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TRAVELING WAVES WITH SINGULARITIES IN A DAMPED HYPERBOLIC MEMS TYPE EQUATION IN THE PRESENCE OF NEGATIVE POWERS NONLINEARITY

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Abstract. We consider traveling waves with singularities in a damped hyperbolic MEMS type equation in the presence of negative powers nonlinearity. We investigate how the existence of traveling waves, their shapes, and asymptotic behavior change with the presence or absence of an inertial term. These are studied by applying the framework that combines Poincaré compactification, classical dynamical systems theory, and geometric methods for the desingularization of vector fields. We report that the presence of this term causes the shapes to change significantly for sufficiently large wave speeds.

1. Introduction

In this paper, we consider the following damped hyperbolic MEMS type equation with negative powers nonlinearity,

$$\varepsilon^2 u_{tt} + u_t = u_{xx} + (1-u)^{-\alpha}, \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

where $\alpha \in 2\mathbb{N}$ and $\varepsilon > 0$. Here, ε is a small constant and the ratio of the interaction due to the inertial and damped terms (see [5, 7, 8, 9] and references therein).

Equation (1.1) is based on the equation

$$u_t = u_{xx} + (1-u)^{-\alpha}, \quad t > 0,\ x \in \mathbb{R},\ \alpha \in \mathbb{N} \tag{1.2}$$

treated in [10, 15], with the term $\varepsilon^2 u_{tt}$ added to the left-hand side. (1.1) is a type of partial differential equation commonly referred to as a damped hyperbolic equation. Since (1.1) has aspects of both parabolic and hyperbolic types, it has recently attracted attention from the viewpoint of partial differential equation theory (see [7, 6]). Guo [7] considers both parabolic and hyperbolic type problem about MEMS, and provides some quenching criteria. For MEMS, see below. In addition, it discusses the global existence of solutions. The previous work [6] is concerned with the behavior of the solutions to the nonlinear damped hyperbolic Allen-Cahn equation with appropriate boundary conditions and initial data in a bounded domain. They argue that reaction-diffusion equations have the lack of inertial and others. There are many ways to overcome these unphysical properties; one of them is to consider hyperbolic reaction-diffusion equations.

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Furthermore, (1.1) and (1.2) are special cases of the generalized MEMS type partial differential equation (see [5, 8, 12] and references therein). The MEMS model is the Micro-Electro Mechanical System devices and used in many machines around us (for instance, see [16]). In general, the MEMS model is known to induce the touchdown phenomenon (mathematically, quenching). Clarification of the structure of singularity formation, such as quench, is one of the most important issues in MEMS type equations, and there have been a lot of studies recently. However, since the nonlinear terms of MEMS equations are not simple, then they have both hyperbolic and parabolic aspects, it is not fully understood what kind of typical solutions exist.

In this article, we investigate how the behavior (shapes and asymptotic behavior) of traveling waves change depending on whether the $\varepsilon^2 u_{tt}$ term is present or absent in the left-hand side of [10]. More precisely, in the traveling wave framework, we compare the family of functions satisfying (1.1) with the family of functions satisfying (1.2) revealed in [10] in terms of the asymptotic behavior and shapes. The reason why we refer to the traveling waves as families of functions satisfying the equations is that they cause singularities at the endpoints of finite intervals despite the equations being defined over the whole domain, which makes subsequent analysis difficult (see [10]). In addition, we are interested in whether the asymptotic behavior obtained from (1.1) and that from (1.2) coincide as $\varepsilon \to 0$. Although it appears to be nothing more than adding $\varepsilon^2 u_{tt}$ to the left-hand side of (1.2), this extension allows us to obtain conclusions from the perspective of traveling waves that cannot be obtained in [10]. To the best of the author's knowledge, there has been no analysis of the existence, shapes and asymptotic behavior of traveling waves in such a type of equation with both hyperbolic and parabolic forms. We believe that this paper will provide this abundant information through a dynamical systems approach and give a new perspective on these types of equations.

To consider the traveling waves of (1.1), we introduce the change of variables

$$\phi(\xi) = 1 - u(t,x), \quad \xi = x - ct, \quad 0 < c \in \mathbb{R}.$$

Then solving (1.1) reduces to solving for $\phi(\xi)$ in

$$(1-\varepsilon^2 c^2)\phi'' = -c\phi' + \phi^{-\alpha}, \quad ('= \frac{d}{d\xi}). \tag{1.3}$$

Equation (1.3) with $\varepsilon = 0$ is discussed in [10]. In (1.3), there is a case classification for $1 - \varepsilon^2 c^2$ that did not appear in [10].

When $1 - \varepsilon^2 c^2 = 0$, i.e., $c = 1/\varepsilon$, from (1.3) we obtain the differential equation

$$0 = -c\phi' + \phi^{-\alpha}.$$

This can be solved by

$$\phi(\xi) = \Big(\frac{\alpha+1}{c}\xi + B\Big)^{\frac{1}{\alpha+1}} \tag{1.4}$$

with a constant $B \in \mathbb{R}$. In other words, we can express $\phi(\xi)$ explicitly in this case. For a discussion of this case, see Remark 2.11.

Hereinafter $1 - \varepsilon^2 c^2 \neq 0$. Then, (1.3) is equivalent to

$$\begin{aligned} \phi' &= \psi, \\ \psi' &= (1-\varepsilon^2 c^2)^{-1}(-c\psi + \phi^{-\alpha}). \end{aligned} \tag{1.5}$$

In (1.5), the dynamics to infinity in the equation with $\varepsilon = 0$ has been studied in [10, 11]. In [11], although the partial differential equations are different, the ordinary differential equations (ODEs for short) derived from them include the ODEs of [10]. As can be seen from these previous studies, (1.5) is not easy to analyze. However, as shown in [10, 11, 14, 15], it is possible to study the dynamics of this ODE to infinity in the framework that combines Poincaré compactification (for instance, see [10, Section 2] and [4, 14, 15] for the details of it), classical dynamical systems theory, and geometric methods for desingularization of vector fields (see [4, Section 3] and references therein). By using these methods, the whole dynamics on the phase space $\mathbb{R}^2$ including infinity (denoted by Poincaré disk) generated by the two-dimensional differential equation (1.5) is obtained. In other words, from these dynamics, we expect to categorize all traveling waves as in these previous studies. Furthermore, the strength of the analysis in this framework is that the existence of connecting orbits in dynamical systems including infinity not only proves the existence of these traveling waves, provides information about their shapes but allows us to study their asymptotic behavior.

This article is organized as follows. In the next section, we reproduce the terminology defined in [10] and the main results obtained, and state the main results of this paper. In Section 3, we obtain the dynamics of (1.5) on the Poincaré disk via Poincaré compactification and basic theory of the dynamical systems. The proof of Theorems will be completed in Section 4. Section 5 is devoted to the concluding remarks.

2. Known and main results

Before we state the main results of this paper, we reproduce the following definitions of quasi traveling waves and quasi traveling waves with quenching. The reason for this is that the main result in this paper will be compared later with that in [10] (see Proposition 2.3 and Theorem 2.5). Here, quenching in ODE (1.5) roughly means that the following holds

$$\phi(\xi) \to 0, \quad |\phi'(\xi)| \to +\infty, \quad \text{as } \xi \to |\xi_*|$$

with $|\xi_*| < +\infty$.

Definition 2.1 (Definition 1, [10])**.** We say that a function $u(t,x) \equiv 1 - \phi(\xi)$ is a quasi traveling wave of (1.2) if the function $\phi(\xi)$ is a solution of (1.3) with $\varepsilon = 0$ on a finite interval or semi-infinite interval.

Definition 2.2 (Definition 2, [10])**.** We say that a function $u(t,x) \equiv 1 - \phi(\xi)$ is a quasi traveling wave with quenching of (1.2) if the function $u(t,x)$ is a quasi traveling wave of (1.2) on a finite interval (resp. semi-infinite interval) such that ϕ reaches 0 and $|\phi'|$ becomes infinite at both ends of the interval (resp. finite end point of the semi-infinite interval). More precisely, we have the following three cases:

(I) The function $\phi(\xi)$ is a solution of (1.3) with $\varepsilon = 0$ on a semi-infinite interval $(-\infty, \xi_*)$ ($\phi(\xi) \in C^2(-\infty, \xi_*) \cap C^0(-\infty, \xi_*]$, $|\xi_*| < \infty$), and satisfies
$$\lim_{\xi \to \xi_* - 0} \phi(\xi) = 0 \quad \text{and} \quad \lim_{\xi \to \xi_* - 0} |\psi(\xi)| = \infty.$$

(II) The function $\phi(\xi)$ is a solution of (1.3) with $\varepsilon = 0$ on a semi-infinite interval $(\xi_*, +\infty)$ ($\phi(\xi) \in C^2(\xi_*, +\infty) \cap C^0[\xi_*, +\infty)$, $|\xi_*| < \infty$), and satisfies
$$\lim_{\xi \to \xi_* + 0} \phi(\xi) = 0 \quad \text{and} \quad \lim_{\xi \to \xi_* + 0} |\psi(\xi)| = \infty.$$

(III) The function $\phi(\xi)$ is a solution of (1.3) with $\varepsilon = 0$ on a finite interval (ξ_-, ξ_+) ($\phi(\xi) \in C^2(\xi_-, \xi_+) \cap C^0[\xi_-, \xi_+]$, $-\infty < \xi_- < \xi_+ < +\infty$), and satisfies the following

$$\lim_{\xi \to \xi_+ - 0} \phi(\xi) = 0, \quad \lim_{\xi \to \xi_- + 0} \phi(\xi) = 0,$$
$$\lim_{\xi \to \xi_+ - 0} |\psi(\xi)| = \infty, \quad \lim_{\xi \to \xi_- + 0} |\psi(\xi)| = \infty.$$

With these definitions, we review the results obtained in [10]. Note that the meaning of the symbol $F(\eta) \sim G(\eta)$ as $\eta \to +\infty$ is

$$\lim_{\eta \to +\infty} \Big| \frac{F(\eta)}{G(\eta)} \Big| = 1.$$

Proposition 2.3 (Theorem 2, [10]). *Assume that $\alpha \in 2\mathbb{N}$. Then* (1.2) *possesses a family of quasi traveling waves with quenching on a finite interval. Moreover, each quasi traveling wave with quenching $u(t,x) = 1 - \phi(\xi)$ satisfies the following:*

- $\lim_{\xi \to \xi_+ - 0} \phi(\xi) = 0$, $\lim_{\xi \to \xi_- + 0} \phi(\xi) = 0$, $\lim_{\xi \to \xi_+ - 0} \psi(\xi) = \infty$, $\lim_{\xi \to \xi_- + 0} \psi(\xi) = -\infty$.
- $\phi(\xi) < 0$ *holds for* $\xi \in (\xi_-, \xi_+)$.
- *There exists a constant $\xi_* \in (\xi_-, \xi_+)$ such that $\psi(\xi) < 0$ for $\xi \in (\xi_-, \xi_*)$, $\psi(\xi_*) = 0$, and $\psi(\xi) > 0$ for $\xi \in (\xi_*, \xi_+)$.*

In addition, the quenching rates are

$$\begin{aligned} \phi(\xi) &\sim -C(\xi_+ - \xi)^{\frac{2}{\alpha+1}} \\ \psi(\xi) &\sim C\,(\xi_+ - \xi)^{-\frac{\alpha-1}{\alpha+1}} \end{aligned} \tag{2.1}$$

as $\xi \to \xi_+ - 0$, and

$$\begin{aligned} \phi(\xi) &\sim -C(\xi - \xi_-)^{\frac{2}{\alpha+1}} \\ \psi(\xi) &\sim -C\,(\xi - \xi_-)^{-\frac{\alpha-1}{\alpha+1}} \end{aligned} \tag{2.2}$$

as $\xi \to \xi_- + 0$, with $C > 0$.

Remark 2.4. Note that the asymptotic behavior for (2.1) and (2.2) in Proposition 2.3 differs in the exponential part from the asymptotic behavior obtained in [10, Theorem 2] and [11, Proposition 1]. The reason for this is that, after the publication of [10, 11], we chose more appropriate principal terms in the computational process of deriving the asymptotic behavior, which resulted in higher accuracy. This improvement is described in detail in Subsection 4.1. Furthermore, this improvement has already been introduced into [12], and the asymptotic behavior, which was previously difficult to derive, has been obtained. However, the underlying idea is similar to the previous ones.

Next, the main results of this paper are described. Figures 1, 2, and 3 show the schematic pictures of traveling waves obtained by each theorem.

Theorem 2.5. *Assume that $\alpha \in 2\mathbb{N}$, $\varepsilon > 0$, and $1 - \varepsilon^2 c^2 > 0$. Then, for a given positive constant $0 < c < 1/\varepsilon$, there exists a family of the functions (which corresponds to a family of the orbits of* (1.5)*) defined on the finite intervals such that each function $u(t,x)$ satisfies equation* (1.1) *on a finite interval (ξ_-, ξ_+) $(-\infty < \xi_- < \xi_+ < +\infty)$. Moreover, each function $u(t,x) \equiv 1 - \phi(\xi)$ satisfies the following:*

- $\lim_{\xi \to \xi_+ - 0} \phi(\xi) = 0$, $\lim_{\xi \to \xi_- + 0} \phi(\xi) = 0$, $\lim_{\xi \to \xi_+ - 0} \psi(\xi) = \infty$, $\lim_{\xi \to \xi_- + 0} \psi(\xi) = -\infty$.

- *$\phi(\xi) < 0$ holds for $\xi \in (\xi_-, \xi_+)$.*
- *There exists a constant $\xi_* \in (\xi_-, \xi_+)$ such that the following holds: $\psi(\xi) < 0$ for $\xi \in (\xi_-, \xi_*)$, $\psi(\xi_*) = 0$ and $\psi(\xi) > 0$ for $\xi \in (\xi_*, \xi_+)$.*

In addition, the asymptotic behavior for $\xi \to \xi_+ - 0$ and $\xi \to \xi_- + 0$ are same as (2.1) *and* (2.2).

On the other hand, assume that $1-\varepsilon^2 c^2 < 0$. Then, for a given positive constant $c > 1/\varepsilon$, there exists a family of the functions (which corresponds to a family of the orbits of (1.5)*) defined on the finite intervals such that each function $u(t, x)$ satisfies equation* (1.1) *on a finite interval (ξ_-, ξ_+) $(-\infty < \xi_- < \xi_+ < +\infty)$. Moreover, each function $u(t, x) \equiv 1 - \phi(\xi)$ satisfies the following:*

- $\lim_{\xi \to \xi_+ - 0} \phi(\xi) = 0$, $\lim_{\xi \to \xi_- + 0} \phi(\xi) = 0$, $\lim_{\xi \to \xi_+ - 0} \psi(\xi) = -\infty$, $\lim_{\xi \to \xi_- + 0} \psi(\xi) = +\infty$.
- *$\phi(\xi) > 0$ holds for $\xi \in (\xi_-, \xi_+)$.*
- *There exists a constant $\xi_* \in (\xi_-, \xi_+)$ such that $\psi(\xi) > 0$ for $\xi \in (\xi_-, \xi_*)$, $\psi(\xi_*) = 0$ and $\psi(\xi) < 0$ for $\xi \in (\xi_*, \xi_+)$.*

In addition, the asymptotic behaviors are

$$\begin{aligned} \phi(\xi) &\sim C(\xi_+ - \xi)^{\frac{2}{\alpha+1}} \\ \psi(\xi) &\sim -C\,(\xi_+ - \xi)^{-\frac{\alpha-1}{\alpha+1}} \end{aligned} \tag{2.3}$$

as $\xi \to \xi_+ - 0$, and

$$\begin{aligned} \phi(\xi) &\sim C(\xi - \xi_-)^{\frac{2}{\alpha+1}} \\ \psi(\xi) &\sim C\,(\xi - \xi_-)^{-\frac{\alpha-1}{\alpha+1}} \end{aligned} \tag{2.4}$$

as $\xi \to \xi_- + 0$, with $C > 0$.

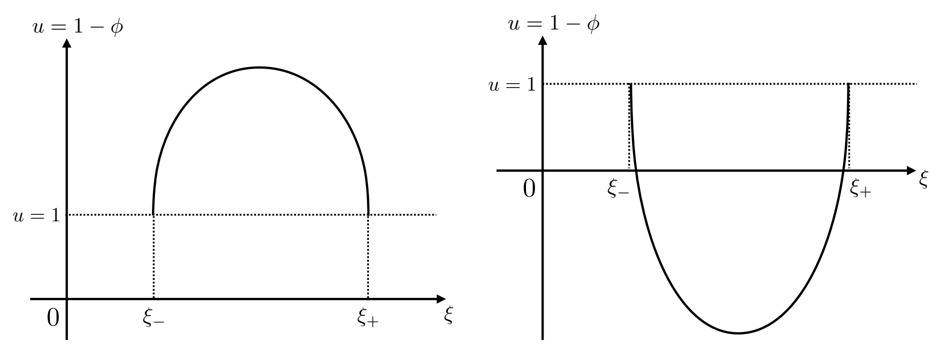

FIGURE 1. Schematic picture of the functions defined on the finite interval such that each function $u(t, x) \equiv 1-\phi(\xi)$ satisfies equation (1.1) on a finite interval (ξ_-, ξ_+) in Theorem 2.5. Here it should be noted that the position of the singular points ξ_- and ξ_+ are not determined in our studies, however, they are shown in this figure for the convenience. [Left: In the case that $1 - \varepsilon^2 c^2 > 0$.] [Right: In the case that $1 - \varepsilon^2 c^2 < 0$.] Note that in the figure on the right, the trajectory in which the minimum of u is below the ξ-axis is chosen from among the infinitely many trajectories that correspond to Theorem 2.5.

Remark 2.6. In Theorem 2.5, the result for the case $1 - \varepsilon^2 c^2 > 0$ is almost the same as in Proposition 2.3. However, since (1.1) is a hyperbolic equation and there is room for consideration in adopting Definition 2.2 as a rigorous discussion of the mathematical formulation of the solution, Theorem 2.5 is phrased as the existence of a family of functions satisfying the equation. Notable points in Theorem 2.5 are as follows:

- (i) In addition, the asymptotic behavior obtained in the above theorem is the same as Proposition 2.3, except for the difference in the sign of the coefficients. This means that the behavior of these as $\varepsilon \to 0$ does not change. This may be due to the fact that the principal part of the derived vector field (3.2) does not change.
- (ii) The most important point to be emphasized in this result is that a condition on the wave speed that is not obtained in [10] appears, and when the wave speed exceeds $c = 1/\varepsilon$, that is, when the wave speed is sufficiently large, traveling waves that are not seen in [10] are observed (see Figure 1 and [10, Figure 1]).

Theorem 2.7. *Assume that $\alpha \in 2\mathbb{N}$, $\varepsilon > 0$, and $1 - \varepsilon^2 c^2 > 0$. Then, for a given positive constant $0 < c < 1/\varepsilon$, there exists a family of the functions (which corresponds to a family of the orbits of (1.5)) defined on the semi-infinite intervals such that each function $u(t,x)$ satisfies (1.1) on a semi-infinite interval $(-\infty, \xi_+)$ $(-\infty < \xi_+ < +\infty)$. Moreover, each function $u(t,x) \equiv 1 - \phi(\xi)$ satisfies the following:*

- $\lim_{\xi \to \xi_+ - 0} \phi(\xi) = 0$, $\lim_{\xi \to -\infty} \phi(\xi) = -\infty$, $\lim_{\xi \to \xi_+ - 0} \psi(\xi) = \infty$.
- $\phi(\xi) < 0$ *holds for* $\xi \in (-\infty, \xi_+)$.

In addition, the asymptotic behavior for $\xi \to \xi_+ - 0$ and $\xi \to -\infty$ are (2.1) and
$$\phi(\xi) \sim -Ce^{-\frac{c}{1-\varepsilon^2 c^2}\xi} \quad as \xi \to -\infty \tag{2.5}$$
with $C > 0$.

On the other hand, assume that $1 - \varepsilon^2 c^2 < 0$. Then, for a given positive constant $c > 1/\varepsilon$, there exists a family of the functions (which corresponds to a family of the orbits of (1.5)) defined on the semi-infinite intervals such that each function $u(t,x)$ satisfies (1.1) on a semi-infinite interval $(\xi_-, +\infty)$ $(-\infty < \xi_- < +\infty)$. Moreover, each function $u(t,x) \equiv 1 - \phi(\xi)$ satisfies the following:

- $\lim_{\xi \to \xi_- + 0} \phi(\xi) = 0$, $\lim_{\xi \to \infty} \phi(\xi) = +\infty$, $\lim_{\xi \to \xi_- + 0} \psi(\xi) = +\infty$.
- $\phi(\xi) > 0$ *holds for* $\xi \in (\xi_-, +\infty)$.

In addition, the asymptotic behavior for $\xi \to \xi_- + 0$ and $\xi \to +\infty$ are (2.4) and
$$\phi(\xi) \sim Ce^{-\frac{c}{1-\varepsilon^2 c^2}\xi} \quad as \ \xi \to +\infty \tag{2.6}$$
with $C > 0$.

Theorem 2.8. *Assume that $\alpha \in 2\mathbb{N}$, $\varepsilon > 0$, and $1 - \varepsilon^2 c^2 > 0$. Then, for a given positive constant $0 < c < 1/\varepsilon$, there exists a family of functions (which corresponds to a family of the orbits of (1.5)) defined on the semi-infinite intervals such that each function $u(t,x)$ satisfies (1.1) on a semi-infinite interval $(-\infty, \xi_+)$ $(-\infty < \xi_+ < +\infty)$. Moreover, each function $u(t,x) \equiv 1 - \phi(\xi)$ satisfies the following:*

- $\lim_{\xi \to \xi_+ - 0} \phi(\xi) = 0$, $\lim_{\xi \to -\infty} \phi(\xi) = -\infty$, $\lim_{\xi \to \xi_+ - 0} \psi(\xi) = \infty$.
- $\phi(\xi) < 0$ *holds for* $\xi \in (-\infty, \xi_+)$.

In addition, the asymptotic behavior for $\xi \to \xi_+ - 0$ and $\xi \to -\infty$ are (2.1) and

$$\begin{aligned} \phi(\xi) &\sim O(\xi^{\frac{1}{\alpha+1}}),\\ \psi(\xi) &\sim O((-\xi)^{-\frac{\alpha}{\alpha+1}}), \end{aligned} \tag{2.7}$$

as $\xi \to -\infty$.

On the other hand, assume that $1-\varepsilon^2c^2 < 0$. Then, for a given positive constant $c > 1/\varepsilon$, there exists a family of the functions (which corresponds to a family of the orbits of (1.5)) defined on the semi-infinite intervals such that each function $u(t,x)$ satisfies equation (1.1) on a semi-infinite interval $(\xi_-, +\infty)$ $(-\infty < \xi_- < +\infty)$. Moreover, each function $u(t,x) \equiv 1-\phi(\xi)$ satisfies the following:

- $\lim_{\xi\to\xi_-+0}\phi(\xi) = 0$, $\lim_{\xi\to+\infty}\phi(\xi) = +\infty$, $\lim_{\xi\to\xi_-+0}\psi(\xi) = +\infty$.
- $\phi(\xi) > 0$ *holds for* $\xi \in (\xi_-, +\infty)$.

In addition, the asymptotic behavior for $\xi \to \xi_- + 0$ and $\xi \to +\infty$ are (2.4) and

$$\begin{aligned} \phi(\xi) &\sim O(\xi^{\frac{1}{\alpha+1}}),\\ \psi(\xi) &\sim O((-\xi)^{-\frac{\alpha}{\alpha+1}}), \end{aligned} \tag{2.8}$$

as $\xi \to +\infty$.

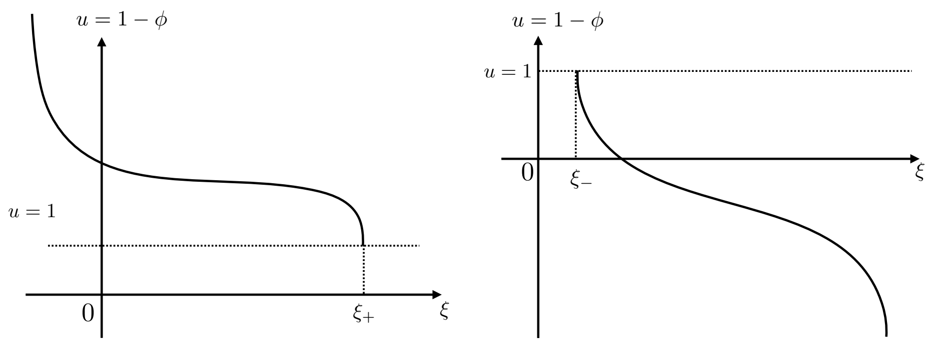

FIGURE 2. Schematic pictures of the functions defined on the semi-infinite interval such that each function $u(t,x) \equiv 1-\phi(\xi)$ satisfies equation (1.1) on a semi-infinite interval in Theorem 2.7 and Theorem 2.8. Here it should be noted that the position of the singular point ξ_+ (or ξ_-) are not determined in our studies, however, they are shown in these figures for the convenience. [Left: In the case that $1-\varepsilon^2c^2 > 0$.] [Right: In the case that $1-\varepsilon^2c^2 < 0$.]

Remark 2.9. *Note that the families of functions satisfying the equations obtained in Theorem 2.7 and Theorem 2.8 are lumped together in a rough form in Figure 2, although their asymptotic behavior is strictly different.*

Theorem 2.10. *Assume that $\alpha \in 2\mathbb{N}$, $\varepsilon > 0$, and $1-\varepsilon^2c^2 > 0$. Then, for a given positive constant $0 < c < 1/\varepsilon$, the equation (1.1) has a family of the traveling wave solutions (which corresponds to a family of the orbits of (1.5)) with singularities at $\xi \to -\infty$ and $\xi \to +\infty$. Moreover, its each function $u(t,x) \equiv 1-\phi(\xi)$ satisfies the following:*

- $\lim_{\xi\to+\infty}\phi(\xi) = +\infty$, $\lim_{\xi\to-\infty}\phi(\xi) = +\infty$.

- *$\phi(\xi) > 0$ holds for $\xi \in \mathbb{R}$.*
- *There exists a constant $\xi_* \in \mathbb{R}$ such that the following holds: $\psi(\xi) < 0$ for $\xi \in (-\infty, \xi_*)$, $\psi(\xi_*) = 0$ and $\psi(\xi) > 0$ for $\xi \in (\xi_*, +\infty)$.*

In addition, the asymptotic behavior for $\xi \to +\infty$ and $\xi \to -\infty$ are

$$\begin{aligned} \phi(\xi) &\sim O(\xi^{\frac{1}{\alpha+1}}), \\ \psi(\xi) &\sim O((-\xi)^{-\frac{\alpha}{\alpha+1}}), \end{aligned} \tag{2.9}$$

as $\xi \to +\infty$, and

$$\phi(\xi) \sim Ce^{-\frac{c}{1-\varepsilon^2c^2}\xi} \quad \text{as } \xi \to -\infty, \tag{2.10}$$

with $C > 0$.

On the other hand, assume that $1-\varepsilon^2c^2 < 0$. Then, for a given positive constant $c > 1/\varepsilon$, the equation (1.1) has a family of the traveling wave solutions (which corresponds to a family of the orbits of (1.5)) with singularities at $\xi \to -\infty$ and $\xi \to +\infty$. Moreover, its each function $u(t,x) \equiv 1 - \phi(\xi)$ satisfies the following:

- $\lim_{\xi\to+\infty} \phi(\xi) = -\infty$, $\lim_{\xi\to-\infty} \phi(\xi) = -\infty$.
- *$\phi(\xi) < 0$ holds for $\xi \in \mathbb{R}$.*
- *There exists a constant $\xi_* \in \mathbb{R}$ such that the following holds: $\psi(\xi) > 0$ for $\xi \in (-\infty, \xi_*)$, $\psi(\xi_*) = 0$ and $\psi(\xi) < 0$ for $\xi \in (\xi_*, +\infty)$.*

In addition, the asymptotic behavior for $\xi \to +\infty$ and $\xi \to -\infty$ are

$$\phi(\xi) \sim -Ce^{-\frac{c}{1-\varepsilon^2c^2}\xi} \quad \text{as } \xi \to +\infty \tag{2.11}$$

with $C > 0$, and

$$\begin{aligned} \phi(\xi) &\sim O(\xi^{\frac{1}{\alpha+1}}), \\ \psi(\xi) &\sim O((-\xi)^{-\frac{\alpha}{\alpha+1}}), \end{aligned} \tag{2.12}$$

as $\xi \to -\infty$.

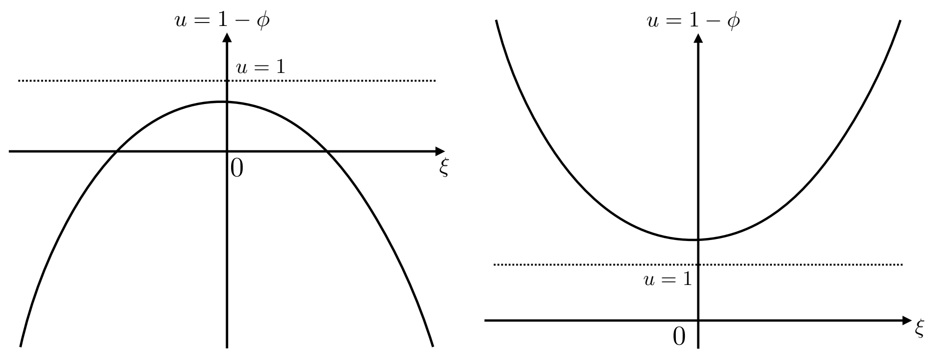

FIGURE 3. Schematic picture of the each traveling wave solutions with the singularities at $\xi \to -\infty$ and $\xi \to +\infty$ in obtained Theorem 2.10. [Left: In the case that $1 - \varepsilon^2c^2 > 0$.] [Right: In the case that $1 - \varepsilon^2c^2 < 0$.] Note that in the figure on the right, the trajectory in which the maximum of u is above the ξ-axis is chosen from among the infinitely many trajectories that correspond to Theorem 2.10.

Remark 2.11. We mentioned that when $1 - \varepsilon^2 c^2 = 0$, $\phi(\xi)$ can be expressed explicitly as in (1.4). This is the same as the number of order of $\phi(\xi)$ in (2.7), (2.8), (2.9), and (2.12). However, the detailed relationships and mathematical meanings of these are not known yet, and will be the subject of future work.

Remark 2.12. Some functions obtained in the above Theorems satisfy the equation only on finite interval or semi-infinite interval. In this paper, we do not discuss the behavior of the solutions of (1.5) after $\psi(\xi)$ becomes infinity (outside of the interval on that $\phi(\xi)$ satisfies (1.3)). It is necessary that more detailed (and hard) analysis in order to study the solutions after $\psi(\xi)$ reaches the singularity. So we leave it open here. It should be noted that equation (1.1) is invariant under translation for spatial coordinates, so many of the same waves connected together should also satisfy the equation, except at the points where the derivatives diverge. However, since our interest in this paper is to study the traveling waves of (1.5) from the viewpoint of dynamical systems, we do not discuss this paper.

3. Dynamics on the Poincaré disk of (1.5)

In this section, we study $\mathbb{R}^2 \cup \{(\phi, \psi) \mid \|(\phi, \psi)\| = +\infty\}$, i.e., the dynamics on the Poincaré disk, by the Poincaré compactification. In order to study the dynamics of (1.5) on the Poincaré disk, we desingularize it by the time-scale desingularization

$$ds/d\xi = \phi^{-\alpha} \quad \text{for } \alpha \in 2\mathbb{N}. \tag{3.1}$$

Since that α is even, the direction of the time does not change via this desingularization. Then, we have

$$\begin{aligned} \phi' &= \phi^\alpha \psi, \\ \psi' &= (1 - \varepsilon^2 c^2)^{-1}(-c\phi^\alpha \psi + 1), \end{aligned} \tag{3.2}$$

where $' = \frac{d}{ds}$, with $1 - \varepsilon^2 c^2 \neq 0$. This system (3.2) does not have equilibria.

It should be noted that the time scale desingularization (3.1) is simply multiplying the vector field by ϕ^α. Then, except the singularity $\{\phi = 0\}$, the solution curves of the system (vector field) remain the same but are parameterized differently. Still, we refer to [13, Section 7.7] and references therein for the analytical treatments of desingularization with the time rescaling. In what follows, we use similar time rescaling (re-parameterization of the solution curves) repeatedly to desingularize the vector fields.

Now we can consider the dynamics of (3.2) on the charts $\overline{U}_j$ and $\overline{V}_j$ $(j = 1, 2)$. See [4, 10, 11] and their references for definitions of these local coordinates. Note that these results described below are consistent with the process shown in [10, Theorem 2] and [11, Proposition 1], assuming $\varepsilon = 0$. For the reader's convenience, the calculation process is described here considering the case where $\varepsilon > 0$.

3.1. Dynamics on the chart $\overline{U}_2$.

To obtain the dynamics on the chart $\overline{U}_2$, we introduce coordinates (λ, x) by the formulas

$$\phi = x/\lambda, \quad \psi = 1/\lambda.$$

In this chart, it corresponds to $\phi \to 0$ and $\psi \to +\infty$ and the direction in which x is positive corresponds to the direction in which ϕ is positive. See [10, Figure 2] for a geometric image. Then, these transformations yield

$$\lambda' = (1 - \varepsilon^2 c^2)^{-1}(c\lambda^{-\alpha+1}x^\alpha - \lambda^2),$$

$$x' = \lambda^{-\alpha}x^{\alpha} + (1-\varepsilon^2c^2)^{-1}(c\lambda^{-\alpha}x^{\alpha+1} - \lambda x),$$

where $' = \frac{d}{ds}$. By using the time-scale desingularization $d\tau/ds = \lambda^{-\alpha}$, we have

$$\begin{gathered}\lambda_\tau = (1-\varepsilon^2c^2)^{-1}(c\lambda x^{\alpha} - \lambda^{\alpha+2}),\\ x_\tau = x^{\alpha} + (1-\varepsilon^2c^2)^{-1}(cx^{\alpha+1} - \lambda^{\alpha+1}x),\end{gathered} \tag{3.3}$$

where $\lambda_\tau = d\lambda/d\tau$ and $x_\tau = dx/d\tau$. System (3.3) has the equilibria

$$E_0^+ : (\lambda, x) = (0,0), \quad E_c : (\lambda, x) = (0, M_1), \quad M_1 = -(1-\varepsilon^2c^2)c^{-1}.$$

The Jacobian matrices of the vector field (3.3) at these equilibria are

$$E_0^+ : \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}, \quad E_c : \begin{pmatrix} M_2 & 0\\ 0 & M_2\end{pmatrix}, \quad M_2 = \frac{(1-\varepsilon^2c^2)^{\alpha-1}}{c^{\alpha-1}}.$$

Therefore, E_c is a source when $1-\varepsilon^2c^2 > 0$, and a sink when $1-\varepsilon^2c^2 < 0$. The equilibrium E_0^+ is not hyperbolic. Thus, to determine the dynamics near E_0^+, we desingularize it by introducing the blow-up coordinates

$$\lambda = r^{\alpha-1}\bar{\lambda}, \quad x = r^{\alpha+1}\bar{x}$$

(see [4]). Since we are interested in the dynamics on the Poincaré disk, we consider the dynamics of blow-up vector fields on the charts $\{\bar{\lambda} = 1\}$ and $\{\bar{x} = \pm 1\}$ (see also [10, 11]).

3.1.1. *Dynamics on the chart* $\{\bar{\lambda} = 1\}$. By the change of coordinates $\lambda = r^{\alpha-1}$, $x = r^{\alpha+1}\bar{x}$, we have

$$\begin{gathered} r_\tau = r(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}(cr^{\alpha(\alpha+1)}\bar{x}^{\alpha} - r^{\alpha^2-1}),\\ \bar{x}_\tau = 2(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}(r^{\alpha^2-1}\bar{x} - cr^{\alpha(\alpha+1)}\bar{x}^{\alpha+1}) + r^{\alpha^2-1}\bar{x}^{\alpha}.\end{gathered}$$

The time-rescaling $d\eta/d\tau = r^{\alpha^2-1}$ yields

$$\begin{gathered} r_\eta = (\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}(cr^{\alpha+2}\bar{x}^{\alpha} - r),\\ \bar{x}_\eta = 2(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}(\bar{x} - cr^{\alpha+1}\bar{x}^{\alpha+1}) + \bar{x}^{\alpha},\end{gathered} \tag{3.4}$$

where $r_\eta = dr/d\eta$ and $\bar{x}_\eta = d\bar{x}/d\eta$. The equilibria of (3.4) on $\{r=0\}$ are

$$\overline{E}_0^+ : (r, \bar{x}) = (0,0), \quad \overline{E}_\alpha^+ : (r, \bar{x}) = (0, M_3), \quad M_3 = [-2(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}]^{\frac{1}{\alpha-1}}.$$

Note that $M_3 < 0$ when $1-\varepsilon^2c^2 > 0$ and $M_3 > 0$ when $1-\varepsilon^2c^2 < 0$. The Jacobian matrices of the vector field (3.4) at these equilibria are

$$\overline{E}_0^+ : \begin{pmatrix} -\frac{1}{(\alpha-1)(1-\varepsilon^2c^2)} & 0\\ 0 & \frac{2}{(\alpha-1)(1-\varepsilon^2c^2)}\end{pmatrix}, \quad \overline{E}_\alpha^+ : \begin{pmatrix} -\frac{1}{(\alpha-1)(1-\varepsilon^2c^2)} & 0\\ 0 & -\frac{2}{1-\varepsilon^2c^2}\end{pmatrix}.$$

Therefore, $\overline{E}_0^+$ is a saddle, and $\overline{E}_\alpha^+$ is a sink in the case that $1-\varepsilon^2c^2 > 0$. In addition, $\overline{E}_0^+$ is a saddle, and $\overline{E}_\alpha^+$ is a source in the case that $1-\varepsilon^2c^2 < 0$.

Furthermore, since $|-(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}| < |-2(1-\varepsilon^2c^2)^{-1}|$ holds, trajectories near $\overline{E}_\alpha^+$ are tangent to $\{\bar{x} = M_3, r \geq 0\}$ as $\eta \to +\infty$. The solutions around $\overline{E}_\alpha^+$ are approximated as

$$\begin{gathered} r(\eta) = C_1 e^{-\frac{1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}(1+o(1)),\\ \bar{x}(\eta) = C_2 e^{-\frac{2}{1-\varepsilon^2c^2}\eta} + M_3(1+o(1)),\end{gathered} \tag{3.5}$$

as $\eta \to +\infty$, with constants C_1 and C_2.

3.1.2. *Dynamics on the chart $\{\bar{x} = 1\}$.* By the change of coordinates $\lambda = r^{\alpha-1}\bar{\lambda}$, $x = r^{\alpha+1}$, and time-rescaling $d\eta/d\tau = r^{\alpha^2-1}$, we have

$$\begin{aligned} r_\eta &= (\alpha+1)^{-1}r + (\alpha+1)^{-1}(1-\varepsilon^2c^2)^{-1}(cr^{\alpha+2} - r\bar{\lambda}^{\alpha+1}), \\ \bar{\lambda}_\eta &= -(\alpha-1)(\alpha+1)^{-1}\bar{\lambda} + (\alpha+1)^{-1}(1-\varepsilon^2c^2)^{-1}(2cr^{\alpha+1}\bar{\lambda} - 2\bar{\lambda}^{\alpha+2}). \end{aligned} \tag{3.6}$$

If $1-\varepsilon^2c^2 > 0$, then the equilibrium on $\{r = 0, \bar{\lambda} \geq 0\}$ is $(r, \bar{\lambda}) = (0, 0)$. The Jacobian matrix of the vector field (3.6) at this equilibrium is

$$(0,0) : \begin{pmatrix} \frac{1}{\alpha+1} & 0 \\ 0 & -\frac{\alpha-1}{\alpha+1} \end{pmatrix}.$$

Therefore, the equilibrium $(0, 0)$ is a saddle.

If $1-\varepsilon^2c^2 < 0$, system (3.6) has the equilibria on $\{r = 0, \bar{\lambda} \geq 0\}$

$$(r, \bar{\lambda}) = (0, 0), \quad (r, \bar{\lambda}) = (0, M_4), \quad M_4 = [-2^{-1}(\alpha-1)(1-\varepsilon^2c^2)]^{\frac{1}{\alpha+1}} > 0.$$

The Jacobian matrices of the vector field (3.6) at these equilibria are

$$(0,0) : \begin{pmatrix} \frac{1}{\alpha+1} & 0 \\ 0 & -\frac{\alpha-1}{\alpha+1} \end{pmatrix}, \quad (0, M_4) : \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \alpha-1 \end{pmatrix}.$$

Therefore, the equilibrium $(0, 0)$ is a saddle, and $(0, M_4)$ is a source.

3.1.3. *Dynamics on the chart $\{\bar{x} = -1\}$.* The change of coordinates $\lambda = r^{\alpha-1}\bar{\lambda}$, $x = -r^{\alpha+1}$, and time-rescaling $d\eta/d\tau = r^{\alpha^2-1}$ yield

$$\begin{aligned} r_\eta &= -(\alpha+1)^{-1}r + (\alpha+1)^{-1}(1-\varepsilon^2c^2)^{-1}(cr^{\alpha+2} - r\bar{\lambda}^{\alpha+1}), \\ \bar{\lambda}_\eta &= (\alpha-1)(\alpha+1)^{-1}\bar{\lambda} + (\alpha+1)^{-1}(1-\varepsilon^2c^2)^{-1}(2cr^{\alpha+1}\bar{\lambda} - 2\bar{\lambda}^{\alpha+2}). \end{aligned} \tag{3.7}$$

If $1-\varepsilon^2c^2 > 0$, the system (3.7) has the equilibria on $\{r = 0, \bar{\lambda} \geq 0\}$

$$(r, \bar{\lambda}) = (0, 0), \quad (r, \bar{\lambda}) = (0, M_5), \quad M_5 = [2^{-1}(\alpha-1)(1-\varepsilon^2c^2)]^{\frac{1}{\alpha+1}} > 0.$$

The Jacobian matrices of the vector field (3.7) at these equilibria are

$$(0,0) : \begin{pmatrix} -\frac{1}{\alpha+1} & 0 \\ 0 & \frac{\alpha-1}{\alpha+1} \end{pmatrix}, \quad (0, M_5) : \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\alpha-1 \end{pmatrix}.$$

Therefore, the equilibrium $(0, 0)$ is a saddle, and $(0, M_5)$ is a sink.

If $1-\varepsilon^2c^2 < 0$, then the equilibrium on $\{r = 0, \bar{\lambda} \geq 0\}$ is $(r, \bar{\lambda}) = (0, 0)$. Eigenvalues of the linearized matrix are $-(\alpha+1)^{-1}$ and $(\alpha-1)(\alpha+1)^{-1}$ with corresponding eigenvectors $(1, 0)$ and $(0, 1)$, respectively. Therefore, the equilibrium $(0, 0)$ is a saddle.

Combining the dynamics on the charts $\{\bar{\lambda} = 1\}$ and $\{\bar{x} = \pm 1\}$, we can obtain the dynamics on $\overline{U}_2$ (see Figure 4). The figure for the case $1-\varepsilon^2c^2 < 0$ can be drawn in the same way as for the case $1-\varepsilon^2c^2 > 0$.

3.2. **Dynamics on the chart $\overline{V}_2$.** In this chart, it corresponds to $\phi \to 0$ and $\psi \to -\infty$ and the direction in which x is negative corresponds to the direction in which ϕ is positive. The change of coordinates

$$\phi = -x/\lambda, \quad \psi = -1/\lambda$$

give the projected dynamics of (3.2) on the chart $\overline{V}_2$:

$$\begin{aligned} \lambda_\tau &= (1-\varepsilon^2c^2)^{-1}(c\lambda x^\alpha + \lambda^{2+\alpha}), \\ x_\tau &= x^\alpha + (1-\varepsilon^2c^2)^{-1}(cx^{\alpha+1} + \lambda^{\alpha+1}x), \end{aligned} \tag{3.8}$$

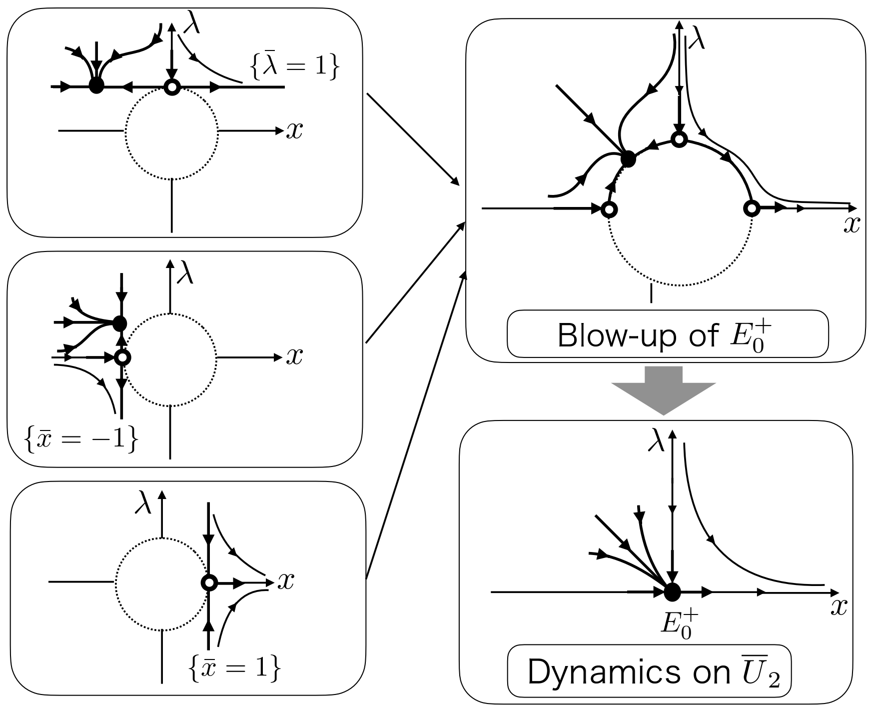

FIGURE 4. Schematic pictures of the dynamics of the blow-up vector fields and $\overline{U}_2$ in the case that $1-\varepsilon^2c^2>0$.

where τ is the new variable introduced by $d\tau/ds=\lambda(s)^{-\alpha}$. The system (3.8) has the equilibria

$$E_0^-:(\lambda,x)=(0,0),\quad E_{c'}:(\lambda,x)=(0,M_1),\quad M_1=-(1-\varepsilon^2c^2)c^{-1}.$$

The Jacobian matrices of the vector field (3.8) at these equilibria are the same as that of $\overline{U}_2$.

The system (3.8) can be transformed into (3.3) by the change of coordinates $(\lambda,x)\mapsto(-\lambda,x)$. Therefore, it is sufficient to consider the blow-up of singularity $E_0^-:(\lambda,x)=(0,0)$ by the formulas

$$\lambda=r^{\alpha-1},\quad x=r^{\alpha+1}\bar{x}\quad\text{with}\quad\bar{\lambda}=1.$$

Then, we have

$$\begin{gathered}r_\eta=(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}(cr^{\alpha+2}\bar{x}^\alpha+r),\\ \bar{x}_\eta=2(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}(-\bar{x}-cr^{\alpha+1}\bar{x}^{\alpha+1})+\bar{x}^\alpha,\end{gathered}\tag{3.9}$$

where η satisfies $d\eta/d\tau=r^{\alpha^2-1}$. The equilibria of (3.9) on $\{r=0\}$ are

$$\overline{E_0^-}:(r,\bar{x})=(0,0),\quad\overline{E_\alpha^-}:(r,\bar{x})=(0,M_6),\quad M_6=[2(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}]^{\frac{1}{\alpha-1}}.$$

Note that $M_6>0$ when $1-\varepsilon^2c^2>0$ and $M_6<0$ when $1-\varepsilon^2c^2<0$. The equilibrium $\overline{E_0^-}$ is a saddle with the eigenvalues $(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}$ and $-2(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}$ whose corresponding eigenvectors are $(1,0)$ and $(0,1)$, respectively for both $1-\varepsilon^2c^2>0$ and $1-\varepsilon^2c^2<0$.

When $1-\varepsilon^2c^2>0$ holds, $\overline{E}_\alpha^-$ is a source with the eigenvalues $(\alpha-1)^{-1}(1-\varepsilon^2c^2)^{-1}$ and $2(1-\varepsilon^2c^2)^{-1}$ whose corresponding eigenvectors are $(1,0)$ and $(0,1)$, respectively. Furthermore, $\overline{E}_\alpha^-$ is a sink in the case that $1-\varepsilon^2c^2<0$.

The solutions around $\overline{E}_\alpha^-$ are approximated as

$$r(\eta)=C_1e^{\frac{1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}(1+o(1)),$$
$$\bar{x}(\eta)=C_2e^{\frac{2}{1-\varepsilon^2c^2}\eta}+M_6(1+o(1)),$$

as $\eta\to-\infty$, with constants C_1 and C_2. This equation will be used later in Subsection 4.1 to derive the asymptotic behavior for $\xi\to\xi_-+0$.

3.3. **Dynamics on the chart $\overline{U}_1$.** Let us study the dynamics on the chart $\overline{U}_1$. In this chart, it corresponds to $\phi\to+\infty$ and $\psi\to 0$. The transformations

$$\phi=1/\lambda,\quad \psi=x/\lambda$$

yield

$$\begin{gathered}\lambda_\tau=-\lambda x,\\ x_\tau=(1-\varepsilon^2c^2)^{-1}(-cx+\lambda^{\alpha+1})-x^2,\end{gathered}\tag{3.10}$$

via time-rescaling $d\tau/ds=\lambda^{-\alpha}$. This system has the equilibria

$$e_0^+:(\lambda,x)=(0,0),\quad e_c:(\lambda,x)=(0,M_7),\quad M_7=-c(1-\varepsilon^2c^2)^{-1}.$$

Note that $M_7>0$ when $1-\varepsilon^2c^2<0$ and $M_7<0$ when $1-\varepsilon^2c^2>0$. The Jacobian matrices of the vector field (3.10) at these equilibria are

$$e_0^+:\begin{pmatrix}0&0\\0&M_7\end{pmatrix},\quad e_c:\begin{pmatrix}-M_7&0\\0&-M_7\end{pmatrix}.$$

Therefore, e_c is a source when $1-\varepsilon^2c^2>0$, and should matches $E_{c'}$. When $1-\varepsilon^2c^2<0$, e_c is sink and should matches E_c.

In a similar way to [10, 11], the dynamics near e_0^+ can be determined by the center manifold theorem (for instance, see [3] for the details of it). The approximation of the (graph of) center manifold can be obtained as follows:

$$\{(\lambda,x)\mid x=\lambda^{\alpha+1}/c+O(\lambda^{2\alpha+2})\}.$$

Further, we can see that the dynamics of (3.10) near $(0,0)$ is topologically equivalent to the dynamics of the following equation:

$$\lambda_\tau=-\lambda^{\alpha+2}/c+O(\lambda^{2\alpha+3}).$$

These results were also obtained in [10, 11]. However, we reproduce them since they will be used in the proof of Theorem 2.10 later.

3.4. **Dynamics on the chart $\overline{V}_1$.** In this chart, it corresponds to $\phi\to-\infty$ and $\psi\to 0$. The transformations

$$\phi=-1/\lambda,\quad \psi=-x/\lambda$$

yield

$$\begin{gathered}\lambda_\tau=-\lambda x,\\ x_\tau=(1-\varepsilon^2c^2)^{-1}(-cx-\lambda^{\alpha+1})-x^2\end{gathered}\tag{3.11}$$

via time-rescaling $d\tau/ds=\lambda^{-\alpha}$. We can see that the system (3.11) can be transformed into the system (3.10) by the change of variables: $(\lambda,x)\mapsto(-\lambda,x)$. Thus, with the exception of $\{\lambda=0\}$, the dynamics of (3.11) is immediately obvious from the dynamics of (3.10), however, we summarize the results for the derivation of the asymptotic behavior (as it is necessary for the proof of Theorem 2.8).

This system has the equilibria

$$e_0^- : (\lambda, x) = (0, 0), \quad e_{c'} : (\lambda, x) = (0, M_7), \quad M_7 = -c(1 - \varepsilon^2 c^2)^{-1}.$$

The Jacobian matrices of the vector field (3.11) at these equilibria are

$$e_0^- : \begin{pmatrix} 0 & 0 \\ 0 & M_7 \end{pmatrix}, \quad e_{c'} : \begin{pmatrix} -M_7 & 0 \\ 0 & -M_7 \end{pmatrix}.$$

Therefore, $e_{c'}$ is a source when $1 - \varepsilon^2 c^2 > 0$, and should matches E_c. When $1 - \varepsilon^2 c^2 < 0$, $e_{c'}$ is sink and should matches $E_{c'}$. The dynamics near e_0^- can be determined by the center manifold theorem. In the same way as above, the approximation of the (graph of) center manifold can be obtained as

$$\{(\lambda, x) \mid x = -\lambda^{\alpha+1}/c + O(\lambda^{2\alpha+2})\}. \tag{3.12}$$

Further, we can see that the dynamics of (3.11) near $(0, 0)$ is topologically equivalent to the dynamics of the equation

$$\lambda_\tau = \lambda^{\alpha+2}/c + O(\lambda^{2\alpha+3}). \tag{3.13}$$

3.5. **Dynamics and connecting orbits on the Poincaré disk.** Combining the dynamics on the charts $\overline{U}_j$ and $\overline{V}_j$ ($j = 1, 2$), we obtain the dynamics on the Poincaré disk that is equivalent to the dynamics of (1.5) (or (3.2)) in the case that α is even as the following Proposition (see also Figure 5). We set the phase space Φ as follows:

$$\Phi = \{(\phi, \psi) : (\phi, \psi) \in \mathbb{R}^2 \cup \{\|(\phi, \psi)\| = +\infty\}\}.$$

Note that in Figure 5, the circumference corresponds to $\{\|(\phi, \psi)\| = +\infty\}$.

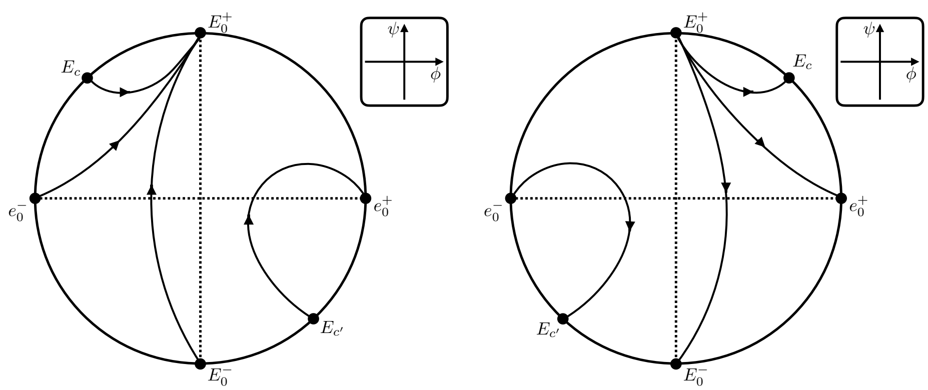

FIGURE 5. Schematic pictures of the dynamics on the Poincaré disk for (1.5) in the case that $\alpha \in 2\mathbb{N}$ and $\varepsilon > 0$. [Left: Case $1 - \varepsilon^2 c^2 > 0$.] [Right: Case $1 - \varepsilon^2 c^2 < 0$.] See also Fig. 4 for the dynamics around E_0^+ for $1 - \varepsilon^2 c^2 > 0$.

Proposition 3.1. *Assume that $\alpha \in 2\mathbb{N}$ and $\varepsilon > 0$. Then, the dynamics on the Poincaré disk of the system* (1.5) *is expressed as Figure 5 in both cases $1 - \varepsilon^2 c^2 > 0$ and $1 - \varepsilon^2 c^2 < 0$.*

Proof. First, the dynamics on the Poincaré disk for the case $1-\varepsilon^2c^2>0$ is immediately shown by the results of [10, 11]. In exactly the same way as [10, 11], it is proved that there exists connecting orbits between E_0^- and E_0^+. Therefore, we can conclude the existence of orbits that connect e_0^- and E_0^+, E_c and E_0^+, and $E_{c'}$ and e_0^+.

Next, we prove it for the case where $1-\varepsilon^2c^2<0$. In (1.5), the transformation of reversing the positive and negative values of $1-\varepsilon^2c^2$ is equivalent to applying the following transformation:

$$\phi\mapsto-\phi,\quad \psi\mapsto\psi,\quad \xi\mapsto-\xi. \tag{3.14}$$

In fact, (1.5) becomes

$$\begin{gathered}\phi'=\psi,\\ \psi'=-(1-\varepsilon^2c^2)^{-1}(-c\psi+\phi^{-\alpha})\end{gathered}$$

by the transformation in (3.14). This equation corresponds to the reversal of the sign of $1-\varepsilon^2c^2$ in (1.5). Thus, the dynamics on the Poincaré disk for $1-\varepsilon^2c^2<0$ is a symmetry transformation of (3.14) over that for $1-\varepsilon^2c^2>0$. This completes the proof. □

4. Proofs of main results

In this section, we prove the main theorems. If the initial data are located on $\Phi\backslash\{\phi=0\}$, the existence of the solutions follows from the standard theory for the ordinary differential equations. Therefore, we consider the existence of the trajectories that connect equilibria and the detailed dynamics near the equilibria on the Poincaré disk and their asymptotic behavior. The table 1 shows the correspondence between each connecting orbit obtained by the proposition and the traveling wave described in the theorem proved below.

Table 1. The correspondence between each connecting orbit obtained by the proposition and the traveling wave described in the theorem proved below.

Theorem	Connecting orbits
Theorem 2.5	between E_0^- and E_0^+
Theorem 2.7	between E_c and E_0^+
Theorem 2.8	between e_0^- and E_0^+ for $1-\varepsilon^2c^2>0$
	between E_0^+ and e_0^+ for $1-\varepsilon^2c^2<0$.
Theorem 2.10	between $E_c^{'}$ and e_0^+ for $1-\varepsilon^2c^2>0$
	between e_0^- and $E_c^{'}$ for $1-\varepsilon^2c^2<0$.

4.1. **Proof of Theorem 2.5.**

Proof. The proof of existence of the connecting orbits between E_0^- and E_0^+ in both cases $1-\varepsilon^2c^2>0$ and $1-\varepsilon^2c^2<0$ is obtained in [10, 11], and Proposition 3.1. Therefore, there exists a family of the functions which corresponds to a family of the orbits of (1.5).

Next, we prove the existence of a constant $\xi_* \in (\xi_-, \xi_+)$. It is sufficient to show the connecting orbits pass through the line $\{\psi = 0\}$. This is evident from the existence of connecting orbits in both cases $1-\varepsilon^2c^2 > 0$ and $1-\varepsilon^2c^2 < 0$. Furthermore, this means that we are giving information about the increase or decrease of ψ.

Finally, we compute the asymptotic behavior of the trajectories near the equilibria E_0^- and E_0^+ as follows. This derivation is a refinement of the discussion in [10, 11]. Note that the basic idea is the same as the previous ones. However, the detailed principal part is chosen as carefully as in [12] (see Remark 2.4).

Assume that $1-\varepsilon^2c^2 > 0$. Using (3.5), we then have

$$\begin{aligned}
\frac{d\eta}{d\xi} &= \frac{ds}{d\xi}\frac{d\tau}{ds}\frac{d\eta}{d\tau} = \phi^{-\alpha}\lambda^{-\alpha}r^{\alpha^2-1} \\
&= r^{-\alpha-1}\bar{x}^{-\alpha}\big\{C_1 e^{-\frac{1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}(1+o(1))\big\}^{-\alpha-1} \\
&\quad\times\big\{C_2 e^{-\frac{2}{1-\varepsilon^2c^2}\eta}(1+o(1)) + M_3\big\}^{-\alpha} \\
&\sim C_3 e^{\frac{\alpha+1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}\big\{C_2 e^{-\frac{2}{1-\varepsilon^2c^2}\eta}(1+o(1)) + M_3\big\}^{-\alpha} \\
&= \frac{C_3 e^{\frac{\alpha+1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}}{\big\{C_2 e^{-\frac{2}{1-\varepsilon^2c^2}\eta}(1+o(1))\big\}^{\alpha} + \alpha\big\{C_2 e^{-\frac{2}{1-\varepsilon^2c^2}\eta}(1+o(1))\big\}^{\alpha-1}M_3 + \dots + (M_3)^\alpha} \\
&\sim C_4 e^{\frac{\alpha+1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta} \quad \text{as } \eta \to +\infty
\end{aligned}$$

with constants C_j. As a note, we emphasize that the last part "$\sim$" corresponds to an improvement from [10, 11] (see Remark 2.4).

From this result, we can obtain

$$\frac{d\xi}{d\eta} = C_5 e^{-\frac{\alpha+1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}(1+o(1)) \quad \text{as } \eta \to +\infty.$$

This yields

$$\xi(\eta) \sim C_6 e^{-\frac{\alpha+1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta} + C_7, \quad C_7 \in \mathbb{R}.$$

Setting $\xi_+ := \lim_{\eta\to+\infty}\xi(\eta)$, we have

$$\xi_+ = \int_0^{+\infty}\frac{d\xi}{d\eta}d\eta = C_5\int_0^{+\infty} e^{-\frac{\alpha+1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}d\eta < +\infty.$$

Therefore,

$$\xi_+ - \xi \sim Ce^{-\frac{\alpha+1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta} \quad \text{as } \eta \to +\infty\,.$$

Finally, we obtain

$$\begin{aligned}
\phi(\xi) &= \frac{x}{\lambda} = \frac{r^{\alpha+1}\bar{x}}{r^{\alpha-1}} = r^2\bar{x} \\
&= \big\{C_1 e^{-\frac{1}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}(1+o(1))\big\}^2\big\{C_2 e^{-\frac{2}{1-\varepsilon^2c^2}\eta}(1+o(1)) + M_3\big\} \\
&\sim C_8 e^{-\frac{2}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}\big\{C_2 e^{-\frac{2}{1-\varepsilon^2c^2}\eta}(1+o(1)) + M_3\big\} \\
&= C_9 e^{-\frac{2\alpha}{(\alpha-1)(1-\varepsilon^2c^2)}\eta} + C_8\cdot M_3 e^{-\frac{2}{(\alpha-1)(1-\varepsilon^2c^2)}\eta} \\
&\sim -Ce^{-\frac{2}{(\alpha-1)(1-\varepsilon^2c^2)}\eta} \quad \text{as } \eta \to +\infty.
\end{aligned}$$

Note that the process here is also different from the process in [10, 11], as we have chosen a more appropriate principal term. Here, in the last relation, since

$$e^{-\frac{2\alpha}{(\alpha-1)(1-\varepsilon^2c^2)}\eta} < e^{-\frac{2}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}$$

is satisfied by $\eta > 0$, we choose the term $e^{-\frac{2}{(\alpha-1)(1-\varepsilon^2c^2)}\eta}$ as $\eta \to +\infty$.

From the above results, we obtain

$$\phi(\xi) \sim -Ce^{-\frac{2}{(\alpha-1)(1-\varepsilon^2c^2)}\eta} \sim -C(\xi_+ - \xi)^{\frac{2}{\alpha+1}} \quad \text{as } \xi \to \xi_+ - 0.$$

Since the trajectories are lying on $\{\phi < 0\}$, it holds that $C > 0$. Similarly, the asymptotic behavior of $\psi(\xi)$ as $\xi \to \xi_+ - 0$ for $1-\varepsilon^2c^2 > 0$ is also derived. Therefore, we can derive (2.1) and (2.2). Furthermore, (2.3) and (2.4) for $1 - \varepsilon^2c^2 < 0$ are derived in exactly the same way. This completes the proof. □

Remark 4.1. Rewriting the process of deriving the asymptotic behavior in the proof above, we can see that

$$\phi'(\xi) \sim \psi(\xi) \quad \text{as } \xi \to \xi_+ - 0\,.$$

This implies that the first equation in (1.5) also holds in the sense of asymptotic behavior. Since this relation does not hold in the results for [10, Theorem 2] and [11, Proposition 1], we believe that this improvement may have improved the accuracy of the asymptotic behavior.

4.2. **Proof of Theorem 2.7.**

Proof. The proof of existence of the connecting orbits between E_c and E_0^+ in both cases $1 - \varepsilon^2c^2 > 0$ and $1 - \varepsilon^2c^2 < 0$ is obtained in Proposition 3.1. That is, in the same way as in the proof of Theorem 2.5, a family of the functions which corresponds to a family of the orbits of (1.5) is shown.

Assume that $1 - \varepsilon^2c^2 > 0$. In this case, all that remains to be shown is to derive (2.5). The solutions at the around $e_{c'}$ on the chart $\overline{V}_1$ (matches E_c) have the form

$$\begin{gathered}\lambda_\tau \sim C_1e^{-M_7\tau}(1 + o(1)),\\ x_\tau \sim C_2e^{-M_7\tau}(1 + o(1)) + M_7,\\ M_7 = -\frac{c}{1 - \varepsilon^2c^2},\end{gathered}$$

where C_1 and C_2 are constants. Then

$$\frac{d\tau}{d\xi} = \frac{d\tau}{ds}\frac{ds}{d\xi} = \lambda^{-\alpha}\phi^{-\alpha} = 1\,.$$

This yields

$$\xi(\tau) = \tau + C_3, \quad (C_3 \in \mathbb{R}).$$

We can see that $\xi \to -\infty$ as $\tau \to -\infty$. This relationship shows that

$$\tau(\xi) = \xi + C_4, \quad (C_3 \in \mathbb{R})\,.$$

Therefore,

$$\begin{aligned}\phi(\xi) &= -\frac{1}{\lambda}\\ &\sim -\Big\{ C_1e^{\frac{c}{1-\varepsilon^2c^2}\tau}(1 + o(1)) \Big\}^{-1}\\ &\sim -C_5e^{-\frac{c}{1-\varepsilon^2c^2}\tau}\end{aligned}$$

$$= -Ce^{-\frac{c}{1-\varepsilon^2c^2}\xi} \quad \text{as } \xi \to -\infty$$

with constants $C_5 > 0$ and $C > 0$. The reason why $C > 0$ ($C_5 > 0$) is the trajectories are lying on $\{\phi < 0\}$. Therefore, (2.5) can be derived.

Furthermore, (2.6) for $1 - \varepsilon^2c^2 < 0$ is derived in exactly the same way. This completes the proof. □

4.3. **Proof of Theorem 2.8.**

Proof. Assume that $1-\varepsilon^2c^2 > 0$. The existence of the orbits connecting e_0^- and E_0^+ is as described in Proposition 3.1 above. Note that the same is true for $1 - \varepsilon^2c^2 < 0$. That is, the existence of the orbits connecting E_0^+ and e_0^+ is as described in Proposition 3.1 above. As in the previous proofs of the Theorems, this implies the existence of a family of the functions which corresponds to a family of the orbits of (1.5).

In this case, all that remains to be shown is to derive (2.7). The proof is almost identical to the proof of Theorem 3 in [11]. However, there are some symbols and parts that are different. We briefly reproduce the proof and describe it below for the reader's convenience.

If the initial value is on the center manifold, the solution at around e_0^- on the chart $\overline{V}_1$ has the form

$$\lambda(\tau) = \sqrt[\alpha+1]{\frac{1}{-\frac{\alpha+1}{c}\tau - (\alpha+1)\cdot A_0}} = O(\tau^{-\frac{1}{\alpha+1}}),$$
$$x(\tau) = \frac{1}{(\alpha+1)\tau + c(\alpha+1)A_0} + O(\lambda^{2\alpha+2}) = O(\tau^{-1})$$

as $\tau \to -\infty$, with a constant A_0. These results are derived (3.12) and (3.13). We then have

$$\frac{d\tau}{d\xi} = \frac{d\tau}{ds}\frac{ds}{d\xi} = \lambda^{-\alpha}\phi^{-\alpha} = 1.$$

This yields $\tau(\xi) = \xi + \tilde{C}$ with a constant $\tilde{C}$.

If $\tilde{\phi}(\xi)$ is a solution of (1.3) (or (1.5)), then $\tilde{\phi}(\xi + \theta)$ is also solution for any $\theta \in \mathbb{R}$. Therefore, there exists a solution $\phi(\xi)$ such that the following holds:

$$\phi(\xi) = -\lambda^{-1} \sim O(\tau^{\frac{1}{\alpha+1}}) \sim O(\xi^{\frac{1}{\alpha+1}}) \quad \text{as } \xi \to -\infty.$$

In addition, we can obtain

$$\psi(\xi) = -x\lambda^{-1} \sim -O(\xi^{\frac{1}{\alpha+1}}) \cdot O(\xi^{-1}) = O(\xi^{-\frac{\alpha}{\alpha+1}}) \quad \text{as } \xi \to -\infty.$$

Therefore, (2.7) can be derived.

Furthermore, (2.8) for $1 - \varepsilon^2c^2 < 0$ are derived in exactly the same way. This completes the proof. □

4.4. **Proof of Theorem 2.10.**

Proof. The existence of the orbits connecting $E_{c'}$ and e_0^+ (resp. e_0^-) in the case that $1-\varepsilon^2c^2 > 0$ (resp. $1-\varepsilon^2c^2 < 0$) is as described in Proposition 3.1. By focusing on $E_{c'}$, (2.10) and (2.11) can be proved in the same way as Theorem 2.7. Furthermore, we assume that $1 - \varepsilon^2c^2 > 0$. By focusing on e_0^+, (2.9) can be proved in the same way as Theorem 2.8. Similarly, by focusing on e_0^- when $1 - \varepsilon^2c^2 < 0$, we obtain (2.12). This completes the proof. □

5. Concluding remarks

The general MEMS type equation is a combination of hyperbolic and parabolic equations as shown in (1.1). In addition, the reaction-diffusion equation with $\varepsilon = 0$ is often considered for convenience of analysis and comparison. In this paper, we studied the existence, information about the shapes, and the asymptotic behavior of traveling waves with the singularity of equation (1.1) by adding $\varepsilon^2 u_{tt}$ to the left-hand side of the equation treated in [10]. Furthermore, by reviewing the process of deriving the asymptotic behavior obtained in [10, Theorem 2] and [11, Proposition 1], and by carefully selecting the principal terms, we were able to obtain a better asymptotic behavior than these results (see Remark 4.1). Even if we add $\varepsilon^2 u_{tt}$, the asymptotic behavior obtained by improving the derivation process does not change, and the condition for the wave speed with respect to the shape, which did not appear in [10], is obtained. In other words, the existence of this term and its coefficients have a significant effect on the wave speed and the shapes of the traveling waves. These are studied by applying the framework that combines Poincaré compactification, classical dynamical systems theory, and geometric methods for the desingularization of vector fields.

Since the addition of this term changes the type of the equation from parabolic to hyperbolic, a rigorous discussion of the mathematical formulation of the solution is necessary. As previously mentioned, since the emphasis of this paper is on discussing how the behavior of traveling waves changes from the viewpoint of dynamical systems, we do not discuss it here and leave it for future work.

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References

[1] M. J. Álvarez, A. Ferragut, X. Jarque; A survey on the blow up technique, *Int. J. Bifurc. Chaos*, **21** (2011), 3108–3118.

[2] M. Brunella; Topological equivalence of a plane vector field with its principal part defined through Newton polyhedra, *J. Differential Equations*, **85** (1990), 338–366.

[3] J. Carr; *Applications of Center Manifold Theory*, Springer, Berlin, 1981.

[4] F. Dumortier, J. Llibre, C. J. Artés; *Qualitative Theory of Planar Differential Systems*, Springer-Verlag, Berlin Heidelberg, 2006.

[5] G. K. Duong, H. Zaag; Profile of a tuchdown solution to a nonlocal MEMS model, *Math. Models Appl. Sci.*, **29** (2019), 1279–1348.

[6] R. Folino, C. Lattanzio, C. Mascia; Motion of interfaces for a damped hyperbolic Allen-Cahn equation, *Commun. Pure Appl. Anal.*, **19**, No. 9 (2020), 4507–4543.

[7] J. S. Guo; Recent developments on a nonlocal problem arising in the micro-electro mechanical system, *Tamkang J. Math.*, **45**, No. 3 (2014), 229–241.

[8] J. S. Guo, B. C. Huang; Hyperbolic quenching problem with damping in the micro-electro mechanical system device, *Discrete Contin. Dyn. System. Ser. B*, **19** (2014), 419–434.

[9] J. S. Guo, P. Souplet; No touchdown at zero points of the permittivity profile for the MEMS problem, *SIAM J. Math. Anal.*, **47**, No. 1 (2015), 614–625.

[10] Y. Ichida, T. O. Sakamoto; Quasi traveling waves with quenching in a reaction-diffusion equation in the presence of negative powers nonlinearity, *Proc. Japan Acad. Ser. A*, **96** (2020), 1–6.

[11] Y. Ichida, T. O. Sakamoto; Radial symmetric stationary solutions for a MEMS type reaction-diffusion equation with spatially dependent nonlinearity, *Jpn. J. Ind. Appl. Math.*, **38** (2021), 297–322.

[12] Y. Ichida, T. O. Sakamoto; Stationary solutions for a 1D pde problem with gradient term and negative powers nonlinearity, *J. Elliptic Parabol. Equ.*, 8 (2022), 885–918.
[13] C. Kuehn; *Multiple Time Scale Dynamics*, Springer International Publishing, 2015.
[14] K. Matsue; On blow-up solutions of differential equations with Poincaré-type compactificaions, *SIAM J. Appl. Dyn. Syst.*, **17** (2018), 2249–2288.
[15] K. Matsue; Geometric treatments and a common mechanism in finite-time singularities for autonomous ODEs, *J. Differential Equations*, **267** (2019), 7313–7368.
[16] Q. Wang, Estimates for the quenching time of a MEMS equation with fringing field, *J. Math. Anal. Appl.*, **405** (2013), 135–147.

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