Electronic Journal of Differential Equations, Vol. 2006(2006), No. 150, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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LOW REGULARITY WELL-POSEDNESS FOR THE ONE-DIMENSIONAL DIRAC-KLEIN-GORDON SYSTEM

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Abstract. Local well-posedness for Dirac-Klein-Gordon equations is proven in one space dimension, where the Dirac part belongs to $H^{-\frac{1}{4}+\epsilon}$ and the Klein-Gordon part to $H^{\frac{1}{4}-\epsilon}$ for $0 < \epsilon < 1/4$, and global well-posedness, if the Dirac part belongs to the charge class L^2 and the Klein-Gordon part to H^k with $0 < k < 1/2$. The proof uses a null structure in both nonlinearities detected by d'Ancona, Foschi and Selberg and bilinear estimates in spaces of Bourgain-Klainerman-Machedon type.

1. Introduction

In this paper we study the Cauchy problem for the Dirac-Klein–Gordon equations in one space dimension

$$-i\beta\frac{\partial}{\partial t}\psi + i\alpha\beta\frac{\partial}{\partial x}\psi + M\psi = g\phi\psi \tag{1.1}$$

$$\frac{\partial^2}{\partial t^2}\phi - \frac{\partial^2}{\partial x^2}\phi + m^2\phi = \langle\beta\psi,\psi\rangle_{\mathbf{C}^2} \tag{1.2}$$

with initial data

$$\psi(x,0) = \psi_0(x)\,, \quad \phi(x,0) = \phi_0(x)\,, \quad \frac{\partial\phi}{\partial t}(x,0) = \phi_1(x)\,. \tag{1.3}$$

Here ψ is a two-spinor field, i.e. ψ has values in $\mathbf{C}^2$, and ϕ is a real-valued function. α and β are hermitian (2×2)-matrices, which fulfill $\alpha^2 = \beta^2 = I$, $\alpha\beta + \beta\alpha = 0$, e.g. we can choose $\alpha = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. M, m and g are real constants.

We are interested in local and global low regularity solutions. The first results were obtained by Chadam and Glassey [5], [6] who proved global well-posedness for data $\psi_0 \in H^1$, $\phi_0 \in H^1$, $\phi_1 \in L^2$. This result was improved by Bournaveas [3] (cf. also Fang [7]) who showed the same results for data $\psi_0 \in L^2$, $\phi_0 \in H^1$, $\phi_1 \in L^2$. Local existence and uniqueness was shown by Fang [8] for data $\psi_0 \in H^{-\frac{1}{4}+\epsilon}$, $\phi_0 \in H^{\frac{1}{2}+\delta}$, $\phi_1 \in H^{-\frac{1}{2}+\delta}$ and $0 < \epsilon \le \frac{1}{4}$, $0 < \delta \le 2\epsilon$. These solutions are global, if $\psi_0 \in L^2$. Finally, Bournaveas and Gibbeson [4] also proved global existence and uniqueness for $\psi_0 \in L^2$, $\phi_0 \in H^k$, $\phi_1 \in H^{k-1}$ for $\frac{1}{4} \le k < \frac{1}{2}$. All these global results

2000 *Mathematics Subject Classification.* 35Q40, 35L70.
Key words and phrases. Dirac-Klein-Gordon system; well-posedness;
Fourier restriction norm method.

Submitted June 28, 2006. Published December 5, 2006.

were obtained by using conservation of charge, namely $\int |\psi|^2\,dx = \int |\psi_0|^2\,dx$. The energy does not help here because it is not positive definite.

In three space dimensions the best result concerning local well-posedness was recently obtained by d'Ancona, Foschi and Selberg [1] for data $\psi_0 \in H^{\epsilon}$, $\phi_0 \in H^{\frac{1}{2}+\epsilon}$, $\phi_1 \in H^{-\frac{1}{2}+\epsilon}$ with $\epsilon > 0$. This result is arbitrarily close to the minimal regularity predicted by scaling ($\epsilon = 0$). Whereas in the above mentioned more recent results a null structure of Klainerman-Machedon type [10] for the nonlinearities was already used in one or the other way, they showed that the null form $\langle \beta\psi, \psi\rangle$ of the wave part is also hidden (by a duality argument) in the Dirac part of the system and both nonlinearities can be treated in a similar way. It was also very helpful to first diagonalize the system by using the eigenspace projections of the Dirac operator (cf. also Beals and Bézard [2]). Of course this local result does not directly imply a global one.

In the present paper we want to improve the local and global results in one space dimension by consequently using this diagonalization of the system and applying the Fourier restriction norm method. We are able to show local existence and uniqueness for data $\psi_0 \in H^{-l}$, $\phi_0 \in H^k$, $\phi_1 \in H^{k-1}$, provided $l < \frac{1}{4}$, $k > 0$, $2l + k < 1$, $l + k \le 1$ and $k \ge |l|$. This means that e.g. $k = l = \frac{1}{4} - \epsilon$ is admissible as well as $l = 0$, $k = \epsilon$, thus improving the above mentioned results of Fang and Bournaveas-Gibbeson. These local results easily imply global ones in the case $\psi_0 \in L^2$, $\phi_0 \in H^k$, $\phi_1 \in H^{k-1}$ for $0 < k < 1/2$, using only charge conservation, also improving Bourneveas-Gibbeson.

This paper is organized as follows. First we rewrite the system as a first order system in time in diagonal form. We split ψ as the sum $\pi_+(D)\psi + \pi_-(D)\psi$, where $\pi_\pm(D)$ are the projections onto the eigenspaces of $-i\alpha\frac{\partial}{\partial x}$, and also split ϕ as the sum $\phi_+ + \phi_-$, where the half waves ϕ_+ and ϕ_- are defined in the usual way. Then we analyze the components of the nonlinearity $\langle \beta\psi, \psi\rangle$, namely $\langle \beta\pi_\pm(D)\psi, \pi_\pm(D)\psi'\rangle$ for all possible combinations of signs by computing its Fourier symbol. It turns out that the symbol is a piecewise constant matrix in Fourier space depending only on the signs of the Fourier variables and especially vanishes in certain regions. Then we examine which bilinear estimates for the nonlinear terms are necessary for local well-posedness in the framework of the $X^{m,b}$-spaces. It turns out that due to duality arguments two similar estimates have to be given for $\langle \beta\pi_\pm(D)\psi, \pi_\pm(D)\psi'\rangle$ in order to treat both nonlinearities. These are given in Lemma 3.2 and Lemma 3.3. The local results are summarized in Theorem 3.4. Global existence is a direct consequence of the local results combined with charge conservation (Theorem 4.1).

We construct our solutions in spaces of the type $X^{m,b}_\varphi$ defined as follows: for an equation of the form $iu_t - \varphi(-i\frac{\partial}{\partial x})u = 0$, where φ is a measurable function, let $X^{m,b}_\varphi$ be the completion of $\mathcal{S}(\mathbb{R}\times\mathbb{R})$ with respect to

$$\|f\|_{X^{m,b}_\varphi} := \|\langle\xi\rangle^m \langle\tau\rangle^b \mathcal{F}(e^{it\varphi(-i\frac{\partial}{\partial x})} f(x,t))\|_{L^2_{\xi\tau}} = \|\langle\xi\rangle^m \langle\tau + \varphi(\xi)\rangle^b \tilde{f}(\xi,\tau)\|_{L^2_{\xi\tau}}$$

where $\langle\cdot\rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$, and $\tilde{f}$ denotes the Fourier transform of f with respect to x and t. We also use the time localized spaces $X^{m,b}_\varphi[0,T]$ defined by

$$\|f\|_{X^{m,b}_\varphi[0,T]} = \inf_{\tilde{f}_{|[0,T]}=f} \|\tilde{f}\|_{X^{m,b}_\varphi}\,.$$

The following fact about these spaces is well-known (cf. , e.g. , [9], section 2): if v is a solution of

$$iv_t - \varphi(-i\frac{\partial}{\partial x})v = F\,, \quad v(0) = f$$

on a time interval $[0,T]$, $T \leq 1$, we have for $b'+1 \geq b \geq 0 \geq b' > -\frac{1}{2}$:

$$\|v\|_{X^{m,b}_{\varphi}[0,T]} \leq c\|f\|_{H^m} + cT^{1+b'-b}\|F\|_{X^{m,b'}_{\varphi}[0,T]}\,. \tag{1.4}$$

2. Preliminaries

First we transform our system (1.1),(1.2) into a first order system (in t) in diagonal form.

Multiplying the Dirac equations from the left by β leads to

$$\begin{gathered}
-i\frac{\partial}{\partial t}\psi - i\alpha\frac{\partial}{\partial x}\psi + M\beta\psi = g\phi\beta\psi \\
\frac{\partial^2}{\partial t^2}\phi - \frac{\partial^2}{\partial x^2}\phi + m^2\phi = \langle\beta\psi,\psi\rangle_{\mathbf{C}^2}\,.
\end{gathered}$$

Following the paper of d'Ancona, Foschi and Selberg we diagonalize the system by defining the projections

$$\pi_{\pm}(\xi) := \frac{1}{2}(I \pm \hat{\xi}\alpha)\,,$$

where $\hat{\xi} := \frac{\xi}{|\xi|}$. Then we have $\psi = \psi_+ + \psi_-$ with $\psi_{\pm} := \pi_{\pm}(D)\psi$, $D := \frac{1}{i}\frac{\partial}{\partial x}$. Using the identity

$$-i\alpha\frac{\partial}{\partial x} = \alpha D = |D|\pi_+(D) - |D|\pi_-(D)$$

and

$$\pi_{\pm}(\xi)\beta = \frac{1}{2}(I \pm \hat{\xi}\alpha)\beta = \frac{1}{2}(\beta \mp \hat{\xi}\beta\alpha) = \beta\pi_{\mp}(\xi) \tag{2.1}$$

we get by application of $\pi_{\pm}(D)$ to the Dirac equation

$$\begin{aligned}
\pi_{\pm}(D)(-i\frac{\partial}{\partial t}\psi - i\alpha\frac{\partial}{\partial x}\psi) &= \pi_{\pm}(D)(-i\frac{\partial}{\partial t}\psi + |D|\pi_+(D)\psi - |D|\pi_-(D)\psi) \\
&= -i\frac{\partial}{\partial t}\pi_{\pm}(D)\psi \pm |D|\pi_{\pm}(D)\psi \\
&= -i\frac{\partial}{\partial t}\psi_{\pm} \pm |D|\psi_{\pm}
\end{aligned}$$

where we also used

$$\pi_{\pm}(\xi)\pi_{\mp}(\xi) = \frac{1}{4}(I \pm \hat{\xi}\alpha)(I \mp \hat{\xi}\alpha) = \frac{1}{4}(I - \hat{\xi}^2\alpha^2) = 0$$

and

$$\pi_{\pm}(\xi)\pi_{\pm}(\xi) = \frac{1}{4}(I \pm \hat{\xi}\alpha)(I \pm \hat{\xi}\alpha) = \frac{1}{4}(I \pm 2\hat{\xi}\alpha + \hat{\xi}^2\alpha^2) = \frac{1}{2}(I \pm \hat{\xi}\alpha) = \pi_{\pm}(\xi)$$

(this also implies especially $\psi_{\pm} = \pi_{\pm}(D)\psi_{\pm}$).

The Dirac equations are thus transformed into

$$\begin{aligned}
(-i\frac{\partial}{\partial t} \pm |D|)\psi_{\pm} &= -M\beta\pi_{\mp}(D)(\psi_+ + \psi_-) + g\pi_{\pm}(D)(\phi\beta\psi) \\
&= -M\beta\psi_{\mp} + g\pi_{\pm}(D)(\phi\beta(\psi_+ + \psi_-))\,.
\end{aligned}$$

We also split the function ϕ into the sum $\phi = \frac{1}{2}(\phi_+ + \phi_-)$, where

$$\phi_\pm := \phi \pm iA^{-\frac{1}{2}}\frac{\partial \phi}{\partial t}\,, \quad A := -\frac{\partial^2}{\partial x^2} + m^2\,.$$

Here we assume $m > 0$ and in fact $m = 1$. Otherwise we artificially add a term $(1 - m^2)\phi$ on both sides of the equation at the expense of having an additional linear term $c_0\phi$ in the inhomogeneous part which can easily be taken care of. We calculate

$$\begin{aligned}
(i\frac{\partial}{\partial t} \mp A^{\frac{1}{2}})\phi_\pm &= (i\frac{\partial}{\partial t} \mp A^{\frac{1}{2}})(I \pm iA^{-\frac{1}{2}}\frac{\partial}{\partial t})\phi \\
&= i\frac{\partial}{\partial t}\phi \mp A^{\frac{1}{2}}\phi \mp A^{-\frac{1}{2}}\frac{\partial^2}{\partial t^2}\phi - i\frac{\partial}{\partial t}\phi \\
&= \mp A^{-\frac{1}{2}}(A\phi + \frac{\partial^2}{\partial t^2}\phi) \\
&= \mp A^{-\frac{1}{2}}(\langle \beta\psi, \psi\rangle_{\mathbf{C}^2} + c_0\phi)\,.
\end{aligned}$$

Thus the Dirac-Klein-Gordon system can be rewritten as

$$(-i\frac{\partial}{\partial t} \pm |D|)\psi_\pm = -M\beta\psi_\mp + g\pi_\pm(D)(\frac{1}{2}(\phi_+ + \phi_-)\beta(\psi_+ + \psi_-)) \tag{2.2}$$

$$(i\frac{\partial}{\partial t} \mp A^{\frac{1}{2}})\phi_\pm = \mp A^{-\frac{1}{2}}(\langle \beta(\psi_+ + \psi_-), \psi_+ + \psi_-\rangle_{\mathbf{C}^2} + c_0(\phi_+ + \phi_-))\,. \tag{2.3}$$

The initial conditions are transformed into

$$\psi_\pm(0,x) = \pi_\pm(D)\psi_0(x)\,, \quad \phi_\pm(0,x) = \phi_0(x) \pm iA^{-\frac{1}{2}}\phi_1(x)\,. \tag{2.4}$$

It turns out that the decisive bilinear form which has to be considered is given by $\langle \beta\pi_{[\pm]}(D)\psi, \pi_\pm(D)\psi'\rangle_{\mathbf{C}^2}$, where $[\pm]$ and $\pm$ denote independent signs. We are going to compute its symbol. One has to treat

$$\begin{aligned}
&\mathcal{F}(\langle \beta\pi_{[\pm]}(D)\psi, \pi_\pm(D)\psi'\rangle_{\mathbf{C}^2})(\xi,\tau) \\
&= \iint_* \langle \beta\pi_{[\pm]}(\xi_1)\tilde{\psi}(\xi_1,\tau_1), \pi_\pm(-\xi_2)\tilde{\psi}'(-\xi_2,-\tau_2)\rangle_{\mathbf{C}^2} d\xi_1 d\tau_1\,,
\end{aligned}$$

where * denotes the region $\xi = \xi_1 + \xi_2$, $\tau = \tau_1 + \tau_2$. Because $\pi_\pm$ are hermitian and by use of (2.1) and $\pi_+(-\xi) = \pi_-(\xi)$ we get

$$\begin{aligned}
&\langle \beta\pi_{[\pm]}(\xi_1)\tilde{\psi}(\xi_1,\tau_1), \pi_\pm(-\xi_2)\tilde{\psi}'(-\xi_2,-\tau_2)\rangle \\
&= \langle \pi_\pm(-\xi_2)\beta\pi_{[\pm]}(\xi_1)\tilde{\psi}(\xi_1,\tau_1), \tilde{\psi}'(-\xi_2,-\tau_2)\rangle \\
&= \langle \beta\pi_\mp(-\xi_2)\pi_{[\pm]}(\xi_1)\tilde{\psi}(\xi_1,\tau_1), \tilde{\psi}'(-\xi_2,-\tau_2)\rangle \\
&= \langle \beta\pi_\pm(\xi_2)\pi_{[\pm]}(\xi_1)\tilde{\psi}(\xi_1,\tau_1), \tilde{\psi}'(-\xi_2,-\tau_2)\rangle\,.
\end{aligned}$$

We compute

$$\begin{aligned}
4\pi_\pm(\xi_2)\pi_+(\xi_1) &= (I \pm \hat{\xi}_2\alpha)(I + \hat{\xi}_1\alpha) \\
&= I \pm \hat{\xi}_1\hat{\xi}_2\alpha^2 + (\hat{\xi}_1 \pm \hat{\xi}_2)\alpha \\
&= (1 \pm \hat{\xi}_1\hat{\xi}_2)I + (\hat{\xi}_1 \pm \hat{\xi}_2)\alpha\,.
\end{aligned}$$

If ξ_1 and ξ_2 have different signs we have $\hat{\xi}_1\hat{\xi}_2 = -1$ and $\hat{\xi}_1 = -\hat{\xi}_2$, thus $\pi_+(\xi_2)\pi_+(\xi_1) = 0$. If ξ_1 and ξ_2 have the same sign we have $\hat{\xi}_1\hat{\xi}_2 = 1$ and $\hat{\xi}_1 = \hat{\xi}_2$, thus $4\pi_+(\xi_2)\pi_+(\xi_1) = 2(I \pm \alpha)$ (+, if $\xi_1, \xi_2 > 0$, and −, if $\xi_1, \xi_2 < 0$). Similarly

$4\pi_-(\xi_2)\pi_+(\xi_1) = 2(I \pm \alpha)$, if ξ_1, ξ_2 have different signs, and $\pi_-(\xi_2)\pi_+(\xi_1) = 0$, if ξ_1, ξ_2 have the same sign. Thus we have

$$\langle \beta\pi_\pm(\xi_2)\pi_{[\pm]}(\xi_1)\tilde{\psi}(\xi_1,\tau_1), \tilde{\psi}'(-\xi_2,-\tau_2)\rangle = \langle \gamma\tilde{\psi}(\xi_1,\tau_1), \tilde{\psi}'(-\xi_2,-\tau_2)\rangle$$

where

- in the $(+,+)$-case and in the $(-,-)$-case: $\gamma = \frac{1}{2}(\beta \pm \beta\alpha)$, if ξ_1, ξ_2 have the same sign, and $\gamma = 0$, if ξ_1, ξ_2 have different signs.
- in the $(+,-)$-case and in the $(-,+)$-case: $\gamma = \frac{1}{2}(\beta \pm \beta\alpha)$, if ξ_1, ξ_2 have different signs, and $\gamma = 0$, if ξ_1, ξ_2 have the same sign.

3. Local solutions

We want to construct solutions $\psi_\pm$ and $\phi_\pm$ in the spaces $X^{s,b}_\pm$ and $Y^{s,b}_\pm$, respectively, defined as follows.

Definition 3.1. $X^{s,b}_\pm$ is the completion of $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|\psi\|_{X^{s,b}_\pm} = \|\langle\xi\rangle^s\langle\tau \pm |\xi|\rangle^b\tilde{\psi}(\xi,\tau)\|_{L^2_{\xi\tau}}$$

for $\mathbf{C}^2$-valued functions ψ. $Y^{s,b}_\pm$ is the same space for $\mathbf{C}$-valued functions ψ. We also use the localized norms

$$\|\psi\|_{X^{s,b}_\pm[0,T]} = \inf_{\hat{\psi}_{|[0,t]}=\psi} \|\hat{\psi}\|_{X^{s,b}_\pm}$$

and similarly $Y^{s,b}_\pm[0,T]$.

We consider the following (slightly modified) system of integral equations which belongs to our Cauchy problem (2.2), (2.3), (2.4).

$$\begin{aligned}\psi_\pm(t) &= e^{\mp it|D|}\psi_\pm(0)\\ &\quad - ig\int_0^t e^{\mp i(t-s)|D|}\pi_\pm(D)(\frac{1}{2}(\phi_+(s)+\phi_-(s))\beta(\pi_+(D)\psi_+(s)\\ &\quad + \pi_-(D)\psi_-(s)))ds + iM\int_0^t e^{\mp i(t-s)|D|}\beta\psi_\mp(s)ds\end{aligned} \tag{3.1}$$

$$\begin{aligned}\phi_\pm(t) &= e^{\mp itA^{\frac12}}\phi_\pm(0)\\ &\quad \pm i\int_0^t e^{\mp i(t-s)A^{\frac12}}A^{-\frac12}\langle\beta(\pi_+(D)\psi_+(s)+\pi_-(D)\psi_-(s)), \pi_+(D)\psi_+(s)\\ &\quad + \pi_-(D)\psi_-(s)\rangle ds \pm ic_0\int_0^t e^{\mp i(t-s)A^{\frac12}}A^{-\frac12}(\phi_+(s)+\phi_-(s))ds\end{aligned} \tag{3.2}$$

We remark that any solution of this system automatically fulfills $\pi_\pm(D)\psi_\pm = \psi_\pm$, because applying $\pi_\pm(D)$ to the right hand side of the equations for $\psi_\pm$ gives $\pi_\pm(D)\psi_\pm(0) = \pi_\pm(D)\pi_\pm(D)\psi_0 = \pi_\pm(D)\psi_0 = \psi_\pm(0)$, and the integral terms also remain unchanged, because $\pi_\pm(D)^2 = \pi_\pm(D)$ and $\pi_\pm(D)\beta\psi_\mp(s) = \beta\pi_\mp(D)\psi_\mp(s) = \beta\psi_\mp(s)$. Thus $\pi_\pm(D)\psi_\pm$ can be replaced by $\psi_\pm$ on the right hand sides, thus the system of integral equations reduces exactly to the one belonging to our Cauchy problem (2.2), (2.3), (2.4).

Let now data be given with

$$\psi_0 \in H^{-l}(\mathbb{R})\,, \quad \phi_0 \in H^k(\mathbb{R})\,, \quad \phi_1 \in H^{k-1}(\mathbb{R})\,.$$

This implies $\psi_\pm(0) \in H^{-l}(\mathbb{R})$ and $\phi_\pm(0) \in H^k(\mathbb{R})$. In order to construct a solution of the integral equations for $t \in [0,T]$ with a suitable $T \le 1$ with $\psi_\pm \in X_\pm^{-l,\frac12+\epsilon'}[0,T]$ and $\phi_\pm \in Y_\pm^{k,\frac12+\epsilon'}[0,T]$ ($\epsilon' > 0$ small) we only have to show the following estimates for the nonlinearities (using standard facts from the theory of $X^{s,b}$ - spaces, especially (1.4)).

Concerning (3.1) we need

$$\|\pi_\pm(D)(\phi\beta\pi_{[\pm]}(D)\psi)\|_{X_\pm^{-l,-\frac12+2\epsilon'}} \le c\|\phi\|_{Y_+^{k,\frac12+\epsilon'}}\|\psi\|_{X_{[\pm]}^{-l,\frac12+\epsilon'}} \tag{3.3}$$

and the same estimates with $\|\phi\|_{Y_+^{k,\frac12+\epsilon'}}$ replaced by $\|\phi\|_{Y_-^{k,\frac12+\epsilon'}}$ on the right hand side. Again $[\pm]$ denotes a sign independent of $\pm$.

Concerning (3.2) we have to show

$$\|\langle\beta\pi_{[\pm]}(D)\psi, \pi_\pm(D)\psi'\rangle\|_{Y_+^{k-1,-\frac12+2\epsilon'}} \le c\|\psi\|_{X_{[\pm]}^{-l,\frac12+\epsilon'}}\|\psi'\|_{X_\pm^{-l,\frac12+\epsilon'}} \tag{3.4}$$

and the same estimate with $Y_+^{k-1,-\frac12+2\epsilon'}$ replaced by $Y_-^{k-1,-\frac12+2\epsilon'}$ on the left hand side.

By duality (3.3) is equivalent to

$$\Big|\iint\langle\pi_\pm(D)(\phi\beta\pi_{[\pm]}(D)\psi), \psi'\rangle dx\,dt\Big| \le c\|\phi\|_{Y_+^{k,\frac12+\epsilon'}}\|\psi\|_{X_{[\pm]}^{-l,\frac12+\epsilon'}}\|\psi'\|_{X_\pm^{l,\frac12-2\epsilon'}}\,.$$

The left hand side equals

$$\Big|\iint\phi\langle\beta\pi_{[\pm]}(D)\psi, \pi_\pm(D)\psi'\rangle dx\,dt\Big|\,,$$

which can be estimated by

$$\|\phi\|_{Y_+^{k,\frac12+\epsilon'}}\|\langle\beta\pi_{[\pm]}(D)\psi, \pi_\pm\psi'\rangle\|_{Y_+^{-k,-\frac12-\epsilon'}}\,.$$

Thus (3.3) is fulfilled if

$$\|\langle\beta\pi_{[\pm]}(D)\psi, \pi_\pm(D)\psi'\rangle\|_{Y_+^{-k,-\frac12-\epsilon'}} \le c\|\psi\|_{X_{[\pm]}^{-l,\frac12+\epsilon'}}\|\psi'\|_{X_\pm^{l,\frac12-2\epsilon'}} \tag{3.5}$$

and the same with $Y_+^{-k,-\frac12-\epsilon'}$ replaced by $Y_-^{-k,-\frac12-\epsilon'}$ on the left hand side.

The linear terms in the integral equations can easily be treated as follows:

$$\begin{aligned}&\|\psi_\mp\|_{X_\mp^{-l,-\frac12+2\epsilon'}[0,T]}\\ &\le \|\psi_\mp\|_{L^2([0,T],H^{-l})} \le T^{\frac12}\|\psi_\mp\|_{L^\infty([0,T],H^{-l})} \le cT^{\frac12}\|\psi_\mp\|_{X_\mp^{-l,\frac12+\epsilon'}[0,T]}\end{aligned}$$

and

$$\begin{aligned}&\|A^{-\frac12}\phi_\pm\|_{Y_{[\pm]}^{k,-\frac12+2\epsilon'}[0,T]}\\ &\le \|\phi_\pm\|_{L^2([0,T],H^{k-1})} \le T^{\frac12}\|\phi_\pm\|_{L^\infty([0,T],H^{k-1})} \le cT^{\frac12}\|\phi_\pm\|_{Y_\pm^{k,\frac12+\epsilon'}[0,T]}\,.\end{aligned}$$

It remains to prove (3.4) and (3.5).

Lemma 3.2. *Assume $l < \frac14$, $2l + k < 1$ and $l + k \le 1$. Then* (3.4) *holds for a sufficiently small $\epsilon' > 0$.*

Proof. We have to show

$$\Big|\iint \langle \beta\pi_{[\pm]}(D)\psi, \pi_\pm(D)\psi'\rangle\overline{\phi} dx\,dt\Big| \le c\|\phi\|_{Y_+^{1-k,\frac12-2\epsilon'}}\|\psi\|_{X_{[\pm]}^{-l,\frac12+\epsilon'}}\|\psi'\|_{X_\pm^{-l,\frac12+\epsilon'}}\,.$$

The left hand side equals (according to the calculation above)

$$\Big|\iint_* \langle \beta\pi_\pm(\xi_2)\pi_{[\pm]}(\xi_1)\tilde{\psi}(\xi_1,\tau_1), \tilde{\psi}'(-\xi_2,-\tau_2)\rangle\overline{\tilde{\phi}}(\xi,\tau)d\xi_1 d\xi_2 d\tau_1 d\tau_2\Big|,$$

where * denotes the region $\xi_1+\xi_2=\xi$, $\tau_1+\tau_2=\tau$. Defining now

$$\begin{aligned}
\tilde{v}_1(\xi_1,\tau_1) &:= \langle\xi_1\rangle^{-l}\langle\tau_1[\pm]|\xi_1|\rangle^{\frac12+\epsilon'}\tilde{\psi}(\xi_1,\tau_1)\\
\tilde{v}_2(\xi_2,\tau_2) &:= \langle\xi_2\rangle^{-l}\langle\tau_2\pm|\xi_2|\rangle^{\frac12+\epsilon'}\tilde{\psi}'(\xi_2,\tau_2)\\
\tilde{\varphi}(\xi,\tau) &:= \langle\xi\rangle^{1-k}\langle\tau+|\xi|\rangle^{\frac12-2\epsilon'}\tilde{\phi}(\xi,\tau),
\end{aligned}$$

we have

$$\|\psi\|_{X_{[\pm]}^{-l,\frac12+\epsilon'}} = \|v_1\|_{L^2_{xt}}\,,\quad \|\psi'\|_{X_\pm^{-l,\frac12+\epsilon'}} = \|v_1\|_{L^2_{xt}}\,,\quad \|\phi\|_{Y_+^{1-k,\frac12-2\epsilon'}} = \|\varphi\|_{L^2_{xt}}\,.$$

Thus we have to show

$$\begin{aligned}
&\Big|\iint_* \frac{\langle \beta\pi_\pm(\xi_2)\pi_{[\pm]}(\xi_1)\tilde{v}_1(\xi_1,\tau_1), \tilde{v}_2(-\xi_2,-\tau_2)\rangle\langle\xi_1\rangle^l\langle\xi_2\rangle^l\overline{\tilde{\varphi}}(\xi,\tau)}{\langle\tau_1[\pm]|\xi_1|\rangle^{\frac12+\epsilon'}\langle\tau_2\mp|\xi_2|\rangle^{\frac12+\epsilon'}\langle\tau+|\xi|\rangle^{\frac12-2\epsilon'}\langle\xi\rangle^{1-k}}d\xi_1 d\xi_2 d\tau_1 d\tau_2\Big|\\
&\le c\|v_1\|_{L^2}\|v_2\|_{L^2}\|\varphi\|_{L^2}\,.
\end{aligned}$$

According to our computations at the end of Section 1 we know: in the $(+,+)$-case or $(-,-)$-case this integral reduces to the region $\xi_1\xi_2>0$, whereas in the $(+,-)$-case or $(-,+)$-case it reduces to $\xi_1\xi_2<0$. In any case $\beta\pi_\pm(\xi_2)\pi_{[\pm]}(\xi_1)$ is a constant matrix in each of the quadrants in the (ξ_1,ξ_2)-plane.

A. Let us first consider the $(+,-)$-case or $(-,+)$-case. Here we have to prove

$$\begin{aligned}
&\iint_{\overset{*}{\xi_1\xi_2<0}} \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|\langle\xi_1\rangle^l\langle\xi_2\rangle^l}{\langle\tau_1\pm|\xi_1|\rangle^{\frac12+\epsilon'}\langle\tau_2\pm|\xi_2|\rangle^{\frac12+\epsilon'}\langle\tau+|\xi|\rangle^{\frac12-2\epsilon'}\langle\xi\rangle^{1-k}}d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le c\|v_1\|_{L^2}\|v_2\|_{L^2}\|\varphi\|_{L^2}\,.
\end{aligned}$$

In this region we have $|\xi| = ||\xi_1|-|\xi_2||$. Define

$$\sigma_1 = \tau_1\pm|\xi_1|\,,\quad \sigma_2=\tau_2\pm|\xi_2|\,,\quad \sigma=\tau+|\xi|\,.$$

Then we get the decisive algebraic inequality:

$$\begin{aligned}
2\min(|\xi_1|,|\xi_2|) &\le |\xi_1|+|\xi_2|\mp||\xi_2|-|\xi_1||\\
&= |\xi_1|+|\xi_2|\mp|\xi|\\
&= \pm(\tau_1\pm|\xi_1|)\pm(\tau_2\pm|\xi_2|)\mp(\tau+|\xi|)\\
&= \pm\sigma_1\pm\sigma_2\mp\sigma\le|\sigma_1|+|\sigma_2|+|\sigma|\,.
\end{aligned} \tag{3.6}$$

Case 1: $|\xi_1| << |\xi_2|$ ($\Rightarrow$ $|\xi|\sim|\xi_2|$) (The case $|\xi_2| << |\xi_1|$ can be handled similarly.) We have

$$\frac{\langle\xi_1\rangle^l\langle\xi_2\rangle^l}{\langle\xi\rangle^{1-k}} \le c\langle\xi_1\rangle^l\langle\xi_2\rangle^{l-1+k}$$

and consider three different cases depending on which of the σ's is dominant.

a. $|\sigma| \geq |\sigma_1|, |\sigma_2|$. By (3.6) we have $\langle\sigma\rangle \geq c\langle\xi_1\rangle$, so that it remains to estimate

$$\begin{aligned}
&\iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|\langle\xi_1\rangle^l\langle\xi_2\rangle^{l-1+k}}{\langle\sigma_1\rangle^{\frac{1}{2}+\epsilon'}\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon'}\langle\xi_1\rangle^{\frac{1}{2}-2\epsilon'}} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
&\leq \iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac{1}{2}+\epsilon'}\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon'}\langle\xi_1\rangle^{-2l+\frac{3}{2}-k-2\epsilon'}} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
&= \iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac{1}{2}+\epsilon'}\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon'}\langle\xi_1\rangle^{\frac{1}{2}+\epsilon}} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \,.
\end{aligned}$$

Here we used the assumptions $l-1+k \leq 0$, $|\xi_1| \leq |\xi_2|$, $2l+k<1$. $\epsilon' > 0$ is sufficiently small and $\epsilon > 0$. Forgetting about the factor $\langle\sigma_1\rangle^{\frac{1}{2}+\epsilon'}$ and using Plancherel and Hölder this is bounded by

$$\begin{aligned}
&\|\mathcal{F}^{-1}(\frac{|\tilde{v}_1|}{\langle\xi_1\rangle^{\frac{1}{2}+\epsilon}})\|_{L^2_t L^\infty_x} \|\mathcal{F}^{-1}(\frac{|\tilde{v}_2(-\xi_2,-\tau_2)|}{\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon'}})\|_{L^\infty_t L^2_x} \|\varphi\|_{L^2_t L^2_x} \\
&\leq c\|v_1\|_{L^2_{xt}}\|v_2\|_{L^2_{xt}}\|\varphi\|_{L^2_{xt}}
\end{aligned}$$

by Sobolev's embedding and $X_{\pm}^{0,\frac{1}{2}+\epsilon'} \subset L^\infty_t L^2_x$.

b. $|\sigma_j|$ ($j=1$ or $j=2$) dominant. This case can be treated similarly by using the estimate $\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon'} \geq c\langle\xi_1\rangle^{\frac{1}{2}+\epsilon'}$.

Case 2: $|\xi_1| \sim |\xi_2|$. We have

$$\frac{\langle\xi_1\rangle^l\langle\xi_2\rangle^l}{\langle\xi\rangle^{1-k}} \sim \frac{\langle\xi_1\rangle^{2l}}{\langle\xi\rangle^{1-k}} \,.$$

a. $|\sigma|$ dominant.

We use (3.6) and get $\langle\sigma\rangle \geq c\langle\xi_1\rangle$, and moreover , $l < \frac{1}{4}$, $\langle\xi_1\rangle \geq c\langle\xi\rangle$, and $2l+k<1$ and estimate as follows:

$$\begin{aligned}
&\iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|\langle\xi_1\rangle^{2l-\frac{1}{2}+2\epsilon'}}{\langle\sigma_1\rangle^{\frac{1}{2}+\epsilon'}\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon'}\langle\xi\rangle^{1-k}} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
&\leq \iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac{1}{2}+\epsilon'}\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon'}\langle\xi\rangle^{\frac{3}{2}-k-2l-2\epsilon'}} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
&= \iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac{1}{2}+\epsilon'}\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon'}\langle\xi\rangle^{\frac{1}{2}+\epsilon}} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
&\leq \|v_1\|_{L^2_{xt}} \|\mathcal{F}^{-1}(\frac{|\tilde{v}_2(-\xi_2,-\tau_2)|}{\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon'}})\|_{L^\infty_t L^2_x} \|\mathcal{F}^{-1}(\frac{|\tilde{\varphi}|}{\langle\xi\rangle^{\frac{1}{2}+\epsilon}})\|_{L^2_t L^\infty_x} \\
&\leq \|v_1\|_{L^2_{xt}}\|v_2\|_{L^2_{xt}}\|\varphi\|_{L^2_{xt}}
\end{aligned}$$

b. The cases $|\sigma_1|$ or $|\sigma_2|$ dominant are handled similarly.

B. Let us next consider the $(+,+)$-case or $(-,-)$-case. We have to prove

$$\begin{aligned}
&\iint_{\substack{* \\ \xi_1\xi_2>0}} \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|\langle\xi_1\rangle^l\langle\xi_2\rangle^l}{\langle\tau_1 \pm |\xi_1|\rangle^{\frac{1}{2}+\epsilon'}\langle\tau_2 \mp |\xi_2|\rangle^{\frac{1}{2}+\epsilon'}\langle\tau+|\xi|\rangle^{\frac{1}{2}-2\epsilon'}\langle\xi\rangle^{1-k}} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
&\leq c\|v_1\|_{L^2}\|v_2\|_{L^2}\|\varphi\|_{L^2} \,.
\end{aligned}$$

In this region we have $|\xi| = |\xi_1| + |\xi_2|$. Assuming without loss of generality $|\xi_2| \geq |\xi_1|$ we have $|\xi| \sim |\xi_2|$ and also

$$\frac{\langle\xi_1\rangle^l\langle\xi_2\rangle^l}{\langle\xi\rangle^{1-k}} \leq \frac{c\langle\xi_1\rangle^l}{\langle\xi_2\rangle^{1-k-l}} \leq c\langle\xi_1\rangle^{2l+k-1} \leq c\langle\xi_1\rangle^{-\epsilon}$$

by our assumptions $l+k \le 1$ and $2l+k<1$. Moreover, defining

$$\sigma_1 = \tau_1 \pm |\xi_1|\,, \quad \sigma_2 = \tau_2 \mp |\xi_2|\,, \quad \sigma = \tau + |\xi|\,,$$

we get

$$\begin{aligned} 2\min(|\xi_1|,|\xi_2|) &\le \mp|\xi_1| \pm |\xi_2| + |\xi_1| + |\xi_2| \\ &= -(\tau_1 \pm |\xi_1|) - (\tau_2 \mp |\xi_2|) + \tau + |\xi| \\ &= -\sigma_1 - \sigma_2 + \sigma \le |\sigma_1| + |\sigma_2| + |\sigma|\,. \end{aligned}$$

a. $|\sigma|$ dominant. This implies $\langle\sigma\rangle \ge c\langle\xi_1\rangle$ so that we estimate for sufficiently small $\epsilon' > 0$:

$$\begin{aligned} &\iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\xi_1\rangle^{\epsilon}\langle\xi_1\rangle^{\frac12-2\epsilon'}\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma_2\rangle^{\frac12+\epsilon'}} \\ &\le \|\mathcal{F}^{-1}\big(\frac{|\tilde{v}_1|}{\langle\xi_1\rangle^{\frac12+\epsilon-2\epsilon'}}\big)\|_{L^2_tL^\infty_x}\|\mathcal{F}^{-1}\big(\frac{|\tilde{v}_2(-\xi_2,-\tau_2)|}{\langle\sigma_2\rangle^{\frac12+\epsilon'}}\big)\|_{L^\infty_tL^2_x}\|\varphi\|_{L^2_tL^2_x} \\ &\le c\|v_1\|_{L^2_{xt}}\|v_2\|_{L^2_{xt}}\|\varphi\|_{L^2_{xt}}\,. \end{aligned}$$

b. The cases $|\sigma_1|$ or $|\sigma_2|$ dominant are handled similarly. □

Remark: The modified estimate (3.4) with $Y_+^{k-1,-\frac12+2\epsilon'}$ replaced by $Y_-^{k-1,-\frac12+2\epsilon'}$ is proven in the same way replacing $\sigma = \tau + |\xi|$ by $\sigma = \tau - |\xi|$ everywhere. One just has to show that the decisive algebraic inequality $2\min(|\xi_1|,|\xi_2|) \le |\sigma_1| + |\sigma_2| + |\sigma|$ still holds true. This can easily be seen as follows: in Part A of the proof we estimate

$$\begin{aligned} 2\min(|\xi_1|,|\xi_2|) &\le |\xi_1| + |\xi_2| \pm ||\xi_1| - |\xi_2|| \\ &= |\xi_1| + |\xi_2| \pm |\xi| \\ &= \pm(\tau_1 \pm |\xi_1|) \pm (\tau_2 \pm |\xi_2|) \mp (\tau - |\xi|) \\ &= \pm\sigma_1 \pm \sigma_2 \mp \sigma \le |\sigma_1| + |\sigma_2| + |\sigma|\,, \end{aligned}$$

and in Part B we get

$$\begin{aligned} 2\min(|\xi_1|,|\xi_2|) \le \pm|\xi_1| \mp |\xi_2| + |\xi_1| + |\xi_2| &= \pm|\xi_1| \mp |\xi_2| + |\xi| \\ &= (\tau_1 \pm |\xi_1|) + (\tau_2 \mp |\xi_2|) - (\tau - |\xi|) \\ &= \sigma_1 + \sigma_2 - \sigma \le |\sigma_1| + |\sigma_2| + |\sigma|\,. \end{aligned}$$

Lemma 3.3. *Assume $k \ge |l|$ and $k > 0$. Then (3.5) holds for a sufficiently small $\epsilon' > 0$.*

Proof. Arguing as in the previous proof we have to show

$$\begin{aligned} &\Big|\iint_* \frac{\langle\beta\pi_\pm(\xi_2)\pi_{[\pm]}(\xi_1)\tilde{v}_1(\xi_1,\tau_1),\tilde{v}_2(-\xi_2,-\tau_2)\rangle\langle\xi_1\rangle^l\overline{\tilde{\varphi}}(\xi,\tau)}{\langle\tau_1[\pm]|\xi_1|\rangle^{\frac12+\epsilon'}\langle\tau_2\mp|\xi_2|\rangle^{\frac12-2\epsilon'}\langle\tau+|\xi|\rangle^{\frac12+\epsilon'}\langle\xi\rangle^k\langle\xi_2\rangle^l}d\xi_1d\xi_2d\tau_1d\tau_2\Big| \\ &\le c\|v_1\|_{L^2}\|v_2\|_{L^2}\|\varphi\|_{L^2}\,. \end{aligned}$$

A: Consider first the $(+,-)$-case or $(-,+)$-case. One has to show

$$\begin{aligned} &\iint_{\xi_1\xi_2<0}^* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|\langle\xi_1\rangle^l}{\langle\tau_1\pm|\xi_1|\rangle^{\frac12+\epsilon'}\langle\tau_2\pm|\xi_2|\rangle^{\frac12-2\epsilon'}\langle\tau+|\xi|\rangle^{\frac12+\epsilon'}\langle\xi\rangle^k\langle\xi_2\rangle^l}d\xi_1d\xi_2d\tau_1d\tau_2 \\ &\le c\|v_1\|_{L^2}\|v_2\|_{L^2}\|\varphi\|_{L^2}\,. \end{aligned}$$

In this region we have $|\xi| = ||\xi_1| - |\xi_2||$. Define

$$\sigma_1 = \tau_1 \pm |\xi_1|\,,\, \sigma_2 = \tau_2 \pm |\xi_2|\,, \quad \sigma = \tau + |\xi|\,.$$

Again as in the previous proof (cf. (3.6)),

$$2\min(|\xi_1|, |\xi_2|) \le |\sigma_1| + |\sigma_2| + |\sigma|\,. \tag{3.7}$$

Case 1: $|\xi_1| << |\xi_2|$ ($\Rightarrow$ $|\xi| \sim |\xi_2|$) We have

$$\frac{\langle\xi_1\rangle^l}{\langle\xi\rangle^k\langle\xi_2\rangle^l} \sim \frac{\langle\xi_1\rangle^l}{\langle\xi_2\rangle^{k+l}}\,.$$

In the $|\sigma_2|$ - dominant case it remains to estimate, using $\langle\xi_1\rangle \le c\langle\sigma_2\rangle$, $k+l \ge 0$, $k > 0$ and $|\xi_2| \ge |\xi_1|$:

$$\begin{aligned}
&\iint_* \frac{|\tilde v_1(\xi_1,\tau_1)||\tilde v_2(-\xi_2,-\tau_2)||\tilde\varphi(\xi,\tau)|\langle\xi_1\rangle^l}{\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma\rangle^{\frac12+\epsilon'}\langle\xi_1\rangle^{\frac12-2\epsilon'}\langle\xi_2\rangle^{k+l}} d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le \iint_* \frac{|\tilde v_1(\xi_1,\tau_1)||\tilde v_2(-\xi_2,-\tau_2)||\tilde\varphi(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma\rangle^{\frac12+\epsilon'}\langle\xi_1\rangle^{k+\frac12-2\epsilon'}} d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le \big\|\mathcal F^{-1}\big(\frac{|\tilde v_1|}{\langle\xi_1\rangle^{k+\frac12-2\epsilon'}}\big)\big\|_{L^2_t L^\infty_x} \|v_2\|_{L^2_t L^2_x} \big\|\mathcal F^{-1}\big(\frac{|\tilde\varphi|}{\langle\sigma\rangle^{\frac12+\epsilon'}}\big)\big\|_{L^\infty_t L^2_x}\\
&\le c\|v_1\|_{L^2_{xt}}\|v_2\|_{L^2_{xt}}\|\varphi\|_{L^2_{xt}}\,.
\end{aligned}$$

The regions where $|\sigma|$ or $|\sigma_1|$ are dominant are treated similarly.
Case 2: $|\xi_2| << |\xi_1|$ ($\Rightarrow$ $|\xi| \sim |\xi_1|$) Using

$$\frac{\langle\xi_1\rangle^l}{\langle\xi\rangle^k\langle\xi_2\rangle^l} \sim \frac{1}{\langle\xi_1\rangle^{k-l}\langle\xi_2\rangle^l}$$

and (3.7) we have to estimate in the $|\sigma_2|$-dominant case, using $k-l \ge 0$, $k > 0$ and $|\xi_1| \ge |\xi_2|$:

$$\begin{aligned}
&\iint_* \frac{|\tilde v_1(\xi_1,\tau_1)||\tilde v_2(-\xi_2,-\tau_2)||\tilde\varphi(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma\rangle^{\frac12+\epsilon'}\langle\xi_1\rangle^{k-l}\langle\xi_2\rangle^{l+\frac12-2\epsilon'}} d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le \iint_* \frac{|\tilde v_1(\xi_1,\tau_1)||\tilde v_2(-\xi_2,-\tau_2)||\tilde\varphi(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma\rangle^{\frac12+\epsilon'}\langle\xi_2\rangle^{k+\frac12-2\epsilon'}} d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le c\|v_1\|_{L^2_{xt}}\|v_2\|_{L^2_{xt}}\|\varphi\|_{L^2_{xt}}
\end{aligned}$$

similarly as in Case 1. The regions where $|\sigma|$ or $|\sigma_1|$ are dominant can be handled similarly.
Case 3: $|\xi_1| \sim |\xi_2|$ ($\Rightarrow$ $|\xi| \le |\xi_1| + |\xi_2| \sim 2|\xi_2|$). We use

$$\frac{\langle\xi_1\rangle^l}{\langle\xi\rangle^k\langle\xi_2\rangle^l} \sim \frac{1}{\langle\xi\rangle^k}$$

and get in the $|\sigma_2|$ - dominant region (the other cases can be treated similarly again) by our assumption $k>0$:

$$\begin{aligned}
&\iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma\rangle^{\frac12+\epsilon'}\langle\xi_2\rangle^{\frac12-2\epsilon'}\langle\xi\rangle^k}d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le \iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma\rangle^{\frac12+\epsilon'}\langle\xi\rangle^{k+\frac12-2\epsilon'}}d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le \Big\|\mathcal{F}^{-1}\Big(\frac{|\tilde{v}_1|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}}\Big)\Big\|_{L^\infty_t L^2_x}\|v_2\|_{L^2_t L^2_x}\Big\|\mathcal{F}^{-1}\Big(\frac{|\tilde{\varphi}|}{\langle\xi\rangle^{k+\frac12-2\epsilon'}}\Big)\Big\|_{L^2_t L^\infty_x}\\
&\le c\|v_1\|_{L^2_{xt}}\|v_2\|_{L^2_{xt}}\|\varphi\|_{L^2_{xt}}\,.
\end{aligned}$$

B: Consider now the $(+,+)$-case or $(-,-)$-case, where one has to estimate

$$\iint_{\xi_1\xi_2>0}^{*} \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|\langle\xi_1\rangle^l}{\langle\tau_1\pm|\xi_1|\rangle^{\frac12+\epsilon'}\langle\tau_2\mp|\xi_2|\rangle^{\frac12-2\epsilon'}\langle\tau+|\xi|\rangle^{\frac12+\epsilon'}\langle\xi\rangle^k\langle\xi_2\rangle^l}d\xi_1 d\xi_2 d\tau_1 d\tau_2\,.$$

One has $|\xi|=|\xi_1|+|\xi_2|$ and with

$$\sigma_1=\tau_1\pm|\xi_1|\,,\ \sigma_2=\tau_2\mp|\xi_2|\,,\ \sigma=\tau+|\xi|$$

one checks again (3.7).

If $|\sigma_2|$ is dominant and $|\xi_1|\ge|\xi_2|$ we have $|\xi|\sim|\xi_1|$ and

$$\frac{\langle\xi_1\rangle^l}{\langle\xi\rangle^k\langle\xi_2\rangle^l}\sim\frac{1}{\langle\xi_1\rangle^{k-l}\langle\xi_2\rangle^l}$$

as well as $\langle\xi_2\rangle\le c\langle\sigma_2\rangle$, so that by use of $k-l\ge0$, $k>0$ and $|\xi_1|\ge|\xi_2|$ we estimate

$$\begin{aligned}
&\iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma\rangle^{\frac12+\epsilon'}\langle\xi_1\rangle^{k-l}\langle\xi_2\rangle^{l+\frac12-2\epsilon'}}d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le \iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma\rangle^{\frac12+\epsilon'}\langle\xi_2\rangle^{k+\frac12-2\epsilon'}}d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le \big\|\mathcal{F}^{-1}\big(\frac{|\tilde{v}_1|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}}\big)\big\|_{L^\infty_t L^2_x}\big\|\mathcal{F}^{-1}\big(\frac{|\tilde{v}_2(-\xi_2,-\tau_2)|}{\langle\xi_2\rangle^{k+\frac12-2\epsilon'}}\big)\big\|_{L^2_t L^\infty_x}\|\varphi\|_{L^2_t L^2_x}\\
&\le c\|v_1\|_{L^2_{xt}}\|v_2\|_{L^2_{xt}}\|\varphi\|_{L^2_{xt}}\,.
\end{aligned}$$

If $|\sigma_2|$ is dominant and $|\xi_2|\ge|\xi_1|$, we have $|\xi|\sim|\xi_2|$ and, using $k+l>0$:

$$\frac{\langle\xi_1\rangle^l}{\langle\xi\rangle^k\langle\xi_2\rangle^l}\sim\frac{\langle\xi_1\rangle^l}{\langle\xi_2\rangle^{k+l}}\le\frac{\langle\xi_1\rangle^l}{\langle\xi_1\rangle^{k+l}}=\frac{1}{\langle\xi_1\rangle^k}$$

and also $\langle\xi_1\rangle\le c\langle\sigma_2\rangle$. Thus, similarly as before we get for $k>0$:

$$\begin{aligned}
&\iint_* \frac{|\tilde{v}_1(\xi_1,\tau_1)||\tilde{v}_2(-\xi_2,-\tau_2)||\tilde{\varphi}(\xi,\tau)|}{\langle\sigma_1\rangle^{\frac12+\epsilon'}\langle\sigma\rangle^{\frac12+\epsilon'}\langle\xi_1\rangle^{k+\frac12-2\epsilon'}}d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\le c\|v_1\|_{L^2_{xt}}\|v_2\|_{L^2_{xt}}\|\varphi\|_{L^2_{xt}}\,.
\end{aligned}$$

The other regions are treated similarly. □

Remark. The modified estimate (3.5) with $Y_+^{-k,-\frac12-\epsilon'}$ replaced by $Y_-^{-k,-\frac12-\epsilon'}$ is proven in the same way. See also the remark to the previous lemma.

We summarize our results in the following theorem.

Theorem 3.4. *Assume $l < 1/4$, $k > 0$, $2l + k < 1$, $l + k \le 1$ and $k \ge |l|$. The Cauchy problem for the Dirac-Klein-Gordon equations* (1.1),(1.2),(1.3) *with data*

$$\psi_0 \in H^{-l}(\mathbb{R})\,, \quad \phi_0 \in H^k(\mathbb{R})\,, \quad \phi_1 \in H^{k-1}(\mathbb{R})$$

has a unique local solution

$$\psi = \psi_+ + \psi_- \quad \text{with} \quad \psi_\pm \in X_\pm^{-l,\frac{1}{2}+\epsilon'}[0,T],$$

and

$$\phi = \frac{1}{2}(\phi_+ + \phi_-)\,, \quad \phi_t = \frac{1}{2i} A^{\frac{1}{2}}(\phi_+ - \phi_-) \quad \text{with} \quad \phi_\pm \in Y_\pm^{k,\frac{1}{2}+\epsilon'}[0,T]\,,$$

where $A = -\frac{\partial^2}{\partial x^2} + 1$. Here $T = T(\|\psi_0\|_{H^{-l}}, \|\phi_0\|_{H^k}, \|\phi_1\|_{H^{k-1}})$ and $\epsilon' > 0$ is sufficiently small. This solution satisfies

$$\psi \in C^0([0,T], H^{-l}(\mathbb{R}))\,, \quad \phi \in C^0([0,T], H^k(\mathbb{R}))\,, \quad \phi_t \in C^0([0,T], H^{k-1}(\mathbb{R}))\,.$$

4. Global existence

The following global existence result is an easy consequence of the local results and conservation of charge.

Theorem 4.1. *Assume $\psi_0 \in L^2(\mathbb{R})$, $\phi_0 \in H^k(\mathbb{R})$, $\phi_1 \in H^{k-1}(\mathbb{R})$, where $0 < k < 1/2$. Then the solution of Theorem 3.4 exists globally in t.*

Proof. We only need an a-priori-bound for $\|\psi(t)\|_{L^2}$ and $\|\phi(t)\|_{H^k} + \|\phi_t(t)\|_{H^{k-1}}$. Charge conservation gives the L^2 - bound of $\psi(t)$ and $\phi(t)$ fulfills the integral equation

$$\begin{aligned}\phi(t) = \cos(A^{\frac{1}{2}}t)\phi_0 + A^{-\frac{1}{2}}\sin(A^{\frac{1}{2}}t)\phi_1 + \int_0^t A^{-\frac{1}{2}}\sin[A^{\frac{1}{2}}(t-s)]\langle\beta\psi(s),\psi(s)\rangle ds \\ + c_0 \int_0^t A^{-\frac{1}{2}}\sin[A^{\frac{1}{2}}(t-s)]\phi(s)ds\,,\end{aligned}$$

where $c_0 = 1 - m^2$. Thus

$$\begin{aligned}&\|\phi(t)\|_{H^k} + \|\phi_t(t)\|_{H^{k-1}} \\ &\le c\big(\|\phi_0\|_{H^k} + \|\phi_1\|_{H^{k-1}} + \int_0^t (\|\langle\beta\psi(s),\psi(s)\rangle\|_{H^{k-1}} + \|\phi(s)\|_{H^{k-1}})\, ds\big)\,.\end{aligned}$$

Using the estimate

$$\|\langle\beta\psi,\psi\rangle\|_{H^{k-1}} \le c\|\psi\|_{L^2}^2 \quad \text{for } k < \frac{1}{2}\,,$$

which follows from (cf. [4])

$$\begin{aligned}\|uv\|_{H^{k-1}}^2 &\le \int \Big|\int \tilde{u}(\eta)\tilde{v}(\xi-\eta)\,d\eta\Big|^2 \langle\xi\rangle^{2(k-1)}\,d\xi \\ &\le \int (\int |\tilde{u}(\eta)|^2 d\eta)(\int |\tilde{v}(\xi-\eta)|^2 d\eta)\langle\xi\rangle^{2(k-1)} d\xi \\ &\le \|u\|_{L^2}^2\|v\|_{L^2}^2 \int \langle\xi\rangle^{2(k-1)} d\xi \le c\|u\|_{L^2}^2\|v\|_{L^2}^2\,,\end{aligned}$$

we arrive at

$$\|\phi(t)\|_{H^k} + \|\phi_t(t)\|_{H^{k-1}} \le c(\|\phi_0\|_{H^k} + \|\phi_1\|_{H^{k-1}} + t\|\psi_0\|_{L^2}^2 + \int_0^t \|\phi(s)\|_{H^k}\, ds)\,,$$

so that Gronwall's lemma gives the desired a-priori-bound. □

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