

EXISTENCE OF SIGN-CHANGING SOLUTIONS FOR RADIALY SYMMETRIC p -LAPLACIAN EQUATIONS WITH VARIOUS POTENTIALS

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ABSTRACT. In this article, we study the nonlinear equation

$$(r^{n-1}|u'(r)|^{p-2}u'(r))' + r^{n-1}w(r)|u(r)|^{q-2}u(r) = 0,$$

where $q > p > 1$. For positive potentials ($w > 0$), we investigate the existence of sign-changing solutions with prescribed number of zeros depending on the increasing initial parameters. For negative potentials, we deduce a finite interval in which the positive solution will tend to infinity. The main methods using in this work are the scaling argument, Prüfer-type substitutions, and some integrals involving the p -Laplacian.

1. INTRODUCTION

The purpose of this article is to investigate some properties related to the radially symmetric problem for

$$-\Delta_p u = g(|x|, u), \quad \text{on } \Omega \subseteq \mathbb{R}^n, \quad (1.1)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$ and $n \geq 1$. The p -Laplacian operator $\Delta_p u$ itself has the originally physical meaning, and also can be treated as a generalization of the Laplacian operator. The quantity p is a characteristic of the medium of non-Newtonian fluids or nonlinear diffusion problems. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. For the above, we refer the readers to [7, 8, 18, 22, 26, 31, 32, 33] and their references. Also, some results for radial solutions related to (1.1) have been obtained in [5, 6, 12, 13, 14, 15, 29, 25, 36, 37]. In [5], the authors extended the eigenvalue theory to the radially symmetric p -Laplacian in \mathbb{R}^n corresponding to Weyl's limit point and Weyl's limit circle theories in the case $p = 2$. In [12], the authors determined the structure of positive radial solutions related to (1.1). In particular, Kabeya et al. [14] deduced the existence of radially *fast-decay* solutions of (1.1) with prescribed number of zeros in $(0, \infty)$. They also considered further boundary problems with similar results. More recently, the authors [6] derived the existence of radial solutions having prescribed number of sign changes on $(0, \infty)$ for $n \geq p > 1$. Another direction is to deduce the existence of *blow-up* solutions. A solution u of (1.1) is called *boundary blow-up* if $\lim_{k \rightarrow \infty} u(x_k) = \infty$ for each

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sequence in Ω which converges to a point on $\partial\Omega$. For $p = 2$ this type of problem has a long history. Bieberbach [3] and Rademacher [27] started to study this theme. Bieberbach was motivated by problems in geometry, Rademacher by a problem in mathematical physics. Later, Keller [16] derived a well-known result. The author gave necessary and sufficient conditions on the the growth of $g(u)$ at infinity to guarantee that such solutions exist. Under more restrictive assumptions, but for general N -dimensional domains, the blow-up problem has been studied by Bandle and Marcus [1, 2], Lazer and McKenna [19, 20] for $p = 2$ and by Diaz and Letelier [9] for general $p > 1$. For a nonlinear radial p -Laplacian, Reichel and Walter [29] employed a strong comparison principle to develop some properties of boundary blow-up solutions. Also, McKenna *et al.* [25] have treated the radial case for $g(u) = |u|^q$ and general $p > 1$. We mention a part of work in [25], where the authors developed the existence of blow-up solutions for the one-dimensional case ($n = 1$) first and applied a crucial inequality to obtain the existence of blow-up solutions to the general case ($n \geq 1$).

By considering radially symmetric solutions to (1.1), we are led to study the nonlinear problem

$$(r^{n-1}|u'(r)|^{p-2}u'(r))' + r^{n-1}w(r)|u(r)|^{q-2}u(r) = 0, \quad (1.2)$$

$$u(0) = \alpha > 0, \quad u'(0) = 0, \quad (1.3)$$

where $r = |x|$ and $' = \frac{d}{dr}$. Motivated by the previous results [6, 14, 17, 25, 35, 37], we study two issues related to (1.2)-(1.3). When $w > 0$, we investigate the existence of sign-changing solutions with the prescribed number of zeros in a finite interval. For this issue, we consider a right endpoint condition

$$u(1) = 0 \quad (1.4)$$

for the sake of simplicity. We denote the solution of (1.2)-(1.3) by $u(r; \alpha)$. The following results (Theorem 1.1 and 1.3) can be treated as the *Sturmian theory* which is related to the existence of solutions having prescribed number of zeros. Some results closely relevant to this issue can be referred to [6, 14, 15, 35, 36, 37] and their bibliographies. In this article we to consider a wide class of potential functions and employ the interesting methods, scaling arguments and Prüfer-type substitutions, to achieve the goal. The method seems to be classical and can be found in [14, 35, 39]. However, this extension is not trivial and need more subtle arguments in the analysis of generalized polar coordinates. Throughout this paper we assume the following conditions:

(A1) $q > p$;

(A2) $w \in C^1(\mathbb{R})$, and $|w| \geq \delta_1$ on $[0, \infty)$ for some $\delta_1 > 0$;

(A3) $\kappa := \max\left\{\frac{|w'(r)|}{w(r)} : r \in [0, 1]\right\}$;

(A4) $p > n$, $w_1 := \max\{|w(r)| : r \in [0, 1]\}$ and $\mathbb{K} := w_1\left(\frac{w(0)}{\delta_1}e^\kappa\right)^{\frac{q-p}{q}}$.

Note that (A1)–(A3) hold for the existence of solutions with prescribed number of zeros to the two-endpoint boundary condition case (Theorem 1.1). (A4) is added to the case of multi-point boundary conditions (Theorem 1.3). Here is our first result.

Theorem 1.1. *Assume that $w > 0$ in (1.2) and (A1)–(A3) hold. Then there exists a strictly increasing sequence of positive numbers $\{\alpha_n\}_{n=1}^\infty$, such that the solution $u(r; \alpha_n)$ is a solution to the BVP (1.2)-(1.4). Moreover, $u(r; \alpha_n)$ has exactly $n - 1$ zeros in $(0, 1)$ for $n \in \mathbb{N}$.*

Remark 1.2. The initial parameter corresponding to the solution with the prescribed number of zeros is not unique usually. For the case of $n = 1$, the author [34] showed that such a sequence is unique, provided $w \in C^2(\mathbb{R})$, $([w(r)]^{-1/p})'' \leq 0$ on \mathbb{R} and $1 < p \leq 2$.

Under the derivation of the existence of sign-changing solutions to the two-point BVP (Theorems 1.1), we observe an application to the case of *multi-point* boundary conditions. The existence of solutions, especially positive solutions, of boundary value problems with multi-point boundary conditions have been studied extensively, see, for example, [11, 23, 24, 28, 38]. Moreover, to the best of the author's knowledge, there is few work done so far on the existence of nodal solutions to problems with the multi-point boundary conditions related to (1.2). Motivated by the idea in [17, 37] and as a byproduct from the derivation of Theorem 1.1, we intend to extend the Sturmian theory to the case of (1.2)-(1.3) coupled with the multi-point boundary conditions. Here we consider a more general situation that the multi-point conditions are dependent on the initial parameter. It is similar to Sturm-Liouville problems coupled with eigendependent boundary conditions. We mention that the Prüfer-type substitution is significant to derive the Sturmian theory. The Prüfer phase is efficient to study the number of sign changes of solutions, and the Prüfer radius estimate leads to satisfy the multi-point boundary condition. The details of Prüfer-type substitutions will be discussed in Section 2. Now we impose

$$u(1) - \sum_{i=1}^d \tau_i e^{-\frac{(\alpha - \frac{q-p}{p-1} \mathbb{K})}{p-1}} u(r_i) = 0, \quad (1.5)$$

where \mathbb{K} is defined in (A4), $\tau_i \in \mathbb{R}$ and $r_i \in (0, 1)$ for $i = 1, 2, 3, \dots, d$ with $d \in \mathbb{N}$. The following is our second result.

Theorem 1.3. *Assume that $w > 0$ in (1.2) and (A1)–(A4) hold. Also assume that*

$$1 - \sum_{i=1}^d |\tau_i| > 0. \quad (1.6)$$

Then there exists a strictly increasing sequence of positive initial values $\{\alpha_n\}_{n=1}^{\infty}$, such that the solution $u(r; \alpha_n)$ is a solution to the multi-point boundary value problem (1.2)-(1.3) and (1.5). Moreover, $u(r; \alpha_n)$ has exactly n or $n + 1$ zeros in $(0, 1)$ for $n \in \mathbb{N}$.

Next, the counterpart of this paper is to investigate the negative potential case ($w < 0$). We intend to discuss the blow-up solutions of (1.2)-(1.3). Motivated by the interesting idea as in [25] and under some minor assumption of w , we plan to discuss the existence of blow-up solutions in a finite interval. We also derive such an interval associated with the initial parameter α precisely by analyzing some integrals involving the p -Laplacian.

Theorem 1.4. *For nonincreasing w with $w < 0$, assume that (A1) and (A2) hold. The nonlinear problem (1.2)-(1.3) has at least one positive blow-up solution $u(r; \alpha)$ in $(0, R_\alpha)$, where*

$$R_\alpha := \sqrt[p]{\frac{nq(p-1)}{p\delta_1\alpha^{q-p}} \left(\frac{1}{p-1} (2^p - 1)^{\frac{p-1}{p}} + \frac{2p}{q-p} \right)}. \quad (1.7)$$

That is, the positive solution $u(r; \alpha)$ tends to infinity as r tends to $R \leq R_\alpha$. Moreover, such a positive blow-up solution can not occur when the problem is considered in a finite interval as $q \leq p$.

The article is organized as follows. Some elementary properties related to (1.2)-(1.3) and the proofs of Theorem 1.1 and 1.3 will be given in Section 2. The existence of blow-up solutions (Theorem 1.4) will be represented in Section 3.

2. PRELIMINARIES AND STURMIAN THEORY

First, the existence of solutions to (1.2)-(1.3) is valid and can be found in [12, 14, 15, 25, 29, 36, 37]. Here we quote the following to coincide with our setting. Also the regularity requirements for a solution u are $u \in C^1$ and $r^{n-1}|u'|^{p-2}u' \in C^1$.

Theorem 2.1 ([25, Theorem EUCD],[29, Theorems 1 and 4],[36, Corollary 2.3]). *Assume conditions (A1) and (A2) hold. For the positive potential function ($w > 0$), there exists a unique local solution $u(r; \alpha)$ of (1.2)-(1.3). Moreover, the solution $u(r; \alpha)$ can be extended to the whole real axis.*

Now, we introduce a Prüfer-type substitution for the solution $u(r; \alpha)$ of (1.2)-(1.3) by using the generalized sine function $S_p(r)$. The generalized sine function S_p has been well studied in the literature (see Lindqvist [21] or [4, 10, 30] with a minor difference in setting). Here we outline some properties for the reader's convenience. The function S_p satisfies

$$|S'_p(r)|^p + \frac{|S_p(r)|^p}{p-1} = 1, \quad (2.1)$$

$$(|S'_p|^{p-2}S'_p)' + |S_p|^{p-2}S_p = 0. \quad (2.2)$$

Moreover,

$$\pi_p \equiv 2 \int_0^{(p-1)^{1/p}} \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}} = \frac{2(p-1)^{1/p}\pi}{p \sin(\pi/p)}$$

is the first zero of S_p in the positive real axis. Similarly, one has $S_p(\frac{\pi_p}{2} + n\pi_p) = (-1)^n$, $S'_p(n\pi_p) = (-1)^n$, $S_p(n\pi_p) = 0$ and $S'_p(\frac{\pi_p}{2} + n\pi_p) = 0$ for $n \in \mathbb{Z}$. With the help of the generalized sine function, we introduce the phase-plane coordinates $\rho > 0$ and θ for the solution $u(r; \alpha)$ of (1.2)-(1.3) as follows:

$$\begin{aligned} |u(r; \alpha)|^{p-2}u(r; \alpha) &= \rho(r; \alpha)|S_p(m\theta(r; \alpha))|^{p-2}S_p(m\theta(r; \alpha)), \\ r^{n-1}|u'(r; \alpha)|^{p-2}u'(r; \alpha) &= g\rho(r; \alpha)|S'_p(m\theta(r; \alpha))|^{p-2}S'_p(m\theta(r; \alpha)), \end{aligned} \quad (2.3)$$

with

$$m\theta(0; \alpha) = \frac{\pi_p}{2} \quad \text{and} \quad \rho(0; \alpha) = \left(\frac{\alpha}{p-1}\right)^{p-1}, \quad (2.4)$$

where m and g are some positive constants that will be specified later. Then

$$(g\rho(r; \alpha))^{\frac{p}{p-1}} = \frac{g^{\frac{p}{p-1}}}{p-1}|u(r; \alpha)|^p + r^{\frac{p(n-1)}{p-1}}|u'(r; \alpha)|^p, \quad (2.5)$$

$$\frac{r^{n-1}|u'|^{p-2}u'}{|u|^{p-2}u} = \frac{g|S'_p|^{p-2}S'_p}{|S_p|^{p-2}S_p}. \quad (2.6)$$

Differentiating both sides of (2.6) with respect to r and employing (1.2) and (2.1)-(2.3), one can obtain

$$\begin{aligned} &mg\theta'(r; \alpha) \\ &= \frac{r^{n-1}}{p-1}w(r)|u(r; \alpha)|^{q-p}|S_p(m\theta(r; \alpha))|^p + g^{\frac{p}{p-1}} \left(r^{\frac{1-n}{p-1}} |S'_p(m\theta(r; \alpha))|^p \right), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \frac{\rho'(r; \alpha)}{\rho(r; \alpha)} &= \left[r^{\frac{1-n}{p-1}} g^{\frac{1}{p-1}} - r^{n-1} g^{-1} w(r) |u(r; \alpha)|^{q-p} \right] \\ &\quad \times |S_p(m\theta(r; \alpha))|^{p-2} S_p(m\theta(r; \alpha)) S'_p(m\theta(r; \alpha)). \end{aligned} \tag{2.8}$$

Employing the above, one can conclude that $u(r; \alpha)$ is the solution of (1.2)-(1.3) if and only if $\{\theta(r; \alpha), \rho(r; \alpha)\}$ satisfies (2.7)-(2.8) and (2.4).

Motivated by a similar idea in [35] (or [14]), we introduce the scaling argument. Assume that $\{\alpha_i\}$ is a positively and strictly increasing sequence which tends to infinity, and define the sequence $\{\mu_i\}$ to satisfy the following relation:

$$\mu_i = \max\{x > 0 : x^p w(x) = \alpha_i^{p-q}\} \tag{2.9}$$

for $i \in \mathbb{N}$. Note that $t^p w(t) = O(t^p)$ and $q > p$. Hence, if $\{\alpha_i\}$ is a positively increasing sequence which tends to infinity, then the corresponding sequence $\{\mu_i\}$ satisfying (2.9) decreases to zero. Then, the scaled function v_i is defined by

$$v_i(r) = \frac{u(\mu_i r; \alpha_i)}{\alpha_i}. \tag{2.10}$$

By (1.2) and (2.9)-(2.10), a direct calculation yields that v_i satisfies

$$\left(r^{n-1} |v'_i(r)|^{p-2} v'_i(r) \right)' + r^{n-1} \frac{w(\mu_i r)}{w(\mu_i)} |v_i(r)|^{q-2} v_i(r) = 0, \tag{2.11}$$

$$v_i(0) = 1, \quad v'_i(0) = 0. \tag{2.12}$$

From Theorem 2.1 and for each fixed i , the function v_i which solves (2.11)-(2.12) exists on $[0, \mu_i^{-1}]$. By the assumption on w , $\frac{w(\mu_i r)}{w(\mu_i)} \rightarrow 1$ as $\mu_i \rightarrow 0$ uniformly on any bounded interval in $[0, \infty)$. Thus v_i converges to a function V uniformly on any bounded interval in $[0, \infty)$, where V solves

$$\begin{aligned} &\left(r^{n-1} |V'|^{p-2} V' \right)' + r^{n-1} |V|^{q-2} V = 0, \\ &V(0) = 1, \quad V'(0) = 0. \end{aligned} \tag{2.13}$$

Next we define an energy functional for the scaled function $v_i(r)$ and deduce an a priori estimate for this energy. Let a functional $E[v_i](r, \alpha)$ be defined by

$$E[v_i](r, \alpha) \equiv \frac{|v'_i(r)|^p}{p} + \frac{w(\mu_i r)}{q(p-1)w(\mu_i)} |v_i(r)|^q \tag{2.14}$$

with

$$E[v_i](0, r) = \frac{w(0)}{q(p-1)w(\mu_i)}. \tag{2.15}$$

Note that from (2.11), one can obtain the following equation by multiplying $v'_i(r)$,

$$\begin{aligned} &(n-1)r^{n-2} |v'_i(r)|^p + (p-1)r^{n-1} |v'_i(r)|^{p-2} v'_i(r) v''_i(r) \\ &+ r^{n-1} \frac{w(\mu_i r)}{w(\mu_i)} |v_i(r)|^{q-2} v_i(r) v'_i(r) = 0. \end{aligned}$$

i.e., for $r \neq 0$,

$$-\frac{|v'_i(r)|^p}{r} = \frac{(p-1)}{(n-1)}|v'_i(r)|^{p-2}v''_i(r)v'_i(r) + \frac{w(\mu_i r)}{(n-1)w(\mu_i)}|v_i(r)|^{q-2}v_i(r)v'_i(r). \quad (2.16)$$

Since there exists a unique solution in $[0, \mu_i^{-1}]$, by Theorem 2.1 and (2.10), all the terms on the right-hand side of (2.16) are bounded in $[0, \mu_i^{-1}]$ and v'_i tends to zero as r vanishes by the initial condition. This implies that

$$\lim_{r \rightarrow 0^+} \frac{|v'_i(r)|^p}{r} = 0 \quad (2.17)$$

and the term $\frac{|v'_i(r)|^p}{r}$ is bounded in $[0, \mu_i^{-1}]$. Then, it follows from (2.11) and (2.16) that for $r \in (0, \mu_i^{-1}]$,

$$\begin{aligned} \frac{d}{dr}E[v_i](r, \alpha) &= E[v_i]'(r, \alpha) \\ &= v'_i(r) \left[|v'_i(r)|^{p-2}v''_i(r) + \frac{w(\mu_i r)}{(p-1)w(\mu_i)}|v_i(r)|^{q-2}v_i(r) \right] \\ &\quad + \frac{\mu_i w'(\mu_i r)}{q(p-1)w(\mu_i)}|v_i(r)|^q \\ &= -\frac{(n-1)}{(p-1)} \cdot \frac{|v'_i(r)|^p}{r} + \frac{\mu_i w'(\mu_i r)}{q(p-1)w(\mu_i)}|v_i(r)|^q \\ &\leq \mu_i \kappa \frac{|u'(r)|^p}{p} + \mu_i \kappa \frac{w(\mu_i r)}{q(p-1)w(\mu_i)}|v_i(r)|^q \\ &= \mu_i \kappa E[v_i](r, \alpha), \end{aligned} \quad (2.18)$$

where $\kappa = \max \left\{ \frac{|w'(\mu_i r)|}{w(\mu_i r)} : r \in [0, \mu_i^{-1}] \right\} = \max \left\{ \frac{|w'(r)|}{w(r)} : r \in [0, 1] \right\}$. In particular, the above inequality holds for the whole interval $[0, \mu_i^{-1}]$ by (2.17). Hence, for any $r \in [0, \mu_i^{-1}]$

$$E[v_i](r, \alpha) \leq E[v_i](0, \alpha)e^{\mu_i \kappa r} = \frac{w(0)e^{\mu_i \kappa r}}{q(p-1)w(\mu_i)} \quad (2.19)$$

by (2.18) and (2.15). This means that both $v_i(r; \alpha)$ and $v'_i(r; \alpha)$ are bounded as long as the solution exists.

Proposition 2.2. *Assume the conditions (A1) and (A2) hold. Let μ_i be defined as in (2.9). Then, v_i satisfies*

$$|v_i(r)| \leq \left(\frac{w(0)}{w(\mu_i r)} e^\kappa \right)^{1/q} \leq \left(\frac{w(0)}{\delta_1} e^\kappa \right)^{1/q} \quad (2.20)$$

for $r \in [0, \mu_i^{-1}]$, where $\kappa = \max \left\{ \frac{|w'(r)|}{w(r)} : r \in [0, 1] \right\}$. Moreover, the function V solving (2.13) satisfies the uniform boundedness,

$$|V(r)| \leq e^{\kappa/q} \quad (2.21)$$

on any bounded interval in $[0, \infty)$.

Now we prove the result related to the Sturmian theory.

Proof of Theorem 1.1. From the Prüfer angular equation (2.7), one has

$$\begin{aligned}
 &mg\theta'(r; \alpha) \\
 &= \frac{r^{n-1}}{p-1}w(r)|u(r; \alpha)|^{q-p}|S_p(m\theta(r; \alpha))|^p + g^{\frac{p}{p-1}}r^{\frac{1-n}{p-1}}|S'_p(m\theta(r; \alpha))|^p.
 \end{aligned}
 \tag{2.22}$$

Applying the scaling argument (2.10) and choosing $m = g^{\frac{1}{p-1}} = \alpha^{\frac{q-p}{p}}$, the phase equation (2.22) can be rewritten as

$$\begin{aligned}
 \theta'(r; \alpha) &= \frac{r^{n-1}}{p-1}w(r)|v(\mu^{-1}r)|^{q-p}|S_p(\alpha^{\frac{q-p}{p}}\theta(r; \alpha))|^p \\
 &\quad + r^{\frac{1-n}{p-1}}|S'_p(\alpha^{\frac{q-p}{p}}\theta(r; \alpha))|^p.
 \end{aligned}
 \tag{2.23}$$

Note that $|S_p(m\theta(r; \alpha))|^p$ and $|S'_p(m\theta(r; \alpha))|^p$ will not vanish at the same point by (2.1). And if $S_p(m\theta(r; \alpha))$ tends to zero, $|S'_p(m\theta(r; \alpha))|$ will approach to one. Furthermore, $v(\mu^{-1}r)$ and $S_p(m\theta(r; \alpha))$ vanish at the same point by (2.3) and (2.10). Integrating the phase equation (2.23) over $[0, r]$ for $r \in (0, 1]$, one can obtain

$$\begin{aligned}
 m\theta(r; \alpha) &= \frac{\pi_p}{2} + m \int_0^r \left(\frac{s^{n-1}}{p-1}w(s)|v(\mu^{-1}s)|^{q-p}|S_p(m\theta(s; \alpha))|^p \right. \\
 &\quad \left. + s^{\frac{1-n}{p-1}}|S'_p(m\theta(s; \alpha))|^p \right) ds,
 \end{aligned}
 \tag{2.24}$$

where $m = \alpha^{\frac{q-p}{p}}$. A detailed analysis similar as in [30, Lemma 3] (or [5]) shows that $m\theta(r; \alpha) - \frac{\pi_p}{2} = O(r^n)$ as $r \rightarrow 0^+$. Hence, we can observe that for any $\alpha > 0$ the integral term in (2.24) is bounded and never vanishes by the above explanation. And the Prüfer phase θ is continuous dependence on α obviously. Then, one can conclude that for $r \in (0, 1]$,

$$\lim_{\alpha \rightarrow 0} \alpha^{\frac{q-p}{p}}\theta(r; \alpha) = \frac{\pi_p}{2}, \quad \lim_{\alpha \rightarrow \infty} \alpha^{\frac{q-p}{p}}\theta(r; \alpha) = \infty.
 \tag{2.25}$$

Now by (2.4) and (2.25), there exists an increasing sequence of of positive numbers $\{\alpha_n\}_{n=1}^\infty$ such that

$$\alpha_n^{\frac{q-p}{p}}\theta(0; \alpha_n) = \frac{\pi_p}{2} \quad \text{and} \quad \alpha_n^{\frac{q-p}{p}}\theta(1; \alpha_n) = n\pi_p.$$

This means that $u(r; \alpha_n)$ is a solution of (1.2)-(1.4) which has exactly $n - 1$ zeros in $(0, 1)$. The proof is complete. \square

Next, we deal with multi-point boundary conditions.

Proof of Theorem 1.3. Recall that $m = g^{\frac{1}{p-1}} = \alpha^{\frac{q-p}{p}}$ are as in the proof of Theorem 1.1. From (2.25) and the continuity of $\theta(r; \alpha)$ in α , there exist a maximal α_n and a minimal α_{n+1} such that

$$\alpha_n^{\frac{q-p}{p}}\theta(1; \alpha_n) = \left(n + \frac{1}{2}\right)\pi_p, \quad \alpha_{n+1}^{\frac{q-p}{p}}\theta(1; \alpha_{n+1}) = \left(n + \frac{3}{2}\right)\pi_p,
 \tag{2.26}$$

$$\left(n + \frac{1}{2}\right)\pi_p < \alpha^{\frac{q-p}{p}}\theta(1; \alpha) < \left(n + \frac{3}{2}\right)\pi_p \quad \text{for } \alpha_n < \alpha < \alpha_{n+1}.
 \tag{2.27}$$

Now by (2.8),(2.10), (A3), (A4), and (2.20), for $r \in (0, 1]$ and $j = n, n + 1$ one can obtain

$$\frac{\rho'(r; \alpha_j)}{\rho(r; \alpha_j)} \geq -\alpha_j^{\frac{q-p}{p}} \left(r^{\frac{1-n}{p-1}} + r^{n-1}w(r)|v(\mu_j^{-1}r)|^{q-p} \right)$$

$$\begin{aligned} &\geq -\alpha_j^{\frac{q-p}{p}} \left(r^{\frac{1-n}{p-1}} + w_1 \left(\frac{w(0)}{\delta_1} e^\kappa \right)^{\frac{q-p}{q}} \right) \\ &= -\alpha_j^{\frac{q-p}{p}} \left(r^{\frac{1-n}{p-1}} + \mathbb{K} \right). \end{aligned}$$

Integrating the above inequality over $[r_i, 1]$ ($1 \leq i \leq d$), by (A4) one can get

$$\begin{aligned} \ln \frac{\rho(1; \alpha_j)}{\rho(r_i; \alpha_j)} &\geq -\alpha_j^{\frac{q-p}{p}} \left(\frac{p-1}{p-n} (1 - r_i^{\frac{p-n}{p-1}}) + \mathbb{K}(1 - r_i) \right) \\ &\geq -\alpha_j^{\frac{q-p}{p}} \left(\frac{p-1}{p-n} + \mathbb{K} \right) > -\alpha_j^{\frac{q-p}{p}} \mathbb{K} \end{aligned}$$

for $j = n, n+1$. Then,

$$\rho(r_i; \alpha_j) < e^{\left(\alpha_j^{\frac{q-p}{p}} \mathbb{K} \right)} \rho(1; \alpha_j), \quad i = 1, 2, 3, \dots, d \quad \text{and} \quad j = n, n+1. \quad (2.28)$$

By the Prüfer-type substitution (2.5) and (2.26), one can observe that

$$\begin{aligned} |u(r_i; \alpha_j)| &\leq \sqrt[p-1]{(p-1)^{\frac{p-1}{p}} \rho(r_i; \alpha_j)}, \\ |u(1; \alpha_j)| &= \sqrt[p-1]{(p-1)^{\frac{p-1}{p}} \rho(1; \alpha_j)}. \end{aligned} \quad (2.29)$$

for $1 \leq i \leq d$ and $j = n, n+1$. Now define

$$\Gamma(\alpha) = u(1; \alpha) - \sum_{i=1}^d \tau_i e^{-\frac{\left(\alpha^{\frac{q-p}{p}} \mathbb{K} \right)}{p-1}} u(r_i; \alpha).$$

Assume that $n = 2k - 1$ for $k \in \mathbb{N}$. Note that

$$u(1; \alpha_{2k-1}) = u(1; \alpha_n) < 0 \quad \text{and} \quad u(1; \alpha_{2k}) = u(1; \alpha_{n+1}) > 0 \quad (2.30)$$

from (2.26) and (2.3). By applying (2.28)-(2.30) and (1.6), one can obtain that

$$\begin{aligned} &\Gamma(\alpha_{2k-1}) \\ &= u(1; \alpha_{2k-1}) - \sum_{i=1}^d \tau_i e^{-\frac{\left(\alpha_{2k-1}^{\frac{q-p}{p}} \mathbb{K} \right)}{p-1}} u(r_i; \alpha_{2k-1}) \\ &\leq -\sqrt[p-1]{(p-1)^{\frac{p-1}{p}} \rho(1; \alpha_{2k-1})} + \sum_{i=1}^d |\tau_i| e^{-\frac{\left(\alpha_{2k-1}^{\frac{q-p}{p}} \mathbb{K} \right)}{p-1}} \sqrt[p-1]{(p-1)^{\frac{p-1}{p}} \rho(r_i; \alpha_{2k-1})} \\ &< -\sqrt[p-1]{(p-1)^{\frac{p-1}{p}} \rho(1; \alpha_{2k-1})} \\ &\quad + \sum_{i=1}^d |\tau_i| e^{-\frac{\left(\alpha_{2k-1}^{\frac{q-p}{p}} \mathbb{K} \right)}{p-1}} \sqrt[p-1]{(p-1)^{\frac{p-1}{p}} e^{\frac{\left(\alpha_{2k-1}^{\frac{q-p}{p}} \mathbb{K} \right)}{p-1}} \rho(1; \alpha_{2k-1})} \\ &= \sqrt[p-1]{(p-1)^{\frac{p-1}{p}} \rho(1; \alpha_{2k-1})} \left(-1 + \sum_{i=1}^d |\tau_i| \right) < 0, \end{aligned}$$

and

$$\begin{aligned} &\Gamma(\alpha_{2k}) \\ &= u(1; \alpha_{2k}) - \sum_{i=1}^d \tau_i e^{-\frac{\left(\alpha_{2k}^{\frac{q-p}{p}} \mathbb{K} \right)}{p-1}} u(r_i; \alpha_{2k}) \end{aligned}$$

$$\begin{aligned}
 &\geq {}^{p-1}\sqrt{(p-1)^{\frac{p-1}{p}} \rho(1; \alpha_{2k})} - \sum_{i=1}^d |\tau_i| e^{-\left(\frac{q-p}{\alpha_{2k}^{\frac{p}{p-1}} \mathbb{K}}\right)} {}^{p-1}\sqrt{(p-1)^{\frac{p-1}{p}} \rho(r_i; \alpha_{2k})} \\
 &> {}^{p-1}\sqrt{(p-1)^{\frac{p-1}{p}} \rho(1; \alpha_{2k})} \\
 &\quad - \sum_{i=1}^d |\tau_i| e^{-\left(\frac{q-p}{\alpha_{2k}^{\frac{p}{p-1}} \mathbb{K}}\right)} {}^{p-1}\sqrt{(p-1)^{\frac{p-1}{p}} e^{\left(\frac{q-p}{\alpha_{2k}^{\frac{p}{p-1}} \mathbb{K}}\right)} \rho(1; \alpha_{2k})} \\
 &= {}^{p-1}\sqrt{(p-1)^{\frac{p-1}{p}} \rho(1; \alpha_{2k})} \left(1 - \sum_{i=1}^d |\tau_i|\right) > 0.
 \end{aligned}$$

By the continuity of $\Gamma(\alpha)$, there exists $\bar{\alpha} \in (\alpha_{2k-1}, \alpha_{2k})$ such that $\Gamma(\bar{\alpha}) = 0$. It is similar to the case of $n = 2k$ with $k \in \mathbb{N}$. Now in both cases, from (2.27) one has

$$\left(n + \frac{1}{2}\right)\pi_p < \bar{\alpha}^{\frac{q-p}{p}} \theta(1; \bar{\alpha}) < \left(n + \frac{3}{2}\right)\pi_p.$$

Hence, the above implies that $u(r; \bar{\alpha})$ has n or $n + 1$ zeros in $(0, 1)$ and satisfies the multi-point boundary condition (1.5). The proof is complete. \square

3. BLOW-UP SOLUTIONS IN FINITE INTERVALS

In this section, we consider the negative potential ($w < 0$) and focus on the issue related to the existence of unbounded solutions in a finite interval. Motivated by the interesting idea raised in [25], we first study the one-dimensional case.

Theorem 3.1. *Let $w < 0$ and assume that (A1) and (A2) hold. Then, the one-dimensional problem $(|u'|^{p-2}u')' + w(r)|u|^{q-2}u = 0$ has at least one blow-up solution in a finite interval. That is, for $u(0) = \alpha > 0$ one positive solution will tend to infinity in the finite interval $(0, \sqrt[p]{n^{-1}}R_\alpha)$, where R_α is defined as in (1.7). For $q \leq p$, such a blow-up solution satisfying this problem can not exist in any finite interval.*

Proof. For $w < 0$, a positive unique local solution $u(r)$ in J with $u(0) = \alpha > 0$ and $u'(0) = 0$ satisfies $u' > 0$, and then

$$\int_0^r (|u'|^{p-2}u')' u' ds = - \int_0^r w(s)|u|^{q-2}uu' ds \geq \delta_1 \int_0^r |u|^{q-2}uu' ds$$

by (A2). The above implies that

$$u'^p - \int_0^r (|u'|^{p-2}u')u'' ds = \frac{p-1}{p} u'^p \geq \frac{\delta_1}{q} (u^q - \alpha^q).$$

i.e.,

$$\frac{u'}{\sqrt[p]{u^q - \alpha^q}} \geq \sqrt[p]{\frac{p\delta_1}{q(p-1)}}.$$

Then,

$$\int_\alpha^{u(r)} \frac{du}{\sqrt[p]{u^q - \alpha^q}} \geq \sqrt[p]{\frac{p\delta_1}{q(p-1)}} r.$$

Hence, if the solution becomes infinite at $r = \ell$, then

$$\begin{aligned}
 \ell &\leq \sqrt[p]{\frac{q(p-1)}{p\delta_1}} \int_{\alpha}^{\infty} \frac{du}{\sqrt[p]{u^q - \alpha^q}} = \sqrt[p]{\frac{q(p-1)}{p\delta_1\alpha^{q-p}}} \int_1^{\infty} \frac{ds}{\sqrt[p]{s^q - 1}} \\
 &= C \left(\int_1^2 + \int_2^{\infty} \right) \frac{ds}{\sqrt[p]{s^q - 1}} \\
 &< C \left(\int_1^2 \frac{ds}{\sqrt[p]{s^p - 1}} + \int_2^{\infty} \frac{ds}{\sqrt[p]{s^q - \frac{1}{2}s^q}} \right) \quad (\text{letting } s^p - 1 = t) \\
 &= C \left(\frac{1}{p} \int_0^{2^p-1} t^{\frac{-1}{p}} (1+t)^{\frac{1-p}{p}} dt + 2^{\frac{q}{p}} \int_2^{\infty} s^{\frac{-q}{p}} ds \right) \\
 &\leq C \left(\frac{1}{p} \int_0^{2^p-1} t^{\frac{-1}{p}} dt + \frac{2p}{q-p} \right) \\
 &= C \left(\frac{1}{p-1} (2^p-1)^{\frac{p-1}{p}} + \frac{2p}{q-p} \right) = \sqrt[p]{n^{-1}} R_{\alpha},
 \end{aligned} \tag{3.1}$$

where $C = \sqrt[p]{\frac{q(p-1)}{p\delta_1\alpha^{q-p}}}$ and R_{α} is defined as in (1.7). This shows that there is at least one positive blow-up solution in $(0, \sqrt[p]{n^{-1}} R_{\alpha})$.

For $q \leq p$ and a positive solution $u(r)$ with $u(0) = \beta > 0$ and $u'(0) = 0$, assume this unique local solution exists in $J = [0, a)$ and let $|w| \leq \delta_a$ in J . Then

$$\int_0^r (|u'|^{p-2} u')' u' ds = - \int_0^r w(s) |u|^{q-2} u u' ds \leq \delta_a \int_0^r |u|^{q-2} u u' ds.$$

Apply the similar argument as in the above case ($q > p$) and let the solution become infinite at $r = R(\beta)$, where

$$R(\beta) \geq \sqrt[p]{q \frac{(p-1)}{p\delta_a}} \int_{\beta}^{\infty} \frac{du}{\sqrt[p]{u^q - \beta^q}} = \tilde{C} \int_1^{\infty} \frac{ds}{\sqrt[p]{s^q - 1}} = \infty,$$

where $\tilde{C} = \sqrt[p]{\frac{q(p-1)}{p\delta_a\beta^{q-p}}}$. This shows that the blow-up solution can not occur when the problem is considered in a finite interval as $q \leq p$. \square

The following is a technical and crucial lemma whose main concept is quoted from [25, Lemma 1]. It represents some elementary properties for solutions of (1.2) and the significant relationship between (1.2) and its corresponding one-dimensional problem ($n = 1$). Here we give the details for the reader's convenience and make it coincide with our setting.

Lemma 3.2. *For $\ell > 0$, assume that $u \in C^1(0, \ell)$ with $|u'|^{p-2} u' \in C^1(0, \ell)$ is a solution of*

$$(r^{n-1} |u'|^{p-2} u')' + r^{n-1} g(u) = 0 \text{ in } (0, \ell), \tag{3.2}$$

where $g \in C(\mathbb{R})$ and u' is bounded near zero. Then $u(0) := \lim_{r \rightarrow 0} u(r)$ exists, and, with this definition,

- (i) $u'(0) = 0$, $u \in C^1[0, \ell)$ and $|u'|^{p-2} u' \in C^1[0, \ell)$;
- (ii) if the function g is negative and nonincreasing, then $u'(r) \geq 0$ for $r > 0$ and

$$(|u'|^{p-2} u')' \leq r^{-n+1} (r^{n-1} |u'|^{p-2} u')' \leq n(|u'|^{p-2} u')';$$

(iii) the function $v(r) = v(r; c; \mu) = cu(\mu r)$ ($c, \mu > 0$) satisfies

$$r^{-n+1} (r^{n-1}|v'|^{p-2}v')' + c^{p-1}\mu^p g(v/c) = 0 \quad \text{in } (0, \ell/\mu).$$

Proof. From (3.2) and the boundedness of u' , one has

$$r^{-n+1}|u'|^{p-2}u' = - \int_0^r s^{n-1}g(u(s))ds = -r^n \int_0^1 t^{n-1}g(u(rt))dt. \quad (3.3)$$

The boundedness of u' near zero implies that $\lim_{r \rightarrow 0^+} u(r) := \alpha$ exists. Letting $\alpha = u(0)$, one can obtain that $\frac{|u'|^{p-2}u'}{r}$ is bounded near zero, hence $u'(0) = 0$. The other properties of (i) are valid by (3.3). For (ii), by the assumption of g one has $u'(r) \geq 0$ from (3.3) obviously. Besides,

$$r^{-n+1} (r^{n-1}|u'|^{p-2}u')' = (n-1)\frac{|u'|^{p-2}u'}{r} + (|u'|^{p-2}u')' \geq (|u'|^{p-2}u')'. \quad (3.4)$$

Employing (3.3) and $g' \leq 0$, one can obtain that

$$\begin{aligned} (|u'|^{p-2}u')' &= - \int_0^1 t^{n-1}g(u(rt))dt - r \left[\int_0^1 t^n g'(u(rt))u'(rt)dt \right] \\ &\geq - \int_0^1 t^{n-1}g(u(rt))dt = \frac{|u'|^{p-2}u'}{r}. \end{aligned} \quad (3.5)$$

Hence, by (3.4)-(3.5)

$$\begin{aligned} (|u'|^{p-2}u')' &\leq r^{-n+1} (r^{n-1}|u'|^{p-2}u')' \\ &\leq (n-1)(|u'|^{p-2}u')' + (|u'|^{p-2}u')' = n(|u'|^{p-2}u')'. \end{aligned}$$

This completes the proof of (ii). Finally, (iii) is valid by a direct substitution. \square

The following is a version of the comparison lemma for the radial p -Laplacian and can be found as a consequence of [25, 29].

Lemma 3.3 (Comparison). *Let $0 \leq a < b$. Assume that $u, v \in C^2[a, b]$ satisfy*

$$r^{-n+1} (r^{n-1}|u'|^{p-2}u')' \leq g(r, u) \quad \text{and} \quad r^{-n+1} (r^{n-1}|v'|^{p-2}v')' \geq g(r, v)$$

in $[a, b]$, $u(a) \leq v(a)$ and $u'(a) \leq v'(a)$, where $g(r, s)$ is increasing in s . Then $u' \leq v'$ in $[a, b]$, which implies $u \leq v$ in $[a, b]$. In addition, if $u(a^+) < v(a^+)$, it follows that $u' \leq v'$ and $u < v$ in $(a, b]$.

Proof of Theorem 1.4. By Theorem 3.1 we assume that v is a positive blow-up solution of

$$(|v'(r)|^{p-2}v'(r))' = n^{-1}w(\mu^{-1}r)|v(r)|^{q-2}v(r) \quad \text{in } (0, R_v)$$

satisfying $v(0) = 1$ and $v'(0) = 0$, where $\mu > 0$ and

$$R_1 = \sqrt[p]{\frac{q(p-1)n}{p\delta_1} \left(\frac{1}{p-1}(2^p-1)^{\frac{p-1}{p}} + \frac{2p}{q-p} \right)}.$$

That is, v becomes infinite as r tends to $\ell \leq R_1$. Now let u_1 be the solution of

$$r^{-n+1} (r^{n-1}|u_1'(r)|^{p-2}u_1'(r))' = w(\mu^{-1}r)|u_1(r)|^{q-2}u_1(r), \quad u_1(0) = 1, \quad u_1'(0) = 0.$$

By Lemma 3.2 (ii), $(|u_1'(r)|^{p-2}u_1'(r))' \geq n^{-1}w(\mu^{-1}r)|u_1(r)|^{q-2}u_1(r)$. Then the comparison lemma gives $u_1 \geq v$. That is, u_1 tends to infinity as $r \rightarrow \ell_1$ with $\ell_1 \leq R_1$. Now we define $u_\alpha(r) = \alpha u_1(\mu r)$ with $\mu = \alpha^{\frac{q-p}{p}}$. Applying Lemma 3.2

(iii), one can obtain that $u_\alpha(r)$ solves (1.2)-(1.3). Hence, u_α has the asymptote $R(\alpha) \leq \alpha^{\frac{p-q}{p}} R_1 = R_\alpha$ (as in (1.7)), which implies that there is at least one positive blow-up solution in $(0, R_\alpha)$. For $q \leq p$, such a blow-up solution of (1.2)-(1.3) can not occur when the problem is considered in a finite interval by applying Theorem 3.1 and Lemma 3.2 (ii) directly. The proof is complete. \square

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