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LIE SYMMETRY ANALYSIS AND CONSERVATION LAWS FOR THE (2+1)-DIMENSIONAL MIKHALËV EQUATION

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Abstract. Lie symmetry analysis is applied to the (2+1)-dimensional Mikhalëv equation, which can be reduced to several (1+1)-dimensional partial differential equations with constant coefficients or variable coefficients. Then we construct exact explicit solutions for part of the above (1+1)-dimensional partial differential equations. Finally, the conservation laws for the (2+1)-dimensional Mikhalëv equation are constructed by means of Ibragimov's method.

1. Introduction

Searching for solutions to partial differential equations (PDEs), which arise from physics, chemistry, economics and other fields, is one of the most fundamental and significant areas. A wealth of solving methods have been developed, such as the Lie symmetry analysis [5, 8, 11, 15], the homogeneous balance method [13, 18], Hirota's bilinear method [10], the Painlev's analysis method [6]. The Lie symmetry analysis is one of the most effective tools for solving partial differential equations and it was firstly traced back to the famous Norwegian mathematician Sophus Lie [12], who was influenced and inspired by the Galois theory founded in the early 18th century. Bluman and Cole proposed similarity theory for differential equations in 1970s [**?**]. Subsequently, the scope of application and theoretical depth of Lie symmetry analysis have been expanded. The (2+1)-dimensional Mikhalëv equation reads [14]

$$u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx} = 0, \tag{1.1}$$

which was first derived by Mikhalëv in 1992. He described a relationship between Poisson-Lie-Berezin-Kirillov brackets and the Mikhalëv system

$$u_y = v_x, \quad v_y + u_t + uv_x - vu_x = 0. \tag{1.2}$$

Pavlov adopts the method of extended Hodograph method to study integrability of exceptional hydrodynamic type systems. The corresponding particular solution of Mikhalëv system [16] is constructed under the condition of three-component case. By constructing new integrable hydrodynamic chains, he describes and integrates all their fluid dynamics, and then extracts new (2+1) integrable hydrodynamic systems from them [17]. Derchyi Wu discussed Cauchy problem of Pavlov's equation and solve the equation by using the backscattering method [19]. Grinevich and

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Santini investigated nonlocality and the inverse scattering transformation for the Mikhalëv equation [9]. Dunajski [7] presented a twistor description of (1.2) and demonstrated that the solutions of (1.2) could be used to construct Lorentzian Einstein-Weyl structures in three dimensions. In this paper, we apply Lie symmetry analysis to the (2+1)-dimensional Mikhalëv equation to present its exactly explicit solutions and construct its conservation laws. The concept of conservation laws is important in nonlinear science. The famous Noether's theorem [1] provides a systematic and effective way of determining conservation laws for Euler-Lagrange differential equations once their Noether symmetries are known. Later, researchers made various generalizations of Noether's theorem. Among these extended methods, the new conservation theorem, also called nonlocal conservation theorem, introduced by Ibragimov, is one of the most frequently used approaches. In this paper we will apply the Ibragimov's method to construct conservation laws for the (2+1)-dimensional Mikhalëv equation.

The paper is organized as follows. In Section 2, we will apply Lie symmetry analysis to the (2+1)-dimensional Mikhalëv equation. In Section 3, we will study some exact explicit solutions for the (2+1)-dimensional Mikhalëv equation based on the similarity reductions. In Section 4, the conservation laws for the (2+1)-dimensional Mikhalëv equation will be established by using Ibragimov's method. In Section 5, we will give some conclusions and discussions.

2. Lie symmetry analysis for the (2+1)-dimensional Mikhalëv equation

First of all, let us consider an one-parameter group of infinitesimal transformation,

$$\begin{aligned} x &\to x + \varepsilon\xi(x,y,t,u) + O(\varepsilon^2),\\ t &\to t + \varepsilon\tau(x,y,t,u) + O(\varepsilon^2),\\ y &\to y + \varepsilon\eta(x,y,t,u) + O(\varepsilon^2),\\ u &\to u + \varepsilon\phi(x,y,t,u) + O(\varepsilon^2), \end{aligned} \tag{2.1}$$

where $\varepsilon \ll 1$ is a group parameter. The vector field associated with the above group of transformation (2.1) is presented

$$V = \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \phi(x,y,t,u)\frac{\partial}{\partial u}. \tag{2.2}$$

Thus, the second prolongation $\mathrm{pr}^{(2)} V$ is

$$\overset{(2)}{\mathrm{Pr}}\, V = V + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} + \phi^{yy}\frac{\partial}{\partial u_{yy}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \phi^{xx}\frac{\partial}{\partial u_{xx}}, \tag{2.3}$$

where

$$\begin{aligned} \phi^y &= D_y(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xy} + \eta u_{yy} + \tau u_{ty},\\ \phi^x &= D_x(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xx} + \eta u_{yx} + \tau u_{tx},\\ \phi^{yy} &= D_y^2(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xyy} + \eta u_{yyy} + \tau u_{tyy},\\ \phi^{xy} &= D_yD_x(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxy} + \eta u_{xyy} + \tau u_{xty},\\ \phi^{xx} &= D_x^2(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxx} + \eta u_{xxy} + \tau u_{xxt},\\ \phi^{xt} &= D_tD_x(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxt} + \eta u_{xyt} + \tau u_{xtt}, \end{aligned} \tag{2.4}$$

and the operators D_x, D_y, D_t are the total derivatives with respect to x, y, t respectively. The determining equation of (1.1) arises from the invariance condition

$$\operatorname{pr}^{(2)} V\big|_{\Delta=0} = 0, \tag{2.5}$$

where $\Delta = u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx} = 0$. Furthermore, we have

$$\phi^{yy} + \phi^{xt} + \phi^{x} u_{xy} + \phi^{xy} u_x - \phi^{y} u_{xx} - \phi^{xx} u_y = 0, \tag{2.6}$$

where the coefficient functions ϕ^y, ϕ^x, ϕ^{yy}, ϕ^{xy}, ϕ^{xx} and ϕ^{xt} are determined in (2.4). Then, the forms of the coefficient functions by calculating the standard symmetry group are obtained

$$\begin{aligned}
&\xi = (F_{1t}(t) + 2c_1)x - \frac{1}{2}F_{1tt}(t)y^2 + \frac{1}{2}(-2F_{2t}(t) + c_2)y - F_3(t) + c_3,\\
&\eta = (F_{1t}(t) + c_1)y + F_2(t),\\
&\tau = F_1(t),\\
&\phi = (F_{1t}(t) + 3c_1)u - (F_{1tt}(t)y - c_2 + F_{2t}(t))x + \frac{1}{6}F_{1ttt}(t)y^3 + \frac{1}{2}F_{2tt}(t)y^2\\
&\qquad + F_{3t}(t)y + F_4(t),
\end{aligned} \tag{2.7}$$

where c_i $(i = 1, 2, 3)$ are arbitrary constants and $F_i(t)$ $(i = 1, 2, 3, 4)$ are arbitrary functions with regard to t. For convenience, we assume that

$$F_1(t) = c_4 t + c_8, \quad F_2(t) = c_5 t + c_9, \quad F_3(t) = c_6 t + c_{10}, \quad F_4(t) = c_7 t + c_{11}. \tag{2.8}$$

Therefore, the Lie algebra of infinitesimal symmetries of equation (1.1) is spanned by the vector field

$$\begin{gathered}
V_1 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3u\frac{\partial}{\partial u}, \quad V_2 = \frac{1}{2}y\frac{\partial}{\partial x} + x\frac{\partial}{\partial u},\\
V_3 = \frac{\partial}{\partial x}, \quad V_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u},\\
V_5 = -y\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} - x\frac{\partial}{\partial u}, \quad V_6 = -t\frac{\partial}{\partial x} + y\frac{\partial}{\partial u},\\
V_7 = t\frac{\partial}{\partial u}, \quad V_8 = \frac{\partial}{\partial t}, \quad V_9 = \frac{\partial}{\partial y}, \quad V_{10} = -\frac{\partial}{\partial x}, \quad V_{11} = \frac{\partial}{\partial u}.
\end{gathered} \tag{2.9}$$

We apply the Lie bracket $[V_i, V_j] = V_iV_j - V_jV_i$, with the (i, j)-th entry representing $[V_i, V_j]$ to get the commutator table listed in Table 1.

Table 1. Lie bracket of equation (1.1)

Lie	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}
V_1	0	$-V_2$	$-2V_3$	0	$-V_5$	$-2V_6$	$-3V_7$	0	$-V_9$	$-2V_{10}$	$-3V_{11}$
V_2	V_2	0	$-V_{11}$	0	$\frac{1}{2}V_6$	V_7	0	0	$\frac{1}{2}V_{10}$	V_{11}	0
V_3	$2V_3$	V_{11}	0	$-V_{10}$	$-V_{11}$	0	0	0	0	0	0
V_4	0	0	$-V_3$	0	0	0	$-V_7$	$-V_8$	$-V_9$	$-V_{10}$	$-V_{11}$
V_5	V_5	$-\frac{1}{2}V_6$	V_{11}	0	0	0	0	$-V_9$	$-V_{10}$	$-V_{11}$	0
V_6	$2V_6$	$-V_7$	0	0	0	0	0	$-V_{10}$	$-V_{11}$	0	0
V_7	$3V_7$	0	0	V_7	0	0	0	$-V_{11}$	0	0	0
V_8	0	0	0	V_8	V_9	V_{10}	V_{11}	0	0	0	0
V_9	V_9	$-\frac{1}{2}V_{10}$	0	V_9	V_{10}	V_{11}	0	0	0	0	0
V_{10}	$-2V_3$	$-V_{11}$	0	V_{10}	V_{11}	0	0	0	0	0	0
V_{11}	$3V_3$	0	0	V_{11}	0	0	0	0	0	0	0

Next, using Table 1 and the Lie series

$$\mathrm{Ad}(\exp(\varepsilon V_i))V_j = V_j - \varepsilon[V_i, V_j] + \frac{1}{2}\varepsilon^2[V_i, [V_i, V_j]] - \dots, \tag{2.10}$$

where ε is a real number and $[\cdot, \cdot]$ is the Lie bracket. The adjoint representation is shown in Table 2.

TABLE 2. Adjoint representation of equation (1.1).

Ad	V_1	V_2	V_3	V_4	V_5	V_6
V_1	V_1	V_2e^{ε}	$V_1e^{2\varepsilon}$	V_4	V_5e^{ε}	$V_6e^{2\varepsilon}$
V_2	$V_1-\varepsilon V_2$	V_2	$V_3+\varepsilon V_{11}$	V_4	$V_5-\frac{\varepsilon}{2}V_6+\frac{\varepsilon^2}{4}V_7$	$V_6-\varepsilon V_7$
V_3	$V_1-2\varepsilon V_3$	$V_2-\varepsilon V_{11}$	V_3	$V_4+\varepsilon V_{10}$	$V_5+\varepsilon V_{11}$	V_6
V_4	V_1	V_2	V_3e^{ε}	V_4	V_5	V_6
V_5	$V_1-\varepsilon V_5$	$V_2+\frac{\varepsilon}{2}V_6$	$V_3-\varepsilon V_{11}$	V_4	V_5	V_6
V_6	$V_1-2\varepsilon V_6$	$V_2+\varepsilon V_7$	V_3	V_4	V_5	V_6
V_7	$V_1-3\varepsilon V_7$	V_2	V_3	$V_4-\varepsilon V_7$	V_5	V_6
V_8	V_1	V_2	V_3	$V_4-\varepsilon V_8$	$V_5-\varepsilon V_9$	$V_6+\varepsilon V_3$
V_9	$V_1-\varepsilon V_9$	$V_2-\frac{1}{2}\varepsilon V_3$	V_3	$V_4-\varepsilon V_9$	$V_5+\varepsilon V_3$	$V_6-\varepsilon V_{11}$
V_{10}	$V_1-2\varepsilon V_{10}$	$V_2+\varepsilon V_{11}$	V_3	$V_4-\varepsilon V_{10}$	$V_5-\varepsilon V_{11}$	V_6
V_{11}	$V_1e^{-3\varepsilon}$	V_2	V_3	$V_4-\varepsilon V_{11}$	V_5	V_6

Ad	V_7	V_8	V_9	V_{10}	V_{11}
V_1	$V_7e^{3\varepsilon}$	V_8	V_9e^{ε}	$V_{10}e^{2\varepsilon}$	$V_{11}e^{3\varepsilon}$
V_2	V_7	V_8	$V_9-\frac{\varepsilon}{2}V_{10}+\frac{\varepsilon^2}{4}V_{11}$	$V_{10}-\varepsilon V_{11}$	V_{11}
V_3	V_7	V_8	V_9	V_{10}	V_{11}
V_4	V_7e^{ε}	V_8e^{ε}	V_9e^{ε}	$V_{10}e^{\varepsilon}$	$V_{11}e^{\varepsilon}$
V_5	V_7	$V_8+\varepsilon V_9+\frac{\varepsilon^2}{2}V_{10}+\frac{\varepsilon^3}{3!}V_{11}$	$V_9+\varepsilon V_{10}+\frac{\varepsilon^2}{2}V_{11}$	$V_{10}+\varepsilon V_{11}$	V_{11}
V_6	V_7	$V_8+\varepsilon V_{10}$	$V_9+\varepsilon V_{11}$	V_{10}	V_{11}
V_7	V_7	$V_8+\varepsilon V_{11}$	V_9	V_{10}	V_{11}
V_8	$V_7-\varepsilon V_{11}$	V_8	V_9	V_{10}	V_{11}
V_9	V_7	V_8	V_9	V_{10}	V_{11}
V_{10}	V_7	V_8	V_9	V_{10}	V_{11}
V_{11}	V_7	V_8	V_9	V_{10}	V_{11}

The one-parameter symmetry groups g_i $(1 \le i \le 11)$ generated by the corresponding infinitesimal generators V_i $(1 \le i \le 11)$ will be obtained

$$\begin{gathered}
g_1 : (x, y, t, u) \to (e^{2\varepsilon}x, e^{\varepsilon}y, t, e^{3\varepsilon}u),\\
g_2 : (x, y, t, u) \to (\frac{1}{2}y\varepsilon + x, y, t, \frac{1}{4}y\varepsilon^2 + x\varepsilon + u),\\
g_3 : (x, y, t, u) \to (x + \varepsilon, y, t, u), \quad g_4 : (x, y, t, u) \to (e^{\varepsilon}x, e^{\varepsilon}y, e^{\varepsilon}t, e^{\varepsilon}u),\\
g_5 : (x, y, t, u) \to (-\frac{\varepsilon^2}{2}t - \varepsilon y + x, \varepsilon t + y, t, \frac{\varepsilon^3}{6}t + \frac{\varepsilon^2}{2}y - \varepsilon x + u),\\
g_6 : (x, y, t, u) \to (x - t\varepsilon, y, t, u + \varepsilon y), \quad g_7 : (x, y, t, u) \to (x, y, t, u + \varepsilon t),\\
g_8 : (x, y, t, u) \to (x, y, t + \varepsilon, u), \quad g_9 : (x, y, t, u) \to (x, y + \varepsilon, t, u),\\
g_{10} : (x, y, t, u) \to (-\varepsilon + x, y, t, u), \ g_{11} : (x, y, t, u) \to (x, y, t, u + \varepsilon),
\end{gathered} \tag{2.11}$$

where g_3, g_9 are space translations, g_8 is a time translation, g_{11} is a dependent variable translation, g_4 is a scaling transformation, and g_5 is a generalized Galilean transformation. According to the above one-parameter symmetry groups g_i $(i = 1, 2, \dots, 11)$, it implies that if $u = f(x, y, t)$ is a solution of (1.1), then $u^{(j)}$ $(1 \le$

$j \le 11$) are also solutions of (1.1)

$$\begin{gathered}
u^{(1)} = e^{3\varepsilon} f(xe^{-2\varepsilon}, ye^{-\varepsilon}, t), \quad u^{(2)} = -\frac{\varepsilon^2}{4} y + x\varepsilon + f(x - \frac{\varepsilon}{2} y, y, t), \\
u^{(3)} = f(x-\varepsilon, y, t), \quad u^{(4)} = e^{\varepsilon} f(xe^{-\varepsilon}, ye^{-\varepsilon}, te^{-\varepsilon}), \\
u^{(5)} = -\varepsilon x - \frac{\varepsilon^2}{2} y + \frac{\varepsilon^3}{6} t + f(x + \varepsilon y - \frac{\varepsilon^2}{2} t, y - \varepsilon t, t), \\
u^{(6)} = \varepsilon y + f(x + t\varepsilon, y, t), \quad u^{(7)} = \varepsilon t + f(x, y, t), \\
u^{(8)} = f(x, y, t-\varepsilon), \quad u^{(9)} = f(x, y-\varepsilon, t), \\
u^{(10)} = f(x+\varepsilon, y, t), \quad u^{(11)} = \varepsilon + f(x, y, t),
\end{gathered} \tag{2.12}$$

where ε is an arbitrary real number.

3. Similarity reductions and exact solutions

The similarity reductions of the given equations can be identified by solving the characteristic equation

$$\begin{aligned}
\frac{dt}{F_1(t)} &= \frac{dx}{(F_{1t}(t) + 2c_1)x - \frac{1}{2} F_{1tt}(t) y^2 + \frac{1}{2}(-2F_{2t}(t) + c_2) y - F_3(t) + c_3} \\
&= \frac{dy}{(F_{1t}(t) + c_1) \cdot y + F_2(t)} \\
&= \Big((F_{1t}(t) + 3c_1) u - (F_{1tt}(t) y - c_2 + F_{2t}(t)) x + \frac{1}{6} F_{1ttt}(t) y^3 \\
&\quad + \frac{1}{2} F_{2tt}(t) y^2 + F_{3t}(t) y + F_4(t) \Big)^{-1} du.
\end{aligned} \tag{3.1}$$

Here, we give the corresponding similarity reduction and provide some exact solutions of the original equation (1.1).

Case 1. Taking $F_1(t) = 0$, $F_2(t) = 0$, $F_3(t) = 0$, $F_4(t) = 0$, $c_1 \neq 0$, $c_2 = 0$, $c_3 = 0$ in (3.2) yields

$$\frac{dt}{0} = \frac{dx}{2c_1 x} = \frac{dy}{c_1 y} = \frac{du}{3c_1 u}, \tag{3.2}$$

where the expression $\frac{dt}{0}$ means that the first integral of time t is a constant. Solving (3.2) provides

$$v = t, \quad w = y x^{-1/2}, \quad u = f(v, w) x^{3/2}. \tag{3.3}$$

Substituting (3.3) into (1.1), we obtain the following (1+1)-dimensional nonlinear PDE with variable coefficients

$$4f_{ww} + 6f_v - 2w f_{wv} + 3f f_w - 3w f f_{ww} + w f_w^2 = 0. \tag{3.4}$$

Case 2. If we take $F_1(t) = 0$, $F_2(t) = 0$, $F_3(t) = 0$, $F_4(t) = 0$, $c_1 = 0$, $c_2 \neq 0$, $c_3 = 0$ in (3.2), then we obtain

$$\frac{dt}{0} = \frac{dx}{\frac{1}{2} c_2 y} = \frac{dy}{0} = \frac{du}{c_2 x}. \tag{3.5}$$

Solving this equation, we obtain the similarity variables and the group-invariant solution

$$v = t, \quad w = y, \quad u = f(w, v) + \frac{x^2}{y}. \tag{3.6}$$

Substituting (3.6) into (1.1), we derive reduced PDE with variable coefficients

$$f_{ww} - 2w^{-1} f_w = 0. \tag{3.7}$$

Solving this equation, we obtain

$$f = F_2(v) w^3 + F_1(v), \tag{3.8}$$

where $F_1(v),\ F_2(v)$ are arbitrary functions of v. Based on (3.6) and (3.8), we obtain the exact solution of (1.1)

$$u = F_2(t) y^3 + F_1(t) + \frac{x^2}{y}, \tag{3.9}$$

where $F_1(t)$, $F_2(t)$ are arbitrary functions of t.

Case 3. Letting $F_1(t) = d_1,\ F_2(t) = d_2,\ F_3(t) = 0,\ F_4(t) = d_4,\ c_1 = 0,\ c_2 = 0,\ c_3 \neq 0$, where d_1, d_2, and d_4 are nonzero constants and we have

$$\frac{dt}{d_1} = \frac{dx}{c_3} = \frac{dy}{d_2} = \frac{du}{d_4}. \tag{3.10}$$

Solving (3.10), we obtain the similarity variables and group-invariant solution

$$v = d_2 x - c_3 y, \quad w = d_1 x - c_3 t, \quad u = \frac{d_4}{c_3} x + f(w, v). \tag{3.11}$$

Substituting (3.11) into (1.1) yields

$$\begin{aligned} &(c_3^2 + d_2 d_4) f_{vv} - c_3 d_1 f_{ww} + (d_1 d_4 + c_3 d_2) f_{vw} + d_1^2 c_3 f_w f_{wv} \\ &- d_1^2 c_3 f_v f_{ww} - d_1 d_2 c_3 f_v f_{wv} + d_1 d_2 c_3 f_w f_{vv} = 0. \end{aligned} \tag{3.12}$$

Letting $d_1 = d_2 = d_4 = c_3 = 1$, we obtain a reduced equation

$$-f_{ww} + 2 f_{vv} + f_w f_{wv} - f_v f_{ww} - f_v f_{wv} + f_w f_{vv} = 0. \tag{3.13}$$

Solving (3.13), the result is obtained

$$f = k_3 \tanh\Big(-\frac{1}{2} k_2 v + k_2 w + k_1 \Big)^3 + k_4 \tanh\Big(-\frac{1}{2} k_2 v + k_2 w + k_1 \Big) + k_5, \tag{3.14}$$

where k_1, k_2, k_3, k_4, k_5 are arbitrary constants. Combining (3.11) and (3.14), one can obtain

$$\begin{aligned} u = x &+ k_3 \tanh\Big(\frac{k_2}{2} x + \frac{k_2}{2} y - k_2 t + k_1 \Big)^3 \\ &+ k_4 \tanh\Big(\frac{k_2}{2} x + \frac{k_2}{2} y - k_2 t + k_1 \Big) + k_5, \end{aligned} \tag{3.15}$$

where k_1, k_2, k_3, k_4, and k_5 are arbitrary constants.

Case 4. If we take $F_1(t) = F_3(t) = 0$, $F_2(t) = d_2$, $F_4(t) = t$, $c_1 = c_2 = 0$, $c_3 \neq 0$ where d_2 and c_3 are nonzero constants. The defining equation is

$$\frac{dt}{0} = \frac{dx}{c_3} = \frac{dy}{d_2} = \frac{du}{t}. \tag{3.16}$$

Solving (3.16), we can obtain the similarity variables and the group-invariant solution

$$v = t, \quad w = d_2 x - c_3 y, \quad u = \frac{t}{c_3} x + f(w, v). \tag{3.17}$$

Substituting (3.17) into (1.1), we obtain the following reduced PDE with variable coefficients

$$c_3^2 f_{ww} + d_2 f_{vv} - d_2 v f_{wv} - c_3 d_2^2 \big(f_v f_{w,v} - f_w f_{vv} \big) + \frac{1}{c_3} = 0 \tag{3.18}$$

Case 5. Taking $F_1(t) = d_1$, $F_2(t) = d_2$, $F_3(t) = d_3$, $F_4(t) = 0$, $c_1 = 0$, $c_2 = 0$, $c_3 \neq 0$, where d_1 and d_2, d_3 are nonzero constants, the characteristic equation becomes

$$\frac{dt}{d_1} = \frac{dx}{-d_3 + c_3} = \frac{dy}{d_2} = \frac{du}{0}. \tag{3.19}$$

Solving this equation, we obtain the corresponding similarity variables and a group-invariant solution

$$v = d_2 t - d_1 y, \quad w = (c_3 - d_3)t - d_1 x, \quad u = f(w, v). \tag{3.20}$$

Substituting (3.20) into (1.1), we have

$$d_1 f_{vv} + (d_3 - c_3) f_{ww} - d_2 f_{wv} - d_1^2 f_w f_{vw} + d_1^2 f_v f_{ww} = 0. \tag{3.21}$$

Solving this equation, we obtain

$$\begin{aligned} f = k_7 \tanh\Big(&\frac{1}{2}\frac{\big(d_2 + \sqrt{-4d_1d_3 + 4d_1c_3 + d_2^2}\big)k_2 v}{d_1} + k_2 w + k_1\Big)^3 \\ &+ k_5 \tanh\Big(\frac{1}{2}\frac{\big(d_2 + \sqrt{-4d_1d_3 + 4d_1c_3 + d_2^2}\big)k_2 v}{d_1} + k_2 w + k_1\Big) + k_4, \end{aligned} \tag{3.22}$$

where k_1, k_2, k_4, k_5, k_7 are arbitrary constants. Combining (3.20) and (3.22), we obtain the exact solution of (1.1),

$$\begin{aligned} u = k_3 \tanh\Big(&\frac{1}{2}\frac{(d_2 + \sqrt{-4d_1d_3 + 4d_1c_3 + d_2^2})k_2(d_2 t - d_1 y)}{d_1} \\ &+ k_2[(c_3 - d_3)t - d_1 x] + k_1\Big)^3 \\ &+ k_5 \tanh\Big(\frac{1}{2}\frac{(d_2 + \sqrt{-4d_1d_3 + 4d_1c_3 + d_2^2})k_2(d_2 t - d_1 y)}{d_1} \\ &+ k_2[(c_3 - d_3)t - d_1 x] + k_1\Big) + k_4, \end{aligned} \tag{3.23}$$

where k_1, k_2, k_3, k_4, k_5 are arbitrary constants.

Case 6. Setting $F_1(t) = 0$, $F_2(t) = 0$, $F_3(t) = 0$, $F_4(t) = 0$, $c_1 \neq 0$, $c_2 = 0$, $c_3 = 0$, the characteristic equation is

$$\frac{dt}{0} = \frac{dx}{2c_1 x} = \frac{dy}{c_1 y} = \frac{du}{3c_1 u}. \tag{3.24}$$

Solving this equation, the similarity variables and a group-invariant solution can be obtained. They are

$$v = xy^{-2}, \quad w = t, \quad u = y^3 f(w, v), \tag{3.25}$$

Substituting (3.25) into (1.1), it is obvious that the reduced nonlinear PDE with variable coefficients is

$$6f - 6v f_v + 4v^2 f_{vv} + f_{vw} + f_v^2 - 3f f_{vv} = 0. \tag{3.26}$$

Case 7. Letting $F_1(t) = 0$, $F_2(t) = d_2$, $F_3(t) = d_3$, $F_4(t) = d_4$, $c_1 = 0$, $c_2 = 0$, $c_3 \neq 0$, where d_2, d_3, d_4 are nonzero constants, then the characteristic equation becomes

$$\frac{dt}{0} = \frac{dx}{-d_3 + c_3} = \frac{dy}{d_2} = \frac{du}{d_4}. \tag{3.27}$$

Solving this equation, we obtain

$$v = (c_3 - d_3)y - d_2x, \quad w = t, \quad u = \frac{d_4}{d_2}y + f(w, v). \tag{3.28}$$

Substituting (3.28) into (1.1) yields a reduced PDE of (1.1) with constant coefficients

$$\big((c_3 - d_3)^2 - d_2d_4\big)f_{vv} - d_2f_{vw} = 0. \tag{3.29}$$

Case 8. Letting $F_1(t) = c_4t + c_5$, $F_2(t) = 0$, $F_3(t) = 0$, $F_4(t) = 0$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 \neq 0$, $c_5 \neq 0$ in (3.1), then we obtain

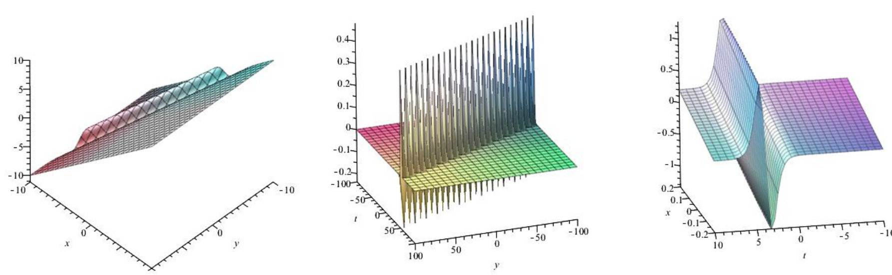
$$\frac{dt}{c_4t + c_5} = \frac{dx}{c_4x} = \frac{dy}{c_4y} = \frac{du}{c_4u}. \tag{3.30}$$

Solving (3.30), we can get the similarity variables and the group-invariant solution

$$v = xy^{-1}, \quad w = (c_4t + c_5)x^{-1}, \quad u = f(v, w)x. \tag{3.31}$$

Substituting (3.31) into (1.1), it is easily to obtain the reduced nonlinear PDE with variable coefficients through a straight calculation

$$\begin{aligned} &2v^3f_v + v^4f_{vv} + c_4vf_{vw} - c_4wf_{ww} - 2v^2ff_v + wv^2ff_{wv} - v^3ff_{vv} \\ &+ 2wv^2f_wf_v - v^2w^2f_wf_{wv} - wv^3f_wf_{vv} - wv^3f_vf_{wv} + w^2v^2f_vf_{ww} = 0. \end{aligned} \tag{3.32}$$



(a) (b) (c)

Figure 1. Propagation of the exact solutions of (1.1) via (3.15) with parameters: $k_1 = 4$, $k_2 = 1$, $k_3 = 3$, $k_4 = -3$, $k_5 = 0$. Perspective of the solutions with: (a) $t = 0$, (b) $x = 0$, (c) $y = 0$.

Case 9. If we set $F_1(t) = c_4$, $F_2(t) = c_5$, $F_3(t) = 0$, $F_4(t) = 0$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 \neq 0$, $c_5 \neq 0$, the defining equation is

$$\frac{dt}{c_4} = \frac{dx}{0} = \frac{dy}{c_5} = \frac{du}{0}. \tag{3.33}$$

Solving this equation, we obtain the similarity variables and the group-invariant solution

$$v = c_5t - c_4y, \quad w = x, \quad u = f(w, v). \tag{3.34}$$

Then, we obtain the reduced nonlinear PDE with constant coefficients

$$c_4^2f_{vv} + c_5f_{wv} - c_4f_wf_{wv} + c_4f_vf_{ww} = 0. \tag{3.35}$$

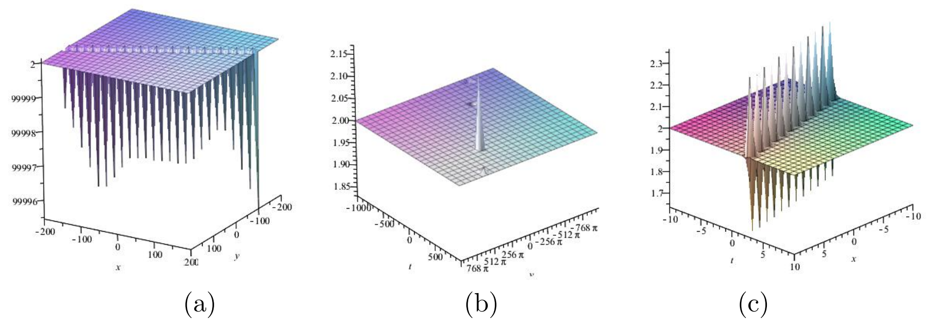


FIGURE 2. Propagation of the exact solutions of (1.1) via (3.35) with parameters: $k_1 = 0$, $k_2 = -1$, $k_3 = 4$, $k_4 = 2$, $k_5 = 1$, $c_4 = 1$, $c_5 = 2$. Perspective of the solutions with: (a) $t = 0$, (b) $x = 0$, (c) $y = 0$.

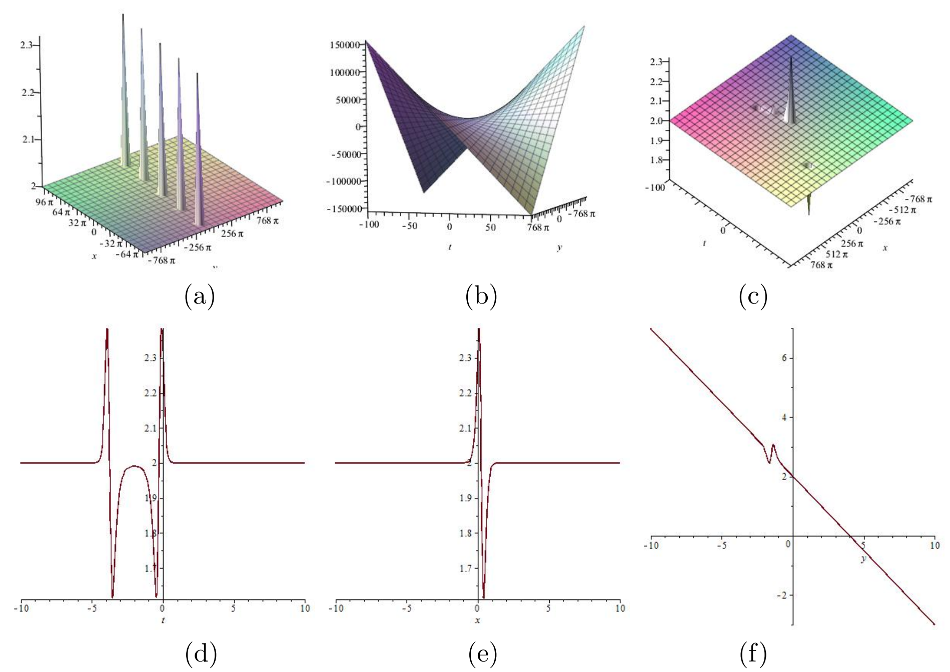


FIGURE 3. Propagation of the exact solutions of (1.1) via (3.48) with parameters: $k_1 = 1$, $k_2 = 4$, $k_3 = -1$, $k_4 = 2$, $k_5 = 1$, $c_4 = -2$, $c_5 = 1$, $c_6 = 2$. Perspective of the solutions with: (a) $t = 0$, (b) $x = 0$, (c) $y = 0$. Wave propagation pattern of the wave along with: (d) the t axis, (e) the x axis, (f) the y axis.

Solving this equation gives

$$f = k_2 \tanh\Big(k_3 v - \frac{k_3 c_4^2}{c_5} w + k_1\Big)^3 + k_5 \tanh\Big(k_3 v - \frac{k_3 c_4^2}{c_5} w + k_1\Big) + k_4, \qquad (3.36)$$

where k_1, k_2, k_3, k_4, k_5 are arbitrary constants. Combining (3.34) and (3.36), the exact solution of (1.1) is presented,

$$\begin{aligned} u = k_2 \tanh\Big(k_3(-c_4y+c_5t) - \frac{k_3c_4^2x}{c_5} + k_1\Big)^3 \\ + k_5 \tanh\Big(k_3(-c_4y+c_5t) - \frac{k_3c_4^2x}{c_5} + k_1\Big) + k_4, \end{aligned} \tag{3.37}$$

where k_1, k_2, k_3, k_4, k_5 are arbitrary constants.

Case 10. If taking $F_1(t) = 0$, $F_2(t) = t$, $F_3(t) = 0$, $F_4(t) = 0$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ in (3.1), then the characteristic equation becomes

$$\frac{dt}{0} = \frac{dx}{-y} = \frac{dy}{t} = \frac{du}{-x}. \tag{3.38}$$

Solving this equation, the similarity variables and the group-invariant solution are presented as follows

$$v = tx + \frac{1}{2}y^2, \quad w = t, \quad u = t^{-1}f(w,v) + \frac{1}{6}t^{-2}y^3 - t^{-1}xy - \frac{1}{2}t^{-2}y^3. \tag{3.39}$$

Then, we obtain the PDE with variable coefficients

$$wf_{vw} + 2vf_{vv} = 0. \tag{3.40}$$

Solving (3.40), we obtain

$$f = F_2(w) + F_1\Big(\frac{v}{w^2}\Big)w^2, \tag{3.41}$$

where $F_1(\frac{v}{w^2})$, $F_2(w)$ are arbitrary functions of variables v and w. Combining (3.39) and (3.41), we obtain the exact solution of (1.1)

$$u = F_2(t)t^{-1} + F_1\Big(\frac{2tx+y^2}{2t^2}\Big)t - xyt^{-1} - \frac{1}{3}y^3t^{-2}, \tag{3.42}$$

where F_1 and F_2 are arbitrary functions of variables x, t and y.

Case 11. Taking $F_1(t) = c_4$, $F_2(t) = t$, $F_3(t) = 0$, $F_4(t) = 0$, $c_1 = 0$, $c_2 = 0$, $c_4 \neq 0$ in (3.1) yields

$$\frac{dt}{c_4} = \frac{dx}{-y} = \frac{dy}{t} = \frac{du}{-x}. \tag{3.43}$$

Solving (3.43), we obtain the similarity variables and the group-invariant solution

$$v = \frac{t^3}{3c_4} - yt - c_4x, \quad w = \frac{t^2}{2} - c_4y, \quad u = f(w,v) + \frac{v}{c_4^2}t + \frac{t^4}{24c_4^3} - \frac{wt^2}{2c_4^3}. \tag{3.44}$$

Substituting (3.44) into (1.1) yields

$$c_4^2f_{ww} - wf_{vv} - c_4^3f_vf_{vw} + c_4^3f_wf_{vv} - \frac{1}{c_4} = 0. \tag{3.45}$$

Case 12. Letting $F_1(t) = c_4$, $F_2(t) = 0$, $F_3(t) = c_5t + c_6$, $F_4(t) = 0$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 \neq 0$, $c_5 \neq 0$, $c_6 \neq 0$ in (3.1), we can obtain

$$\frac{dt}{c_4} = \frac{dx}{-c_5t - c_6} = \frac{dy}{0} = \frac{du}{c_5y}. \tag{3.46}$$

Solving this equation we obtain the similarity variables and the group-invariant solution

$$v = -\frac{c_5}{2}t^2 - c_6t - c_4x, \quad w = y, \quad u = f(w,v) + \frac{c_5}{c_4}yt. \tag{3.47}$$

Substituting (3.47) into (1.1) yields nonlinear PDE with constant coefficients

$$f_{ww}+c_4c_6f_{vv}+c_4^2f_vf_{vw}-c_4^2f_wf_{vv}=0. \tag{3.48}$$

Solving this equation we have

$$\begin{aligned} f=&k_3\tanh\Big(-\frac{k_2v}{\sqrt{-c_4c_6}}+k_2w+k_1\Big)^3\\ &+k_5\tanh\Big(-\frac{k_2v}{\sqrt{-c_4c_6}}+k_2w+k_1\Big)+k_4, \end{aligned} \tag{3.49}$$

where k_1,k_2,k_3,k_4,k_5 are arbitrary constants. Combining (3.47) and (3.48), we obtain the exact solutions of (1.1)

$$\begin{aligned} u=&k_3\tanh\Big(-\frac{k_2(-\frac{c_5}{2}t^2-c_6t-c_4x)}{\sqrt{-c_4c_6}}+k_2y+k_1\Big)^3\\ &+k_5\tanh\Big(-\frac{k_2(-\frac{c_5}{2}t^2-c_6t-c_4x)}{\sqrt{-c_4c_6}}+k_2y+k_1\Big)+k_4+\frac{c_5}{c_4}yt, \end{aligned} \tag{3.50}$$

where k_1,k_2,k_3,k_4,k_5 are arbitrary constants. The illustrative examples of exact solutions to case 3, case 9 and case 12 are presented graphically.

4. Construction of conservation laws

In this section, we will construct conservation laws for the (2+1)-dimensional Mikhalëv equation (1.1). The formal Lagrangian form of (1.1) is present by

$$\psi=v(u_{yy}+u_{xt}+u_xu_{xy}-u_yu_{xx}). \tag{4.1}$$

Furthermore, the adjoint equation is written in this form

$$F^*=-2v_xu_{xy}+2v_yu_{xx}+v_{xy}u_x-v_{xx}u_y+v_{yy}+v_{xt}=0. \tag{4.2}$$

Let us consider a Lie point symmetry generator,

$$X=7x\frac{\partial}{\partial x}+6y\frac{\partial}{\partial y}+5t\frac{\partial}{\partial t}+8u\frac{\partial}{\partial u}. \tag{4.3}$$

Thus, the extension of (4.3) to v has the form

$$Y=7x\frac{\partial}{\partial x}+6y\frac{\partial}{\partial y}+5t\frac{\partial}{\partial t}+8u\frac{\partial}{\partial u}-14v\frac{\partial}{\partial v}. \tag{4.4}$$

Theorem 4.1. *Any infinitesimal symmetry*

$$X=\xi^i(x,u,u_{(1)},\dots)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u,u_{(1)},\dots)\frac{\partial}{\partial u^\alpha} \tag{4.5}$$

of a nonlinearly self-adjoint system to differential equation (1.1) *produces a conservation law for this system,*

$$[D_i(C^i)]_{(1.1)}=0 \tag{4.6}$$

The components of the conserved vector are given by

$$\begin{aligned} C^i=&\xi^i\psi+W^\alpha\Big[\frac{\partial\psi}{\partial u_i^\alpha}-D_j\Big(\frac{\partial\psi}{\partial u_{ij}^\alpha}\Big)+D_jD_k(\frac{\partial\psi}{\partial u_{ijk}^\alpha})-\cdots\Big]\\ &+D_j(W^\alpha)\Big[\frac{\partial\psi}{\partial u_{ij}^\alpha}-D_k\Big(\frac{\partial\psi}{\partial u_{ijk}^\alpha}\Big)+\cdots\Big]+D_jD_k(W^\alpha)\Big[\frac{\partial\psi}{\partial u_{ijk}^\alpha}-\cdots\Big], \end{aligned} \tag{4.7}$$

where

$$W^\alpha=\eta^\alpha-\xi^ju_j^\alpha, \tag{4.8}$$

and ψ is the formal Lagrangian.

In this case, we obtain the conservation laws

$$D_x(C^1) + D_t(C^2) + D_y(C^3) = 0, \tag{4.9}$$

with the components of conserved vector $C = (C^1, C^2, C^3)$, where

$$\begin{aligned} C^1 &= 7xv(u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx}) + (3u_t - 7xu_{xt} - 5tu_{tt} - 6yu_{ty})v \\ &\quad - (u_x - 7xu_{xx} - 5tu_{xt} - 6yu_{xy})(vu_y) + (2u_y - 7xu_{xy} - 5tu_{ty} \\ &\quad - 6yu_{yy})(vu_x) + (8u - 7xu_x - 5tu_t - 6yu_y)(vu_{xy} \\ &\quad + v_x u_y - v_y u_x - v_t), \end{aligned} \tag{4.10}$$

$$\begin{aligned} C^2 &= 5tv(u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx}) - 8uv_x + 7xu_x v_x + 5tu_t v_x \\ &\quad + 6yu_y v_x + vu_x - 7xvu_{xx} - 5tvu_{xt} - 6yvu_{xy}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} C^3 &= 6yv(u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx}) + (2u_y - 7xu_{xy} - 5tu_{yt} - 6yu_{yy})(v) \\ &\quad + (8u - 7xu_x - 5tu_t - 6yu_y)(-2vu_{xx} + v_y - v_x u_x) \\ &\quad + (u_x - 7xu_{xx} - 5tu_{tx} - 6yu_{yx})(vu_x). \end{aligned} \tag{4.12}$$

This conserved vector includes an arbitrary solution v of the adjoint equation $F^* = -2v_x u_{xy} + 2v_y u_{xx} + v_{xy} u_x - v_{xx} u_y + v_{yy} + v_{xt} = 0$, and it can derive infinitely many conservation laws. For convenience, let us take $v = t$, then the components of the conserved vector are simplified to the form

$$\begin{aligned} C^1 &= 7xt(u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx}) + (8u - 7xu_x - 5tu_t - 6yu_y)(tu_{xy}) \\ &\quad - (u_x - 7xu_{xx} - 5tu_{xt} - 6yu_{xy})(tu_y) + (2u_y - 7xu_{xy} - 5tu_{ty} \\ &\quad - 6yu_{yy})(tu_x) + (3u_t - 7xu_{xt} - 5tu_{tt} - 6yu_{ty})t, \end{aligned} \tag{4.13}$$

$$C^2 = 5t^2(u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx}) + tu_x - 7xtu_{xx} - 5t^2 u_{xt} - 6ytu_{xy}, \tag{4.14}$$

$$\begin{aligned} C^3 &= 6yt(u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx}) + (8u - 7xu_x - 5tu_t - 6yu_y)(-2tu_{xx}) \\ &\quad + (t)(2u_y - 7xu_{xy} - 5tu_{yt} - 6yu_{yy}) + (u_x - 7xu_{xx} - 5tu_{tx} - 6yu_{yx})(tu_x). \end{aligned} \tag{4.15}$$

Then, we consider the point symmetry for the (2+1)-dimensional Mikhalëv equation (1.1),

$$X = \frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \tag{4.16}$$

and we obtain the conserved vector

$$\begin{aligned} C^1 &= (-u_y - u_t)(v_x u_y + vu_{xy} - v_y u_x - v_t) + (u_{xy} + u_{xt})(vu_y) \\ &\quad - (u_{yy} + u_{yt})(vu_x) - v(u_{ty} + u_{tt}), \end{aligned} \tag{4.17}$$

$$C^2 = (u_y + u_t)(v_x) + (u_{yy} + u_x u_{xy} - u_y u_{xx} - u_{yx})(v), \tag{4.18}$$

$$\begin{aligned} C^3 &= u_y v_y + u_t v_y + (u_x u_y + u_x u_t) v_x + (u_y u_{xx} \\ &\quad + 2u_t u_{xx} - u_x u_{xt} - u_{ty} + u_{xt}) v. \end{aligned} \tag{4.19}$$

Similarly, we take $v = -1$ and get simplified conserved vector

$$C^1 = u_t u_{xy} - u_{xt} u_y + u_{yt} + u_{tt} + (u_{ty} + u_{yy}) u_x, \tag{4.20}$$

$$C^2 = u_{xt} + u_{yx}, \tag{4.21}$$

$$C^3 = -u_y u_{xx} - 2u_t u_{xx} + u_x u_{xt} + u_{ty} - u_{xt}. \tag{4.22}$$

We study a point symmetry for the (2+1)-dimensional Mikhalëv equation (1.1)

$$X = \frac{\partial}{\partial x}, \tag{4.23}$$

and the conserved vector

$$C^1 = (-2u_x u_{xy} - u_{xt} + u_y u_{xx})v + u_x v_t + u_x^2 v_y - u_x u_y v_x, \tag{4.24}$$

$$C^2 = u_x v_x - u_{xx} v, \tag{4.25}$$

$$C^3 = (u_x u_{xx} - u_{xy})v + u_x^2 v_x + u_x v_y. \tag{4.26}$$

Taking the solution $v = -1$ of (4.2), the following vector can be obtained

$$C^1 = (2u_x u_{xy} + u_{xt} - u_y u_{xx}) = u_x u_{xy} - u_{yy}, \tag{4.27}$$

$$C^2 = u_{xx}, \tag{4.28}$$

$$C^3 = (u_x u_{xx} + u_{xy} - 2u_x u_{xx}) = -u_x u_{xx} + u_{xy}. \tag{4.29}$$

Specially, the conservation laws for the vector (4.27)-(4.29) have the form

$$\begin{aligned} &D_x(C^1) + D_t(C^2) + D_y(C^3) \\ &= u_x u_{xxy} + 2u_{xxt} - u_y u_{xxx} + u_{xyy} = (F)_x + u_{xxt} = 0. \end{aligned} \tag{4.30}$$

5. Conclusions and discussions

In this paper, we have presented the Lie symmetry analysis for the (2+1)-dimensional Mikhalëv equation and applied the Ibragimov's method to construct its conservation laws. We have taken $F_1(t)$, $F_2(t)$, $F_3(t)$ and $F_4(t)$ as linear functions and systematically shown the Lie bracket and the adjoint representation to the Mikhalëv equation. Compared with [2], we have obtained several partial differential equations with variable coefficients, such as, (3.7), (3.18), (3.40) and get their solutions. Meanwhile, we also have derived the solutions of partial differential equations with constant coefficients such as equations (3.12), (3.21), (3.35), (3.48). Illustrative examples of solutions for the (2+1)-dimensional Mikhalëv equation are exhibited.

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