have that for every $\varepsilon > 0$ there exists some $M > 0$ such that for $|u| > M$ we have $\frac{G(x,u)}{|u|^{p^1}} \le \varepsilon$ for all $x \in \Omega$. Thus,

$$\begin{aligned}
&\int_\Omega \frac{G(x, u_n(x))}{\|u_n\|_{1,\Phi}^{p^1}} dx \\
&\le \int_{\{x\in\Omega:|u_n(x)|\le M\}} \frac{G(x, u_n(x))}{\|u_n\|_{1,\Phi}^{p^1}} dx + \int_{\{x\in\Omega:|u_n(x)|\ge M\}} \varepsilon|y_n(x)|^{p^1} dx.
\end{aligned}$$

Note that $p^1 \le p^0 \le a$ so we have that $W^1L_\Phi \hookrightarrow L^{p^1}$. From that we obtain

$$\int_\Omega \frac{G(x, u_n(x))}{\|u_n\|_{1,\Phi}^{p^1}} dx \le \int_{\{x\in\Omega:|u_n(x)|\le M\}} \frac{G(x, u_n(x))}{\|u_n\|_{1,\Phi}^{p^1}} dx + \varepsilon c\|y_n\|_{1,\Phi}^{p^1}.$$

Finally, note that $\|y_n\|_{1,\Phi} = 1$ so we have proved our claim.

Now $\int_\Omega \Phi(|\nabla y_n(x)|dx \to 0$ thus, $\|\nabla y_n\|_\Phi \to 0$. Since

$$\|\nabla y\|_\Phi \le \liminf_{n\to\infty} \|\nabla y_n\|_\Phi \to 0,$$

so $\|\nabla y_n\|_\Phi \to \|\nabla y\|_\Phi$ and moreover $y_n \to y$ weakly in X, thus from the uniform convexity of X we deduce that $y_n \to y$ strongly in X. Note that $\|y_n\|_{1,\Phi} = 1$ so,

$y \neq 0$ and from the fact that $\|\nabla y\|_\Phi = 0$ we have that $y = c \in \mathbb{R}$ with $c \neq 0$. From this we obtain that $|u_n(x)| \to \infty$.

Choosing now $\phi = u_n$ in (1.2) and substituting with (1.3), we arrive at

$$\begin{aligned}&\int_\Omega p^1 G(x, u_n(x)) - g(x, u_n(x))u_n(x)dx + \int_\Omega \phi(|\nabla u_n|)|\nabla u_n| - p^1\Phi(|\nabla u_n|)dx \\ &\leq M + \varepsilon_n \frac{\|u_n\|_{1,\Phi}}{1 + \|u_n\|_{1,\Phi}}.\end{aligned}$$

From the definition of p^1 we have $p^1\Phi(t) \leq t\phi(t)$. Using this fact and dividing the last inequality with $h(\|u_n\|_{1,\Phi})$ we obtain

$$\begin{aligned}&\int_\Omega \frac{p^1 G(x, u_n(x)) - g(x, u_n(x))u_n(x)}{h(|u_n(x)|)} \frac{h(|y_n(x)|\|u_n\|_{1,\Phi})}{h(\|u_n\|_{1,\Phi})}dx \\ &\leq \frac{M + \varepsilon_n \frac{\|u_n\|_{1,\Phi}}{1+\|u_n\|_{1,\Phi}}}{h(\|u_n\|_{1,\Phi})}.\end{aligned}$$

From this we can see that

$$\liminf_{n\to\infty} \int_\Omega \frac{p^1 G(x, u_n(x)) - g(x, u_n(x))u_n(x)}{h(|u_n(x)|)} \frac{h(|y_n(x)|\|u_n\|_{1,\Phi})}{h(\|u_n\|_{1,\Phi})}dx \leq 0.$$

Using Fatou's lemma and $(H2)(iii)$ we obtain the contradiction. That is u_n is bounded. So, we can say, at least for a subsequence, that $u_n \to u$ weakly in X and $u_n \to u$ strongly in $L_a(\Omega)$.

To show the strong convergence we going back to (1.2) and choose $\phi = u_n - u$. Thus, we obtain

$$\begin{aligned}&\Big|\int_\Omega \Big(\alpha(|\nabla u_n|)\nabla u_n - \alpha(|\nabla u|)\nabla u\Big)\Big(\nabla u_n - \nabla u\Big)dx\Big| \\ &\leq \int_\Omega g(x, u_n)(u_n - u)dx + \varepsilon_n\|u_n - u\|_{1,\Phi} - \int_\Omega \alpha(|\nabla u|)\nabla u(\nabla u_n - \nabla u)dx.\end{aligned}$$

Using the compact imbedding $X \hookrightarrow L^a(\Omega)$ and the fact that $u_n \to u$ weakly in X we arrive at $\int_\Omega \big(a(|\nabla u_n|)\nabla u_n - a(|\nabla u|)\nabla u\big)\big(\nabla u_n - \nabla u\big)dx \to 0$ and using [6, Theorem 4] we obtain the strong convergence of u_n. □

Lemma 1.4. *If hypotheses $(H1)(ii), (H2)$ holds, then there exists some $e \in X$ with $I(e) \leq 0$.*

Proof. We will show that there exists some $a \in \mathbb{R}$ such that $I(a) \leq 0$. Suppose that this is not the case. Then there exists a sequence $a_n \in \mathbb{R}$ with $a_n \to \infty$ and $I(a_n) \geq c > 0$. We can easily see that

$$\begin{aligned}(-\frac{G(x,u)}{u^{p^1}})' &= \frac{p^1 G(x,u) - g(x,u)u}{u^{p^1+1}} \\ &= \frac{p^1 G(x,u) - g(x,u)u}{h(|u|)} \frac{h(|u|)}{u^{p^1+1}} \\ &\geq (k(x) - \varepsilon)\frac{1}{u^{p^1+1}} = \frac{k(x) - \varepsilon}{p^1}(-\frac{1}{u^{p^1}})',\end{aligned}$$

for a large enough $u \in \mathbb{R}$. We can say then

$$\int_t^s \big(-\frac{G(x,u)}{u^{p^1}}\big)'du \geq \int_t^s \frac{k(x) - \varepsilon}{p^1}\big(-\frac{1}{u^{p^1}}\big)'du.$$

Take now $s \to \infty$ and using (H2)(iii), we obtain

$$G(x,t) \geq \frac{k(x)}{p^1},$$

for large enough $t \in \mathbb{R}$. From this we obtain

$$\limsup_{a_n \to \infty} I(a_n) \geq \liminf_{a_n \to \infty} I(a_n) \geq 0$$

implies

$$\limsup_{a_n \to \infty} \int_\Omega -G(x,a_n)dx \geq 0$$

which implies $\int_\Omega \frac{-k(x)}{p^1}dx \geq 0$. Then using (H2)(iii) we obtain the contradiction. □

Lemma 1.5. *If (H1)(ii) and (H2) hold, then there exists some $\rho > 0$ such that for all $u \in X$ with $\|u\|_\Phi = \rho$ we have that $I(u) > \eta > 0$.*

Proof. ¿From (H2)(ii) we have that for every $\varepsilon > 0$ there exists some $u^* \leq 1$ such that for every $|u| \leq u^*$ we have $G(x,u) \leq (-\mu+\varepsilon)\Phi(|u|) \leq k(-\mu+\varepsilon)|u|^{p^0}$ with $k > 0$. On the other hand there exists $c_1, c_2 > 0$ such that $|G(x,u)| \leq c_1|u|^{\frac{Np^1}{N-p^1}} + c_2$ for every $u \in \mathbb{R}$. Recall that $p^0 < \frac{Np^1}{N-p^1}$ so we can find some $\gamma > 0$ such that $G(x,u) \leq k(-\mu+\varepsilon)|u|^{p^0} + \gamma|u|^{\frac{Np^1}{N-p^1}}$. Indeed, we can choose

$$\gamma \geq c_1 + \frac{c_2}{|u^*|^{\frac{Np^1}{N-p^1}}} + k(\mu-\varepsilon)\frac{|u^*|^{p^0}}{|u^*|^{\frac{Np^1}{N-p^1}}}.$$

Take now a sequence $\{u_n\} \in X$ such that $\|u_n\|_{1,\Phi} \to 0$. Thus, we can see that

$$I(u_n) \geq \int_\Omega \Phi(|\nabla u_n|)dx + k(\mu-\varepsilon)\|u_n\|_{p^0}^{p^0} - \gamma\|u_n\|_{\frac{Np^1}{N-p^1}}^{\frac{Np^1}{N-p^1}}$$

implies

$$I(u_n) \geq c\||\nabla u_n|\|_\Phi^{p^0} + k(\mu-\varepsilon)\|u_n\|_\Phi^{p^0} - \gamma\|u_n\|_{\frac{Np^1}{N-p^1}}^{\frac{Np^1}{N-p^1}}$$

which implies

$$I(u_n) \geq C\|u_n\|_{1,\Phi}^{p^0} - \gamma\|u_n\|_{1,\Phi}^{\frac{Np^1}{N-p^1}}.$$

Here we have used the fact that $L^{p^0}(\Omega)$ imbeds continuously in $L_\Phi(\Omega)$ and the fact that $L^{Np^1/(N-p^1)}$ imbeds continuously in W^1L_Φ. Finally we have $C = \min\{c, k(\mu-\varepsilon)\}$. Thus, for big enough $n \in \mathbb{N}$ and noting that $p^0 < \frac{Np^1}{N-p^1}$ we deduce that there exists some $\rho > 0$ such that for all $u \in X$ with $\|u\|_\Phi = \rho$ we have that $I(u) > \eta > 0$. The Lemma is proved. □

The existence theorem follows from the Mountain-Pass theorem. Note that we also extend the recently results of Tang [10] for Neumann problems because the author there needs $h(u) = u$.

2. Superlinear Case

In this section we consider problem (1.1) with a superlinear right hand side. We assume the following conditions on g,

(H3) The funciton $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following hypotheses:
  (i) There exists nonnegative constants a_1, a_2 such that $|g(x, s)| \le a_1 + a_2|s|^{a-1}$, for all $(x, s) \in \Omega \times \mathbb{R}$, with $p^0 \le a < \frac{Np^1}{N-p^1}$, .
  (ii) There exists some $q > 0$ such that for all $x \in \Omega$,

$$\limsup_{u\to 0} \frac{G(x,u)}{\Phi(|u|)} < -k < 0 \qquad \lim_{u\to\infty} \frac{G(x,u)}{|u|^q} = 0, \quad 0 < \beta \le \liminf_{|s|\to\infty} \frac{G(x,s)}{\Phi(s)}$$

  (iii) There exists $\mu > N/p^1(q - p^1)$ such that

$$\liminf_{|u|\to\infty} \frac{g(x,u)u - p^1G(x,u)}{|u|^\mu} \ge m > 0.$$

  with $G(x, u) = \int_0^u g(x, r)dr$.

Theorem 2.1. *If hypotheses (H1)(ii) and (H3) hold, then problem* (1.1) *has a nontrivial solution $u \in X$.*

Proof. Let us denote first by $N(u) = \int_\Omega G(x, u)dx$. Suppose that there exists a sequence $\{u_n\} \subseteq X$ such that $I(u_n) \to c$ and $| < I'(u_n), y > | \le \varepsilon_n \frac{\|y\|_{1,\Phi}}{1+\|u_n\|_{1,\Phi}}$ for all $y \in X$. We are going to show that $\|u_n\|_{1,\Phi}$ is bounded in X. Suppose not. Then there exists a subsequence such that $\|u_n\|_{1,\Phi} \to \infty$.

Using the definition of p^1 it is easy to see that $|\langle I'(u), u\rangle - p^1 I(u)| \ge |\langle N'(u), u\rangle - p^1 N(u)|$ and using (H3)(iii), we arrive at $\|u_n\|_\mu^\mu \le C$.

Next, we use the interpolation inequality, namely

$$\|u\|_q \le \|u\|_\mu^{1-t}\|u\|^t_{\frac{Np^1}{N-p^1}},$$

where $0 < \mu \le q \le \frac{Np^1}{N-p^1}$, $t \in [0, 1]$. Using the fact that X imbeds continuously in $L^{\frac{Np^1}{N-p^1}}$ we have

$$\begin{aligned}\int_\Omega \Phi(|\nabla u_n|)dx &= I(u_n) + N(u_n)\\ &\le c_1\|u_n\|_q^q + c_2\\ &\le \|u_n\|_\mu^{(1-t)q}\|u_n\|^{qt}_{\frac{Np^1}{N-p^1}}\\ &\le c_1\|u_n\|^{qt}_{1,\Phi} + c_2,\end{aligned} \tag{2.1}$$

here we have used the second assertion of (H3)(ii). From the relation $|I(u_n)| \le M$ we obtain

$$\int_\Omega G(x, u_n)dx \le \int_\Omega \Phi(|\nabla u_n|)dx + M$$

and

$$\beta \int_\Omega \Phi(u_n)dx \le \int_\Omega \Phi(|\nabla u_n|)dx + M\,.$$

We have used here the third assertion of (H3)(ii). Adding $\beta \int_\Omega \Phi(|\nabla u_n|)dx$ to the last inequality, we obtain

$$\beta(\int_\Omega \Phi(u_n)dx + \int_\Omega \Phi(|\nabla u_n|)dx) \leq C \int_\Omega \Phi(|\nabla u_n|)dx + M. \tag{2.2}$$

We can prove that $\Phi(t) \geq \rho^{p^1} \Phi(t/\rho)$ for $\rho \geq 1$ and combining (2.1) and (2.2), we arrive at

$$c_1 \|u_n\|_{1,\Phi}^{p^1} - c_2 \leq \int_\Omega \Phi(|\nabla u_n|)dx \leq c_1 \|u_n\|_{1,\Phi}^{qt} + c_2.$$

for some $c_1, c_2 > 0$. Choosing $qt < p^1$ (or equivalently $\mu > N/p^1(q - p^1)$) we obtain a contradiction. Thus, $\{u_n\} \subseteq X$ is bounded and using the same arguments as in Lemma 1.3 we can prove that in fact $\{u_n\}$ has a strongly convergent subsequence in X.

Next we prove that there exists some $e \in X$ such that $I(e) \leq 0$. Indeed, take a sequence $t_n \to \infty$, then

$$I(t_n) = -\int_\Omega G(x, t_n)dx \leq -\beta \int_\Omega \Phi(t_n)dx + C.$$

It is clear now that for big enough $n \in \mathbb{N}$ we have $I(t_n) \leq 0$. Using Lemma 1.5 and the Mountain-Pass theorem, we obtain a nontrivial solution. $\square$

As an example of functions that satisfy the above hypotheses, we have $\Phi(u) = \log(1 + |u|)|u|^2$ and $G(u) = \log(1 + |u|)\Phi(u)$.

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Nikolaos Halidias
University of the Aegean, Department of Statistics and Actuarial Science, Karlovassi, 83200, Samos, Greece
E-mail address: nick@aegean.gr