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EXISTENCE OF SOLUTIONS TO NONLINEAR PARABOLIC PROBLEMS WITH DELAY

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ABSTRACT. We prove the existence and uniqueness of global solutions to semilinear parabolic equations with a nonlinear delay term. We study these problems in the whole space $\mathbb{R}^n$, obtain classic solutions, and give a mass decay result of the solution.

1. INTRODUCTION

This paper is devoted to the study of existence and uniqueness of solutions to parabolic problems with delay in unbounded domains. Let $\mathbb{R}^n$ be Euclidean space, $n \geq 1$ and consider the problem

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u + \mu|\nabla u|^p + f(t,x,u_t) \quad x \in \mathbb{R}^n \\ u(x,s) = \phi(x,s) \quad -r \leq s \leq 0,\ x \in \mathbb{R}^n \end{gathered} \tag{1.1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \times C \to \mathbb{R}$ is a locally Lipschitz functions with respect to u_t, C denotes the phase space will be defined in section 2, $r > 0$, $\mu \neq 0$, and $u_t = u(x, t+\theta)$, for $-r \leq \theta \leq 0$, denotes the delay term, $\phi(x,s) \in C^2(\mathbb{R}^n \times [-r,0])$.

In the recent years many authors have been concerned with the nonlinear partial differential equations involving delay, for the examples we refer to the book [8] and references cited therein. Generally speaking, this type of problems can be described as an abstract nonlinear partial functional differential equation

$$\begin{gathered} \frac{du}{dt} = A_T u(t) + F(t,u,u_t) \\ u(\theta) = \phi(\theta), \quad \text{for } -r \leq \theta \leq 0 \end{gathered}$$

and then it can be written as an integral equation

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)F(s,u,u_s)ds$$

where $T(t)$ is a strongly continuous semigroup of bounded linear operators with A_T its infinitesimal generator. Then many methods similar to those adopted in ordinary equations can be used to study this type of problems in abstract space, for the details we refer to [8]. However, the problem (1.1) we investigate here

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cannot be carried on along this line for at least in two reasons: first, the operator Δ on $\mathbb{R}^n$ is not compact, so the related semigroup bear some more complexity than the compact cases; second, the general nonlinear term $F(t,u,u_t)$ depend on ∇u, which may cause some difficulty for the study of our problem. Many mechanic and physical problems reduced to the problem (1.1), for the case of $f(t,x,u_t)=0$, (1.1) has emerged in recent years in a number of interesting, and quite different models, for example in the one-dimensional case and $1<p<2$, it has been known as the KPZ-equation, serving as a physical model of growth of surfaces, see Krug and Spohn [6], [7]. In [1] the authors studied this problem in high dimension and proved the existence results, and get some useful mass decay results. There also has been a strong interest in the related equation

$$u_t = \Delta u + \mu|\nabla u|^p + u^q \tag{1.2}$$

where $p,q>1$, $\mu<0$, both in bounded and unbounded domains, see [4], [8] and the references cited therein. In this paper, we focus on studying problem (1.1) by using some estimates in parabolic equation involving the nonlinearity $|\nabla u|^p$,which is called the damping nonlinear gradient term. To our knowledge there are no results about the problem we present here, even if in the case of $\mu=0$ the solution of (1.1) may blow up in finite time. This paper is motivated by the recent results in [1] and [4]. Our main results are the following theorem.

Theorem 1.1. *Let $\phi\in C$. Assume that $p\geq 1$, f satisfies a Lipschitz-type condition,*

$$|\nabla_y f(t,x,y)| \leq L|y| \tag{1.3}$$

for some constant $L>0$, then there exists a unique classical solution of (1.1) *in Ω_{T_0}, where $T_0=\min\left\{[2^{p+1}C_e]^{-2},\frac{1}{2}\right\}>0$, and C_e will be defined later.*

Theorem 1.2. *Let $f(t,x,u_t)=g(t,u(t-r)),\phi\in C$, here $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the strong Lipschitz-condition*

$$|g_u(t,u)| \leq L \tag{1.4}$$

for any $(t,u)\in\mathbb{R}\times\mathbb{R}$. Then (1.1) *has a unique classical solution $u(x,t)$ in $\mathbb{R}^n\times[0,T]$, where $T>0$ is any real number, and for every $t\in[0,T]$, $u(\cdot,t)\in C_b^2(\mathbb{R}^n)$.*

Theorem 1.3. *Assume that $0\leq\phi(0,x)\in C_b^2(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$, $\mu<0$, $p<(n+2)/(n+1)$ and $f(t,x,u)\leq 0$ for any $(t,u)\in\mathbb{R}^+\times C$. If $u(x,t)$ is a C^2 solution of* (1.1) *in $\mathbb{R}^n\times[0,\infty)$, then it decays as $t\to+\infty$ in the sense that*

$$\lim_{t\to+\infty}\int_{\mathbb{R}^n} u(x,t)dx = 0.$$

2. Proof of Main Theorems

In what follows we set $C=\{u(\cdot,t)\in C_b^2(\mathbb{R}^n),\text{ for any } t\in[-r,0]\}$ as the phase space with the norm

$$\|u(x,t)\|_C = \max_t\{\|u(x,t)\|_{L^\infty}+\|\nabla u(x,t)\|_{L^\infty}\}$$

where

$$C_b^2(\mathbb{R}^n)=\{u:u\in C^2(\mathbb{R}^n),u,\nabla u,\nabla^2 u\in L^\infty(\mathbb{R}^n)\},$$
$$\Omega_t=\mathbb{R}^n\times[-r,t],\quad t\geq 0\,.$$

To prove our theorems, we begin by establishing the existence of the solution for a short time. The idea of the proof is inspired by the papers of Amour and Ben-Artzi [5] and [1].

Proof of Theorem 1.1. Define

$$u^0(x,t)=\begin{cases}\phi(x,t) & \text{for } t\in[-r,0]\\ \int_{\mathbb{R}^n} G(x-y,t)\phi(y,0)dy & \text{for } t\in(0,T_0)\end{cases}$$

where the heat kernel is,

$$G(x,t)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}},\quad t>0,$$

which satisfies,

$$\int_{\mathbb{R}^n} G(x,t)dx=1,\quad \int_{\mathbb{R}^n}|\nabla G(x,t)|dx=\beta t^{-1/2}, \tag{2.1}$$

with $\beta=\int_{\mathbb{R}^n}|\nabla G(x,1)|dx$. Then we may solve by iterations the following linear heat equation, with delay,

$$\begin{gathered}\frac{\partial u^k}{\partial t}-\Delta u^k=\mu|\nabla u^{k-1}|^p+f(t,x,u_t^{k-1})\\ u^k(x,s)=\phi(x,s)\quad\text{for }-r\le s\le 0\end{gathered} \tag{2.2}$$

$k=1,2,3,\dots$. By Duhamel's principle from (2.2), we get

$$\begin{aligned}u^k(x,t)&=\int_{\mathbb{R}^n} G(x-y,t)\phi(y,0)dy+\mu\int_{\mathbb{R}^n}\int_0^t G(x-y,t-s)|\nabla u^{k-1}(y,s)|^p\,dy\,ds\\ &\quad+\int_{\mathbb{R}^n}\int_0^t G(x-y,t-s)f(s,x,u_s^{k-1})\,dy\,ds\\ u^k(x,t)&=\phi(x,t)\quad\text{for } t\in[-r,0]\end{aligned} \tag{2.3}$$

$$\begin{aligned}\nabla u^k(x,t)&=\int_{\mathbb{R}^n} G(x-y,t)\nabla\phi(y,0)dy+\mu\int_{\mathbb{R}^n}\int_0^t \nabla_x G(x-y,t-s)|\nabla u^{k-1}(y,s)|^p\,dy\,ds\\ &\quad+\int_{\mathbb{R}^n}\int_0^t \nabla_x G(x-y,t-s)f(s,x,u_s^{k-1})\,dy\,ds\end{aligned} \tag{2.4}$$

Setting $M_k(t)=\sup_{\Omega_t}|\nabla u^k(x,t)|$, and $U_k(t)=\sup_{\Omega_t}|u^k(x,t)|$, in view of (2.1), (2.3), and (2.4), we have

$$M_k(t)\le M_0(t)+\beta|\mu|\int_0^t(t-s)^{-1/2}M_{k-1}^p(s)ds+L\int_0^t(t-s)^{-1/2}U_{k-1}(s)ds \tag{2.5}$$

$$U_k(t)\le U_0(t)+\beta|\mu|\int_0^t M_{k-1}^p(s)ds+L\int_0^t U_{k-1}(s)ds\,. \tag{2.6}$$

Since $M_0(t)\le\|\nabla\phi(x,t)\|_C$ and $U_k(t)\le\|\phi(x,t)\|_C$, it follows inductively from (2.5) that for $t\le T_0$,

$$\begin{gathered}\|\nabla u^k(x,t)\|_{L^\infty(\mathbb{R}^n)}\le 2\|\phi(x,t)\|_C,\\ \|u^k(x,t)\|_{L^\infty(\mathbb{R}^n)}\le 2\|\phi(x,t)\|_C,\quad k=0,1,2,\dots\end{gathered} \tag{2.7}$$

To prove the convergence of the iterations, we need the following inequality

$$\big||\nabla u^k(x,t)|^p - |\nabla u^{k-1}(x,t)|^p\big| \le C_p \|\phi(x,t)\|_C^{p-1} |\nabla u^k(x,t) - \nabla u^{k-1}(x,t)| \quad (2.8)$$

which can be derived by using (2.7) and the classical inequality: $a^p - b^p \le C_p(a-b)(a^{p-1} - b^{p-1})$, $a>0$, $b>0$. Now setting

$$N_k(t) = \sup_{\Omega_t} |\nabla u^k(x,t) - \nabla u^{k-1}(x,t)|, \quad V_k(t) = \sup_{\Omega_t} |u^k(x,t) - u^{k-1}(x,t)|,$$

then from (2.3), (2.4) and (2.8), we have

$$\begin{aligned} &N_k(t) \\ &\le |\mu| \sup \int_0^t \int_{\mathbb{R}^n} ||\nabla u^{k-1}(y,s)|^p - |\nabla u^{k-2}(y,s)|^p||\nabla_x G(x-y,t-s)|\,dy\,ds \\ &\quad + L \sup \int_0^t \int_{\mathbb{R}^n} |\nabla_x G(x-y,t-s)||u^k(y,s) - u^{k-1}(y,s)|\,dy\,ds \\ &\le C_p|\mu|\|\phi(x,t)\|_C^{p-1}\beta \int_0^t (t-s)^{-1/2} N_{k-1}(s)ds + L\beta \int_0^t (t-s)^{-1/2} V_{k-1}(s)ds\,. \end{aligned} \quad (2.9)$$

$$\begin{aligned} V_k(t) &\le C_p|\mu|\|\phi(x,t)\|_C^{p-1} \int_0^t N_{k-1}(s)ds + L\int_0^t V_{k-1}(s)ds \\ &\le C_p|\mu|\|\phi(x,t)\|_C^{p-1} \int_0^t (t-s)^{-1/2} N_{k-1}(s)ds + L\int_0^t (t-s)^{-1/2} V_{k-1}(s)ds \end{aligned} \quad (2.10)$$

Choosing $C_e = \max\{C_p|\mu|\|\phi(x,t)\|_C^{p-1}\beta, L\beta, C_p|\mu|\|\phi(x,t)\|_C^{p-1}, L\} > 0$, from (2.9), (2.10), we have

$$N_k(t) + V_k(t) \le C_e \int_0^t (t-s)^{-1/2}[N_{k-1}(s) + V_{k-1}(s)]ds \quad (2.11)$$

then it follows inductively that (see, [10, Chapter 3]),

$$N_k(t) + V_k(t) \le C_e^k t^{\frac{k}{2}} \Gamma\big(\frac{k+2}{2}\big)^{-1}. \quad (2.12)$$

In particular, $\sum_k (N_k(T_0) + V_k(T_0)) < \infty$, we conclude that $\{\nabla u^k\}_{k=1}^\infty$ and $\{u^k\}_{k=1}^\infty$ converge uniformly in Ω_{T_0}. Then we set

$$u(x,t) = \lim_{k\to\infty} u^k(x,t) \quad (2.13)$$

Next, we prove the uniform boundedness of $\{\nabla^2 u^k\}_{k=1}^\infty$ in Ω_{T_0}, $f \in C^2$ one has $|\nabla|\nabla f|| \le C_n|\nabla^2 f|$, hence also $|\nabla|\nabla f|^p| \le C_{p,n}|\nabla f|^{p-1}|\nabla^2 f|$, here C_n and $C_{p,n}$ depend only on n and p, n respectively. Denoting

$$L_k(t) = \sup_{\Omega_t} |\nabla^2 u^k(x,t)|$$

it follows from Duhamel's principle and previous estimates that

$$\begin{aligned} \nabla^2 u^k(x,t) &= \int_{\mathbb{R}^n} G(x-y,t)\nabla^2\phi(y,0)dy \\ &\quad + \mu\int_{\mathbb{R}^n}\int_0^t \nabla_x G(x-y,t-s)\nabla|\nabla u^{k-1}(y,s)|^p\,dy\,ds \\ &\quad + \int_{\mathbb{R}^n}\int_0^t \nabla_x G(x-y,t-s)\nabla f(s,x,u_s^{k-1})\,dy\,ds \end{aligned} \tag{2.14}$$

then

$$L_k(t) \le L_0(t) + C_{p,n}\|\nabla\phi\|_C^{p-1}\beta|\mu|\int_0^t (t-s)^{-1/2}L_{k-1}(s)ds + 2L\beta\|\phi\|_C t^{1/2}. \tag{2.15}$$

If $\Lambda > 0$ is large so that

$$C_{p,n}\|\phi\|_C^{p-1}\beta|\mu|\int_0^{T_0} s^{-1/2}e^{-\Lambda s}ds < \frac{1}{2},$$

then it follows inductively, using $L_0(t) \le \|\nabla^2\phi\|_C$ that

$$L_k(t) \le 2\|\nabla^2\phi\|_C \tag{2.16}$$

for $t \in [-r, T_0]$, $k = 0, 1, 2, \dots$.

Then the same argument as in the proof of [2, Proposition 2.4] shows that $\{\nabla^2 u^k\}_{k=1}^\infty$ is equicontinuous in Ω_{T_0}. Using the Arzela-Ascoli theorem and the convergence of (2.13) we conclude from the standard regularity results of parabolic equation that the solution of (2.13) satisfies the (1.1) in classic sense in Ω_{T_0}.

If $v(\cdot,t) \in C^2$ is another classical solution in Ω_{T_0}, $v(x,t) = \phi(x,t)$ for $-r \le t \le 0$, then setting $N_1(t) = \sup_{\Omega_t}|\nabla u - \nabla v|$, $N_2(t) = \sup_{\Omega_t}|u - v|$, we obtain as in (2.11) that

$$N_1(t) + N_2(t) \le C\int_0^t (t-s)^{-1/2}(N_1(s) + N_2(s))ds,$$

for a sufficiently large constant C; this implies $N_1(t) + N_2(t) \equiv 0$ and $u \equiv v$. This completes the proof of the Lemma. □

Proof of Theorem 1.2. By Theorem 1.1, there exists a local solution to (1.1). Hence we just need to get an estimate of ∇u. Taking differential on both side we have

$$\frac{\partial u_j}{\partial t} - \Delta u_j - g_u(t, u(t-r))u_j(t-r) = \sum_{i=1}^n \psi_i(x,t)\frac{\partial u_j}{\partial x_j}$$

where $u_j = \frac{\partial u}{\partial x_j}$, and $\psi_i(x,t) = \mu p|\nabla u|^{p-2}\frac{\partial u}{\partial x_j} \in L^\infty(\Omega_T)$. By a maximum principle of linear parabolic equation which was proved in the appendix of [1], we can obtain, for $0 \le t \le r$,

$$|u_j(t,x)| \le C\big(|u_j(0,x)| + \max_{t\in[-r,0]}|\phi_j(t,x)|\big)$$

where C depends only on L, p, n. By reiterating the procedure above on $[r, 2r]$, $[2r, 3r]$, …, we conclude that, for $t \in [0, T]$

$$|u_j(t,x)| \le C_3 e^{rT}\big(|u_j(0,x)| + \max_{t\in[-r,0]}|\phi_j(t,x)|\big),$$

that is

$$|\nabla u(t,x)| \le C_3 e^{rT}\big(|\nabla u(0,x)| + \max_{t\in[-r,0]}|\nabla\phi(t,x)|\big).$$

So from Theorem 1, we get the existence of a global solution, and this completes the proof. □

Proof of Theorem 1.3. Let $Q(t) = \int_{\mathbb{R}^n} u(x,t)dx$. Integrating (1.1) and using the Gauss formula, we have

$$\frac{d}{dt}Q(t) \le 0. \tag{2.17}$$

Let $\widetilde{u}(x,t)$ be the solution of the heat equation $\frac{\partial \widetilde{u}}{\partial t} - \Delta \widetilde{u} = 0$, having the same initial data $\widetilde{u}(x,0) = \phi(x,0)$. Since $\mu < 0$, and $f(t,x,u_t) \le 0$, it follows by standard comparison principle that

$$0 \le u(x,t) \le \widetilde{u}(x,t). \tag{2.18}$$

Integrating (1.1) with respect to x and t yield

$$Q(T) = Q(0) + \mu \int_0^T \int_{\mathbb{R}^n} |\nabla u(y,s)|^p \, dy \, ds + \int_0^T \int_{\mathbb{R}^n} f(s,x,u_s) \, dy \, ds$$

by (2.17) and $f(t,x,u) \le 0$ we can easily conclude that

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla u(x,t)|^p \, dx \, dt < \infty. \tag{2.19}$$

Fix $\varepsilon > 0$. It follows from (2.19) that there exists a sequence $1 < t_1 < t_2 < \dots < t_m \to +\infty$ such that

$$\int_{\mathbb{R}^n} |\nabla u(x,t_j)|^p dx \le \varepsilon t_j^{-1},$$

$j = 1, 2, \dots$. Using the Sobolev inequality we have

$$\int_{\mathbb{R}^n} u(x,t_j)^{p^\star} dx \le C_p(\varepsilon t_j^{-1})^{p^\star/p} \tag{2.20}$$

where $p^\star = np/(n-p)$. Now using the condition $p < (n+2)/(n+1)$ it is easy to see that one can choose $\delta > 0$ such that

$$-\frac{1}{p} + (\frac{1}{2} + \delta)n(1 - \frac{1}{p^\star}) < 0, \tag{2.21}$$

then the Hölder inequality and (2.20) imply

$$\int_{|x| \le t_j^{\frac{1}{2}+\delta}} u(x,t_j)dx \le C_{p,n} \varepsilon^{\frac{1}{p}} t_j^{-\frac{1}{p}} t_j^{(\frac{1}{2}+\delta)n(1-\frac{1}{p^\star})},$$

$j = 1, 2, \dots$, so that by (2.21) we have

$$\int_{|x| \le t_j^{\frac{1}{2}+\delta}} u(x,t_j)dx \le C_{p,n}\varepsilon^{1/p}. \tag{2.22}$$

By standard linear parabolic theory, it is easy to see that

$$\int_{|x| \le t_j^{\frac{1}{2}+\delta}} \widetilde{u}(x,t_j)dx \to 0 \tag{2.23}$$

as $j \to \infty$, which in conjunction with (2.17), (2.18) and (2.22) gives

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} u(x,t)dx = 0\,.$$

This completes the proof. □

Remark 2.1. In the paper [4], the authors give the decay results in the critical case $p = (n+2)/(n+1)$ when $f(t,u) \equiv 0$. We think the same result will be true in the critical case, we will study this problem in a later work.

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