*Electronic Journal of Differential Equations*, Vol. 2022 (2022), No. 75, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# POSITIVE SOLUTIONS FOR KIRCHHOFF-SCHRÖDINGER EQUATIONS VIA POHOZAEV MANIFOLD

XIAN HU, YONG-YI LAN

Abstract. In this article we consider the Kirchhoff-Schrödinger equation

$$-\Big((a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\Big)\Delta u+\lambda u=k(x)f(u),\quad x\in\mathbb{R}^3,$$

where $u\in H^1(\mathbb{R}^3)$, $\lambda>0$, $a>0$, $b\geq 0$ are real constants, $k:\mathbb{R}^3\to\mathbb{R}$ and $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$. To overcome the difficulties that $k$ is non-symmetric and the non-linear, and that $f$ is non-homogeneous, we prove the existence a positive solution using projections on a general Pohozaev type manifold, and the linking theorem.

## 1. Introduction and main results

This article concerns the Kirchhoff-Schrödinger equation

$$-\Big((a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\Big)\Delta u+\lambda u=k(x)f(u),\quad x\in\mathbb{R}^3,\tag{1.1}$$

where $u\in H^1(\mathbb{R}^3)$, $\lambda>0$, $a>0$, $b\geq 0$ real constants, $k:\mathbb{R}^3\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$. We use the following assumptions:

- (H1) $k\in\mathcal{C}^1(\mathbb{R}^3,[0,\infty])$, with $k_0=\inf_{x\in\mathbb{R}^3}k(x)>0$;
- (H2) $k_\infty=\lim_{|y|\to\infty}k(y)<\infty$;
- (H3) $t\mapsto k(tx)+\frac{1}{3}\nabla k(tx)\cdot(tx)$ is nondecreasing on $(0,\infty)$ for all $x\in\mathbb{R}^3$;
- (H4) $\nabla k(x)\cdot x\geq 0$ and $k(x)+\frac{1}{3}\nabla k(x)\cdot x\leq(\not\equiv)k_\infty$, for all $x\in\mathbb{R}^3$;
- (H5) $\sup_{\mathbb{R}^3}|k_\infty-k(x)|\leq\beta_0(\int_{\mathbb{R}^3}F(w)\,\mathrm{d}x)^{-1}$, where $\beta_0$ is the unique positive root of the equation
$$t^{2/3}+2(m^\infty)^{1/3}t=(m^\infty)^{2/3};$$
- (H6) $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$, $tf(t)\geq 0$, and there exist $q\in(2,6)$ such that $\lim_{|t|\to\infty}f(t)/|t|^{q-1}=0$;
- (H7) $\lim_{t\to 0}f(t)/t=0$;
- (H8) $f(t)t-4F(t)\geq 0$ for all $t\in\mathbb{R}\backslash\{0\}$, where $F(t)=\int_0^t f(s)\,\mathrm{d}s$.

We look for the weak solutions of (1.1) which are the same as the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2+\lambda u^2)\,\mathrm{d}x+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\Big)^2-\int_{\mathbb{R}^3}k(x)F(u)\,\mathrm{d}x.\tag{1.2}$$

---

2020 *Mathematics Subject Classification.* 35J35, 35B38, 35J92.
*Key words and phrases.* Kirchhoff-Schrödinger equation; Pohozaev manifold; Cerami sequence; linking theorem.

Submitted March 21, 2022. Published November 17, 2022.

If $k(x) \equiv k_\infty$, then (1.1) reduces to the autonomous form

$$-\Big((a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\Big)\Delta u+\lambda u=k_\infty f(u),\quad x\in\mathbb{R}^3, \tag{1.3}$$

with $u\in H^1(\mathbb{R}^3)$. Its energy functional is

$$I^\infty(u)=\frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2+\lambda u^2)\,\mathrm{d}x+\frac{b}{4}(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x)^2-k_\infty\int_{\mathbb{R}^3}F(u)\,\mathrm{d}x. \tag{1.4}$$

Problem (1.1) is related to the stationary analogue of the equation

$$u_{tt}-(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x)\triangle u=0$$

which was proposed by Kirchhoff [8] as an extension of classical D'Alembert's wave equation. It has been applied widely to model various physics problems and appears in some biological systems. The nonlocal term $(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x)\triangle u$, arises in various models of physical and biological systems, and the research for related issues gives rise to more mathematical difficulties and challenges; for more details and backgrounds, we refer the reader to [1, 3, 6] and references therein. After the pioneer work of Lions [12], Kirchhoff type problems began to attract the attention of mathematicians, see for example [10, 11, 21].

Recently, a lots of interesting results for problem (1.1) or similar problems have been obtained, see for example [2, 12, 16, 17, 18, 20] for the radial symmetry case, and [4, 5, 9, 13, 14, 19, 22, 23] for the non-radial symmetry case. As we known, the radial symmetry plays a crucial role since which can restore the compactness of the (PS)-sequence for the energy functional $I$. Salvatore [16] established the existence of multiple radially symmetric solutions with the radially symmetric case where $V$ depends on $|x|$. Wang et al [20] obtained a least-energy sign-changing (or nodal) solution by using constraint variational method and the quantitative deformation lemma. When $b=0$, the existence of solution was obtain by Strauss [17] and Lions [12] if $f$ is superlinear at infinity, also in [2, 18] if $f$ is asymptotically linear at infinity.

For non-radial symmetry case, problem (1.1) with $k(x)>k_\infty>0$ was also solved in [13] by constrained minimization and concentration-compactness arguments. There the role played by the inequality $k(x)>k_\infty$ in restoring compactness in $\mathbb{R}^N$ is used. However, in case $k(x)\le(\not\equiv)k_\infty$ and $f$ is superlinear at infinity, nonsymmetric problem (1.1) cannot be solved by minimization [4]. Che and Chen [5] considered existence and multiplicity of positive solutions by using the Nehari manifold technique and the Ljusternik Schnirelmann category theory. Under proper assumptions, Wang and Zhang [23] obtained a ground state solution for the above problem with the help of Nehari manifold. In [14, 19], the authors studied the existence of ground state solutions of Nehari-Pohozaev type. When $b=0$, [9, 22] studied a class of nonlinear Schrödinger equations by using concentration compactness arguments and projections on a general Pohozaev type manifold.

Motivated by [9, 14, 19, 22], we investigate the existence of nontrivial solutions of problem (1.1). In this article, the main obstacle is that the geometrical hypotheses on the potential $k(x)$ does not allow us to use concentration compactness arguments as in [4, 13]. In general, this difficulty is circumvented by assuming symmetry properties of $k(x)$. Our objective is to prove the existence of a positive solution of (1.1) under $k(x)\le(\not\equiv)k_\infty$ and $k_\infty=\lim_{|x|\to\infty}k(x)$, but not requiring any symmetry properties. Another obstacle is that the nonlinear term in (1.1) is non-homogeneous and non-autonomous. Projections on Nehari manifold are not possible in general, thus one is motivated to use the more suitable projections on the set of points which satisfy the Pohozaev identity [15], the so-called the Pohozaev manifold of (1.1).

Let $a > 0$ and $b \geq 0$ be fixed. Throughout the paper we use the following notation:

$H^1(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the norm

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^3} (a|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x.$$

$L^s(\mathbb{R}^3)$ $(1 \leq s < \infty)$ denotes the Lebesgue space with the norm

$$\|u\|_s^s = \int_{\mathbb{R}^3} |u|^s\,\mathrm{d}x.$$

For $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, $u_t(x) = u(x/t)$ for $t > 0$. For $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) = \{y \in \mathbb{R}^3 : |y - x| < r\}$. We denote various positive constants as $c, c_i, C, C_i$ $(i = 0, 1, 2, 3, \dots)$.

To state our results, we define two functionals on $H^1(\mathbb{R}^3)$ as follows:

$$\begin{aligned} P(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x + \frac{3\lambda}{2}\int_{\mathbb{R}^3} \lambda u^2\,\mathrm{d}x + \frac{b}{2}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\Big)^2 \\ - \int_{\mathbb{R}^3} [3k(x) + \nabla k(x)\cdot x] F(u)\,\mathrm{d}x\,, \end{aligned} \tag{1.5}$$

$$\begin{aligned} P^\infty(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x + \frac{3\lambda}{2}\int_{\mathbb{R}^3} u^2\,\mathrm{d}x + \frac{b}{2}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\Big)^2 \\ - 3k_\infty \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x. \end{aligned} \tag{1.6}$$

We define the Pohozaev manifold associated with (1.1) and (1.3) by

$$\mathcal{M} = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : P(u) = 0\}, \tag{1.7}$$

$$\mathcal{M}^\infty = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : P^\infty(u) = 0\}. \tag{1.8}$$

We are now in position to state and prove our main result.

**Theorem 1.1.** *Under assumptions* (H1)–(H8), *problem* (1.1) *has a positive solution* $u \in H^1(\mathbb{R}^3)\backslash\{0\}$.

## 2. Proof of Theorem 1.1

**Lemma 2.1.** *Suppose that* $\int_{\mathbb{R}^3}[\frac{\lambda u^2}{2} - k_\infty F(u)]\,\mathrm{d}x < 0$. *Then there exists unique* $t_u > 0$ *and* $t_{u^*} > 0$ *such that* $u_{t_u} \in \mathcal{M}$ *and* $u_{t_{u^*}} \in \mathcal{M}^\infty$.

*Proof.* First we define the function

$$\begin{aligned} \psi(t) &= I(u_t) \\ &= \frac{at}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x + \frac{\lambda t^3}{2}\int_{\mathbb{R}^3} u^2\,\mathrm{d}x + \frac{bt^2}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\Big)^2 \\ &\quad - t^3 \int_{\mathbb{R}^3} k(tx) F(u)\,\mathrm{d}x. \end{aligned}$$

Taking the derivative of $\psi(t)$, we obtain

$$\begin{aligned}\psi'(t) &= \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x + \frac{3\lambda t^2}{2}\int_{\mathbb{R}^3}u^2\,\mathrm{d}x + \frac{bt}{2}\Big(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\Big)^2\\ &\quad - 3t^2\int_{\mathbb{R}^3}k(tx)F(u)\,\mathrm{d}x - t^3\int_{\mathbb{R}^3}\nabla k(tx)\cdot xF(u)\,\mathrm{d}x\\ &= \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x + \frac{bt}{2}\Big(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\Big)^2 + 3t^2\int_{\mathbb{R}^3}[\frac{\lambda u^2}{2} - k(tx)F(u)]\,\mathrm{d}x\\ &\quad - t^3\int_{\mathbb{R}^3}\nabla k(tx)\cdot xF(u)\,\mathrm{d}x.\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{t\to\infty}\int_{\mathbb{R}^3}[\frac{\lambda u^2}{2} - k(tx)F(u)]\,\mathrm{d}x = \int_{\mathbb{R}^3}[\frac{\lambda u^2}{2} - k_\infty F(u)]\,\mathrm{d}x < 0.$$

By (H2) and (H4), we have

$$\nabla k(x)\cdot x \to 0, \quad \text{as } |x|\to\infty. \tag{2.1}$$

Using again the Lebesgue Dominated Convergence Theorem,

$$\lim_{t\to\infty}\int_{\mathbb{R}^3}\nabla k(tx)\cdot(tx)F(u)\,\mathrm{d}x = 0.$$

where we have used (H6) and (H7). Therefore, if $t > 0$ is sufficiently large, then $\psi'(t) < 0$. On the other hand, taking $t > 0$ sufficiently small in the expression of $\psi'(t)$, we obtain $\psi'(t) > 0$. Since $\psi'$ is continuous, there exists at least one $t_u > 0$ such that $\psi'(t_u) = 0$. Then $P(u_{t_u}) = t\psi'(t_u) = 0$ so that $u_{t_u}\in\mathcal{M}$.

Moreover (H3) implies that

$$3t^3[k(x) - k(tx)] + (t^3-1)\nabla k(x)\cdot x \le 0, \quad \forall t\ge 0,\ x\in\mathbb{R}^3. \tag{2.2}$$

By this inequality, (H6) and (H7), for any $u\in H^1(\mathbb{R}^3), t>0$, one has

$$\begin{aligned}&I(u) - I(u_t)\\ &= \frac{a(1-t)}{2}\|\nabla u\|_2^2 + \frac{\lambda(1-t^3)}{2}\|u\|_2^2 + \frac{b(1-t^2)}{4}\|\nabla u\|_2^4\\ &\quad - \int_{\mathbb{R}^3}[k(x) - t^3k(tx)]F(u)\,\mathrm{d}x\\ &= \frac{1-t^3}{3}P(u) + \frac{a(t^3-3t+2)}{6}\|\nabla u\|_2^2 + \frac{b(2t^3-3t^2+1)}{12}\|\nabla u\|_2^4\\ &\quad - \frac{1}{3}\int_{\mathbb{R}^3}[3t^3(k(x)-k(tx)) + (t^3-1)\nabla k(x)\cdot x]F(u)\,\mathrm{d}x\\ &\ge \frac{1-t^3}{3}P(u) + \frac{a(t^3-3t+2)}{6}\|\nabla u\|_2^2 + \frac{b(2t^3-3t^2+1)}{12}\|\nabla u\|_2^4.\end{aligned} \tag{2.3}$$

Next we claim that $t_u$ is unique. In fact, for any given $u$ satisfies $\int_{\mathbb{R}^3}[\frac{\lambda u^2}{2} - k_\infty F(u)]\,\mathrm{d}x < 0$. Let $t_1, t_2 > 0$ such that $u_{t_1}, u_{t_2}\in\mathcal{M}$. Then $P(u_{t_1}) = P(u_{t_2}) = 0$. From this and (2.3), we have

$$\begin{aligned}I(u_{t_1}) &\ge I(u_{t_2}) + \frac{t_1^3 - t_2^3}{3t_1^3}P(u_{t_1}) + \frac{a(2t_1^3 - 3t_1^2t_2 + t_2^3)}{6t_1^3}\|\nabla u_{t_1}\|_2^2\\ &\quad + \frac{b(3t_1^4 - 3t_1^2t_2^2 - 2t_1^3 + 2t_2^3)}{12t_1^2}\|\nabla u_{t_1}\|_2^4\end{aligned}$$

$$= I(u_{t_2}) + \frac{a(2t_1^3 - 3t_1^2 t_2 + t_2^3)}{6t_1^3}\|\nabla u_{t_1}\|_2^2 + \frac{b(3t_1^4 - 3t_1^2 t_2^2 - 2t_1^3 + 2t_2^3)}{12t_1^2}\|\nabla u_{t_1}\|_2^4$$

and

$$\begin{aligned}
I(u_{t_2}) &\geq I(u_{t_1}) + \frac{t_2^3 - t_1^3}{3t_2^3} P(u_{t_2}) + \frac{a(2t_2^3 - 3t_2^2 t_1 + t_1^3)}{6t_2^3}\|\nabla u_{t_2}\|_2^2 \\
&\quad + \frac{b(3t_2^4 - 3t_2^2 t_1^2 - 2t_2^3 + 2t_1^3)}{12t_2^2}\|\nabla u_{t_2}\|_2^4 \\
&= I(u_{t_1}) + \frac{a(2t_2^3 - 3t_2^2 t_1 + t_1^3)}{6t_2^3}\|\nabla u_{t_2}\|_2^2 + \frac{b(3t_2^4 - 3t_2^2 t_1^2 - 2t_2^3 + 2t_1^3)}{12t_2^2}\|\nabla u_{t_2}\|_2^4.
\end{aligned}$$

These inequalities above imply $t_1 = t_2$. Therefore, $t_u > 0$ is unique.

Similarly, we define the function

$$\begin{aligned}
\varphi(t) &= I^\infty(u_t) \\
&= \frac{at}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \frac{\lambda t^3}{2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{bt^2}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\Big)^2 - k_\infty t^3 \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x.
\end{aligned}$$

Taking the derivative of $\psi(t)$, we obtain

$$\begin{aligned}
\varphi'(t) &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \frac{3\lambda t^2}{2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{bt}{2}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\Big)^2 - 3t^2 k_\infty \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x \\
&= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \frac{bt}{2}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\Big)^2 + 3t^2 \int_{\mathbb{R}^3} [\frac{\lambda u^2}{2} - k_\infty F(u)]\,\mathrm{d}x.
\end{aligned}$$

Therefore, if $t > 0$ is sufficiently large, then $\varphi'(t) < 0$. Taking $t > 0$ sufficiently small, we obtain $\varphi'(t) > 0$. Since $\varphi'$ is continuous, there exists at least one $t_{u^*} > 0$ such that $\varphi'(t_{u^*}) = 0$. Then $P^\infty(u_{t_{u^*}}) = t\varphi'(t_{u^*}) = 0$ so that $u_{t_{u^*}} \in \mathcal{M}^\infty$. For any $u \in H^1(\mathbb{R}^3), t > 0$, one has

$$\begin{aligned}
&I^\infty(u) - I^\infty(u_t) \\
&= \frac{a(1-t)}{2}\|\nabla u\|_2^2 + \frac{\lambda(1-t^3)}{2}\|u\|_2^2 + \frac{b(1-t^2)}{4}\|\nabla u\|_2^4 \\
&\quad - k_\infty(1-t^3)\int_{\mathbb{R}^3} F(u)\,\mathrm{d}x \\
&= \frac{1-t^3}{3}P^\infty(u) + \frac{a(t^3 - 3t + 2)}{6}\|\nabla u\|_2^2 + \frac{b(2t^3 - 3t^2 + 1)}{12}\|\nabla u\|_2^4 \\
&= \frac{1-t^3}{3}P^\infty(u) + \frac{a(t^3 - 3t + 2)}{6}\|\nabla u\|_2^2 + \frac{b(2t^3 - 3t^2 + 1)}{12}\|\nabla u\|_2^4.
\end{aligned} \tag{2.4}$$

Now we claim that $t_{u^*}$ is unique. In fact, each $u$ satisfies $\int_{\mathbb{R}^3}[\frac{\lambda u^2}{2} - k_\infty F(u)]\,\mathrm{d}x < 0$. Let $t_3, t_4 > 0$ such that $u_{t_3}, u_{t_4} \in \mathcal{M}^\infty$. Then $P^\infty(u_{t_3}) = P^\infty(u_{t_4}) = 0$. From this and (2.4), we have

$$\begin{aligned}
&I^\infty(u_{t_3}) \\
&= I^\infty(u_{t_4}) + \frac{t_3^3 - t_4^3}{3t_3^3}P^\infty(u_{t_3}) + \frac{a(2t_3^3 - 3t_3^2 t_4 + t_4^3)}{6t_3^3}\|\nabla u_{t_3}\|_2^2 \\
&\quad + \frac{b(3t_3^4 - 3t_3^2 t_4^2 - 2t_3^3 + 2t_4^3)}{12t_3^2}\|\nabla u_{t_3}\|_2^4 \\
&= I^\infty(u_{t_4}) + \frac{a(2t_3^3 - 3t_3^2 t_4 + t_4^3)}{6t_3^3}\|\nabla u_{t_3}\|_2^2 + \frac{b(3t_3^4 - 3t_3^2 t_4^2 - 2t_3^3 + 2t_4^3)}{12t_3^2}\|\nabla u_{t_3}\|_2^4
\end{aligned}$$

and

$$
\begin{aligned}
&I^\infty(u_{t_4})\\
&= I^\infty(u_{t_3}) + \frac{t_4^3 - t_3^3}{3t_4^3} P^\infty(u_{t_4}) + \frac{a(2t_4^3 - 3t_4^2 t_3 + t_3^3)}{6t_4^3} \|\nabla u_{t_4}\|_2^2\\
&\quad + \frac{b(3t_4^4 - 3t_4^2 t_3^2 - 2t_4^3 + 2t_3^3)}{12t_4^2} \|\nabla u_{t_4}\|_2^4\\
&= I^\infty(u_{t_3}) + \frac{a(2t_4^3 - 3t_4^2 t_3 + t_3^3)}{6t_4^3} \|\nabla u_{t_4}\|_2^2 + \frac{b(3t_4^4 - 3t_4^2 t_3^2 - 2t_4^3 + 2t_3^3)}{12t_4^2} \|\nabla u_{t_4}\|_2^4.
\end{aligned}
$$

The two inequalities above imply $t_3 = t_4$. Therefore, $t_{u^*} > 0$ is unique. $\square$

**Lemma 2.2.** *If $u \in \mathcal{M}^\infty$, then there exists $t_u \geq 1$ such that $u_{t_u} \in \mathcal{M}$.*

*Proof.* Since $u \in \mathcal{M}^\infty$, we have

$$
P^\infty(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{3\lambda}{2}\|u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 - 3k_\infty \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x = 0. \tag{2.5}
$$

In view of Lemma 2.1, there exists $t_u > 0$ such that $u_{t_u} \in \mathcal{M}$. From (H4), (H6), and (H7), one has

$$
\begin{aligned}
0 &= P(u_{t_u})\\
&= \frac{at_u}{2}\|\nabla u\|_2^2 + \frac{3\lambda t_u^3}{2}\|u\|_2^2 + \frac{bt_u^2}{2}\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} [3k(t_u x) + \nabla k(t_u x)\cdot(t_u x)]F(u)\,\mathrm{d}x\\
&= \frac{at_u}{2}\|\nabla u\|_2^2 + t_u^3\Big(-\frac{a}{2}\|\nabla u\|_2^2 - \frac{b}{2}\|\nabla u\|_2^4 + 3k_\infty \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x\Big)\\
&\quad + \frac{bt_u^2}{2}\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} [3k(t_u x) + \nabla k(t_u x)\cdot(t_u x)]F(u)\,\mathrm{d}x\\
&= \frac{a(t_u - t_u^3)}{2}\|\nabla u\|_2^2 + \frac{b(t_u^2 - t_u^3)}{2}\|\nabla u\|_2^4\\
&\quad + t_u^3 \int_{\mathbb{R}^3} [3(k_\infty - k(t_u x)) - \nabla k(t_u x)\cdot(t_u x)]F(u)\,\mathrm{d}x\\
&\geq \frac{a(t_u - t_u^3)}{2}\|\nabla u\|_2^2 + \frac{b(t_u^2 - t_u^3)}{2}\|\nabla u\|_2^4,
\end{aligned}
$$

which implies $t_u \geq 1$. $\square$

**Lemma 2.3.** *If $u \in \mathcal{M}$, then there exists $t_u \in (0,1]$ such that $u_{t_u} \in \mathcal{M}^\infty$.*

*Proof.* Since $u \in \mathcal{M}$, we have

$$
P(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{3\lambda}{2}\|u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} [3k(x) + \nabla k(x)\cdot x]F(u)\,\mathrm{d}x = 0.
$$

In view of Lemma 2.1, there exists $t_u > 0$ such that $u_{t_u} \in \mathcal{M}^\infty$. From (H4), (H6) and (H7), one has

$$
\begin{aligned}
0 &= P^\infty(u_{t_u})\\
&= \frac{at_u}{2}\|\nabla u\|_2^2 + \frac{3\lambda t_u^3}{2}\|u\|_2^2 + \frac{bt_u^2}{2}\|\nabla u\|_2^4 - 3k_\infty \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x\\
&= \frac{at_u}{2}\|\nabla u\|_2^2 + t_u^3\Big(-\frac{a}{2}\|\nabla u\|_2^2 - \frac{b}{2}\|\nabla u\|_2^4 + \int_{\mathbb{R}^3} [3k(x) + \nabla k(x)\cdot x]F(u)\,\mathrm{d}x\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{bt_u^2}{2}\|\nabla u\|_2^4 - 3k_\infty \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x \\
&= \frac{a(t_u - t_u^3)}{2}\|\nabla u\|_2^2 + \frac{b(t_u^2 - t_u^3)}{2}\|\nabla u\|_2^4 \\
&+ t_u^3 \int_{\mathbb{R}^3} [3(k(x) - k_\infty) + \nabla k(x)\cdot x] F(u)\,\mathrm{d}x \\
&\le \frac{a(t_u - t_u^3)}{2}\|\nabla u\|_2^2 + \frac{b(t_u^2 - t_u^3)}{2}\|\nabla u\|_2^4
\end{aligned}
$$

which implies $t_u \le 1$. Therefore $t_u \in (0,1]$. □

**Lemma 2.4.** *If $u \in \mathcal{M}^\infty$, then $u(\cdot - y) \in \mathcal{M}^\infty$ for all $y \in \mathbb{R}^3$. Moreover, for every $y \in \mathbb{R}^3$, there exists $t_y \ge 1$ such that $u_{t_y}(\cdot - y) \in \mathcal{M}$ and $\lim_{|y|\to\infty} t_y = 1$.*

*Proof.* If $u \in \mathcal{M}^\infty$, then from the translation invariance of $I^\infty$ it follows that $u(\cdot - y) \in \mathcal{M}^\infty$ for all $y \in \mathbb{R}^3$. Furthermore, from Lemma 2.2 there exists $t_y \ge 1$ such that $u_{t_y}(\cdot - y) \in \mathcal{M}$. By (2.1) and the Lebesgue Dominated Convergence Theorem, we have

$$
\begin{aligned}
0 &= \liminf_{|y|\to\infty} t_y^{-3} P(u_{t_y}(\cdot - y)) \\
&= \liminf_{|y|\to\infty} [\frac{at_y^{-2}}{2}\|\nabla u\|_2^2 + \frac{3\lambda}{2}\|u\|_2^2 + \frac{bt_y^{-1}}{2}\|\nabla u\|_2^4] \\
&\quad - \liminf_{|y|\to\infty} \int_{\mathbb{R}^3} [3k(t_y x + y) + \nabla k(t_y x + y)\cdot (t_y x + y)] F(u)\,\mathrm{d}x \\
&= \liminf_{|y|\to\infty} [\frac{at_y^{-2}}{2}\|\nabla u\|_2^2 + \frac{bt_y^{-1}}{2}\|\nabla u\|_2^4 - \frac{a}{2}\|\nabla u\|_2^2 - \frac{b}{2}\|\nabla u\|_2^4 + 3\int_{\mathbb{R}^3} k_\infty(x) F(u)\,\mathrm{d}x] \\
&\quad - \liminf_{|y|\to\infty} \int_{\mathbb{R}^3} [3k(t_y x + y) + \nabla k(t_y x + y)\cdot (t_y x + y)] F(u)\,\mathrm{d}x \\
&= \liminf_{|y|\to\infty} [\frac{a(t_y^{-2} - 1)}{2}\|\nabla u\|_2^2 + \frac{b(t_y^{-1} - 1)}{2}\|\nabla u\|_2^4] \\
&\quad + \liminf_{|y|\to\infty} \int_{\mathbb{R}^3} 3[k_\infty - k(t_y x + y) - \frac{1}{3}\nabla k(t_y x + y)\cdot (t_y x + y)] F(u)\,\mathrm{d}x \\
&= \frac{a}{2}(\liminf_{|y|\to\infty} t_y^{-2} - 1)\|\nabla u\|_2^2 + \frac{b}{2}(\liminf_{|y|\to\infty} t_y^{-1} - 1)\|\nabla u\|_2^4
\end{aligned}
$$

which implies $\limsup_{|y|\to\infty} t_y = 1$, and so $\lim_{|y|\to\infty} t_y = 1$. □

From Jeanjean and Tanaka [7] have that

$$
\inf_{u \in \mathcal{M}^\infty} I^\infty(u) = m^\infty.
$$

**Lemma 2.5.** $m = m^\infty$.

*Proof.* Let $u \in H^1(\mathbb{R}^3)$ be the ground state solution (which is positive and radially symmetric) of the problem at infinity, $u \in \mathcal{M}^\infty$ and $I^\infty(u) = m^\infty$. From the translation invariance of the integrals, given any $y \in \mathbb{R}^3$ such that $u(\cdot - y) \in \mathcal{M}^\infty$, $I^\infty(u(\cdot - y)) = m^\infty$. From Lemma 2.4, for any $y \in \mathbb{R}^3$, there exists a $t_y \ge 1$ such that $u_{t_y}(\cdot - y) \in \mathcal{M}$. Therefore,

$$
|I(u_{t_y}\cdot(-y)) - m^\infty|
$$

$$
\begin{aligned}
&= |I(u_{t_y}\cdot(-y)) - I^\infty(u\cdot(-y))| \\
&= |\frac{a(t_y-1)}{2}\|\nabla u\|_2^2 + \frac{b(t_y^2-1)}{4}\|\nabla u\|_2^4 + \frac{\lambda(t_y^3-1)}{2}\int_{\mathbb{R}^3} u^2\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}(k_\infty - t_y^3 k(t_y x+y))F(u)\,\mathrm{d}x| \\
&\le |\frac{a(t_y-1)}{2}\|\nabla u\|_2^2| + |\frac{\lambda(t_y^3-1)}{2}\int_{\mathbb{R}^3} u^2\,\mathrm{d}x| + \int_{\mathbb{R}^3}|k_\infty - t_y^3 k(t_y x+y)||F(u)|\,\mathrm{d}x.
\end{aligned}
$$

Since $t_y \to 1$ as $|y|\to\infty$, it follows that

$$
|I(u_{t_y}\cdot(-y)) - m^\infty| \le o_y(1) + o_y(1) + \int_{\mathbb{R}^3}|k_\infty - k(x+y)||F(u)|\,\mathrm{d}x.
$$

and since $k(x+y)\to k_\infty$ as $|y|\to\infty$, it follows that

$$
\lim_{|y|\to\infty} I(u_{t_y}\cdot(-y)) = m^\infty.
$$

Therefore, $m = \inf_{u\in\mathcal{M}} I(u) \le m^\infty$.

On the other hand, we consider $u\in\mathcal{M}$ and $0<t_y\le 1$ such that $u_{t_y}\in\mathcal{M}^\infty$. Since $u\in\mathcal{M}$, then $P(u)=0$ and $u$ satisfies

$$
\begin{aligned}
m = I(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{\lambda}{2}\|u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_{\mathbb{R}^3} k(x)F(u)\,\mathrm{d}x \\
&= \frac{1}{3}P(u) + \frac{a}{3}\|\nabla u\|_2^2 + \frac{b}{12}\|\nabla u\|_2^4 + \frac{1}{3}\int_{\mathbb{R}^3}\nabla k(x)\cdot xF(u)\,\mathrm{d}x \\
&\ge \frac{at_y}{3}\|\nabla u\|_2^2 + \frac{bt_y^2}{12}\|\nabla u\|_2^4 \\
&\ge I^\infty(u_{t_y}) - \frac{1}{3}P^\infty(u_{t_y}) \\
&= I^\infty(u_{t_y}) \\
&\ge m^\infty
\end{aligned}
$$

where we have used (H4) and (H6). Thus, for any $u\in\mathcal{M}$, $I(u)\ge m^\infty$ and hence $\inf_{u\in\mathcal{M}} I(u)\ge m^\infty$. We conclude that $m=m^\infty$. □

**Lemma 2.6.** *The functional $I$ satisfies condition $(Ce)$ at level $d\in(m^\infty, 2m^\infty)$.*

*Proof.* Since $\{u_n\}\subset H^1(\mathbb{R}^3)$ is a Cerami sequence $(Ce)_d$, by (H8), we have

$$
\begin{aligned}
d + o(1) &= I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^3} a|\nabla u_n|^2 + \lambda u_n^2\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} a(f(u_n)u_n - 4F(u_n))\,\mathrm{d}x \\
&\ge \frac{1}{4}\|u_n\|_\lambda^2.
\end{aligned}
$$

This shows $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Applying the splitting lemma cite[Lemma 4.6]l1, up to subsequences, we have

$$
u_n - \sum_{j=1}^k u^j(x-y_n^j) \to u \quad \text{in } H^1(\mathbb{R}^3),
$$

where $u^j$ is a weak solution of the problem at infinity, $|y_n^j| \to \infty$ and $u$ is a weak solution of (1.1). Moreover,

$$I(u_n) = I(u) + \sum_{j=1}^{k} I^\infty(u_j) + o_n(1).$$

Since $d < 2m^\infty$, it follows that $k < 2$. If $k = 1$, we have two cases to distinguish:

(1) $u \neq 0$, which implies $I(u) \geq m^\infty$ and hence $I(u_n) \geq 2m^\infty$.
(2) $u = 0$, which yields $I(u_n) \to I^\infty(u_1)$.

In both cases we arrive at a contradiction with the fact that $d \in (m^\infty, 2m^\infty)$. Therefore, we must have $k = 0$ and the convergence $u_n \to u$ follows. □

**Definition 2.7.** Define the barycenter function of a given function $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ as follows: let

$$\mu(u)(x) = \frac{1}{|B_1|} \int_{B_1(x)} |u(y)| \,\mathrm{d}y,$$

with $\mu(u) \in L^\infty(\mathbb{R}^3))$ and $\mu$ is a continuous function. Subsequently, take

$$\hat{\mu}(u)(x) = [\mu(u)(x) - \frac{1}{2} \max \mu(u)]^+.$$

It follows that $\hat{u} \in \mathcal{C}_0(\mathbb{R}^3)$. Now define the barycenter of $u$ by

$$\beta(u)(x) = \frac{1}{\|\hat{u}\|} \int_{\mathbb{R}^3} x\hat{u}(x) \,\mathrm{d}x \in \mathbb{R}^3.$$

Since $\hat{u}$ has compact support, by definition, $\beta(u)$ is well defined.

Now we define

$$b = \inf\{I(u) : u \in \mathcal{M},\ \beta(u) = 0\}.$$

It is clear that $b \geq m^\infty$.

**Lemma 2.8.** *$b > m^\infty$.*

*Proof.* By contradiction, suppose that $b = m^\infty$. By the definition of $b$, there exists a (minimizing) sequence $\{u_n\} \in \{u \in \mathcal{M},\ \beta(u) = 0\}$ such that $I(u_n) \to b$. By Lemma 2.8, the sequence $\{u_n\}$ is bounded. Since $m = m^\infty$ by Lemma 2.6, then $\{u_n\}$ is also a minimizing sequence of $I$ on $\mathcal{M}$. By Ekeland Variational Principle [24, Theorem 8.5] there exists another sequence $\{\tilde{u}_n\} \in \mathcal{M}$ such that:

(i) $I(\tilde{u}_n) \to m$;
(ii) $I'(\tilde{u}_n) \to 0$;
(iii) $\|\tilde{u}_n - u_n\| \to 0$.

Moreover, $\{u_n\}$ is bounded, $\beta(u_n) = 0$ and $\|\tilde{u}_n - u_n\| \to 0$ imply that the sequence $\{\tilde{u}_n\}$ is bounded and $|\beta(\tilde{u}_n) - \beta(u_n)| \to 0$, since $\beta$ is a continuous function. So we have that $\beta(\tilde{u}_n)$ is bounded.

Therefore, the sequence $\{\tilde{u}_n\}$ satisfies the assumptions of [9, Corollary 4.8] and since $m = m^\infty$ and is not attained, then the splitting lemma holds with $k = 1$. This yields

$$\tilde{u}_n(x) \to u^1(x - y_n),$$

where $y_n \in \mathbb{R}^3$, $|y| \to \infty$, and $u^1$ is a solution of the problem at infinity. By making a translation, we obtain

$$\tilde{u}_n(x + y_n) = u^1(x) + o_n(1).$$

Calculating the barycenter function on both sides, we have

$$\beta(\tilde{u}_n(x+y_n)) = \beta(\tilde{u}_n) - y_n,$$

where $\beta(\tilde{u}_n)$ is bounded and

$$\beta(u^1(x) + o_n(1)) \to \beta(u^1(x)),$$

since $\beta$ is a continuous function. On one side, $\beta(u^1(x))$ is a fixed real value and, on the other, $|y_n| \to \infty$ so we arrive at a contradiction. Therefore, we must have $b > m^\infty$. □

Inspired by [9], let $w \in H^1(\mathbb{R}^3)$ be the positive, radially symmetric, ground state solution of (1.3). We define the operator $\Pi : \mathbb{R}^3 \to \mathcal{M}$ by

$$\Pi[y](x) = w(\frac{x-y}{t_y}) = w_{t_y}(x-y).$$

*Proof of Theorem 1.1.* By Lemma 2.2, for any $w \in \mathcal{M}^\infty$, then there exists $t_y \geq 1$ such that $w_{t_y} = \Pi[y] \in \mathcal{M}$. Therefore $P(\Pi[y]) = 0$ for any $y \in \mathbb{R}^3$, and we have

$$\begin{aligned}
&I(\Pi[y]) \\
&= \frac{a}{2}\|\nabla\Pi[y]\|_2^2 + \frac{\lambda}{2}\|\Pi[y]\|_2^2 + \frac{b}{4}\|\nabla\Pi[y]\|_2^4 - \int_{\mathbb{R}^3} k(x)F(\Pi[y])\,\mathrm{d}x \\
&= \frac{1}{3}P(\Pi[y]) + \frac{a}{3}\|\nabla\Pi[y]\|_2^2 + \frac{b}{12}\|\nabla\Pi[y]\|_2^4 + \frac{1}{3}\int_{\mathbb{R}^3} \nabla k(x)\cdot xF(\Pi[y])\,\mathrm{d}x \\
&= \frac{a}{3}\|\nabla\Pi[y]\|_2^2 + \frac{b}{12}\|\nabla\Pi[y]\|_2^4 + \frac{1}{3}\int_{\mathbb{R}^3} \nabla k(x)\cdot xF(\Pi[y])\,\mathrm{d}x \\
&= \frac{at_y}{3}\|\nabla w\|_2^2 + \frac{bt_y^2}{12}\|\nabla w\|_2^4 + \frac{t_y^3}{3}\int_{\mathbb{R}^3} \nabla k(t_yx+y)\cdot(t_yx+y)F(w)\,\mathrm{d}x.
\end{aligned} \tag{2.6}$$

Moreover, since $w \in \mathcal{M}^\infty$, we have

$$\begin{aligned}
I^\infty &= \frac{1}{3}P^\infty(w) + \frac{at_y}{3}\|\nabla w\|_2^2 + \frac{bt_y^2}{12}\|\nabla w\|_2^4 \\
&= \frac{at_y}{3}\|\nabla w\|_2^2 + \frac{bt_y^2}{12}\|\nabla w\|_2^4.
\end{aligned}$$

Combing (2.6) and the above equality yields

$$I(\Pi[y]) = I^\infty + \frac{t_y^3}{3}\int_{\mathbb{R}^3} \nabla k(t_yx+y)\cdot(t_yx+y)F(w)\,\mathrm{d}x.$$

By (2.2), it follows that $I(\Pi[y]) \to m^\infty$, as $|y| \to \infty$. In view of Lemma 2.8, we have $b > m^\infty$. Then there exists $\bar{\rho} > 0$ such that for every $\rho \geq \bar{\rho}$,

$$m^\infty < \max_{|y|=\rho} I(\Pi[y]) < b.$$

To apply the Linking Theorem, we take $Q = \Pi(B_{\bar{\rho}}(0))$ and $S = \{u \in \mathcal{M} : \beta(u) = 0\}$. From [9, Lemma 4.13], we have

$$\beta(\Pi[y](x)) = y, \quad \forall y \in \mathbb{R}^3.$$

If $u \in S$, then $\beta(u) = 0$, and if $u \in \partial Q$, then $\beta(u) = y \neq 0$, because of equality $|y| = \bar{\rho}$; therefore $\partial Q \cap S = \emptyset$.

For any $h \in \mathcal{H} = \{h \in C(Q, \mathcal{M}) : h|_{\partial Q} = id\}$, we define $\mathcal{T} : B_{\bar{\rho}(0)} \to \mathbb{R}^3$ as $T[y] = \beta \circ h \circ \Pi[y]$. The function $\mathcal{T}$ is continuous. Moreover, for any $|y| = \bar{\rho}$, we have $\Pi[y] \in \partial Q$, thus $h \circ \Pi[y] = \Pi[y], \mathcal{T}(y) = \beta(\Pi[y]) = y$. By Brouwer's Fixed Point Theorem we conclude that there exists $\tilde{y} \in B_{\bar{\rho}}(0)$ such that $\mathcal{T}(\tilde{y}) = 0$, which implies $h(\Pi[\tilde{y}]) \in S$. Therefore $h(Q) \cap S \neq \emptyset$ and $S$ and $\partial Q$ link.

If $h$ is fixed, then there exists $z \in S$ such that $z$ also belongs to $h(Q)$, which means that $z = h(v)$ form some $v \in \Pi(B_{\bar{\rho}}(0))$. Therefore,

$$I(z) \geq \inf_{u \in S} I(u) \text{ and } \max_{u \in Q} I(h(u)) \geq I(h(v)).$$

This gives

$$\max_{u \in Q} I(h(u)) \geq I(h(v)) = I(z) \geq \inf_{u \in S} I(u) = b,$$

and hence

$$d = \inf_{h \in \mathcal{H}} \max_{u \in Q} I(h(u)) \geq b > m^\infty.$$

Since $w \in \mathcal{M}^\infty$ and $m^\infty = I^\infty(w)$, it follows that $m^\infty = \frac{a}{3}\|\nabla w\|_2^2 + \frac{b}{12}\|\nabla w\|_2^4$, and

$$P^\infty(w) = \frac{a}{2}\|\nabla w\|_2^2 + \frac{3\lambda}{2}\|w\|_2^2 + \frac{b}{2}\|\nabla w\|_2^4 - 3k_\infty \int_{\mathbb{R}^3} F(w)\,\mathrm{d}x = 0.$$

We set

$$t_* = [\frac{m^\infty}{m^\infty - 2\beta_0}]^{1/2}.$$

Since $\beta_0$ is the unique positive root of (H5), then $1 < t_* < \infty$. Hence

$$\begin{aligned}
I(\Pi[y]) &= \frac{at_y}{2}\|\nabla w\|_2^2 + \frac{\lambda t_y^3}{2} + \frac{bt_y^2}{4}\|\nabla w\|_2^4 - t_y^3 \int_{\mathbb{R}^3} k(t_y x + y)F(w)\,\mathrm{d}x \\
&= t_y^3 P^\infty(w) + \frac{a(3t_y - t_y^3)}{6}\|\nabla w\|_2^2 + \frac{b(3t_y^2 - 2t_y^3)}{12}\|\nabla w\|_2^4 \\
&\quad + t_y^3 \int_{\mathbb{R}^3} [k_\infty - k(t_y x + y)]F(w)\,\mathrm{d}x \\
&\leq \frac{a(3t_y - t_y^3)}{6}\|\nabla w\|_2^2 + \frac{b(3t_y^2 - 2t_y^3)}{12}\|\nabla w\|_2^4 + \beta_0 t_y^3 \\
&\leq \frac{a(3t_* - t_*^3)}{6}\|\nabla w\|_2^2 + \frac{b(3t_*^2 - 2t_*^3)}{12}\|\nabla w\|_2^4 + \beta_0 t_*^3 \\
&< \frac{a}{3}\|\nabla w\|_2^2 + \frac{b}{12}\|\nabla w\|_2^4 + \beta_0[\frac{m^\infty}{m^\infty - 2\beta_0}]^{3/2} \\
&= 2m^\infty.
\end{aligned}$$

Furthermore, if we take $h = id$, then

$$d = \inf_{h \in \mathcal{H}} \max_{u \in Q} I(h(u)) < \max_{u \in Q} I(u) < 2m^\infty.$$

Then we have $d \in (m^\infty, 2m^\infty)$, thus from Lemma 2.6, (Ce) condition is satisfied at level $d$. Therefore, we can apply the Linking Theorem and conclude that $d$ is a critical level for the functional $I$ . This guarantees the existence of a nontrivial solution $u \in H^1(\mathbb{R}^3)$ of (1.1). Reasoning as usual, because of the hypotheses on $f$, and using the maximum principle we may conclude that $u$ is positive, which implies the proof. □

**Acknowledgments.** This work was supported by the Natural Science Foundation of Fujian Province (No. 2022J01339; No. 2020J01708), and by the National Foundation Training Program of Jimei University (ZP2020057).

## References


[1] A. Arosio, S. Panizzi; *On the well-posedness of the Kirchhoff string*, Trans. Am. Math. Soc., 348 (1996), 305–330.

[2] A. Azzollini, A. Pomponio; *On the Schrödinger equation in* $\mathbb{R}^N$ *under the effect of a general nonlinear term*, Indiana Univ. Math. J., 58(3) (2009), 1361–1378.

[3] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano; *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation*, Adv. Differ. Equ., 6 (2001), 701–730.

[4] G. Cerami; *Some nonlinear elliptic problems in unbounded domains*, Milan J. Math., 74 (2006), 47–77.

[5] G. F. Che, H. B. Chen; *Existence and multiplicity of positive solutions for Kirchhoff-Schrödinger-Poisson system with critical growth*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A, Matematicas, 114 (2020), 78.

[6] M. Chipot, B. Lovat; *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal., 30 (7) (1997), 4619–4627.

[7] L. Jeanjean, K. Tanaka; *A remark on least energy solutions in* $\mathbb{R}^N$, Proc. Amer. Math. Soc.,131(8) (2002), 2399–2408.

[8] G. Kirchhoff, K. Hensel; *Vorlesungen Uber mathematische Physik*, Vol.1, Teubner, Leipzig, 1883.

[9] R. Lehrer, L. A. Maia; *Positive solutions of asymptotically linear equations via Pohozaev manifold*, J. Funct. Anal., 266 (2014), 213–246.

[10] Y. H. Li, F. Y. Li, J. P. Shi; *Existence of a positive solution to Kirchhoff type problems without compactness conditions*, J. Differ. Equ., 253 (2012), 2285–2294.

[11] G. B. Li, H. Y. Ye; *Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in* $\mathbb{R}^3$, J. Differ. Equ., 257 (2014), 566–600.

[12] J. L. Lions; *On some questions in boundary value problems of mathematical physics.* North-Holland Mathematics Studies, 30 (1978), 284–346.

[13] P. L. Lions; *The concentration-compactness principle in the calculus of variations*, The locally compact case, Ann. Inst. H. Poincare Anal. Non Lineaire, 1(2) (1984), 109–145.

[14] A. M. Mao, S. Mo; *Ground state solutions to a class of critical Schrödinger problem*, Adv. Nonlinear Anal.,11(1) (2021), 96–127.

[15] S. I. Pohozaev; *Eigenfunctions of the equation* $\triangle u + \lambda f(u) = 0$, Sov. Math. Dokl., 5 (1965), 1408–1411.

[16] A. Salvatore; *Multiple solitary waves for a non-homogeneous Schrödinger-Maxwell system in* $\mathbb{R}^3$, Adv. Nonlinear Stud., 6(2) (2006), 157–169.

[17] W. A. Strauss; *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., 55 (1977), 149–162.

[18] C. A. Stuart, H. S. Zhou; *Applying the mountain pass theorem to an asymptotically linear elliptic equation on* $\mathbb{R}^N$, Commun. Partial Differ. Equ., 9-10 (1999), 1731–1758.

[19] X. H. Tang, S. T. Chen; *Ground state solutions of Nehari-Pohozaev type for Kirchhoff-type problems with general potentials*, Calc. Var., 56 (2017), 110.

[20] D. B. Wang, T. J. Li, X. N. Hao; *Least-energy sign-changing solutions for Kirchhoff-Schrödinger-Poisson systems in* $\mathbb{R}^3$, Bound. Value Probl., 2019 (2019), 75.

[21] X. P. Wang, F. F. Liao; *Nontrivial solutions for a nonlinear Schrödinger equation with nonsymmetric coefficients*, Nonlinear Anal., 195 (2020), 111755.

[22] J. Wang, L. X. Tian, J. X. Xu, F. B. Zhang; *Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth*, J. Differ. Equ., 253 (2012), 2314–2351.

[23] Y. Wang, Z. H. Zhang; *Ground State Solutions for Kirchhoff-Schrödinger-Poisson System with Sign-Changing Potentials*, Bull. Malays. Math. Sci. Soc., 44 (2021), 2319–2333.

[24] M. Willem; *Minimax Theorems*, vol. 24, Birkhauser, Boston, 1996.



Xian Hu
School of Sciences, Jimei University, Xiamen 361021, China
*Email address*: huxian19972021@163.com

Yong-Yi Lan (corresponding author)
School of Sciences, Jimei University, Xiamen 361021, China
*Email address*: lanyongyi@jmu.edu.cn