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Optimal control and homogenization in a mixture of fluids separated by a rapidly oscillating interface *

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Abstract

We study the limiting behaviour of the solution to optimal control problems in a mathematical mixture of two homogeneous viscous fluids. These fluids are separated by a rapidly oscillating periodic interface with constant amplitude. We show that the limit of the optimal control is the optimal control for the limiting problem

1 Introduction

The aim of this paper is to study the optimal control problem in a mixture of fluids. More precisely, we consider a mixture of two viscous, homogeneous and incompressible fluids occupying sub-domains of a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ ($n = 1$ or 2). These fluids are separated by a given interface whose form is determined using a rapidly oscillating function of period $\varepsilon > 0$ and constant amplitude $h_1 > 0$.

We assume that the velocity and pressure of both fluids satisfy the Stokes equations. On the interface, we assume that the velocity is continuous and that the normal forces that the fluids exert each other are equal in magnitude and opposite in direction (hence, surface tension effects are neglected).

We associate an optimal control problem to these equations and our aim is to study the limiting behaviour of the solutions when the oscillating period ε tends to 0. To do so, we use some homogenization tools (see Bensoussan-Lions-Papanicolaou [4] and Sanchez-Palencia [15]) and Murat's compactness result [11].

This work is based on the mathematical framework of Baffico & Conca [3] for the Stokes problem, of Brizzi [5] for the transmission problem, and of Baffico & Conca [1, 2] for the transmission problem in elasticity.

The plan of this paper is as follows. In Section 2, we present the domain with the rapidly oscillating interface, the Stokes problem posed in this domain

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and the definition of the associated optimal control problem. In Section 3, we introduce the adjoint problem and present related convergence results. In Section 4, we prove the results announced in the previous section. In Section 5, we sketch the proof of the convergence results concerning the optimal control. In Section 6, we present the case $\Omega \subset \mathbb{R}^2$.

2 Setting of the problem

For $n = 1$ or 2, let $Y =]0,1[^n$ and $\widetilde{\Omega} =]0, L_i[^n$ with $L_i > 0$, $i = 1, \dots, n$. Let $h : \bar{Y} \to \mathbb{R}, h \geq 0$ be a smooth function such that

i) $h|_{\partial Y} = h_1$ where $h_1 = \max\{h(y) : y \in \bar{Y}\}$ and $h_1 > 0$.

ii) Exists $y_0 \in Y$ such that $h(y_0) = 0$ and $\nabla_y h(y_0) = 0$.

For $z_o \in \mathbb{R}^+$, define $\Omega \subset \mathbb{R}^{n+1}$ by $\Omega = \widetilde{\Omega}\times] - z_0, z_0[$ and Γ the boundary Ω by $\Gamma = \widetilde{\Omega} \times \{-z_0\} \cup \partial\widetilde{\Omega}\times] - z_0, z_0[\cup \widetilde{\Omega} \times \{z_0\}$.

To define the reference cell, we introduce the sub-domains:

$$\begin{gathered}\Omega_1^1 = \{(y,z) \in Y \times \mathbb{R} : h(y) < z < z_0\}\\ \Omega_2^1 = \{(y,z) \in Y \times \mathbb{R} : -z_0 < z < h(y)\},\end{gathered}$$

which are separated by the interface

$$\Gamma^1 = \{(y,z) \in Y \times \mathbb{R} : h(y) = z\}.$$

So that, we have the decomposition of the reference cell Λ (as in figure 2)

$$\Lambda = \Omega_1^1 \cup \Gamma^1 \cup \Omega_2^1 = (Y\times] - z_0, z_0[)$$

When we intersect Λ with the hyperplane $\{Z = z\}(0 < z < h_1)$ we obtain $Y \times \{z\}$ and the following decomposition for Y (as in figure 2)

$$Y = Y^\star(z) \cup \gamma(z) \cup O(z),$$

where

$$\begin{gathered}Y^\star(z) = \{y \in Y : h(y) > z\}, \quad O(z) = \{y \in Y : h(y) < z\},\\ \gamma(z) = \{y \in Y : h(y) = z\}.\end{gathered}$$

Let $\varepsilon > 0$ be a small positive parameter. We extend h by Y-periodicity to $\mathbb{R}^n$, we restrict this function to $\widetilde{\Omega}$ (this function is still denoted by h). Let

$$h^\varepsilon(x) = h(\frac{x}{\varepsilon}) \quad x \in \widetilde{\Omega}.$$

Now we introduce

$$\Omega_1^\varepsilon = \{(x,z) \in \widetilde{\Omega} \times \mathbb{R} : h^\varepsilon(x) < z < z_0\}$$

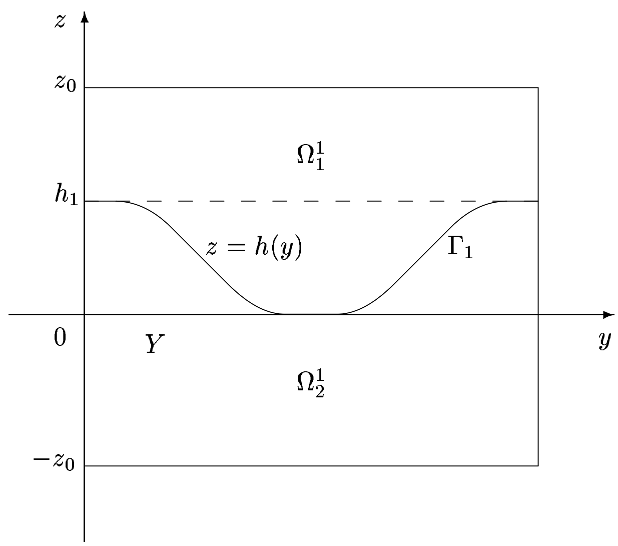

Figure 1: The reference cell Λ

$$\Omega_2^\varepsilon = \{(x,z) \in \widetilde{\Omega} \times \mathbb{R} : -z_0 < z < h^\varepsilon(x)\}$$

and the rapidly oscillating interface is therefore defined by

$$\Gamma^\varepsilon = \{(x,z) \in \widetilde{\Omega} \times \mathbb{R} : h^\varepsilon(x) = z\}.$$

So that, we obtain the following decomposition of Ω (see figure 2):

$$\Omega = \Omega_1^\varepsilon \cup \Gamma^\varepsilon \cup \Omega_2^\varepsilon.$$

Finally, as in figure 2, we set $\Omega_1 = \widetilde{\Omega} \times]h_1, z_0[$, $\Omega_m = \widetilde{\Omega} \times]0, h_1[$, $\Omega_2 = \widetilde{\Omega} \times]-z_0, 0[$. We notice that $\Omega = \Omega_1 \cup \Omega_m \cup \Omega_2$.

The Stokes Problem

Let the viscosity of the problem defined by

$$\mu^\varepsilon = \mu_1 \chi_{\Omega_1^\varepsilon} + \mu_2 \chi_{\Omega_2^\varepsilon},$$

where $\mu_1, \mu_2 > 0$, $\mu_1 \neq \mu_2$, and $\chi_{\Omega_i^\varepsilon}$ correspond to the characteristic functions of Ω_i^ε $(i = 1, 2)$.

We denote by $\vec{v} = (\underline{v}, v_{n+1})$, a vector of $\mathbb{R}^{n+1}$. Throughout this paper, C denotes various real positive constants independent of ε. We also denote by $|\cdot|$ the n-dimensional Lebesgue measure and by $(\underline{e}_k)_{1 \le k \le n}$ the canonical basis of $\mathbb{R}^n$ (y_k is the k-th component of $y \in \mathbb{R}^n$ in this basis).

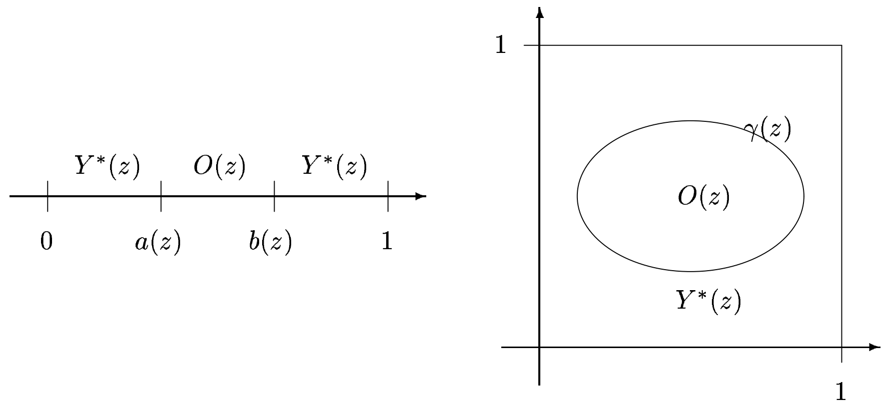

Figure 2: Decomposition of Y when $n = 1$ (left) and when $n = 2$ (rigut)

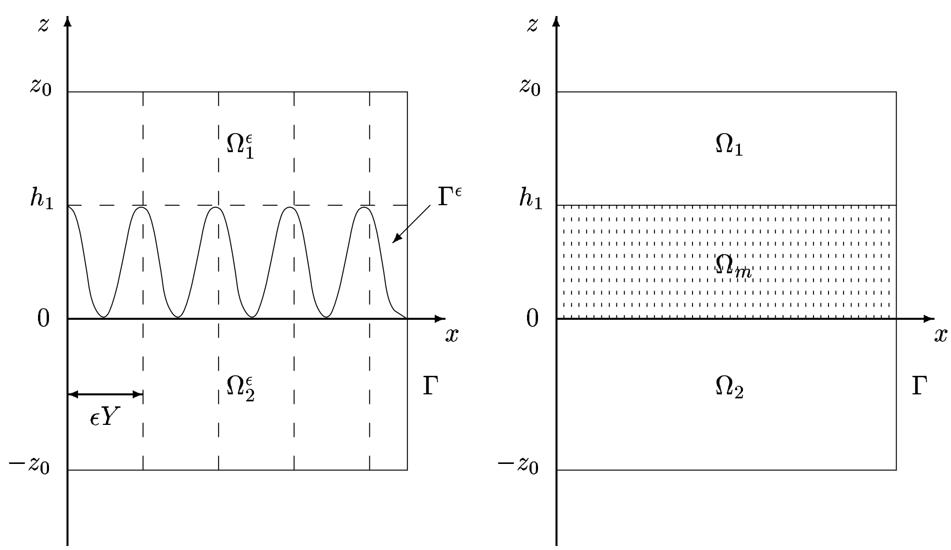

Figure 3: Rapidly oscillating interface (left) and its homogenized version (right)

We define the optimal control as follows. Let $\mathcal{U}_{ad} \subset L^2(\Omega)^{n+1}$ be a closed non empty convex subset. Let $N > 0$ be a given constant. For $\vec{\theta} \in \mathcal{U}_{ad}$ the state equation is given by the Stokes problem

$$\begin{aligned} -\operatorname{div}(2\mu^\varepsilon e(\vec{u}^\varepsilon)) &= \vec{f}^\varepsilon - \nabla p^\varepsilon + \vec{\theta} \quad \text{in } \Omega \\ \operatorname{div} \vec{u}^\varepsilon &= 0 \quad \text{in } \Omega \\ \vec{u}^\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

where $(\vec{u}^\varepsilon, p^\varepsilon)$ are respectively the velocity and the pressure of the fluid, $\vec{\theta}$ is the control and $\vec{f}^\varepsilon$ is a density of external forces defined by $\vec{f}^\varepsilon = \vec{f}^1 \chi_{\Omega_1^\varepsilon} + \vec{f}^2 \chi_{\Omega_2^\varepsilon}$ with $\vec{f}^i \in L^2(\Omega)^{n+1} (i = 1, 2)$. The rate-of-strain tensor $e(\vec{u}^\varepsilon)$ is

$$e(\vec{u}^\varepsilon) = \frac{1}{2}\big(\nabla\vec{u}^\varepsilon +^t \nabla\vec{u}^\varepsilon\big).$$

The cost function is

$$J_\varepsilon(\vec{\theta}) = \frac{1}{2}\int_\Omega e(\vec{u}^\varepsilon) : e(\vec{u}^\varepsilon)dx + \frac{N}{2}\int_\Omega |\vec{\theta}|^2 dx. \tag{2.2}$$

The optimal control $\vec{\theta}_\star^\varepsilon$ is the function in $\mathcal{U}_{ad}$ which minimizes $J_\varepsilon(\vec{\theta})$ for $\vec{\theta} \in \mathcal{U}_{ad}$, in other words

$$J_\varepsilon(\vec{\theta}_\star^\varepsilon) = \min_{\vec{\theta} \in \mathcal{U}_{ad}} J_\varepsilon(\vec{\theta}). \tag{2.3}$$

This problem is standard and admits an unique optimal solution $\vec{\theta}_\star^\varepsilon \in \mathcal{U}_{ad}$ (see Lions [10]). Our aim is to study the limiting behaviour of $\vec{\theta}_\star^\varepsilon$ as $\varepsilon \to 0$. In particular, it can be shown that (for a subsequence)

$$\vec{\theta}_\star^\varepsilon \rightharpoonup \vec{\theta}_\star \quad \text{weakly in } L^2(\Omega)^{n+1}.$$

Our objective is to characterize $\vec{\theta}_\star$ as the optimal control of a similar problem with limiting tensors $\mathcal{A}$ and $\mathcal{B}$ and to identify these tensors. The homogenization of the problem (2.1) is thanks to Baffico & Conca [3]. About the optimal control, Saint Jean Paulin & Zoubairi [12] studied the problem of a mixture of two fluids periodically distributed one in the other. Also this type of problem has been studied by Kesavan & Vanninathan [9] in the periodic case for a problem which the state equation is a second order elliptic problem with rapidly oscillating coefficients and by Kesavan & Saint Jean Paulin [7] and [8] in the general case (with H-convergence).

In this paper, we adapt these methods to the Stokes problem following the technique used by Baffico & Conca [2] and [3], by Kesavan & Saint Jean Paulin [7] and [8], and by Saint Jean Paulin & Zoubairi [12].

Definition We define $L_0^2(\Omega)$ by

$$L_0^2(\Omega) = \Big\{g \in L^2(\Omega) : \int_\Omega g(y)dy = 0\Big\}.$$

The problem (2.1)–(2.3) can be reduced to a system of equations by introducing the adjoint state $(\vec{v}^\varepsilon, p'^\varepsilon) \in H^1(\Omega)^{n+1} \times L^2_0(\Omega)$. Thus we get

$$\begin{gathered} -\operatorname{div}(2\mu^\varepsilon e(\vec{u}^\varepsilon)) = \vec{f}^\varepsilon - \nabla p^\varepsilon + \vec{\theta} \quad \text{in } \Omega \\ \operatorname{div}(2\mu^\varepsilon e(\vec{v}^\varepsilon) - e(\vec{u}^\varepsilon)) = -\nabla p'^\varepsilon \quad \text{in } \Omega \\ \operatorname{div} \vec{u}^\varepsilon = \operatorname{div} \vec{v}^\varepsilon = 0 \quad \text{in } \Omega \\ \vec{u}^\varepsilon = \vec{v}^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.4}$$

the optimal control $\vec{\theta}^\varepsilon_\star$ is characterized by the variationnal inequality

$$\vec{\theta}^\varepsilon_\star \in \mathcal{U}_{ad} \text{ and } \int_\Omega (\vec{v}^\varepsilon + N\vec{\theta}^\varepsilon_\star).(\vec{\theta} - \vec{\theta}^\varepsilon_\star) dx \geq 0 \forall \vec{\theta} \in \mathcal{U}_{ad}.$$

3 Convergence results

The homogenized adjoint problem

Let $\mu = \mu(y,z)$ (where $y \in Y$ and $z \in]0, h_1[$), be the variable viscosity given by

$$\mu(y,z) = \mu_1 \chi_{O(z)}(y) + \mu_2 \chi_{Y^\star(z)}(y).$$

Let us introduce some functions as solution of the Stokes problem defined on Y. These functions, introduced by Baffico & Conca [3], are associated to the state equation.

Let $1 \leq k, l \leq n$, and let $(\underline{\chi}^{kl}, r_1^{kl})$ be the solution of

$$\begin{gathered} -\operatorname*{div}_y(2\mu e_y(-\underline{\chi}^{kl} + \underline{P}^{kl})) = -\nabla_y r_1^{kl} \quad \text{in } Y \\ \operatorname*{div}_y \underline{\chi}^{kl} = 0 \quad \text{in } Y \\ \underline{\chi}^{kl}, r_1^{kl} \quad Y\text{-periodic}, \end{gathered} \tag{3.1}$$

where $\underline{P}^{kl} = \frac{1}{2}\big(y_k \underline{e}_l + y_l \underline{e}_k\big)$. We define M^{kl} the $n \times n$ matrix by $M^{kl} = e_y(\underline{P}^{kl})$. We notice that $[M^{kl}]_{ij} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ for all $1 \leq i,j,k,l \leq n$. We also define the $n+1 \times n+1$ matrix

$$[E^{kl}]_{ij} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad \forall 1 \leq i,j,k,l \leq n+1.$$

For each $z \in]0, h_1[$, problem (3.1) admits an unique solution in $\big(H^1_\sharp(Y)^n/\mathbb{R}\big) \times L^2_0(Y)$ (see Sanchez-Palencia [15] or Baffico & Conca [6]).

Now, we consider the periodic problem

$$\begin{gathered} -\operatorname*{div}_y(\mu \nabla_y(-\varphi^k + 2y_k)) = 0 \quad \text{in } Y \\ \varphi^k \quad Y\text{-periodic}. \end{gathered} \tag{3.2}$$

For each $z \in]0, h_1[$ fixed, problem (3.2) has an unique solution in $H^1(Y)$ up to an additive constant.

Let $\mathcal{A}(z)$ be the fourth-order tensor whose coefficients are defined by

$$a_{ijkl} = \begin{cases} 2\mu_1[E^{kl}]_{ij} & h_1 < z < z_0 \\ \widetilde{a_{ijkl}} & 0 < z < h_1 \\ 2\mu_2[E^{kl}]_{ij} & -z_0 < z < h_1 \end{cases} \quad 1 \le i,j,k,l \le n+1 \tag{3.3}$$

with

$$\widetilde{a_{ijkl}} = \begin{cases} \int_Y 2\mu\big[e_y(-\underline{\chi}^{kl} + \underline{P}^{kl})\big]_{ij} dy & 1 \le i,j,k,l \le n \\ \frac{1}{2}\int_Y \mu \frac{\partial(-\varphi^k + 2y_k)}{\partial y_i} dy & 1 \le i,k \le n \quad j,l = n+1 \\ \frac{1}{2}\int_Y \mu \frac{\partial(-\varphi^l + 2y_l)}{\partial y_i} dy & 1 \le i,l \le n \quad j,k = n+1 \\ \frac{1}{2}\int_Y \mu \frac{\partial(-\varphi^k + 2y_k)}{\partial y_j} dy & 1 \le j,k \le n \quad i,l = n+1 \\ \frac{1}{2}\int_Y \mu \frac{\partial(-\varphi^l + 2y_l)}{\partial y_j} dy & 1 \le j,l \le n \quad i,k = n+1 \\ \frac{1}{2}\int_Y 2\mu dy & i,j,k,l = n+1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

This tensor (introduced by Baffico & Conca [3]) is the homogenized tensor associated to the state equation. By the same way, we introduce other test functions which will be associated to the adjoint state.

Let $(\underline{\psi}^{kl}, r_2^{kl})$ be the solution of

$$\begin{gathered} -\operatorname*{div}_y(2\mu e_y(\underline{\psi}^{kl}) + e_y(-\underline{\chi}^{kl} + \underline{P}^{kl})) = -\nabla_y r_2^{kl} \quad \text{in } Y \\ \operatorname*{div}_y \underline{\psi}^{kl} = 0 \quad \text{in } Y \\ \underline{\psi}^{kl}, r_2^{kl} \quad Y\text{-periodic}, \end{gathered} \tag{3.5}$$

For each $z \in]0, h_1[$, problem (3.5) has an unique solution in $\big(H^1_\sharp(Y)^n/\mathbb{R}\big) \times L^2_0(Y)$.

As we did for the problem (3.2), we introduce the scalar problem

$$\begin{gathered} -\operatorname*{div}_y(2\mu\nabla_y\psi^k + \nabla_y(-\varphi^k + 2y_k)) = 0 \quad \text{in } Y \\ \psi^k \quad Y\text{-periodic.} \end{gathered} \tag{3.6}$$

For $z \in]0, h_1[$ fixed, this problem admits an unique solution in $H^1(Y)$ up to an additive constant.

Let $\mathcal{B}(z)$ be the fourth order tensor which coefficients are given by

$$b_{ijkl} = \begin{cases} [E^{kl}]_{ij} & h_1 < z < z_0 \\ \widetilde{b_{ijkl}} & 0 < z < h_1 \\ [E^{kl}]_{ij} & -z_0 < z < h_1 \end{cases} \quad 1 \le i,j,k,l \le n+1 \tag{3.7}$$

with

$$\widetilde{b_{ijkl}} = \begin{cases} \int_Y \big(2\mu[e_y(\underline{\psi}^{kl})]_{ij} + [e_y(-\underline{\chi}^{kl} + \underline{P}^{kl})]_{ij}\big)dy & 1 \le i,j,k,l \le n \\ \frac{1}{4}\int_Y \big(2\mu\frac{\partial\psi^k}{\partial y_i} + \frac{\partial(-\varphi^k+2y_k)}{\partial y_i}\big)dy & 1 \le i,k \le n \quad j,l = n+1 \\ \frac{1}{4}\int_Y \big(2\mu\frac{\partial\psi^l}{\partial y_i} + \frac{\partial(-\varphi^l+2y_l)}{\partial y_i}\big)dy & 1 \le i,l \le n \quad j,k = n+1 \\ \frac{1}{4}\int_Y \big(2\mu\frac{\partial\psi^k}{\partial y_j} + \frac{\partial(-\varphi^k+2y_k)}{\partial y_j}\big)dy & 1 \le j,k \le n \quad i,l = n+1 \\ \frac{1}{4}\int_Y \big(2\mu\frac{\partial\psi^l}{\partial y_j} + \frac{\partial(-\varphi^l+2y_l)}{\partial y_j}\big)dy & 1 \le j,l \le n \quad i,k = n+1 \\ 1 & i,j,k,l = n+1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

Now we give a result concerning some properties of tensor $\mathcal{A}$.

Proposition 3.1 (Baffico & Conca [3]) *The coefficients of $\mathcal{A}$ in (3.3) satisfy:*
a) $a_{ijkl}(z) = a_{klij}(z) = a_{ijlk}(z)$ $\forall 1 \le i,j,k,l \le n+1, \quad \forall z \in]-z_0, z_0[$
b) there exists $\alpha > 0$ such that for all ξ, $n+1 \times n+1$ symmetric matrix,

$$\mathcal{A}(z)\xi : \xi \ge \alpha\xi : \xi \quad \forall z \in]-z_0, z_0[.$$

Now we give a result concerning some symmetry and ellipticity properties of the tensor $\mathcal{B}$.

Proposition 3.2 *The coefficients of $\mathcal{B}$ (voir (3.7)) are such that:*
a) $b_{ijkl}(z) = b_{klij}(z) = b_{ijlk}(z)$ for $1 \le i,j,k,l \le n+1$, for all $z \in]-z_0, z_0[$
b) There exists $\beta > 0$ such that for all ξ, the $n+1 \times n+1$ symmetric matrix,

$$\mathcal{B}(z)\xi : \xi \ge \beta\xi : \xi \quad \forall z \in]-z_0, z_0[.$$

Proof. Throughout this proof, we adopt the convention of summation over repeated indices. To prove a), we first study the coefficients of tensor $\mathcal{B}$ with indexes $1 \le i,j,k,l \le n$. The symmetry of these coefficients is evident when $z \in]h_1, z_0[$ and $z \in]-z_0, h_1[$.

Let study the case where $z \in]0, h_1[$. In this case (cf (3.8))

$$b_{ijkl} = \widetilde{b_{ijkl}} = \int_Y \big(2\mu\big[e_y(\underline{\psi}^{kl})\big]_{ij} + \big[e_y(-\underline{\chi}^{kl} + \underline{P}^{kl})\big]_{ij}\big)dy. \tag{3.9}$$

Following the ideas in [7, 12, 14], we transform the above expression to obtain a symmetric form. Let $(\underline{Y}^{kl}, r_3^{kl})$ be the solution of

$$\begin{gathered} -\operatorname*{div}_y(2\mu e_y(-\underline{Y}^{kl} + \underline{P}^{kl})) = -\nabla_y r_3^{kl} \quad \text{in } Y \\ \operatorname*{div}_y \underline{Y}^{kl} = 0 \quad \text{in } Y \\ \underline{Y}^{kl}, r_1^{kl} \quad Y\text{-periodic,} \end{gathered} \tag{3.10}$$

Introducing $\underline{Y}^{kl}$ the solution of the previous local problem, the coefficients (3.9) can be rewritten as

$$\widetilde{b_{ijkl}} = \int_Y \big[e_y(-\underline{Y}^{kl}+\underline{P}^{kl})\big]_{ij}dy + \int_Y \big(2\mu\big[e_y(\underline{\psi}^{kl})\big]_{ij} - \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{ij}\big)dy. \tag{3.11}$$

The first term of the second integral of the right-hand side of this equation is evaluated as follows (using the fact that $\big[e_y(\underline{\psi}^{kl})\big]_{ij} = \big[e_y(\underline{\psi}^{kl})\big]_{ji}$)

$$\begin{aligned}
\int_Y 2\mu\big[e_y(\underline{\psi}^{kl})\big]_{ij}dy &= \int_Y 2\mu\big[e_y(\underline{\psi}^{kl})\big]_{\beta m}\delta_{\beta i}\delta_{mj}dy \\
&= \int_Y \mu\big[e_y(\underline{\psi}^{kl})\big]_{\beta m}(\delta_{\beta i}\delta_{mj}+\delta_{\beta j}\delta_{mi})dy \\
&= \int_Y 2\mu\big[e_y(\underline{\psi}^{kl})\big]_{\beta m}\big[e_y(\underline{P}^{ij})\big]_{\beta m}dy.
\end{aligned}$$

Using successively (3.1),(3.5) and (3.10), we have

$$\begin{aligned}
\int_Y 2\mu\big[e_y(\underline{\psi}^{kl})\big]_{\beta m}\big[e_y(\underline{P}^{ij})\big]_{\beta m}dy &= \int_Y 2\mu\big[e_y(\underline{\psi}^{kl})\big]_{\beta m}\big[e_y(\underline{\chi}^{ij})\big]_{\beta m}dy \\
&= \int_Y \big[e_y(\underline{\chi}^{kl}-\underline{P}^{kl})\big]_{\beta m}\big[e_y(\underline{\chi}^{ij})\big]_{\beta m}dy \\
&= \int_Y \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{\beta m}\big[e_y(\underline{\chi}^{ij})\big]_{\beta m}dy.
\end{aligned}$$

Moreover using (3.10), we can rewrite the last integral as

$$\begin{aligned}
&\int_Y \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{\beta m}\big[e_y(\underline{\chi}^{ij})\big]_{\beta m}dy \\
&= \int_Y \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{\beta m}\big[e_y(\underline{\chi}^{ij}-\underline{Y}^{ij})\big]_{\beta m}dy \\
&\quad + \int_Y \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{\beta m}\big[e_y(\underline{Y}^{ij})\big]_{\beta m}dy \\
&= \int_Y \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{\beta m}\big[e_y(\underline{\chi}^{ij}-\underline{Y}^{ij})\big]_{\beta m}dy \\
&\quad + \int_Y \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{\beta m}\big[e_y(\underline{P}^{ij})\big]_{\beta m}dy \\
&= \int_Y \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{\beta m}\big[e_y(\underline{\chi}^{ij}-\underline{Y}^{ij})\big]_{\beta m}dy + \int_Y \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{ij}dy
\end{aligned}$$

Thus the second integral of the right-hand side of (3.11) can be rewritten as

$$\begin{aligned}
&\int_Y \big(2\mu\big[e_y(\underline{\psi}^{kl})\big]_{ij} - \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{ij}\big)dy\big] \\
&\qquad = \int_Y \big[e_y(\underline{\chi}^{kl}-\underline{Y}^{kl})\big]_{\beta m}\big[e_y(\underline{\chi}^{ij}-\underline{Y}^{ij})\big]_{\beta m}dy.
\end{aligned}$$

Let us now consider the first integral in (3.11). Multiplying the first equation of (3.10) by Y^{ij} and integrating by parts we have

$$\int_Y \big[e_y(-\underline{Y}^{kl}+\underline{P}^{kl})\big]_{\beta m}\big[e_y(\underline{Y}^{ij})\big]_{\beta m}dy=0.$$

Using the fact that $\big[e_y(\underline{P}^{ij})\big]_{\beta m}=\frac{1}{2}\big(\delta_{\beta i}\delta_{mj}+\delta_{\beta j}\delta_{mi}\big)$, we obtain

$$\begin{aligned}
&\int_Y \big[e_y(-\underline{Y}^{kl}+\underline{P}^{kl})\big]_{\beta m}\big[e_y(-\underline{Y}^{ij}+\underline{P}^{ij})\big]_{\beta m}dy\\
&=\int_Y \big[e_y(-\underline{Y}^{kl}+\underline{P}^{kl})\big]_{\beta m}\big[e_y(\underline{P}^{ij})\big]_{\beta m}dy\\
&=\int_Y \big[e_y(-\underline{Y}^{kl}+\underline{P}^{kl})\big]_{ij}dy.
\end{aligned}$$

Then using definition (3.11), we derive

$$\widetilde{b_{ijkl}}=\int_Y e_y(-\underline{Y}^{kl}+\underline{P}^{kl}):e_y(-\underline{Y}^{ij}+\underline{P}^{ij})dy+\int_Y e_y(\underline{\chi}^{kl}-\underline{Y}^{kl}):e_y(\underline{\chi}^{ij}-\underline{Y}^{ij})dy. \tag{3.12}$$

It is immediate from the above form that the coefficients of $\mathcal{B}$ satisfy $\widetilde{b_{ijkl}}=\widetilde{b_{klij}}$. On the other hand, since $e_y(\underline{P}^{kl})=e_y(\underline{P}^{lk})$ then by uniqueness of problem (3.1), we have $\chi^{kl}=\chi^{lk}$ (up to an additive constant) and then $b_{ijkl}=b_{ijlk}$.

We now study the coefficients b_{ijkl} with $i=k=n+1$ and $1\le j,l\le n$. From the definition of $\mathcal{B}$ (cf (3.7)), these coefficients are symmetric when $z\in]h_1,z_0[$ and $z\in]-z_0,h_1[$. To prove the symmetry when $z\in]0,h_1[$, we proceed as we did before. These coefficients are as follows

$$\widetilde{b_{n+1jn+1l}}=\frac{1}{4}\int_Y\big(\frac{\partial(-\varphi^l+2y_l)}{\partial y_j}+2\mu\frac{\partial\psi^l}{\partial y_j}\big)dy. \tag{3.13}$$

Let τ^k be the solution of

$$\begin{gathered}
-\Delta_y(-\tau^k+2y_k)=0\quad\text{in } Y\\
\tau^k\quad Y\text{-périodic.}
\end{gathered} \tag{3.14}$$

The expression (3.13) can be rewritten as follows (using τ^l):

$$\widetilde{b_{n+1jn+1l}}=\frac{1}{4}\int_Y\frac{\partial(-\tau^l+2y_l)}{\partial y_j}dy+\frac{1}{4}\int_Y\big(2\mu\frac{\partial\psi^l}{\partial y_j}-\frac{\partial(\varphi^l-\tau^l)}{\partial y_j}\big)dy. \tag{3.15}$$

Using exactly the same technique used above, we obtain (using (3.2), (3.6) and (3.14))

$$\int_Y 2\mu\frac{\partial\psi^l}{\partial y_j}=\frac{1}{2}\int_Y\frac{\partial(\varphi^l-\tau^l)}{\partial y_k}\frac{\partial(\varphi^j-\tau^j)}{\partial y_k}dy+\int_Y\frac{\partial(\varphi^l-\tau^l)}{\partial y_j}dy.$$

Therefore,

$$\int_Y \big(2\mu\frac{\partial\psi^l}{\partial y_j} - \frac{\partial(\varphi^l-\tau^l)}{\partial y_j}\big)dy = \frac{1}{2}\int_Y \frac{\partial(\varphi^l-\tau^l)}{\partial y_k}\frac{\partial(\varphi^j-\tau^j)}{\partial y_k}dy.$$

Concerning the first integral in (3.15), we consider τ^l the solution of (3.14): multiplying the first equation by τ^j and integrating by parts, we have

$$\int_Y \frac{\partial(-\tau^l+2y_l)}{\partial y_k}\frac{\partial\tau^j}{\partial y_k}dy = 0,$$

so we derive

$$\int_Y \frac{\partial(-\tau^l+2y_l)}{\partial y_k}\frac{\partial(-\tau^j+2y_j)}{\partial y_k}dy = 2\int_Y \frac{\partial(-\tau^l+2y_l)}{\partial y_j}dy.$$

Finally, we obtain the following expression

$$\widetilde{b_{n+1jn+1l}} = \frac{1}{8}\int_Y \nabla(-\tau^l+2y_l).\nabla(-\tau^j+2y_j)dy + \frac{1}{8}\int_Y \nabla(\varphi^l-\tau^l).\nabla(\varphi^j-\tau^j)dy. \tag{3.16}$$

From the form of (3.16), it is evident that $\widetilde{b_{n+1jn+1l}} = \widetilde{b_{n+1ln+1j}}$. By construction of $\mathcal{B}$, we also have $\widetilde{b_{n+1ln+1j}} = \widetilde{b_{n+1ljn+1}}$. For the other nonzero terms, the same method can be used to obtain

$$\begin{gathered}
\widetilde{b_{in+1kn+1}} = \widetilde{b_{kn+1in+1}} = \widetilde{b_{in+1n+1k}} \\
\widetilde{b_{in+1n+1l}} = \widetilde{b_{in+1ln+1}} = \widetilde{b_{n+1lin+1}} \\
\widetilde{b_{n+1jkn+1}} = \widetilde{b_{n+1jn+1k}} = \widetilde{b_{kn+1n+1j}}.
\end{gathered}$$

To prove part b), we first notice that the coerciveness of $\mathcal{B}$ when $z \in]h_1, z_0[$ and $z \in]-z_0, h_1[$ is evident. When $z \in]0, h_1[$, from the form of $\widetilde{b_{ijkl}} 1 \le i,j,k,l \le n$ (see (3.12)) and the form of $\widetilde{b_{n+1jn+1l}} 1 \le j,l \le n$ (see (3.16), we have that $\mathcal{B}$ is elliptic. $\square$

Now we introduce the homogenized problem. Let $(\vec{u},p)$ and $(\vec{v},p')$ be in $\big(H^1(\Omega)^{n+1} \times L_0^2(\Omega)\big)^2$ and be the solution of

$$\begin{gathered}
-\operatorname{div}\big(\mathcal{A}e(\vec{u})\big) = \vec{f} - \nabla p + \vec{\theta} \quad \text{in } \Omega \\
\operatorname{div}\big(\mathcal{A}e(\vec{v}) - \mathcal{B}e(\vec{u})\big) = -\nabla p' \quad \text{in } \Omega \\
\operatorname{div}\vec{u} = \operatorname{div}\vec{v} = 0 \quad \text{in } \Omega \\
\vec{u} = \vec{v} = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.17}$$

where $\vec{f}$ is the weak limit of $\vec{f^\varepsilon}$ in $L^2(\Omega)^{n+1}$ and we precise it later, (4.10).

Remark 3.3 Since Propositions 3.1 and 3.2 hold, problem (3.17) admits an unique solution.

Now we state the main result of this paper, and we will prove it in the next section.

Theorem 3.3 *Under some regularity hypotheses concerning the solutions of (3.1), (3.2), (3.5), and (3.6) (detailed in Section 4) the solutions* $(\vec{u}^\varepsilon, p^\varepsilon)$ *and* $(\vec{v}^\varepsilon, p'^\varepsilon)$ *of (2.4) are such that* $\vec{u}^\varepsilon \rightharpoonup \vec{u}$ *weakly in* $H^1(\Omega)^{n+1}$, $\vec{v}^\varepsilon \rightharpoonup \vec{v}$ *weakly in* $H^1(\Omega)^{n+1}$, $p^\varepsilon \rightharpoonup p$ *weakly in* $L^2_0(\Omega)$, $p'^\varepsilon \rightharpoonup p'$ *weakly in* $L^2_0(\Omega)$, *where* $(\vec{u}, p)$ *and* $(\vec{v}, p')$ *are the unique solutions of (3.17).*

4 Proof of the convergence result

A priori estimates Let

$$\xi^\varepsilon = 2\mu^\varepsilon e(\vec{u}^\varepsilon), \tag{4.1}$$

$$q^\varepsilon = 2\mu^\varepsilon e(\vec{v}^\varepsilon) - e(\vec{u}^\varepsilon). \tag{4.2}$$

Proposition 4.1 *The sequences* $(\vec{u}^\varepsilon, p^\varepsilon)$, $(\vec{v}^\varepsilon, p'^\varepsilon)$, ξ^ε *and* q^ε *are such that (up to subsequences)*

$$\begin{aligned} \vec{u}^\varepsilon \rightharpoonup \vec{u} \text{ weakly in } H^1(\Omega)^{n+1} &\qquad \vec{v}^\varepsilon \rightharpoonup \vec{v} \text{ weakly in } H^1(\Omega)^{n+1} \\ p^\varepsilon \rightharpoonup p \text{ weakly in } L^2_0(\Omega) &\qquad p'^\varepsilon \rightharpoonup p' \text{ weakly in } L^2_0(\Omega) \\ \xi^\varepsilon \rightharpoonup \xi \text{ weakly in } L^2(\Omega)^{n+1\times n+1} &\qquad q^\varepsilon \rightharpoonup q \text{ weakly in } L^2(\Omega)^{n+1\times n+1}. \end{aligned}$$

Proof. Using $\vec{u}^\varepsilon$ as a test function in the first equation of (2.4), we can easily see that there exists a constant $C > 0$ independent of ε such that

$$\|\vec{u}^\varepsilon\|_{H^1(\Omega)^{n+1}} \leq C; \tag{4.3}$$

therefore, we have for a subsequence (still denoted by ε)

$$\vec{u}^\varepsilon \rightharpoonup \vec{u} \text{ weakly in } H^1(\Omega)^{n+1}. \tag{4.4}$$

Similarly, multiplying the second equation of (2.4) by $\vec{v}^\varepsilon$, integrating by parts and using (4.3), we obtain

$$\|\vec{v}^\varepsilon\|_{H^1(\Omega)^{n+1}} \leq C,$$

so we have (for a subsequence)

$$\vec{v}^\varepsilon \rightharpoonup \vec{v} \text{ weakly in } H^1(\Omega)^{n+1}. \tag{4.5}$$

Now since $\|\text{div } (2\mu^\varepsilon e(\vec{u}^\varepsilon))\|_{H^{-1}(\Omega)^{n+1}}$ is bounded, we have $\|\nabla p^\varepsilon\|_{H^{-1}(\Omega)^{n+1}} \leq C$, this implies (see Temam [16]) $|p^\varepsilon|_{L^2_0(\Omega)} \leq C$, we derive (for a subsequence),

$$p^\varepsilon \rightharpoonup p \text{ weakly in } L^2_0(\Omega).$$

Also by similar arguments, we get

$$p'^{\varepsilon} \rightharpoonup p' \text{ weakly in } L^2_0(\Omega).$$

The boundedness of $\|\xi^{\varepsilon}\|_{L^2(\Omega)^{(n+1)^2}}$ provides from (4.3) and we derive by extraction of subsequences

$$\xi^{\varepsilon} \rightharpoonup \xi \text{ weakly in } L^2(\Omega)^{(n+1)^2}.$$

Similarly, the boundedness of $\|q^{\varepsilon}\|_{L^2(\Omega)^{(n+1)^2}}$ provides from (4.3) and (4.1.6), so we can extract a subsequence such that

$$q^{\varepsilon} \rightharpoonup q \text{ weakly in } L^2(\Omega)^{(n+1)^2}. \tag{4.6}$$

□

Since $(\vec{u}^{\varepsilon}, p^{\varepsilon})$ and $(\vec{v}^{\varepsilon}, p'^{\varepsilon})$ are solutions of (2.4) and since Proposition 4.1 holds, we obtain that $(\vec{u}, p), (\vec{v}, p'), \xi$ and q satisfy in the distribution sense

$$\begin{gathered}
-\operatorname{div}\big(\xi\big) = \vec{f} - \nabla p + \vec{\theta} \quad \text{in } \Omega \\
\operatorname{div}\big(q\big) = -\nabla p' \text{ in } \Omega \\
\operatorname{div}\vec{u} = \operatorname{div}\vec{v} = 0 \quad \text{in } \Omega \\
\vec{u} = \vec{v} = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{4.7}$$

where $\vec{f}$ is the weak limit of $\vec{f^{\varepsilon}}$ in $L^2(\Omega)^{n+1}$. This limit can be identified explicitly (cf [3] or [5]). Indeed, the characteristic functions $\chi_{\Omega_1^{\varepsilon}}$ $(i = 1, 2)$ are such that

$$\chi_{\Omega_i^{\varepsilon}} \rightharpoonup \rho \text{ and } \chi_{\Omega_2^{\varepsilon}} \rightharpoonup (1-\rho) \text{ weakly } \star \quad \text{in } L^{\infty}(\Omega), \tag{4.8}$$

where

$$\rho(x,z) = \begin{cases} 1 & \text{in } \Omega_1 \\ \frac{|O(z)|}{|Y|} & \text{in } \Omega_m \\ 0 & \text{in } \Omega_2. \end{cases} \tag{4.9}$$

Then we have

$$\vec{f^{\varepsilon}} \rightharpoonup \vec{f} = \vec{f^1}\rho + \vec{f^2}(1-\rho) \text{ weakly in } L^2(\Omega)^{n+1}. \tag{4.10}$$

Proposition 4.2 (Baffico & Conca [3]) *Under the hypotheses (4.12), (4.15) and (4.28) (introduced in the next subsections), $\xi = \mathcal{A}e(\vec{u})$, where $\mathcal{A}$ is defined by (3.4).*

To prove Theorem 3.3, we have to show that $q, \vec{u}$ and $\vec{v}$ are related by

$$q = \mathcal{A}e(\vec{v}) - \mathcal{B}e(\vec{u}). \tag{4.11}$$

Using the same method as Baffico & Conca[3], we show that the identification of q is carried out in Ω_1, Ω_m and Ω_2 independently. In Ω_1 and Ω_2, this identification will pose no particular problem. In Ω_m, following the ideas of Baffico & Conca ([3]), there is three steps: we first identify the components $[q]_{ij}$ of q for $1 \le i,j \le n$, and then $[q]_{n+1j}$ for $1 \le j \le n$ and finally we identify $[q]_{n+1n+1}$. To do so, we use some suitable test functions and the energy method (cf Bensoussan, Lions & Papanicolaou [4] or Sanchez-Palencia [15]).

Identification of $[q]_{ij} 1 \le i,j \le n$ in Ω_m

In what follows, we construct at first the test functions which allow us the identification of $[q]_{ij}$, then we introduce the regularity conditions that these functions must satisfy and finally we establish the identification.

Let $\underline{w}^{kl} = -\underline{\chi}^{kl} + \underline{P}^{kl}$ and $\sigma^{kl} = 2\mu e_y(\underline{w}^{kl})$ where $(\underline{\chi}^{kl}, r_1^{kl})$ be the solution of (3.1). We assume that $(\underline{\chi}^{kl}, r_1^{kl})$, as function of $(y,z) \in Y \times]0,h_1[$, satisfies the regularity hypothesis

$$\begin{aligned} &a) \quad \underline{\chi}^{kl} \in L^2_{\mathrm{loc}}(0,h_1,H^1_\sharp(Y)^n) \cap (L^2_{\mathrm{loc}}(]0,h_1[\times\mathbb{R}^n))^n \\ &b) \quad \frac{\partial}{\partial z}\big((\underline{\chi}^{kl})_i\big) \in L^2_{\mathrm{loc}}(0,h_1,L^2_\sharp(Y)^n) \cap L^2_{\mathrm{loc}}(]0,h_1[\times\mathbb{R}) \end{aligned} \qquad 1 \le i,j \le n \tag{4.12}$$

We define the following functions by extension by Y-periodicity to $\mathbb{R}^{n+1}$ and by restriction to Ω_m:

$$\begin{aligned} \underline{w}^{\varepsilon,kl}(x,z) &= \varepsilon \underline{w}^{kl}(\frac{x}{\varepsilon}, z) \\ r_1^{\varepsilon,kl}(x,z) &= r_1^{kl}(\frac{x}{\varepsilon}, z) \\ \sigma^{\varepsilon,kl}(x,z) &= \underline{\sigma}^{kl}(\frac{x}{\varepsilon}, z) \end{aligned} \tag{4.13}$$

It is easy to check that

$$\begin{aligned} \operatorname*{div}_x(\sigma^{\varepsilon,kl}) &= -\nabla_x r_1^{\varepsilon,kl} \quad \text{in } \Omega_m \\ \operatorname*{div}_x(\underline{w}^{\varepsilon,kl}) &= \operatorname*{div}_x(\underline{P}^{kl}) = \delta_{kl} \quad \text{in } \Omega_m. \end{aligned} \tag{4.14}$$

We also need the Murat's compactness result [11].

Lemma 4.3 *If the sequence $(g_n)_n$ belongs to a bounded subset of $W^{-1,p}(\Omega)$ for some $p > 2$, and $(g_n)_n \ge 0$ in the following sense i.e., for all $\phi \in \mathcal{D}(\Omega)$ such that $\phi \ge 0$ then for all $n > 0 \langle g_n, \phi\rangle \ge 0$. Then $(g_n)_n$ belongs to a compact subset of $H^{-1}(\Omega)$.*

If we suppose that $r_1^{\varepsilon,kl}$ satisfy

$$\begin{aligned} &a) \quad r_1^{\varepsilon,kl} \in L^p_{\mathrm{loc}}(\Omega_m) \text{ for some } p > 2, \text{ locally bounded} \\ &b) \quad \frac{\partial}{\partial z}\big(r_1^{\varepsilon,kl}\big) \ge 0 \text{ in distribution sense,} \end{aligned} \tag{4.15}$$

Then using Lemma 4.3 and hypothesis (4.12), we have the following result.

Proposition 4.4 (Baffico & Conca [3]) *If (4.12) and (4.15) hold. Then for all $\Omega' \subset\subset \Omega_m$, we have the following convergence*

$$\begin{aligned} &a)\ \underline{w}^{\varepsilon,kl} \rightharpoonup \underline{P}^{kl} \text{ weakly in } H^1(\Omega')^n \\ &b)\ \tfrac{\partial}{\partial z}\big((\underline{w}^{\varepsilon,kl})_i\big) \to 0 \text{ strongly in } L^2(\Omega')^n,\ 1 \le i \le n \\ &c)\ r_1^{\varepsilon,kl} \to 0 \text{ weakly in } L^2(\Omega')^n \\ &d)\ \tfrac{\partial}{\partial z}\big(r_1^{\varepsilon,kl}\big) \to 0 \text{ strongly in } H^{-1}(\Omega')^n,\ 1 \le i \le n \\ &e)\ \sigma^{\varepsilon,kl} \rightharpoonup \sigma^{kl} = m_Y(2\mu e_y(\underline{w}^{kl}) \text{ weakly in } L^2(\Omega')^{n\times n}. \end{aligned} \tag{4.16}$$

In the same way, we assume that the solution $(\underline{\psi}^{kl}, r_2^{kl})$ of (3.5) satisfies the following convergence hypothesis:

$$\begin{aligned} &a) \quad \underline{\psi}^{kl} \in L^2_{\text{loc}}(0, h_1, H^1_\sharp(Y)^n) \cap (L^2_{\text{loc}}(]0, h_1[\times \mathbb{R}^n))^n \\ &b) \quad \frac{\partial}{\partial z}\big((\underline{\psi}^{kl})_i\big) \in L^2_{\text{loc}}(0, h_1, L^2_\sharp(Y)^n) \cap L^2_{\text{loc}}(]0, h_1[\times \mathbb{R}) \end{aligned} \qquad 1 \le i, j \le n \tag{4.17}$$

We define

$$\underline{\psi}^{\varepsilon,kl}(x,z) = \varepsilon \underline{\psi}^{kl}(\frac{x}{\varepsilon}, z), \quad \text{and} \quad r_2^{\varepsilon,kl}(x,z) = r_2^{kl}(\frac{x}{\varepsilon}, z) \tag{4.18}$$

so we obtain

$$\begin{gathered} -\operatorname*{div}_x \big(2\mu^\varepsilon e_x(\underline{\psi}^{\varepsilon,kl}) + e_x(\underline{w}^{\varepsilon,kl})\big) = -\nabla_x r_2^{\varepsilon,kl} \quad \text{in } \Omega_m \\ \operatorname*{div}_x \underline{\psi}^{\varepsilon,kl} = 0 \quad \text{in } \Omega_m \end{gathered} \tag{4.19}$$

If we suppose that $r_2^{\varepsilon,kl}$ satisfy

$$\begin{aligned} &a) \quad r_2^{\varepsilon,kl} \in L^p_{\text{loc}}(\Omega_m) \text{ for some } p > 2, \text{ locally bounded} \\ &b) \quad \frac{\partial}{\partial z}\big(r_2^{\varepsilon,kl}\big) \ge 0 \text{ in distribution sense,} \end{aligned} \tag{4.20}$$

then using Lemma 4.3, we derive

$$\frac{\partial}{\partial z}\big(r_2^{\varepsilon,kl}\big) \to 0 \text{ strongly in } H^{-1}(\Omega')^n . 1 \le i \le n$$

We have the following result concerning these functions

Proposition 4.5 *If (4.12) and (4.15) hold. Then for all $\Omega' \subset\subset \Omega_m$, we have*

$$\begin{aligned} &a) \quad \underline{\psi}^{\varepsilon,kl} \rightharpoonup 0 \textit{ weakly in } H^1(\Omega')^n \\ &b) \quad \frac{\partial}{\partial z}\big((\underline{\psi}^{\varepsilon,kl})_i\big) \to 0 \textit{ strongly in } L^2(\Omega')^n, \quad 1 \le i \le n \\ &c) \quad r_2^{\varepsilon,kl} \rightharpoonup 0 \textit{ weakly in } L^2(\Omega')^n. \end{aligned} \tag{4.21}$$

To prove this proposition, we use well-known results concerning the convergence of periodic functions.

We shall prove now the principal result of this section

Proposition 4.6 *If (4.12), (4.15), (4.17), and (4.20) hold, then $[q^\varepsilon]_{kl} \rightharpoonup [q]_{kl}$ weakly in $L^2(\Omega_m)$ (up to a subsequence) for all $1 \le k, l \le n$ where*

$$\begin{aligned} [q]_{kl} &= \frac{1}{|Y|} \sum_{i,j=1}^n \Big\{ \int_Y 2\mu [e_y(-\underline{\chi}^{ij} + \underline{P}^{ij})]_{kl} dy \Big\} \big[e(\vec{v})\big]_{ij} \\ &- \frac{1}{|Y|} \sum_{i,j=1}^n \Big\{ \int_Y \big(2\mu [e_y(-\underline{\psi}^{ij})]_{kl} + [e_y(-\underline{\chi}^{ij} + \underline{P}^{ij})]_{kl}\big) dy \Big\} \big[e(\vec{u})\big]_{ij} \end{aligned}$$

Proof. Let $\phi \in \mathcal{D}(\Omega_m)$ and $\vec{w}^{\varepsilon,kl} = (\underline{w}^{\varepsilon,kl}, 0)$. Multiplying the second equation of (2.4) by $\phi\vec{w}^{\varepsilon,kl}$, integrating by parts and using (4.2), we obtain

$$\begin{aligned} -\int_{\Omega_m} &\nabla p'^{\varepsilon}.(\phi\vec{w}^{\varepsilon,kl})dxdz \\ &= -\int_{\Omega_m} (q^{\varepsilon}\nabla\phi).\vec{w}^{\varepsilon,kl}dxdz + \int_{\Omega_m} e(\vec{u}^{\varepsilon}) : \nabla\vec{w}^{\varepsilon,kl}\phi dxdz \\ &\quad - \int_{\Omega_m} 2\mu^{\varepsilon} e(\vec{v}^{\varepsilon}) : \nabla\vec{w}^{\varepsilon,kl}\phi dxdz. \end{aligned}$$

Developing the second and third integral of the right-hand side of the above equation,

$$\begin{aligned} -\int_{\Omega_m} &\nabla p'^{\varepsilon}.\phi\vec{w}^{\varepsilon,kl}dxdz \\ &= -\int_{\Omega_m} (q^{\varepsilon}\nabla\phi).\vec{w}^{\varepsilon,kl}dxdz + \int_{\Omega_m} e_x(\underline{u}^{\varepsilon}) : e_x(\underline{w}^{\varepsilon,kl})\phi dxdz \\ &\quad - \int_{\Omega_m} 2\mu^{\varepsilon} e_x(\underline{v}^{\varepsilon}) : e_x(\underline{w}^{\varepsilon,kl})\phi dxdz - \int_{\Omega_m}\sum_{j=1}^{n}[q^{\varepsilon}]_{n+1j}\frac{\partial}{\partial z}\big((\underline{w}^{\varepsilon,kl})_j\big)\phi dxdz. \end{aligned} \tag{4.22}$$

Let $\vec{\psi}^{\varepsilon,kl} = (\underline{\psi}^{\varepsilon,kl}, 0)$. Multiplying the first equation of (2.4) by $\phi\vec{\psi}^{\varepsilon,kl}$, integrating by parts and using definition (4.1), we obtain (after algebraic developments)

$$\begin{aligned} \int_{\Omega_m} &(\vec{f}^{\varepsilon} - \nabla p^{\varepsilon} + \vec{\theta}).\phi\vec{\psi}^{\varepsilon,kl}dxdz \\ &= \int_{\Omega_m} (\xi^{\varepsilon}\nabla\phi).\vec{\psi}^{\varepsilon,kl} + \int_{\Omega_m} 2\mu^{\varepsilon} e_x(\underline{u}^{\varepsilon}) : e_x(\underline{\psi}^{\varepsilon,kl})\phi dxdz \\ &\quad + \int_{\Omega_m}\sum_{j=1}^{n}[\xi^{\varepsilon}]_{n+1j}\frac{\partial}{\partial z}\big((\underline{\psi}^{\varepsilon,kl})_j\big)\phi dxdz. \end{aligned}$$

Integrating by parts now the second integral of the right-hand side, we have

$$\begin{aligned} \int_{\Omega_m} &(\vec{f}^{\varepsilon} - \nabla p^{\varepsilon} + \vec{\theta}).\phi\vec{\psi}^{\varepsilon,kl}dxdz \\ &= \int_{\Omega_m} (\xi^{\varepsilon}\nabla\phi).\vec{\psi}^{\varepsilon,kl} - \int_{\Omega_m} \operatorname*{div}_x\big(2\mu^{\varepsilon} e_x(\underline{\psi}^{\varepsilon,kl})\big).(\underline{u}^{\varepsilon}\phi)dxdz \\ &\quad - \int_{\Omega_m} (2\mu^{\varepsilon} e_x(\underline{\psi}^{\varepsilon,kl})\nabla_x\phi).\underline{u}^{\varepsilon}dxdz + \int_{\Omega_m}\sum_{j=1}^{n}[\xi^{\varepsilon}]_{n+1j}\frac{\partial}{\partial z}\big((\underline{\psi}^{\varepsilon,kl})_j\big)\phi dxdz. \end{aligned}$$

Using the equation that $\underline{\psi}^{\varepsilon,kl}$ satisfy (see (4.19)),

$$\int_{\Omega_m} (\vec{f}^{\varepsilon} - \nabla p^{\varepsilon} + \vec{\theta}).\phi\vec{\psi}^{\varepsilon,kl}dxdz$$

$$
\begin{aligned}
&= \int_{\Omega_m} (\xi^\varepsilon \nabla\phi).\vec{\psi}^{\varepsilon,kl} - \int_{\Omega_m} \underline{u}^\varepsilon \operatorname*{div}_x \big(e_x(\underline{w}^{\varepsilon,kl})\big)\phi dxdz + \int_{\Omega_m} \nabla_x r_2^{\varepsilon,kl}.(\phi\underline{u}^\varepsilon)dxdz \\
&\quad - \int_{\Omega_m} \big(2\mu^\varepsilon e_x(\underline{\psi}^{\varepsilon,kl})\nabla_x\phi\big).\underline{u}^\varepsilon dxdz + \int_{\Omega_m} \sum_{j=1}^n [\xi^\varepsilon]_{n+1j} \frac{\partial}{\partial z}\big((\underline{\psi}^{\varepsilon,kl})_j\big)\phi dxdz.
\end{aligned}
$$

Integrating again by parts the second integral,

$$
\begin{aligned}
&\int_{\Omega_m} (\vec{f}^\varepsilon + \vec{\theta} - \nabla p^\varepsilon).\phi\vec{\psi}^{\varepsilon,kl} dxdz \\
&\quad = \int_{\Omega_m} (\xi^\varepsilon \nabla\phi).\vec{\psi}^{\varepsilon,kl} - \int_{\Omega_m} e_x(\underline{u}^\varepsilon) : e_x(\underline{w}^{\varepsilon,kl})\phi dxdz \\
&\qquad + \int_{\Omega_m} \nabla_x r_2^{\varepsilon,kl}.(\phi\underline{u}^\varepsilon)dxdz - \int_{\Omega_m} \big(2\mu^\varepsilon e_x(\underline{\psi}^{\varepsilon,kl})\nabla_x\phi\big).\underline{u}^\varepsilon dxdz \\
&\qquad - \int_{\Omega_m} \big(e_x(\underline{w}^{\varepsilon,kl})\nabla_x\phi\big).\underline{u}^\varepsilon dxdz + \int_{\Omega_m} \sum_{j=1}^n [\xi^\varepsilon]_{n+1j} \frac{\partial}{\partial z}\big((\underline{\psi}^{\varepsilon,kl})_j\big)\phi dxdz.
\end{aligned}
$$

Adding (4.22) and the above equation, we obtain

$$
\begin{aligned}
&\int_{\Omega_m} (\vec{f}^\varepsilon + \vec{\theta}).\phi\vec{\psi}^{\varepsilon,kl} dxdz - \int_{\Omega_m} \nabla p^\varepsilon.\phi\vec{\psi}^{\varepsilon,kl} dxdz - \int_{\Omega_m} \nabla p'^\varepsilon.\phi\vec{w}^{\varepsilon,kl} dxdz \\
&\quad = - \int_{\Omega_m} (q^\varepsilon\nabla\phi).\vec{w}^{\varepsilon,kl} dxdz - \int_{\Omega_m} 2\mu^\varepsilon e_x(\underline{v}^\varepsilon) : e_x(\underline{w}^{\varepsilon,kl})\phi dxdz \\
&\qquad - \int_{\Omega_m} \sum_{j=1}^n [q^\varepsilon]_{n+1j} \frac{\partial}{\partial z}\big((\underline{w}^{\varepsilon,kl})_j\big)\phi dxdz + \int_{\Omega_m} (\xi^\varepsilon \nabla\phi).\vec{\psi}^{\varepsilon,kl} \\
&\qquad + \int_{\Omega_m} \nabla_x r_2^{\varepsilon,kl}.(\phi\underline{u}^\varepsilon)dxdz - \int_{\Omega_m} (b^{\varepsilon,kl}\nabla_x\phi).\underline{u}^\varepsilon dxdz \\
&\qquad + \int_{\Omega_m} \sum_{j=1}^n [\xi^\varepsilon]_{n+1j} \frac{\partial}{\partial z}\big((\underline{\psi}^{\varepsilon,kl})_j\big)\phi dxdz.
\end{aligned}
\tag{4.23}
$$

where

$$
b^{\varepsilon,kl} = 2\mu^\varepsilon e_x(\underline{\psi}^{\varepsilon,kl}) + e_x(\underline{w}^{\varepsilon,kl}). \tag{4.24}
$$

We obtain easily that (using Problem (3.5))

$$
\operatorname*{div}_x(b^{\varepsilon,kl}) = -\nabla_x r_2^{\varepsilon,kl} \quad \text{in } \Omega_m.
$$

We now pass to the limit in (4.23) as ε tends to 0. In order to do so, we need some preliminaries results.

By Definition (4.24) and classical arguments concerning the convergence of periodic functions, we conclude that for all $\Omega' \subset\subset \Omega_m$,

$$
\begin{gathered}
b^{\varepsilon,kl} \rightharpoonup b^{kl} = \mathrm{m}_Y(2\mu e_y(\underline{\psi}^{kl}) + e_y(\underline{w}^{kl})) \text{ weakly in } L^2(\Omega')^{n\times n} \\
\text{and} \quad \operatorname*{div}_x(b^{kl}) = 0 \quad \text{in } \Omega_m.
\end{gathered}
\tag{4.25}
$$

By the convergence (4.16) a), we have

$$\vec{w}^{\varepsilon,kl} \to \vec{P}^{kl} = (\underline{P}^{kl}, 0) \text{ strongly in } L^2(\Omega')^{n+1}.$$

Also by (4.21) a), we get

$$\vec{\psi}^{\varepsilon,kl} \to 0 \text{ strongly in } L^2(\Omega')^{n+1}.$$

Now passing to the limit in (4.23) taking into account the precedent convergence results, we obtain

$$\begin{aligned} -\int_{\Omega_m} &\nabla p'.\phi \vec{P}^{kl} dxdz \\ &= -\int_{\Omega_m} (q\nabla\phi).\vec{P}^{kl} dxdz - \int_{\Omega_m} \sigma^{kl} : e_x(\underline{v})\phi dxdz - \int_{\Omega_m} (b^{kl}\nabla_x\phi).\underline{u} dxdz \end{aligned}$$

Integrating by parts the right-hand side of the above expression, using the second equation of (4.1.14) and the expression (4.25), we obtain

$$0 = \int_{\Omega_m} q : e(\vec{P}^{kl})\phi dxdz - \int_{\Omega_m} \sigma^{kl} : e_x(\underline{v})\phi dxdz + \int_{\Omega_m} e_x(\underline{u}) : b^{kl}\phi dxdz$$

Since $\left[e(\vec{P}^{kl})\right]_{ij} = \left[M^{kl}\right]_{ij}$, then we obtain in the distribution sense

$$[q]_{kl} = \sum_{i,j=1}^{n} \left[\sigma^{kl}\right]_{ij} \left[e_x(\underline{v})\right]_{ij} - \sum_{i,j=1}^{n} \left[b^{kl}\right]_{ij} \left[e_x(\underline{u})\right]_{ij}.$$

Now since (4.16) e), (4.25) hold and since $\left[e_x(\underline{v})\right]_{ij} = \left[e(\vec{v})\right]_{ij}$ and $\left[e_x(\underline{u})\right]_{ij} = \left[e(\vec{u})\right]_{ij}$ for all $1 \le i, j \le n$, we get

$$\begin{aligned} [q]_{kl} =& \frac{1}{|Y|} \sum_{i,j=1}^{n} \Big\{ \int_Y 2\mu[e_y(-\underline{\chi}^{kl} + \underline{P}^{kl})]_{ij} dy \Big\} \left[e(\vec{v})\right]_{ij} \\ &- \frac{1}{|Y|} \sum_{i,j=1}^{n} \Big\{ \int_Y \big(2\mu[e_y(-\underline{\psi}^{kl})]_{ij} + [e_y(-\underline{\chi}^{kl} + \underline{P}^{kl})]_{ij}\big) dy \Big\} \left[e(\vec{u})\right]_{ij}. \end{aligned} \tag{4.26}$$

Also since the following symmetry property holds (see Proposition 3.1)

$$\int_Y 2\mu[e_y(-\underline{\chi}^{kl} + \underline{P}^{kl})]_{ij} dy = \int_Y 2\mu[e_y(-\underline{\chi}^{ij} + \underline{P}^{ij})]_{kl} dy,$$

and the following's one holds too (see Proposition 3.2)

$$\begin{aligned} \int_Y &\big(2\mu\left[e_y(-\underline{\psi}^{kl}) + e_y(-\underline{\chi}^{kl} + \underline{P}^{kl})\right]_{ij}\big) dy \\ &= \int_Y \big(2\mu\left[e_y(-\underline{\psi}^{ij}) + e_y(-\underline{\chi}^{ij} + \underline{P}^{ij})\right]_{kl}\big) dy, \end{aligned} \tag{4.27}$$

we conclude that

$$\begin{aligned}[q]_{kl} =&\frac{1}{|Y|}\sum_{i,j=1}^{n}\Big\{\int_Y 2\mu[e_y(-\underline{\chi}^{ij}+\underline{P}^{ij})]_{kl}dy\Big\}\big[e(\vec{v})\big]_{ij}\\ &-\frac{1}{|Y|}\sum_{i,j=1}^{n}\Big\{\int_Y \big(2\mu[e_y(-\underline{\psi}^{ij})]_{kl}+[e_y(-\underline{\chi}^{ij}+\underline{P}^{ij})]_{kl}\big)dy\Big\}\big[e(\vec{u})\big]_{ij}\end{aligned}$$

This completes the proof. □

Identification of $[q]_{n+1j}$, $1 \le j \le n$ in Ω_m

Let φ^k be the solution of Problem (3.2). We assume that $\varphi^k = \varphi^k(y,z)$ satisfy the regularity hypotheses

$$\begin{aligned} &\text{a)}\quad \varphi^k \in L^2_{\text{loc}}(0,h_1,H^1_\sharp(Y)) \cap L^2_{\text{loc}}(]0,h_1[\times\mathbb{R}^n)\\ &\text{b)}\quad \tfrac{\partial\varphi^k}{\partial z} \in L^2_{\text{loc}}(0,h_1,L^2_\sharp(Y)) \cap L^2_{\text{loc}}(]0,h_1[\times\mathbb{R}^n)\end{aligned} \tag{4.28}$$

Let us define $\zeta^k = -\varphi^k + 2y_k$ and $\underline{\eta}^k = \mu\nabla_y\zeta^k$. We also define the following functions by Y-periodicity:

$$\zeta^{\varepsilon,k}(x,z) = \varepsilon\zeta^k(\frac{x}{\varepsilon},z), \quad \underline{\eta}^{\varepsilon,k}(x,z) = \underline{\eta}^k(\frac{x}{\varepsilon},z) \tag{4.29}$$

It is easy to see that $-\operatorname{div}_x \underline{\eta}^{\varepsilon,k} = 0$ in Ω_m. We introduce a supplementary hypothesis concerning $\underline{\eta}^{\varepsilon,k}$:

$$\begin{aligned} &\text{a)}\quad \{\big(\underline{\eta}^{\varepsilon,k}\big)_j\}_{\varepsilon>0} \subset L^p_{\text{loc}}(\Omega_m) \text{ for some } p>2, \text{ locally bounded}\\ &\text{b)}\quad \frac{\partial}{\partial z}\big(\underline{\eta}^{\varepsilon,k}\big)_j \ge 0 \text{ in the distribution sense.}\end{aligned} \tag{4.30}$$

Then we have the following result.

Proposition 4.7 (Baffico & Conca [3]) *Assume (4.28) and (4.30). Then for all $\Omega' \subset\subset \Omega_m$, we have*

$$\begin{aligned} &a)\ \zeta^{\varepsilon,k} \rightharpoonup 2y_k \text{ weakly in } H^1(\Omega')\\ &b)\ \tfrac{\partial}{\partial z}\big(\zeta^{\varepsilon,k}\big) \to 0 \text{ strongly in } L^2(\Omega')\\ &c)\ \underline{\eta}^{\varepsilon,k} \rightharpoonup \underline{\eta}^k = m_Y(\underline{\eta}^k) \text{ weakly in } L^2(\Omega')^n\\ &d)\ \tfrac{\partial}{\partial z}\big(\underline{\eta}^{\varepsilon,k}\big)_j \to \tfrac{\partial}{\partial z}\big(\underline{\eta}^k\big)_j \text{ strongly in } H^{-1}(\Omega'),\ 1 \le j \le n.\end{aligned} \tag{4.31}$$

In view of (4.3.5) c) and (4.3.3), we get $-\operatorname{div}_x \underline{\eta}^k = 0$ in Ω_m. Similarly we assume that ψ^k, the solution of (3.6), satisfies

$$\begin{aligned} &\text{a)}\quad \psi^k \in L^2_{\text{loc}}(0,h_1,H^1_\sharp(Y)) \cap L^2_{\text{loc}}(]0,h_1[\times\mathbb{R}^n)\\ &\text{b)}\quad \frac{\partial\psi^k}{\partial z} \in L^2_{\text{loc}}(0,h_1,L^2_\sharp(Y)) \cap L^2_{\text{loc}}(]0,h_1[\times\mathbb{R}^n)\end{aligned} \tag{4.32}$$

Let

$$\psi^{\varepsilon,k}(x,z)=\psi^k(\frac{x}{\varepsilon},z) \quad \text{and} \quad \underline{d}^{\varepsilon,k}=2\mu^\varepsilon\nabla_x\psi^{\varepsilon,k}+\nabla_x\zeta^{\varepsilon,k}, \tag{4.33}$$

then using (3.6), (4.29) and (4.33), we get

$$-\operatorname*{div}_x \underline{d}^{\varepsilon,k}=0 \quad \text{in } \Omega_m. \tag{4.34}$$

We assume that $\underline{d}^{\varepsilon,k}$ satisfies the regularity conditions:

$$\begin{aligned} &a) \quad \big\{\big(\underline{d}^{\varepsilon,k}\big)_j\big\}_{\varepsilon>0}\subset L^p_{\text{loc}}(\Omega_m) \text{ for some } p>2, \text{ locally bounded}\\ &b) \quad \frac{\partial}{\partial z}\big(\underline{d}^{\varepsilon,k}\big)_j\geq 0 \text{ in the distribution sense.}\end{aligned} \tag{4.35}$$

We have the following result.

Proposition 4.8 *Assume hypotheses (4.3.7) and (4.35) hold. Then for all $\Omega'\subset\subset\Omega_m$, we have*

$$\begin{aligned} &a) \quad \psi^{\varepsilon,k}\rightharpoonup 0 \text{ weakly in } H^1(\Omega')\\ &b) \quad \frac{\partial}{\partial z}\big(\psi^{\varepsilon,k}\big)\to 0 \text{ strongly in } L^2(\Omega')\\ &c) \quad \underline{d}^{\varepsilon,k}\rightharpoonup\underline{d}^k=m_Y(2\mu\nabla_y\psi^k+\nabla_y\zeta^k) \text{ weakly in } L^2(\Omega')^n\\ &d) \quad \frac{\partial}{\partial z}\big(\underline{d}^{\varepsilon,k}\big)_j\to\frac{\partial}{\partial z}\big(\underline{d}^k\big)_j \text{ strongly in } H^{-1}(\Omega'),\ 1\leq j\leq n.\end{aligned} \tag{4.36}$$

Remark 4.9 From (4.34) and (4.36) c), we have $-\operatorname{div}_x\underline{d}^k=0$ in Ω_m.

Proof of Proposition 4.8 Using classical arguments concerning convergence of periodic functions, we can prove the three first assertions. For the last one, we use the compactness Lemma 4.3 (the hypotheses of this lemma hold since we suppose that (4.35) is satisfied). $\square$

Proposition 4.10 *If (4.28), (4.30), (4.32) and (4.35) hold, then up to a subsequence, we have $[q^\varepsilon]_{n+1k}\rightharpoonup[q]_{n+1k}$ weakly in $L^2(\Omega_m)$ $\forall 1\leq j\leq n$, where*

$$\begin{aligned}[q]_{n+1k}=&\frac{1}{|Y|}\sum_{i=1}^n\Big\{\int_Y\mu\frac{\partial(-\varphi^i+2y_i)}{\partial y_k}dy\Big\}\big[e(\vec v)\big]_{n+1i}\\ &-\frac{1}{2|Y|}\sum_{i=1}^n\Big\{\int_Y\big(2\mu\frac{\partial\psi^i}{\partial y_k}+\frac{\partial(-\varphi^i+2y_i)}{\partial y_k}dy\Big\}\big[e(\vec u)\big]_{n+1i}.\end{aligned}$$

Proof. Let $\phi\in\mathcal{D}(\Omega_m)$ and $\vec\zeta^{\varepsilon,k}=(\underline{0},\zeta^{\varepsilon,k})$. Multiplying the second equation of (2.4) by $\phi\vec\zeta^{\varepsilon,k}$, integrating by parts and using (4.2), we obtain

$$-\int_{\Omega_m}\nabla p'^\varepsilon.\vec\zeta^{\varepsilon,k}\phi dxdz=-\int_{\Omega_m}(q^\varepsilon\nabla\phi).\vec\zeta^{\varepsilon,k}dxdz+\int_{\Omega_m}e(\vec u^\varepsilon):\nabla\vec\zeta^{\varepsilon,k}\phi dxdz$$

$$-\int_{\Omega_m} 2\mu^\varepsilon e(\vec{v}^\varepsilon) : \nabla \vec{\zeta}^{\varepsilon,k} \phi dxdz.$$

After some elementary computations on the second and third integral of the right-hand side of the above equation,

$$\begin{aligned}
&-\int_{\Omega_m} \nabla p'^\varepsilon . \vec{\zeta}^{\varepsilon,k} \phi dxdz \\
&\quad = -\int_{\Omega_m} (q^\varepsilon \nabla \phi). \vec{\zeta}^{\varepsilon,k} dxdz - \int_{\Omega_m} \mu^\varepsilon \nabla_x v^\varepsilon_{n+1} . \nabla_x \zeta^{\varepsilon,k} \phi dxdz \\
&\qquad - \int_{\Omega_m} \sum_{i=1}^n \mu^\varepsilon \frac{\partial v_i^\varepsilon}{\partial z} \frac{\partial \zeta^{\varepsilon,k}}{\partial x_i} \phi dxdz + \frac{1}{2} \int_{\Omega_m} \nabla_x u^\varepsilon_{n+1} . \nabla_x \zeta^{\varepsilon,k} \phi dxdz \\
&\qquad - \int_{\Omega_m} \sum_{j=1}^n [q^\varepsilon]_{n+1n+1} \frac{\partial \zeta^{\varepsilon,k}}{\partial z} \phi dxdz + \frac{1}{2} \int_{\Omega_m} \sum_{i=1}^n \mu^\varepsilon \frac{\partial u_i^\varepsilon}{\partial z} \frac{\partial \zeta^{\varepsilon,k}}{\partial x_i} \phi dxdz.
\end{aligned} \tag{4.37}$$

Let $\vec{\psi}^{\varepsilon,k} = (\underline{0}, \psi^{\varepsilon,k})$. Multiplying the first equation of (2.4) by $\phi \vec{\psi}^{\varepsilon,k}$, integrating by parts and using Definition (4.1), we get

$$\int_{\Omega_m} (\vec{f}^\varepsilon - \nabla p^\varepsilon + \vec{\theta}). \vec{\psi}^{\varepsilon,k} \phi dxdz = \int_{\Omega_m} (\xi^\varepsilon \nabla \phi). \vec{\psi}^{\varepsilon,k} + \int_{\Omega_m} 2\mu^\varepsilon e(\vec{u}^\varepsilon) : \nabla \vec{\psi}^{\varepsilon,k} \phi dxdz$$

Rewriting the second integral of the right-hand side differently, the above expression becomes

$$\begin{aligned}
&\int_{\Omega_m} (\vec{f}^\varepsilon - \nabla p^\varepsilon + \vec{\theta}). \vec{\psi}^{\varepsilon,k} \phi dxdz \\
&\quad = \int_{\Omega_m} (\xi^\varepsilon \nabla \phi). \vec{\psi}^{\varepsilon,k} + \int_{\Omega_m} \mu^\varepsilon \nabla_x u^\varepsilon_{n+1} . \nabla_x \psi^{\varepsilon,k} \phi dxdz \\
&\qquad + \int_{\Omega_m} \sum_{i=1}^n (\mu^\varepsilon \frac{\partial u_i^\varepsilon}{\partial z}) \frac{\partial \psi^{\varepsilon,k}}{\partial x_i} \phi dxdz + \int_{\Omega_m} \sum_{i=1}^n [\xi^\varepsilon]_{n+1n+1} \frac{\partial \psi^{\varepsilon,k}}{\partial z} \phi dxdz.
\end{aligned} \tag{4.38}$$

Integrating by parts the second integral of the right-hand side of the above equation and using (4.33) and(4.3.10), we derive

$$\begin{aligned}
&\int_{\Omega_m} \mu^\varepsilon \nabla_x u^\varepsilon_{n+1} . \nabla_x \psi^{\varepsilon,k} \phi dxdz \\
&\quad = -\frac{1}{2} \int_{\Omega_m} u^\varepsilon_{n+1} \operatorname*{div}_x (2\mu^\varepsilon \nabla_x \psi^{\varepsilon,k}) \phi dxdz - \int_{\Omega_m} \mu^\varepsilon u^\varepsilon_{n+1} \nabla_x \phi . \nabla_x \psi^{\varepsilon,k} dxdz \\
&\quad = \frac{1}{2} \int_{\Omega_m} u^\varepsilon_{n+1} \operatorname*{div}_x (\nabla_x \zeta^{\varepsilon,k}) \phi dxdz - \int_{\Omega_m} \mu^\varepsilon u^\varepsilon_{n+1} \nabla_x \phi . \nabla_x \psi^{\varepsilon,k} dxdz
\end{aligned}$$

Now, integrating by parts the first integral of the right-hand side of the above equation and using (4.33), we get

$$\int_{\Omega_m} \mu^\varepsilon \nabla_x u^\varepsilon_{n+1} . \nabla_x \psi^{\varepsilon,k} \phi dxdz$$

$$= -\frac{1}{2}\int_{\Omega_m} \nabla_x u^\varepsilon_{n+1}.\nabla_x \zeta^{\varepsilon,k}\phi dxdz - \frac{1}{2}\int_{\Omega_m} u^\varepsilon_{n+1}(\underline{d}^{\varepsilon,k}.\nabla_x\phi)dxdz$$

Therefore the expression (4.38) becomes

$$\begin{aligned}
&\int_{\Omega_m}(\vec{f}^\varepsilon - \nabla p^\varepsilon + \vec{\theta}).\vec{\psi}^{\varepsilon,k}\phi dxdz\\
&= \int_{\Omega_m}(\xi^\varepsilon\nabla\phi).\vec{\psi}^{\varepsilon,k} - \frac{1}{2}\int_{\Omega_m}\nabla_x u^\varepsilon_{n+1}.\nabla_x\zeta^{\varepsilon,k}\phi dxdz\\
&\quad - \frac{1}{2}\int_{\Omega_m} u^\varepsilon_{n+1}(\underline{d}^{\varepsilon,k}.\nabla_x\phi)dxdz + \int_{\Omega_m}\sum_{i=1}^n(\mu^\varepsilon\frac{\partial u^\varepsilon_i}{\partial z})\frac{\partial\psi^{\varepsilon,k}}{\partial x_i}\phi dxdz\\
&\quad + \int_{\Omega_m}\sum_{i=1}^n[\xi^\varepsilon]_{n+1n+1}\frac{\partial\psi^{\varepsilon,k}}{\partial z}\phi dxdz.
\end{aligned}$$

Adding (4.37) and the above equation, we have (using Definition (4.33))

$$\begin{aligned}
&\int_{\Omega_m}(\vec{f}^\varepsilon - \nabla p^\varepsilon + \vec{\theta}).\vec{\psi}^{\varepsilon,k}\phi dxdz - \int_{\Omega_m}\nabla p'^\varepsilon.\vec{\zeta}^{\varepsilon,k}\phi dxdz\\
&= -\int_{\Omega_m}(q^\varepsilon\nabla\phi).\vec{\zeta}^{\varepsilon,k}dxdz - \int_{\Omega_m}\underline{\eta}^{\varepsilon,k}.\nabla_x v^\varepsilon_{n+1}\phi dxdz\\
&\quad - \int_{\Omega_m}\sum_{i=1}^n\mu^\varepsilon\frac{\partial q^\varepsilon_i}{\partial z}\frac{\partial\zeta^{\varepsilon,k}}{\partial x_i}\phi dxdz - \int_{\Omega_m}\sum_{j=1}^n[q^\varepsilon]_{n+1n+1}\frac{\partial\zeta^{\varepsilon,k}}{\partial z}\phi dxdz \\
&\quad + \frac{1}{2}\int_{\Omega_m}\sum_{i=1}^n\frac{\partial u^\varepsilon_i}{\partial z}(\underline{d}^{\varepsilon,k})_i\phi dxdz + \int_{\Omega_m}(\xi^\varepsilon\nabla\phi).\vec{\psi}^{\varepsilon,k}\\
&\quad - \frac{1}{2}\int_{\Omega_m}u^\varepsilon_{n+1}(\underline{d}^{\varepsilon,k}.\nabla_x\phi)dxdz + \int_{\Omega_m}\sum_{i=1}^n[\xi^\varepsilon]_{n+1n+1}\frac{\partial\psi^{\varepsilon,k}}{\partial z}\phi dxdz.
\end{aligned}\tag{4.39}$$

We now pass to the limit in the above equation as $\varepsilon \to 0$. To do so, we need some preliminary results. From convergence (4.3.5) a), we have

$$\vec{\zeta}^{\varepsilon,k} \to \vec{\zeta}^k = (\underline{0}, 2y_k) \text{ weakly in } L^2(\Omega_m)^{n+1}.$$

and from (4.3.12) a), we get $\vec{\psi}^{\varepsilon,k} \to 0$ weakly in $L^2(\Omega_m)^{n+1}$.

Now passing to the limit in (4.3.22) taking into account the precedent convergence results, we obtain

$$\begin{aligned}
\int_{\Omega_m}\nabla p'.\vec{\zeta}^k\phi dxdz &= -\int_{\Omega_m}(q\nabla\phi).\vec{\zeta}^k dxdz - \int_{\Omega_m}\underline{\eta}^k.\nabla_x v_{n+1}\phi dxdz\\
&\quad - \int_{\Omega_m}\sum_{i=1}^n(\underline{\eta}^k)_i\frac{\partial v_i}{\partial z}\phi dxdz - \frac{1}{2}\int_{\Omega_m}u_{n+1}(\underline{d}^k.\nabla_x\phi)dxdz\\
&\quad + \frac{1}{2}\int_{\Omega_m}\sum_{i=1}^n(\underline{d}^k)_i\frac{\partial u_i}{\partial z}\phi dxdz.
\end{aligned}$$

Integrating by parts the above expression, we derive

$$\begin{aligned}0 = &\int_{\Omega_m} q : e(\vec{\zeta}^k)\phi dxdz - \int_{\Omega_m}\sum_{i=1}^{n} 2(\underline{\eta}^k)_i \big[e(\vec{v})\big]_{n+1i}\phi dxdz \\ &+ \int_{\Omega_m}\sum_{i=1}^{n}(\underline{d}^k)_i \big[e(\vec{u})\big]_{n+1i}\phi dxdz.\end{aligned}$$

Using the fact that $q : e(\vec{\zeta}^k) = 2[q]_{n+1k}$, we obtain, in the distribution sense,

$$[q]_{n+1k} = \sum_{i=1}^{n}(\underline{\eta}^k)_i\big[e(\vec{v})\big]_{n+1i} - \frac{1}{2}\sum_{i=1}^{n}(\underline{d}^k)_i\big[e(\vec{u})\big]_{n+1i}.$$

Also by Definition (4.31) c) and (4.36) c) of $\underline{\eta}^k$ and of $\underline{d}^k$, we have

$$\begin{aligned}{}_{n+1k} = &\frac{1}{|Y|}\sum_{i=1}^{n}\big\{\int_Y \mu\frac{\partial(-\varphi^k + 2y_k)}{\partial y_i}dy\big\}\big[e(\vec{v})\big]_{n+1i} \\ -&\frac{1}{2|Y|}\sum_{i=1}^{n}\big\{\int_Y \big(2\mu\frac{\partial\psi^k}{\partial y_i} + \frac{\partial(-\varphi^k + 2y_k)}{\partial y_i}\big)dy\big\}\big[e(\vec{u})\big]_{n+1i}.\end{aligned} \tag{4.40}$$

Hence, since Proposition 3.1 holds,

$$\int_Y \mu\frac{\partial(-\varphi^k + 2y_k)}{\partial y_i}dy = \int_Y \mu\frac{\partial(-\varphi^i + 2y_i)}{\partial y_k}dy, \tag{4.41}$$

and since Proposition 3.2 holds,

$$\int_Y \big(2\mu\frac{\partial\psi^k}{\partial y_i} + \frac{\partial(-\varphi^k + 2y_k)}{\partial y_i}\big)dy = \int_Y \big(2\mu\frac{\partial\psi^i}{\partial y_k} + \frac{\partial(-\varphi^i + 2y_i)}{\partial y_k}\big)dy. \tag{4.42}$$

Finally using (4.41) and (4.42) in (4.40) we obtain the announced result, i.e.,

$$\begin{aligned}[q]_{n+1k} = &\frac{1}{|Y|}\sum_{i=1}^{n}\big\{\int_Y \mu\frac{\partial(-\varphi^i + 2y_i)}{\partial y_k}dy\big\}\big[e(\vec{v})\big]_{n+1i} \\ &- \frac{1}{2|Y|}\sum_{i=1}^{n}\big\{\int_Y \big(2\mu\frac{\partial\psi^i}{\partial y_k} + \frac{\partial(-\varphi^i + 2y_i)}{\partial y_k}dy\big\}\big[e(\vec{u})\big]_{n+1i}.\end{aligned}$$

Since the matrix q^ε is symmetric, this implies that q is symmetric. Hence $[q]_{n+1k} = [q]_{kn+1}$ which completes the proof. □

Identification of $[q]_{n+1n+1}$ in Ω_m

The following proposition gives a result concerning the identification of the last component of q.

Proposition 4.11 $[q^\varepsilon]_{n+1n+1} \to [q]_{n+1n+1}$ *weakly in* $L^2(\Omega_m)$ *(up to a subsequence), where*

$$[q]_{n+1n+1} = \big\{\frac{1}{|Y|}\int_Y 2\mu dy\big\}\big[e(\vec{v})\big]_{n+1n+1} - \big[e(\vec{u})\big]_{n+1n+1}.$$

Proof. To prove this result, we proceed as in [1, 2]; We use the method of Brizzi [5]. From (4.2) we have $[q^\varepsilon]_{n+1n+1} = 2\mu^\varepsilon \frac{\partial v^\varepsilon_{n+1}}{\partial z} - \frac{\partial u^\varepsilon_{n+1}}{\partial z}$ and $\mu^\varepsilon = \mu_1 \chi_{\Omega_1^\varepsilon \cap \Omega_m} + \mu_2 \chi_{\Omega_2^\varepsilon \cap \Omega_m}$, hence

$$
\begin{aligned}
&[q^\varepsilon]_{n+1n+1} \\
&= 2\mu_1 P_1\big(\frac{\partial (v^\varepsilon_{n+1})_1}{\partial z}\big) + 2\mu_2 P_2\big(\frac{\partial (v^\varepsilon_{n+1})_2}{\partial z}\big) - P_1\big(\frac{\partial (u^\varepsilon_{n+1})_1}{\partial z}\big) - P_2\big(\frac{\partial (u^\varepsilon_{n+1})_2}{\partial z}\big),
\end{aligned}
$$

where $(u^\varepsilon_{n+1})_i = u^\varepsilon_{n+1}|_{\Omega_i^\varepsilon}, (v^\varepsilon_{n+1})_i = v^\varepsilon_{n+1}|_{\Omega_i^\varepsilon} (i = 1, 2)$ and P_i represent the extension by 0 in $\Omega \setminus \Omega_i^\varepsilon (i = 1, 2)$.

It is easy to see that $P_i\big(\frac{\partial (u^\varepsilon_{n+1})_i}{\partial z}\big)$ and $P_i\big(\frac{\partial (v^\varepsilon_{n+1})_i}{\partial z}\big)$ are bounded in $L^2(\Omega_m)$. Therefore (up to a subsequence), there exists ν_i and $\gamma_i \in L^2(\Omega_m)$ such that

$$
P_i\big(\frac{\partial (u^\varepsilon_{n+1})_i}{\partial z}\big) \rightharpoonup \nu_i \text{ weakly in } L^2(\Omega_m), \tag{4.43}
$$

$$
P_i\big(\frac{\partial (v^\varepsilon_{n+1})_i}{\partial z}\big) \rightharpoonup \gamma_i \text{ weakly in } L^2(\Omega_m). \tag{4.44}
$$

We now proceed to identify these limits. Concerning ν_i, we have

$$
\nu_1 = \rho(x, z)\frac{\partial u_{n+1}}{\partial z}, \quad \nu_2 = (1 - \rho(x, z))\frac{\partial u_{n+1}}{\partial z} \tag{4.45}
$$

where ρ is defined by (4.9) (see Baffico & Conca [3]).

In the same way, we can find explicitly γ_i: Let $\phi \in \mathcal{D}(\Omega)$, then

$$
\begin{aligned}
&{}_{H^{-1}(\Omega')}\langle \frac{\partial}{\partial z}(\chi_{\Omega_1^\varepsilon \cap \Omega_m}), \phi v^\varepsilon_{n+1}\rangle_{H_0^1(\Omega')} \\
&= -\int_{\Omega_m} \chi_{\Omega_1^\varepsilon \cap \Omega_m} v^\varepsilon_{n+1} \frac{\partial \phi}{\partial z} dxdz - \int_{\Omega_m} P_i\big(\frac{\partial (v^\varepsilon_{n+1})_i}{\partial z}\big)\phi dxdz.
\end{aligned} \tag{4.46}
$$

Using Lemma 4.3 for the sequence $\big\{\frac{\partial}{\partial z}(\chi_{\Omega_1^\varepsilon \cap \Omega_m})\big\}_{\varepsilon > 0}$, it is shown that this sequence satisfies the hypotheses required [1, 5]. Using (4.5), we can pass to the limit in the left-hand side of (4.46). For the right-hand side, from (4.4), (4.8) and (4.44), in the limit, we have

$$
\begin{aligned}
&H^{-1}(\Omega')\langle \frac{\partial}{\partial z}\big(\frac{|O(z)|}{|Y|}\big), \phi v_{n+1}\rangle_{H_0^1(\Omega')} \\
&\qquad = -\int_{\Omega_m} \big(\frac{|O(z)|}{|Y|}\big) v_{n+1} \frac{\partial \phi}{\partial z} dxdz - \int_{\Omega_m} \gamma_1 \phi dxdz
\end{aligned}
$$

and developing the duality product in the last equation, we obtain in the distribution sense the identity

$$
\gamma_1 = \rho(x, z)\frac{\partial v_{n+1}}{\partial z}. \tag{4.47}
$$

In the same way, passing to the limit in $\langle\frac{\partial}{\partial z}(\chi_{\Omega_2^\varepsilon\cap\Omega_m}), \phi v^\varepsilon_{n+1}\rangle$ and using again the compactness Lemma, we get

$$\gamma_2 = (1-\rho(x,z))\frac{\partial v_{n+1}}{\partial z}. \tag{4.48}$$

From (4.43) and (4.44), we have $[q]_{n+1n+1} = 2\mu_1\gamma_1 + 2\mu_2\gamma_2 - \nu_1 - \nu_2$. Hence using (4.4.5), (4.47) and (4.48), we conclude

$$[q]_{n+1n+1} = 2\big(\mu_1\rho(x,z) + \mu_2\big(1-\rho(x,z)\big)\big)\frac{\partial v_{n+1}}{\partial z} - \frac{\partial u_{n+1}}{\partial z},$$

which gives the announced result, that is

$$[q]_{n+1n+1} = \{\frac{1}{|Y|}\int_Y 2\mu dy\}\big[e(\vec{v})\big]_{n+1n+1} - \big[e(\vec{u})\big]_{n+1n+1}.$$

Hence Proposition 4.11 is proved. □

Identification of q in Ω_1 and Ω_2

Proposition 4.12 *For $i = 1,2$ $q^\varepsilon|_{\Omega_i} \rightharpoonup q|_{\Omega_i}$ weakly in $L^2(\Omega_m)^{n+1^2}$ (up to a subsequence), where*

$$q|_{\Omega_i} = 2\mu_i e(\vec{v}|_{\Omega_i}) - e(\vec{u}|_{\Omega_i}).$$

Proof. From (4.2), we have

$$q^\varepsilon|_{\Omega_i} = 2\mu_i e(\vec{v}^\varepsilon|_{\Omega_i}) - e(\vec{u}^\varepsilon|_{\Omega_i}).$$

From the convergence (4.4), (4.5), and (4.6), we easily obtain the result of Proposition 4.12. □

Conclusion From Propositions 4.6, 4.10, 4.11, and 4.12, from the definition of tensors $\mathcal{A}$ and $\mathcal{B}$, we conclude that q, $\vec{u}$, and $\vec{v}$ are related by (4.11). Therefore, since q satisfies (4.7) by Proposition 4.2, we have that $(\vec{u},p)$ and $(\vec{v},p)$ are solutions of (3.16).

From the properties of the tensors $\mathcal{A}$ and $\mathcal{B}$, problem (3.16) admits a unique solution and hence, the whole sequence $\vec{u}^\varepsilon$ and $\vec{v}^\varepsilon$ converge weakly to $\vec{u}$ and $\vec{v}$ respectively. This completes the proof of Theorem 3.3. □

5 Optimal control

The following theorem gives a convergence result of the optimal control.

Theorem 5.1 *For θ fixed in $\mathcal{U}_{ad}$, we consider $(\vec{u},p)$ as the solution of*

$$\begin{gathered} -\operatorname{div}(\mathcal{A}e(\vec{u})) = \vec{f} - \nabla p + \vec{\theta} \quad \text{in } \Omega \\ \operatorname{div}\vec{u} = 0 \quad \text{in } \Omega \\ \vec{u} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{5.1}$$

The cost function is

$$J_0(\vec{\theta}) = \frac{1}{2}\int_\Omega \mathcal{B}e(\vec{u}) : e(\vec{u})dxdz + \frac{N}{2}\int_\Omega |\vec{\theta}|^2 dxdz.$$

Then the optimal control $\vec{\theta}^\varepsilon_\star$ *of problem (2.1)–(2.3) satisfies*

$$\vec{\theta}^\varepsilon_\star \to \vec{\theta}_\star \text{ strongly in } L^2(\Omega)^{n+1}$$

and $\vec{\theta}_\star$ *satisfies the optimality condition*

$$J_0(\vec{\theta}_\star) = \min_{\vec{\theta}\in\mathcal{U}_{ad}} J_0(\vec{\theta}), \tag{5.2}$$

Furthermore there is convergence of the minimal cost, i.e.,

$$\lim_{\varepsilon\to 0} J_\varepsilon(\vec{\theta}^\varepsilon_\star) = J_0(\vec{\theta}_\star). \tag{5.3}$$

Proof. **Step 1: A priori estimates** By the optimal control definition, we have for all $\vec{\theta}$ in $\mathcal{U}_{ad}$

$$\frac{N}{2}\|\vec{\theta}^\varepsilon_\star\|_{L^2(\Omega)^{n+1}} \le J_\varepsilon(\vec{\theta}^\varepsilon_\star) \le J_\varepsilon(\vec{\theta}) \le C,$$

so $\vec{\theta}^\varepsilon_\star$ is bounded in $L^2(\Omega)^n$ and is such that (for a subsequence)

$$\vec{\theta}^\varepsilon_\star \rightharpoonup \vec{\theta}_\star \text{ weakly in } L^2(\Omega)^{n+1}. \tag{5.4}$$

We will show later in the proof that the above weakly convergence is in fact a strong convergence (cf 5.14). Let $(\vec{u}^\varepsilon_\star, p^\varepsilon_\star)$ and $(\vec{v}^\varepsilon_\star, p^{'\varepsilon}_\star)$ the optimal state and the corresponding adjoint state respectively associated to $\vec{\theta}^\varepsilon_\star$. By the same arguments as those used in the proof of Theorem 3.3 (the fact that $\vec{\theta}$ is replaced by $\vec{\theta}^\varepsilon_\star$ poses no problem), we get

$$\begin{gathered}\vec{u}^\varepsilon_\star \rightharpoonup \vec{u}_\star \text{ weakly in } H^1(\Omega)^{n+1}, \quad p^\varepsilon_\star \rightharpoonup p_\star \text{ weakly in } L^2_0(\Omega),\\ \vec{v}^\varepsilon_\star \rightharpoonup \vec{v}_\star \text{ weakly in } H^1(\Omega)^{n+1}, \quad p^{'\varepsilon}_\star \rightharpoonup p'_\star \text{ weakly in } L^2_0(\Omega)\end{gathered} \tag{5.5}$$

where $(\vec{u}_\star, p_\star)$ and $(\vec{v}_\star, p'_\star)$ satisfy

$$\begin{gathered}-\operatorname{div}\big(\mathcal{A}e(\vec{u}_\star)\big) = \vec{f} - \nabla p_\star + \vec{\theta}_\star \quad \text{in } \Omega\\ \operatorname{div}\big(\mathcal{A}e(\vec{v}_\star) - \mathcal{B}e(\vec{u}_\star)\big) = -\nabla p'_\star \text{ dans } \Omega\\ \operatorname{div}\vec{u}_\star = \operatorname{div}\vec{v}_\star = 0 \quad \text{in } \Omega\\ \vec{u}_\star = \vec{v}_\star = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{5.6}$$

the optimal control $\vec{\theta}_\star$ is characterized by the variationnal inequality

$$\vec{\theta}_\star \in \mathcal{U}_{ad} \quad \text{and} \quad \int_\Omega (\vec{v}_\star + N\vec{\theta}_\star).(\vec{\theta} - \vec{\theta}_\star) \ge 0\ \forall\vec{\theta} \in \mathcal{U}_{ad}.$$

Step 2: Energy convergence Using integration by parts and the first equation of (2.4), we have

$$\begin{aligned}\int_\Omega e(\vec{u}^\varepsilon_\star):e(\vec{u}^\varepsilon_\star)dxdz &= -\int_\Omega \operatorname{div}(e(\vec{u}^\varepsilon_\star)).\vec{u}^\varepsilon_\star dxdz\\ &= -\int_\Omega \operatorname{div}(2\mu^\varepsilon e(\vec{v}^\varepsilon_\star)).\vec{u}^\varepsilon_\star dxdz - \int_\Omega \nabla {p'}^\varepsilon_\star.\vec{u}^\varepsilon_\star dxdz.\end{aligned} \tag{5.7}$$

Integrating now by parts the right-hand side of the above equation and using the second equation of (2.4),

$$\begin{aligned}\int_\Omega e(\vec{u}^\varepsilon_\star):e(\vec{u}^\varepsilon_\star)dxdz &= \int_\Omega 2\mu^\varepsilon e(\vec{v}^\varepsilon_\star):e(\vec{u}^\varepsilon_\star)dxdz\\ &= \int_\Omega 2\mu^\varepsilon e(\vec{u}^\varepsilon_\star):e(\vec{v}^\varepsilon_\star)dxdz\\ &= -\int_\Omega \operatorname{div}(2\mu^\varepsilon e(\vec{u}^\varepsilon_\star)).\vec{v}^\varepsilon_\star dxdz\\ &= \int_\Omega (\vec{f}^\varepsilon - \nabla p^\varepsilon_\star + \vec{\theta}^\varepsilon_\star).\vec{v}^\varepsilon_\star dxdz.\end{aligned} \tag{5.8}$$

Using (4.10),(5.5) and (5.6), we have

$$\int_\Omega e(\vec{u}^\varepsilon_\star):e(\vec{u}^\varepsilon_\star)dxdz \to \int_\Omega (\vec{f} - \nabla p_\star + \vec{\theta}_\star).\vec{v}_\star dxdz.$$

Using the first equation of (5.6) and integrating by parts in the right-hand side of the above equation, we get (using the symmetry properties of $\mathcal{A}$),

$$\begin{aligned}\int_\Omega (\vec{f} - \nabla p_\star + \vec{\theta}_\star).\vec{v}_\star dxdz &= -\int_\Omega \operatorname{div}(\mathcal{A}e(\vec{u}_\star)).\vec{v}_\star dxdz\\ &= -\int_\Omega \mathcal{A}e(\vec{u}_\star):e(\vec{v}_\star)dxdz\\ &= -\int_\Omega e(\vec{u}_\star):\mathcal{A}e(\vec{v}_\star)dxdz\\ &= -\int_\Omega \operatorname{div}(\mathcal{A}e(\vec{v}_\star)).\vec{u}_\star dxdz.\end{aligned} \tag{5.9}$$

Using now the second equation of (5.7) and integrating by parts in the last integral of (5.9),

$$\begin{aligned}-\int_\Omega \operatorname{div}(\mathcal{A}e(\vec{v}_\star)).\vec{u}_\star dxdz &= -\int_\Omega \operatorname{div}(\mathcal{B}e(\vec{u}_\star)).\vec{u}_\star dxdz + \int_\Omega \nabla p'_\star.\vec{u}_\star dxdz\\ &= -\int_\Omega \operatorname{div}(\mathcal{B}e(\vec{u}_\star)).\vec{u}_\star dxdz\\ &= \int_\Omega \mathcal{B}e(\vec{u}_\star):e(\vec{u}_\star)dxdz.\end{aligned} \tag{5.10}$$

Finally, by (5.8)-(5.10), we have the energy convergence

$$\int_\Omega e(\vec{u}_\star^\varepsilon) : e(\vec{u}_\star^\varepsilon) dxdz \to \int_\Omega \mathcal{B}e(\vec{u}_\star) : e(\vec{u}_\star) dxdz. \tag{5.11}$$

Step 3 Taking into account the definition of optimal control (5.2), we have

$$\forall \vec{\theta} \in \mathcal{U}_{ad} \quad J_\varepsilon(\vec{\theta}) \geq J_\varepsilon(\vec{\theta}_\star^\varepsilon).$$

Now passing to the limit in inequality (4.38) and using (4.37), we get

$$J_0(\vec{\theta}) \geq \frac{1}{2} \int_\Omega \mathcal{B}e(\vec{u}_\star) : e(\vec{u}_\star) dxdz + \frac{N}{2} \limsup_{\varepsilon \to 0} \int_\Omega |\vec{\theta}_\star^\varepsilon|^2 dxdz. \tag{5.12}$$

Thus taking $\vec{\theta} = \vec{\theta}_\star$ in the above equation, we have

$$\limsup_{\varepsilon \to 0} \int_\Omega |\vec{\theta}_\star^\varepsilon|^2 dxdz \leq \int_\Omega |\vec{\theta}_\star|^2 dxdz.$$

By (5.4) and the above equation, we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega |\vec{\theta}_\star^\varepsilon|^2 dxdz = \int_\Omega |\vec{\theta}_\star|^2 dxdz. \tag{5.13}$$

From (5.12) and the above equation, we have (5.2). Now from (5.11) and the above equation, we get (5.3).

Finally from (5.4) and (5.13), we derive

$$\vec{\theta}_\star^\varepsilon \to \vec{\theta}_\star \quad \text{strongly in } L^2(\Omega)^{n+1}. \tag{5.14}$$

This completes the proof. $\square$

6 The case $\Omega \subset \mathbb{R}^2$

When $\Omega \subset \mathbb{R}^2$, it is possible to find explicit functions which are solutions of (3.1), (3.2), (3.5), and (3.6) and satisfies hypotheses (4.12), (4.15), (4.17), (4.20), (4.28), (4.30), (4.32) and (4.35).

In this case $Y =]0,1[$ is as in figure 2 (left) and Problems (3.1) and (3.2) become

$$\begin{gathered} -\frac{d}{dy}\big(2\mu \frac{d}{dy}(-\chi + y)\big) = -\frac{dr_1}{dy} \quad \text{in }]0,1[\\ \frac{d\chi}{dy} = 0 \quad \text{in }]0,1[\\ \chi(0) = \chi(1), r_1(0) = r_1(1) \end{gathered} \tag{6.1}$$

and

$$\begin{gathered} -\frac{d}{dy}\big(\mu \frac{d}{dy}(-\varphi + 2y)\big) = 0 \quad \text{in }]0,1[\\ \varphi(0) = \varphi(1). \end{gathered} \tag{6.2}$$

We have the following result (see Baffico & Conca [3]):

Proposition 6.1 *For $z \in [0, h_1[$ fixed. Let $\chi = 0$, $r_1 = 2\mu$ and*

$$\varphi(y) = \begin{cases} (2 - \frac{C}{\mu_2})y & \text{if } y \in]0, a(z)[\\ (2 - \frac{C}{\mu_1})y + a(\frac{C}{\mu_1} - \frac{C}{\mu_2}) & \text{if } y \in]a(z), b(z)[\\ (2 - \frac{C}{\mu_2})y + (a-b)(\frac{C}{\mu_1} - \frac{C}{\mu_2}) & \text{if } y \in]b(z), 1[\end{cases}$$

with $a = a(z)$, $b = b(z)$, and

$$C = C(z) = 2\big(\frac{1}{|Y|}\int_Y \frac{1}{\mu} dy\big)^{-1}.$$

Then (χ, r_1) and φ are the unique solution (up to an additive constant) of Problems (6.1) and (6.2). □

Remark 6.2 We can calculate the value of $C(z)$ (see [2, 3]), we get

$$C(z) = \frac{2\mu_1\mu_2}{(\mu_2 - \mu_1)(b(z) - a(z)) + \mu_1}.$$

Studying the regularity of the solutions of Problems (6.1) and (6.2) and supposing that $\mu_2 < \mu_1$, hypotheses (4.12), (4.15), (4.28) and (4.30) are satisfied (cf [3]).

Let study now Problems (3.5) and (3.6), they becomes

$$\begin{gathered} -\frac{d}{dy}\big(2\mu\frac{d\lambda}{dy} + \frac{d}{dy}(-\chi + y)\big) = -\frac{dr_2}{dy} \quad \text{in }]0,1[\\ \frac{\lambda}{dy} = 0 \quad \text{in }]0,1[\\ \lambda(0) = \lambda(1), r_2(0) = r_2(1) \end{gathered} \tag{6.3}$$

and

$$\begin{gathered} -\frac{d}{dy}\big(\mu\frac{d\psi}{dy} + \frac{d}{dy}(-\varphi + 2y)\big) = 0 \quad \text{in }]0,1[\\ \psi(0) = \psi(1). \end{gathered} \tag{6.4}$$

Proposition 6.3 *For z fixed in $[0, h_1[$, Let $\lambda = c_1$, $r_2 = c_2$, where c_1 and c_2 are constants, and let*

$$\psi(y) = \begin{cases} \frac{1}{2\mu_2}(K - \frac{C}{\mu_2})y & \text{if } y \in]0, a(z)[\\ \frac{1}{2\mu_1}(K - \frac{C}{\mu_1})y + a\big((\frac{K}{2\mu_2} - \frac{K}{2\mu_1}) - (\frac{C}{2\mu_2^2} - \frac{C}{2\mu_1^2})\big) & \text{if } y \in]a(z), b(z)[\\ \frac{1}{2\mu_2}(K - \frac{C}{\mu_1})y & \\ +(a-b)\big((\frac{K}{2\mu_2} - \frac{K}{2\mu_1}) - (\frac{C}{2\mu_2^2} - \frac{C}{2\mu_1^2})\big) & \text{if } y \in]b(z), 1[\end{cases}$$

with

$$K = K(z) = \frac{C}{2\mu_1\mu_2}\big(2(\mu_1 + \mu_2) - C\big)$$

and the constant C defined in Proposition 6.1. Then (λ, r_2) and ψ are the unique solutions (up to an additive constant) of Problems (6.3) and (6.4).

Remark 6.4 The hypotheses (4.17), (4.20), (4.32) and (4.35) are satisfied as shown next.

From Proposition 6.3, it is easy to check that $\lambda = c_1$ verifies hypothesis (4.17).

It is no longer necessary to suppose hypothesis (4.20), since we can pass to the limit in $\langle \frac{\partial}{\partial z} r_2^\varepsilon, u_{n+1}^\varepsilon \rangle$ using the explicit form of r_2 and r_2^ε.

By the form of ψ (see Proposition 6.3), and since the function h satisfies i) and ii) (see Section 2), we show that $\psi = \psi(y,z) \in H^1_{\text{loc}}(Y \times [0, h_1[)$ (using the implicit function theorem). Therefore, Hypothesis (4.3.7) is fulfilled.

Concerning Hypothesis (4.35), since we can calculate explicitly $\frac{\partial \psi}{\partial y}$ and $\frac{\partial \varphi}{\partial y}$, the expression (4.33) becomes $d^\varepsilon = K(z)$ where K is defined in Proposition 6.3.

Part a) of (4.35) follows from regularity of $K(z)$ which follows from the regularity of $C(z)$.

Concerning part b), we must show that $\frac{\partial d^\varepsilon}{\partial z} \geq 0$ in the distributions sense, i.e

$$\langle \frac{\partial d^\varepsilon}{\partial z}, \phi \rangle \geq 0. \quad \forall \phi \in \mathcal{D}(\Omega), \phi \geq 0$$

Indeed, from the form of K and after elementary computations, we get

$$\langle \frac{\partial d^\varepsilon}{\partial z}, \phi \rangle = -\langle C(z)\big(\frac{1}{\mu_1} + \frac{1}{\mu_2}\big) - \frac{C(z)^2}{2\mu_1\mu_2}, \frac{\partial \phi}{\partial z} \rangle.$$

From [3], if $\mu_2 < \mu_1$, we have $-\langle C(z), \frac{\partial \phi}{\partial z} \rangle \geq 0$, and since $\mu_1, \mu_2 > 0$, hence

$$-\langle C(z)\big(\frac{1}{\mu_1} + \frac{1}{\mu_2}\big), \frac{\partial \phi}{\partial z} \rangle \geq 0.$$

Also $\langle \frac{C(z)^2}{2\mu_1\mu_2}, \frac{\partial \phi}{\partial z} \rangle \geq 0$; therefore, $\frac{\partial d^\varepsilon}{\partial z} \geq 0$ in the distribution sense. $\square$

Remark 6.5 Propositions 6.1 and 6.3 allow us to compute explicitly the coefficients of the tensors $\mathcal{A}$ and $\mathcal{B}$.

For tensor $\mathcal{A}$ (cf [3]), we have

$$a_{1111} = a_{2222} = \begin{cases} 2\mu_1 & \text{if } h_1 < z < z_0 \\ 2\mu^\star & \text{if } 0 < z < h_1 \\ 2\mu_2 & \text{if } -z_0 < z < 0 \end{cases}$$

$$a_{1212} = a_{1221} = a_{2112} = a_{2121} \begin{cases} \mu_1 & \text{if } h_1 < z < z_0 \\ \mu^+ & \text{if } 0 < z < h_1 \\ \mu_2 & \text{if } -z_0 < z < 0 \end{cases}$$

where $\mu^\star = \frac{1}{|Y|}\int_Y \mu dy$, $\mu^+ = \big(\frac{1}{|Y|}\int_Y \frac{1}{\mu} dy\big)^{-1} = \frac{C(z)}{2}$ and the rest of the coefficients of $\mathcal{A}$ are 0.

For tensor $\mathcal{B}$, we derive

$$b_{1111} = b_{2222} = 1 \quad \forall z \in]-z_0, z_0[$$

$$b_{1212} = b_{1221} = b_{2112} = b_{2121} \begin{cases} 1 & \text{if } h_1 < z < z_0 \\ \frac{K(z)}{4} & \text{if } 0 < z < h_1 \\ 1 & \text{if } -z_0 < z < 0 \end{cases}$$

and the rest of the coefficients of $\mathcal{B}$ are 0.

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