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DISCUSSION OF A UNIQUENESS RESULT IN "EQUILIBRIUM CONFIGURATIONS FOR A FLOATING DROP"

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Abstract. We analyze a uniqueness result presented by Elcrat, Neel, and Siegel [1] for unbounded liquid bridges, and show that the proof they presented is incorrect. We add a hypothesis to their stated theorem and prove that their result holds under this condition. Then we use Chebyshev spectral methods to approximate solutions to certain boundary value problems used to check this hypothesis holds at least on a range of cases.

1. Introduction

In 2004 Elcrat, Neel, and Siegel published a collection of results on the floating drop problem and the related floating bubble problem [1]. Physically, one can visualize a drop of oil resting on a reservoir of water. The resulting free boundary problem will not be described in detail here. This work has been held in high esteem in the field of capillarity, which is evident in the review Robert Finn wrote in Math Reviews for the paper [2]. We offer a select quote from that review here:

> The problem of characterizing the configuration of a drop of liquid floating in equilibrium on the surface of an infinite bath of another liquid appeared initially in the second supplément to the tenth book of Laplace's Mécanique Céleste, in 1806, without detailed treatment. It was later studied by Poisson in his "Nouvelle Théorie..."(1831), and discussed by Bowditch (1839) in his English translation of Laplace's treatise. These works were remarkable for their time but far from complete, and there appears to be no further mathematical discussion in the literature, prior to the present study. That the problem was ignored so long despite its evident theoretical and also practical interest is perhaps indicative of the technical obstacles that have impeded a full formal description.
>
> Also the present authors have not solved the problem completely, but what they offer is impressive, with a direct hands-on approach. The authors make clever use of new results that appeared in other contexts during the past quarter century.

Given this perspective, it is unfortunate that the subject of this current paper is a flaw in one of the proofs presented in that work. The theorem stated as [1,

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Theorem 3.2] treats the existence and uniqueness of a boundary value problem for unbounded liquid bridges, and the proof of uniqueness is the topic of discussion here. We state the theorem here as Conjecture 2.2. This flaw is seemingly in the proof alone, and does not seem to be in the stated result. In what follows we will analyze the presented proof and show how the published approach is seemingly not able to be repaired. Then we offer an alternative approach and we give numerical evidence that this alternate approach yields the (still unproven) result found in that paper.

Before we proceed to the details, a short comment on the impact of this flaw is appropriate. We note that the strongest results found in [1] are not effected by this flaw, but those stronger results do not hold for the general cases of all physical configurations of fluids. Specifically, they found that under some restrictions on the associated surfaces tensions the floating drop problem is solved for any given drop volume. This assumption implies all of the component surfaces for a floating drop can be shown to be a graph over a base domain, and under this restriction all of the results in [1] still hold. This assumption is discussed there, and with the goal of avoiding the quite technical description of the floating drop problem, we refer the reader to that work for the detailed criterion. It is when the problem was generalized to admit all possible physical configurations that the result described below was used. The results just described in that work are Theorem 4.1, Theorem 5.3, Corollary 5.1 which have gaps, and Theorem 4.2 having no gap.

Finally, the layout of this paper is as follows. In Section 2 we present the unbounded liquid bridge and the theorem from [1]. In Section 3 we collect preliminary results found in a paper by Vogel [7]. In Section 4 we analyze the proof given in [1], and in Section 5 we prove a theorem on uniqueness with the addition of a hypothesis. Then in Section 6 we describe numerical results that verify that this new hypothesis holds over a subset of the total values needed.

2. Uniqueness of solutions for a boundary value problem involving unbounded liquid bridges

We consider here the family of radially symmetric unbounded liquid bridges. These bridges are modeled by solutions of the ordinary differential equations

$$\frac{dr}{d\phi} = \frac{-r\cos\phi}{ru+\sin\phi}, \tag{2.1}$$

$$\frac{du}{d\phi} = \frac{-r\sin\phi}{ru+\sin\phi} \tag{2.2}$$

over the range $\phi \in [0,\pi)$, that are subject to various initial values

$$r(\phi_0) = r_0,$$
$$u(\phi_0) = u_0$$

with $0 \le \phi_0 < \pi$ and $r_0, u_0 > 0$. Here the radial value is $r = r(\phi)$ and the height of the interface above a fixed reference level is $u = u(\phi)$. A first observation is that not all such initial values lead to an unbounded liquid bridge. Indeed, a special curve of initial values

$$S = \{(r_0, u_0) = (\sigma, T(\sigma)) : \sigma > 0\}$$

has been identified and studied in the papers of Siegel [3], Turkington [6], Vogel [7], and Elcrat, Neel, and Siegel [1]. These initial values are precisely the initial values for which the following hold for $\phi_0 = \pi/2$:

(1) $r = r(\phi)$ is (strictly) increasing for $\pi/2 \le \phi < \pi$ with
$$\lim_{\phi \nearrow \pi} r(\phi) = \infty,$$
and

(2) $u = u(\phi)$ is (strictly) decreasing for $\pi/2 \le \phi < \pi$ with
$$\lim_{\phi \nearrow \pi} u(\phi) = 0.$$

An example of one such unbounded liquid bridge is shown in Figure 1.

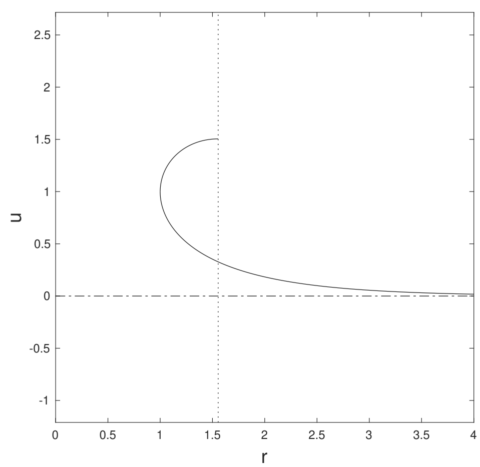

Figure 1. An example unbounded liquid bridge with vertical point at $\sigma = 1$, where the generating curve for the radially symmetric surface is graphed. The reference height is indicated at $u = 0$ and the vertical line corresponds to the radius where the boundary condition $\phi_0 = 0$ holds.

The function $T = T(\sigma)$ is differentiable and increasing for $0 < \sigma < \infty$ with
$$\lim_{\sigma \searrow 0} T(\sigma) = 0, \quad \lim_{\sigma \searrow 0} T'(\sigma) = \infty, \quad \lim_{\sigma \to \infty} T(\sigma) = \sqrt{2},$$
as illustrated in Figure 2.

Each solution of the system (2.1)-(2.2) subject to the initial conditions
$$r(\pi/2) = \sigma,$$
$$u(\pi/2) = T(\sigma)$$
is defined for $\phi \in [0, \pi)$, and each solution is called an **unbounded liquid bridge** solution. Considering a fixed $\sigma > 0$ and for each $\phi_0 \in [0, \pi)$, one can use these initial conditions to find the unique point $(r_0, u_0) = (r(\phi_0), u(\phi_0))$ along the solution curve. This then leads to the initial value problem of (2.1)-(2.2) with
$$r(\phi_0) = r_0,$$

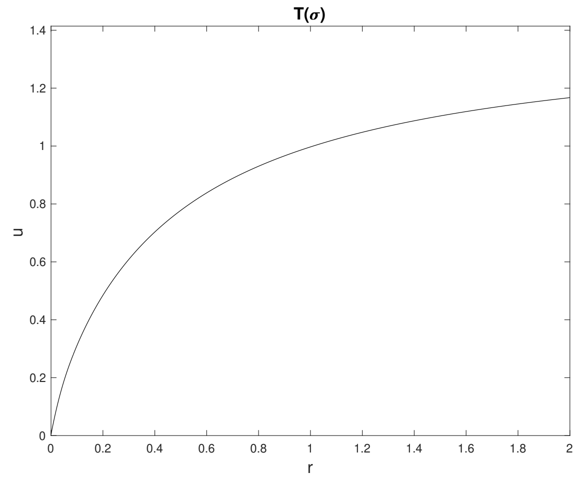

FIGURE 2. The function $T(\sigma)$ computed by the algorithm described in Section 6 for $\sigma \geq 0.1$ and smoothly interpolated from the asymptotic behavior for smaller σ. This asymptotic expansion was derived in [6] and can be found more simply stated in [7], and plays no explicit role in the current work.

$$u(\phi_0) = u_0$$

for $\phi \in [0, \pi)$, and this initial value problem is then uniquely solved by the same unbounded liquid bridge we used to find those initial conditions.

One may consider multiple unbounded liquid bridges, leading to the possibilities of other uniqueness results. Vogel [7] showed following result.

Theorem 2.1. *Consider the solutions of* (2.1)-(2.2) *subject to*

$$\begin{aligned} r(\phi_0) &= r_1, \\ u(\phi_0) &= u_0, \end{aligned}$$

and also the solutions of (2.1)-(2.2) *subject to*

$$\begin{aligned} r(\phi_0) &= r_2, \\ u(\phi_0) &= u_0. \end{aligned}$$

If both solutions are unbounded liquid bridges, then $r_1 = r_2$.

The following was claimed in ENS.

Conjecture 2.2 ([1, Theorem 3.2])**.** *If the solutions of* (2.1)-(2.2) *subject to*

$$\begin{aligned} r(\phi_0) &= r_0, \\ u(\phi_0) &= u_1, \end{aligned}$$

and the solutions of (2.1)-(2.2) *subject to*

$$\begin{aligned} r(\phi_0) &= r_0, \\ u(\phi_0) &= u_2 \end{aligned}$$

are both unbounded liquid bridges, then $u_1 = u_2$.

We follow the proof presented in [1], noting a specific inequality that was not adequately justified. We will then follow with an observation that the reverse inequality seems to have been verified in [1].

3. Preliminaries

The term **profile curve** will now be used to refer to the parametric image

$$\{(r(\phi), u(\phi)) : 0 \le \phi < \pi\}$$

associated with an unbounded liquid bridge solution. The following results are from Vogel [7].

Lemma 3.1. *Let $\Gamma = (r(\phi), u(\phi))$ be a particular profile curve. Pick $\phi_0 \in [0, \pi)$, and let $r_0 = r(\phi_0)$. Let A be the set obtained by rotating the region bounded by $r = r_0$ and Γ, and let B be the set obtained by rotating the unbounded region between Γ and the r-axis from $r = r_0$ to $r = \infty$ around the u-axis. Let $|A|$ and $|B|$ denote the volume of each set. Then*

$$|B| - |A| = 2\pi r_0 \sin\phi_0. \tag{3.1}$$

This lemma is illustrated in Figure 3.

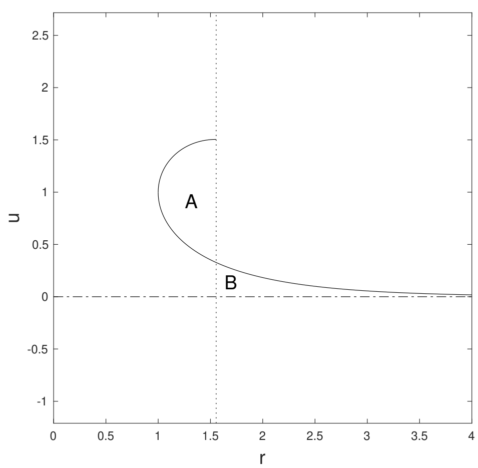

Figure 3. Generating curves describing the regions A and B from Lemma 3.1.

Lemma 3.2. *No two distinct profile curves can cross twice.*

The following result is contained in a remark in Vogel's paper, and also is extended by his Theorem 2.1.

Lemma 3.3. *Given two profile curves with vertical points at radii σ_1 and σ_2, if $\sigma_2 > \sigma_1$, then $u(\phi_0; \sigma_2) > u(\phi_0; \sigma_1)$. Conversely, if $u(\phi_0; \sigma_2) > u(\phi_0; \sigma_1)$, then $\sigma_2 > \sigma_1$.*

Next we will consider how the system (2.1)-(2.2) behaves as the location of the vertical point moves by differentiating with respect to that parameter σ, the results of which we denote by $\dot{r}$ and $\dot{u}$:

$$\frac{d\dot{r}}{d\phi} = -\cos\phi \frac{\dot{u}r^2 - \dot{r}\sin\phi}{(ru+\sin\phi)^2}, \tag{3.2}$$

$$\frac{d\dot{u}}{d\phi} = -\sin\phi \frac{\dot{u}r^2 - \dot{r}\sin\phi}{(ru+\sin\phi)^2}. \tag{3.3}$$

The conditions that $r(\pi/2;\sigma) = \sigma$ and $u(\pi/2,\sigma) = T(\sigma)$ imply

$$\dot{r}(\pi/2,\sigma) = 1 \text{ and} \tag{3.4}$$

$$\dot{u}(\pi/2,\sigma) = T'(\sigma). \tag{3.5}$$

Vogel shows solutions of this initial value problem satisfy:

Lemma 3.4. *If $\dot{u}(0;\sigma) = 0$, then $\dot{r}(0;\sigma) < 0$.*

Lemma 3.5. *If $\phi \in (0,\pi)$, then $\dot{u}(\phi;\sigma) > 0$.*

4. The proof presented

The proof is by contradiction. Let Γ_1 and Γ_2 be two profile curves as above such that $r_1(\phi_0) = r_2(\phi_0)$ for some $\phi_0 \in [0,\pi/2)$. We will assume that these curves exist and are distinct. Upon possibly relabeling, we have $u_1(\phi_0) > u_2(\phi_0)$.

Suppose ϕ_0 is the largest such value of ϕ such that $r_1(\phi_0) = r_2(\phi_0)$ and denote that radius by ρ_0. These are the leftmost such points, and ρ_0 is the smallest such radius. Let A_1, B_1, A_2, and B_2 be the corresponding sets from Lemma 3.1 for these profile curves.

Consider the intersection points of these curves with $r = \rho_0$. Define $\alpha_U = u_1(\phi_0)$ and $\beta_U = u_2(\phi_0)$ to be the upper intersection points, and define α_L and β_L to be the lower intersection points. Define $\phi_1^- \in (\pi/2,\pi)$ to be the inclination angle of Γ_1 at the crossing of $r = \rho_0$ where $u(\phi_1^-) = \alpha_L$, and similarly define $\phi_2^- \in (\pi/2,\pi)$ to be the angle where Γ_2 crosses $r = \rho_0$ and $u_2(\phi_2^-) = \beta_L$. See Figure 4. We will have some cases to consider, depending on these values.

Step 1 (The case that $\alpha_L \le \beta_L$) This means $\alpha_U > \beta_U > \beta_L \ge \alpha_L$. Since Γ_1 and Γ_2 cannot cross more than once, in this case they cannot cross at all. Thus Γ_1 lies entirely outside the region bounded by Γ_2 and the line $r = \rho_0$. Thus $\sigma_1 < \sigma_2$ and it follows that $T(\sigma_1) < T(\sigma_2)$. This then implies that there is some $\phi_1 \in (\phi_0,\pi/2)$ such that $u_1(\phi_1) = u_2(\phi_1)$. This contradicts Theorem 2.1 and eliminates the case that $\alpha_L \le \beta_L$.

Step 2 (The case that $\alpha_L > \beta_L$) As above, if $T(\sigma_1) \le T(\sigma_2)$, we contradict Theorem 2.1. Thus $T(\sigma_1) > T(\sigma_2)$ and $\sigma_1 > \sigma_2$. The goal here is to contradict Lemma 3.1. To this end, we consider two more steps, the first of which is to show that $|B_1| > |B_2|$ and the second is to show that $|A_1| < |A_2|$.

Step 3 If $\alpha_L \le \beta_U$, then Γ_1 and Γ_2 must cross somewhere above α_L. Then, as they cannot cross twice, they cannot cross below α_L. Then, as $\alpha_L > \beta_L$, this implies the lower portion of Γ_1 lies completely above the lower portion of Γ_2 when $r > \rho_0$. Thus $\alpha_L \le \beta_U$ implies $|B_1| > |B_2|$.

If $\alpha_L \ge \beta_U$, then the part of Γ_1 with $r < \rho_0$ lies completely above the corresponding part of Γ_2. Thus, at $u = \beta_U$, Γ_1 has an inclination angle greater than $\pi/2$, while Γ_2 has an inclination angle less than $\pi/2$. Also, at $r = \rho_0$, the lower

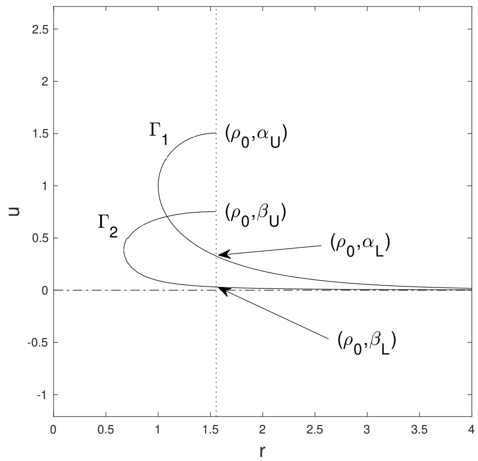

FIGURE 4. The two (potentially) distinct curves Γ_1 and Γ_2.

part of Γ_1 is above the lower part of Γ_2. the goal is to show that Γ_1 remains above Γ_2 for all $r > \rho_0$. Assume the opposite: the two curves cross somewhere below $u = \beta_U$. At this crossing point the inclination angle of Γ_2 would be less than that of Γ_2. At $u = \beta_U$ the inequality was reversed. This implies that there is a value of u between β_U and the value of u at the crossing point where both curves have the same inclination angle. This leads to a contradiction $\Gamma_1 \equiv \Gamma_2$ using Siegel's uniqueness theorem [3] or alternatively Vogel's Theorem 2.1. Thus the curves cannot cross, and the lower part of Γ_1 lies above the lower part of Γ_2 for all $r > \rho_0$. Thus $\alpha_L \geq \beta_U$ implies $|B_1| > |B_2|$, and we have established that $|B_1| > |B_2|$ holds for all possibilities in the case that $\alpha_L > \beta_L$.

Step 4 The argument of [1] rests at this point on showing the volumes $|A_1|$ and $|A_2|$ satisfy the inequality

$$|A_1| \leq |A_2|.$$

It has been pointed out by one of the referees, however, that it is shown in [1] itself that the angles ϕ_1^- and ϕ_2^- satisfy $\pi/2 < \phi_1^- < \phi_2^- < \pi$ with

$$|A_1| = 2\pi\rho_0(\sin\phi_1^- - \sin\phi_0) \text{ and } |A_2| = 2\pi\rho_0(\sin\phi_2^- - \sin\phi_0). \tag{4.1}$$

Consequently, direct comparison of the quantities in (4.1) gives that the reverse inequality

$$|A_1| > |A_2|$$

actually holds. In light of this, it was suggested by the other referee that we remove the remaining detailed analysis of the proof presented in [1] where we showed precisely where their argument breaks down.

5. A different approach

We first claim that given any $\phi_0 \in [0, \pi/2]$ and any $\rho_0 > 0$, there exists at least one infinite bridge solution with $r(\phi_0) = \rho_0$. We know from Vogel [7] that any such solution must correspond to a unique $\sigma > 0$ so the initial conditions $r(\pi/2) = \sigma$

and $u(\pi/2) = T(\sigma)$ give the same solution. Moreover the entire family of infinite liquid bridge solutions depends smoothly on σ for $\sigma > 0$. Vogel shows

$$\lim_{\sigma \to 0} \frac{u(0;\sigma)}{r(0;\sigma)} = 0.$$

This along with the estimate

$$r(0;\sigma) = \mathcal{O}\Big(\frac{1}{\sqrt{\log(1/\sigma)}}\Big)$$

implies $u(0;\sigma) \to 0$ as $\sigma \to 0$. Vogel also shows

$$\sqrt{\frac{\sigma}{T(\sigma)} + \sigma^2} \le r(0;\sigma) \le \sqrt{\frac{2\sigma}{T(\sigma)} + \sigma^2},$$

which implies $r(0;\sigma) < \infty$ for $\sigma < \infty$. We also have an upper bound on T by $\sqrt{2}$.

Starting with $\sigma_1 << 1$ so that $r(0;\sigma_1) < \rho_0$ and smoothly increasing σ until $\sigma = \rho_0$ and thus $r(\phi;\rho_0) > \rho_0$ for all $\phi \neq \pi/2$, there is at least one $\sigma \in (\sigma_1, \rho_0)$ for which the desired condition $r(\phi_0;\sigma) = \rho_0$ holds.

If there exist two distinct values σ_2 and σ_3 for which $r(\phi_0;\sigma_2) = r(\phi_0;\sigma_3)$, then it follows that there exists some $\sigma \in (0,\infty)$ for which $\dot{r}(\phi;\sigma) = 0$. We conclude the following result.

Theorem 5.1. *If every solution of* (2.1)-(2.2) *with*

$$r(\pi/2) = \sigma, \tag{5.1}$$
$$u(\pi/2) = T(\sigma), \tag{5.2}$$

for $\sigma > 0$ satisfies

$$\dot{r}(\phi_0;\sigma) > 0 \quad \text{for } 0 \le \phi \le \pi/2, \tag{5.3}$$

then the uniqueness assertion of [1] *holds. That is, Conjecture 2.2 holds with this additional hypothesis that $\dot{r}(\phi;\sigma) > 0$.*

This author has not found a way to rigorously establish this new condition. In the course of the rest of this paper we will give some numerical evidence that this criteria holds.

6. Numerical study of $\dot{r}$

Given a radius $\sigma > 0$, we will first find the height $T(\sigma)$ of the vertical point on the unbounded liquid bridge there. In order to find this height, we will need to solve (2.1)-(2.2) for $\phi \in [\pi/2, \pi)$ so that the solution has the required height decay at infinity. We will adapt a recently developed Chebyshev spectral method to achieve this. See [5] for an exposition of the methods used.

The equations (2.1)-(2.2) can be written as an equivalent system of three nonlinear ordinary differential equations, parametrized by the arclength s:

$$\begin{aligned} \frac{dr}{ds} &= \cos\psi, \\ \frac{du}{ds} &= \sin\psi, \\ \frac{d\psi}{ds} &= u - \frac{\sin\psi}{r}, \end{aligned}$$

where we still have r as the radius and u as the height of the interface, and we introduce the inclination angle ψ, which merely satisfies $\psi = \phi - \pi$. We will specify boundary conditions at $s = 0$ and at some arclength $s = \ell > 0$. The radius $r(\ell)$ meets a prescribed value $b > 0$, and the inclination angle $\psi(\ell) = 0$ is required there. However, this value of the arclength ℓ is unknown, and the methods in [5] find this arclength as part of solving the boundary value problem. The boundary conditions are

$$r(0) = \sigma, \tag{6.1}$$

$$r(\ell) = b, \tag{6.2}$$

$$\psi(0) = -\pi/2, \tag{6.3}$$

$$\psi(\ell) = 0. \tag{6.4}$$

To summarize, $s = 0$ is the arclength where we prescribe the vertical point corresponding to $r = \sigma$, and we prescribe a horizontal slope at the right endpoint $r = b$ where $s = \ell$.

The application of the Chebyshev spectral methods in [5] numerically solve this nonlinear boundary value problem. We are primarily interested in the height $T_b(\sigma) := u(0; b)$ at the left endpoint.

We incrementally increase the prescribed value of b until the height T_b has incremental changes within a small tolerance. We require 13 digits of relative error for the boundary value problem, and the underlying Newton's method requires 14 digits of relative error. Then we use this converged value of $T_b(\sigma)$ as an approximation of $T(\sigma)$. This left endpoint is at the radius $r = \sigma$, so we can sweep over a range of those values to form the graph $(\sigma, T(\sigma))$. Our algorithm works well for values of σ down to $\sigma = 0.1$, but smaller than that the problem becomes multi-scale in nature and more specialized methods are required.

In light of the initial condition (3.5), we need a careful approach in our approximation of $T'(\sigma)$. The interpolation of data points by Chebyshev polynomials is known to be near optimal among polynomial interpolants. The basis of this interpolation is to pick the data points at so-called Chebyshev points. Roughly, if the domain is $[-1, 1]$, then those points are $x_j = \cos(\theta_j)$ where the angles θ_j are evenly spaced in $[0, \pi]$. For domains such as the ones we are considering, as over a bounded range of positive σ values, one translates and scales the problem. Then one can compute the derivative of the interpolating polynomial with spectral accuracy by using a so-called Chebyshev differentiation matrix D. If the grid values for $T(\sigma)$ are found at these Chebyshev points, and stored in a column vector $\mathbf{T}$, then the grid values of the derivative $T'(\sigma)$ are found as $\mathbf{T}' = D\mathbf{T}$. See Trefethen [4] for background.

Now we have the components necessary to construct numerical approximations of the solutions of (3.2)-(3.3) with the boundary conditions (3.4)-(3.5) where we need to append that system with the original (2.1)-(2.2) with the conditions that $r(\pi/2, \sigma) = \sigma$ and $u(\pi/2, \sigma) = T(\sigma)$. Then we use Matlab's ode45 to solve this system from $\phi = \pi/2$ to $\phi = 0$ to obtain the "top" portion of the solution. Here we ask for 11 digits of accuracy in both the absolute and relative error. We use all values of σ considered and sweep out a region of the $\phi\dot{r}-$plane in Figure 5. This numerical experiment shows that the entirety of the foliation lies completely in the upper half-plane, and thus $\dot{r}$ is never negative for those σ values. We also visualize this phenomenon by graphing the endpoint $\dot{r}(0, \sigma)$ as a function of σ.

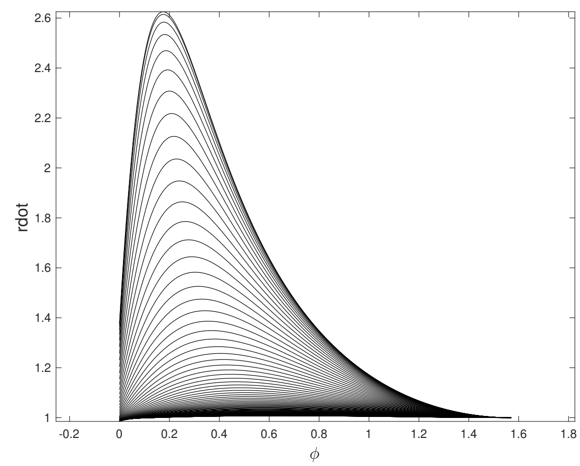

FIGURE 5. The solutions $\dot{r}(\phi)$ foliate a region of the $\phi\dot{r}-$plane and do not enter the lower half-plane.

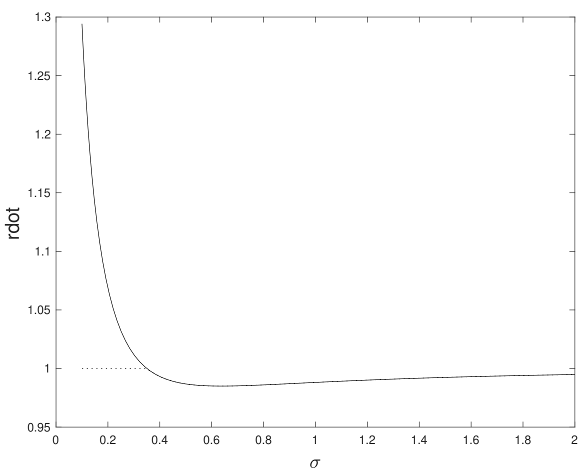

FIGURE 6. The endpoint $\dot{r}(0, \sigma)$ is graphed as a function of σ. This is the lowest part of the curve $\dot{r}(\phi, \sigma)$ in many cases, but for smaller values of σ the lowest part of the curve occurs at $\phi = \pi/2$, and this is graphed with a dotted line when relevant.

For many cases, this endpoint is the lowest part of the curve $\dot{r}(\phi, \sigma)$, however for smaller values of σ the lowest part of the curve occurs at $\phi = \pi/2$, and this is also included in Figure 6. This figure gathers the most interesting parts of the graph of $\min_\phi \dot{r}(\phi, \sigma)$, though we have computed this function out to values of σ at 200, and the function continues to be increasing and seemingly asymptotic to a height of 1.

We have released the matlab code for these computations under an open source license and hosted it on a software repository found at
`https://github.com/raytreinen/Unbounded-Liquid-Bridges.git`

Finally, the referees offered keen insight into the problem at hand, and the author is grateful for their comments.

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