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EXPONENTIAL STABILITY OF A DAMPED BEAM-STRING-BEAM TRANSMISSION PROBLEM

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Abstract. We consider a beam-string-beam transmission problem, where two structurally damped or undamped beams are coupled with a frictionally damped string by transmission conditions. We show that for this structure, the dissipation produced by the frictional part is strong enough to produce exponential decay of the solution no matter how small is its size. For the exponential stability in the damped-damped-damped situation we use energy method. For the undamped-damped-undamped situation we use a frequency domain method from semigroups theory, which combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent. Additionally, we show that the solution first defined by the weak formulation has higher Sobolev space regularity.

1. Introduction

Recent advances in material science have provided new means for suppressing vibrations from elastic multi-link structures, for instance, by applying some type of local or total damping. These structures consisting of connected flexible elements such as strings, beams, plates and shells have many applications in engineering areas such as in robot arms, frames, solar panels, aircrafts, satellite antennae, bridges and so on (see [6, 7, 15] and the references therein). In this context we consider a coupled beam-string-beam system, where we assume structural damping/no-damping for the beams and frictional damping for the string. More precisely, we consider an elastic structure composed by three parts. The first and the third parts are structurally damped or undamped beams in the open intervals $I_1 := (l_0, l_1)$ and $I_3 := (l_2, l_3)$, respectively, and the second is a frictionally damped string, occupying in equilibrium the open interval $I_2 := (l_1, l_2)$, where $l_0 < l_1 < l_2 < l_3$ as shown in Figure 1.

We denote by $u = u(x,t)$, $w = w(x,t)$, and $v = v(x,t)$ the vertical displacements of the points on the two beams and on the string with coordinates x at time t, respectively.

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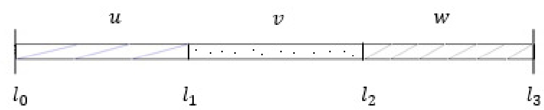

FIGURE 1. Beam-string-beam system

The mathematical model for the structure is given by the equations

$$u_{tt} + u_{xxxx} - \rho_1 u_{txx} = 0, \quad \text{in } (l_0, l_1) \times (0, \infty), \tag{1.1}$$

$$v_{tt} - v_{xx} + \beta v_t = 0, \quad \text{in } (l_1, l_2) \times (0, \infty), \tag{1.2}$$

$$w_{tt} + w_{xxxx} - \rho_2 w_{txx} = 0, \quad \text{in } (l_2, l_3) \times (0, \infty), \tag{1.3}$$

where $\rho_i \geq 0$, $i = 1, 2$, and $\beta \geq 0$ are fixed constants. The coefficients $\rho_1 \geq 0$ and $\rho_2 \geq 0$ describe the structural damping (or the absence of damping) for the beam equations (1.1) and (1.3), whereas $\beta > 0$ in (1.2) describes a frictional damping on the string. On the endpoints l_0, l_3 of the beams, we impose clamped (Dirichlet) boundary conditions

$$u(l_0, t) = u_x(l_0, t) = w(l_3, t) = w_x(l_3, t) = 0, \quad t \in (0, \infty). \tag{1.4}$$

On the interface $\{l_1, l_2\}$, we have transmission conditions

$$u(l_1, t) = v(l_1, t) \text{ and } v(l_2, t) = w(l_2, t), \quad t \in (0, \infty), \tag{1.5}$$

$$u_{xxx}(l_1, t) - \rho_1 u_{tx}(l_1, t) + v_x(l_1, t) = 0, \quad t \in (0, \infty), \tag{1.6}$$

$$w_{xxx}(l_2, t) - \rho_2 w_{tx}(l_2, t) + v_x(l_2, t) = 0, \quad t \in (0, \infty), \tag{1.7}$$

$$u_{xx}(l_1, t) = 0, \quad t \in (0, \infty), \tag{1.8}$$

$$w_{xx}(l_2, t) = 0, \quad t \in (0, \infty). \tag{1.9}$$

Condition (1.5) is known as the continuity transmission condition. (1.6) and (1.7) mean that the two forces which are the shear force of the beams and the stress of the string are such that one cancels the other. And (1.8) and (1.9) describe the fact that the beams present possible inflection point on l_1 and l_2 (compare with [14, p. 1934]).

Finally, the boundary-transmission problem (1.1)–(1.9) is endowed the with initial conditions:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (l_0, l_1), \tag{1.10}$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in (l_1, l_2), \tag{1.11}$$

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad x \in (l_2, l_3). \tag{1.12}$$

The aim of this article is to study the well-posedness, regularity, and exponential stability of the solution of (1.1)-(1.12).

In recent years, the large-time behavior of structures consisting of elastic strings and beams with different damping has been studied. We refer the reader to [17, 19, 24], where structures formed by beams were studied. For instance, Shel in [24] showed, under certain conditions, the exponential stability of a network of elastic and thermoelastic Euler-Bernoulli beams. For transmission problems between strings, see for example, [1, 9, 13, 18, 22, 23]. Alves, Muñoz Rivera et al. considered in [1] a transmission problem of a material composed of three components;

one of them is a Kelvin-Voigt viscoelastic material, the second is an elastic material (no dissipation) and the third is an elastic material with a frictional damping mechanism. They proved exponential stability of the solution the if viscoelastic component is not in the middle of the material. Rissel and Wang established in [23] exponential stability for a coupled system composed of three parts: the first and third are purely elastic and the second thermo-elastic. For elastic structures composed of string and beam the reader is referred to [2, 3, 4, 5, 14, 16, 25, 26, 27]. Ammari et al. [2, 3, 4, 5] considered the nodal feedback stabilization for networks of strings and beams. They obtained that the decay rate of a closed-loop system depends on the positions of the nodal feedback controllers. Hassine in [14] studied a elastic transmission wave/beam systems with a local Kelvin-Voigt damping. He showed that the energy of this coupled system decays polynomially as the time variable goes to infinity if the damping (which is locally distributed) acts through one part of the structure. Li, Han and Xu in [16] obtained polynomial stability for a string-frictionally damped beam system and exponential stability for a frictionally damped string-beam system. Shel in [25] considered transmission problems for a coupling of a string and a beam with at least one of them being thermoelastic and established that the associated semigroup is exponentially stable when the string is thermoelastic and polynomial stables when only the beam is thermoelastic and satisfies certain additional condition. Wang in [26] obtained the strong stability of the semigroup associated to a frictionally damped string-beam system. F. Wang and J. M. Wang [27] established exponential stability for a beam-frictionally damped string system with some feedback at the interface point.

In this article, we study the well-posedness of problem (1.1)-(1.12), higher regularity of the solution, and exponential stability of the energy of the system, depending on the dampings. Using the energy method, we prove the exponential stability of (1.1)-(1.12) if the beams and the string are damped (i.e. if ρ_1, ρ_2 and β are positive), our proof does not need higher regularity of the solution. Moreover, by a frequency domain method from the semigroup theory, we show the exponential stability of (1.1)-(1.12) in the undamped-damped-undamped situation; i.e. $\rho_1 = \rho_2 = 0$ and $\beta > 0$. We use the higher Sobolev space regularity of the solutions, which implies that the transmission conditions, first defined by the weak formulation, hold in the classical sense. For this we follow ideas developed in Section 4 of [8].

This article is organized as follows: In Section 2, we define the basic spaces and operators. In Section 3 we show the generation of a C_0s-emigroup of contractions (and therefore the well-posedness of (1.1)-(1.12)). Exponential stability for the cases damped-damped-damped and undamped-damped-undamped, as well as higher regularity of the solutions, are shown in Section 4.

Let us set some notation. Derivatives with respect to t of a function will be denoted by a "dot" over the name of the function. So $\dot{\phi}$ will denote the derivative of ϕ with respect to t. We also use ψ', ψ'', ψ''', or in general $\psi^{(n)}$ for $n \geq 4$, for the derivatives of ψ with respect to the one-dimensional spatial variable x.

2. Base Spaces

We let A be the matrix differential operator

$$A := \begin{pmatrix} 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ -\frac{\partial^4}{\partial x^4} & 0 & 0 & \rho_1 \frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & \frac{\partial^2}{\partial x^2} & 0 & 0 & -\beta I & 0 \\ 0 & 0 & -\frac{\partial^4}{\partial x^4} & 0 & 0 & \rho_2 \frac{\partial^2}{\partial x^2} \end{pmatrix}.$$

Then, if $z_t := \frac{\partial z}{\partial t}$, $z_{tt} := \frac{\partial^2 z}{\partial t^2}$, and $U := (u, v, w, u_t, v_t, w_t)^\top$, we have that $U_t = (u_t, v_t, w_t, u_{tt}, v_{tt}, w_{tt})^\top$ and (1.1)-(1.3) can be written as

$$U_t = \begin{pmatrix} u_t \\ v_t \\ w_t \\ u_{tt} \\ v_{tt} \\ w_{tt} \end{pmatrix} = \begin{pmatrix} u_t \\ v_t \\ w_t \\ -\frac{\partial^4 u}{\partial x^4} + \rho_1 \frac{\partial^2 u_t}{\partial x^2} \\ \frac{\partial^2 v}{\partial x^2} - \beta v_t \\ -\frac{\partial^4 w}{\partial x^4} + \rho_2 \frac{\partial^2 w_t}{\partial x^2} \end{pmatrix} = AU.$$

Because of the initial condition (1.10)–(1.12),

$$U_t(t) = AU(t) \quad (t > 0), \quad U(0) = U_0, \tag{2.1}$$

where $U_0 := (u_0, v_0, w_0, u_1, v_1, w_1)^\top$.

On the intervals $I_1 = (l_0, l_1)$, $I_2 = (l_1, l_2)$, and $I_3 = (l_2, l_3)$, we define the following spaces

$$H^2_{l_0} := \{u \in H^2(I_1) : u(l_0) = u'(l_0) = 0\},$$
$$H^2_{l_3} := \{w \in H^2(I_3) : w(l_3) = w'(l_3) = 0\},$$

with the inner products

$$\langle u, \tilde{u} \rangle_{H^2_{l_0}} := \langle u'', \tilde{u}'' \rangle_{L^2(I_1)}, \tag{2.2}$$
$$\langle w, \tilde{w} \rangle_{H^2_{l_3}} := \langle w'', \tilde{w}'' \rangle_{L^2(I_3)}. \tag{2.3}$$

From the generalized Poincaré inequality, the induced norms $\|\cdot\|_{H^2_{l_0}}$ and $\|\cdot\|_{H^2_{l_3}}$ are equivalent to the standard norms $\|\cdot\|_{H^2(I_1)}$ and $\|\cdot\|_{H^2(I_3)}$ on $H^2_{l_0}$ and $H^2_{l_3}$, respectively.

Now, let us define the spaces

$$\mathbb{H} := \big\{(u, v, w)^\top \in H^2_{l_0} \times H^1(I_2) \times H^2_{l_3} : u(l_1) = v(l_1) \text{ and } v(l_2) = w(l_2)\big\},$$
$$\mathbb{L} := L^2(I_1) \times L^2(I_2) \times L^2(I_3),$$

equipped with the inner products

$$\langle (u, v, w)^\top, (\tilde{u}, \tilde{v}, \tilde{w})^\top \rangle_{\mathbb{H}} := \langle u, \tilde{u} \rangle_{H^2_{l_0}} + \langle v', \tilde{v}' \rangle_{L^2(I_2)} + \langle w, \tilde{w} \rangle_{H^2_{l_3}}, \tag{2.4}$$
$$\langle (u, v, w)^\top, (\tilde{u}, \tilde{v}, \tilde{w})^\top \rangle_{\mathbb{L}} := \langle u, \tilde{u} \rangle_{L^2(I_1)} + \langle v, \tilde{v} \rangle_{L^2(I_2)} + \langle w, \tilde{w} \rangle_{L^2(I_3)}. \tag{2.5}$$

Again, by Poincaré's inequality, the norm in $\mathbb{H}$, induced by the inner product (2.4), is equivalent to the standard norm in the product space $H^2(I_1) \times H^1(I_2) \times H^2(I_3)$. From the continuity of the trace operator, $\mathbb{H}$ is a closed subspace of $H^2(I_1) \times H^1(I_2) \times H^2(I_3)$ and therefore $\big(\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}}\big)$ is a Hilbert space.

Also, we define the Hilbert space $\mathscr{H} := \mathbb{H} \times \mathbb{L}$, with the inner product

$$\begin{aligned}
\langle U, \tilde{U}\rangle_{\mathscr{H}} &:= \langle (u_1, v_1, w_1)^\top, (\tilde{u}_1, \tilde{v}_1, \tilde{w}_1)^\top\rangle_{\mathbb{H}} + \langle (u_2, v_2, w_2)^\top, (\tilde{u}_2, \tilde{v}_2, \tilde{w}_2)^\top\rangle_{\mathbb{L}} \\
&= \langle u_1, \tilde{u}_1\rangle_{H^2_{l_0}} + \langle v_1', \tilde{v}_1'\rangle_{L^2(I_2)} + \langle w_1, \tilde{w}_1\rangle_{H^2_{l_3}} + \langle u_2, \tilde{u}_2\rangle_{L^2(I_1)} \\
&\quad + \langle v_2, \tilde{v}_2\rangle_{L^2(I_2)} + \langle w_2, \tilde{w}_2\rangle_{L^2(I_3)},
\end{aligned}$$

for all $U = (u_1, v_1, w_1, u_2, v_2, w_2)^\top$, $\tilde{U} = (\tilde{u}_1, \tilde{v}_1, \tilde{w}_1, \tilde{u}_2, \tilde{v}_2, \tilde{w}_2)^\top \in \mathscr{H}$.

Since the functions in the spaces defined above are not regular enough to satisfy the transmission conditions (1.6)-(1.9) in the classic sense, or even in the trace sense, we interpret this transmission condition first in a "weak" sense. For this, we consider $U = (u_1, v_1, w_1, u_2, v_2, w_2)^\top \in \mathscr{H}$ and $\tilde{U} = (\tilde{u}_1, \tilde{v}_1, \tilde{w}_1, \tilde{u}_2, \tilde{v}_2, \tilde{w}_2)^\top \in \mathbb{H}\times\mathbb{H}$ sufficiently smooth, such that the following calculations make sense. Applying integration by parts we obtain

$$\langle AU, \tilde{U}\rangle_{\mathscr{H}} = a(U, \tilde{U}) + b(U, (\tilde{u}_2, \tilde{v}_2, \tilde{w}_2)^\top),$$

where

$$\begin{aligned}
a(U, \tilde{U}) &:= \langle u_2, \tilde{u}_1\rangle_{H^2_{l_0}} + \langle v_2', \tilde{v}_1'\rangle_{L^2(I_2)} + \langle w_2, \tilde{w}_1\rangle_{H^2_{l_3}} - \langle u_1'' \tilde{u}_2''\rangle_{L^2(I_1)} \\
&\quad - \rho_1\langle u_2', \tilde{u}_2'\rangle_{L^2(I_1)} - \beta\langle v_2, \tilde{v}_2\rangle_{L^2(I_2)} - \langle v_1', \tilde{v}_2'\rangle_{L^2(I_2)} - \langle w_1'', \tilde{w}_2''\rangle_{L^2(I_3)} \\
&\quad - \rho_2\langle w_2', \tilde{w}_2'\rangle_{L^2(I_3)}
\end{aligned}$$

and

$$\begin{aligned}
&b(U, (\tilde{u}_2, \tilde{v}_2, \tilde{w}_2)^\top) \\
&:= -u_1^{(3)}(l_1)\tilde{u}_2(l_1) + u_1^{(3)}(l_0)\tilde{u}_2(l_0) + u_1''(l_1)\tilde{u}_2'(l_1) - u_1''(l_0)\tilde{u}_2'(l_0) \\
&\quad + \rho_1 u_2'(l_1)\tilde{u}_2(l_1) - \rho_1 u_2'(l_0)\tilde{u}_2(l_0) + v_1'(l_2)\tilde{v}_2(l_2) - v_1'(l_1)\tilde{v}_2(l_1) \\
&\quad - w_1^{(3)}(l_3)\tilde{w}_2(l_3) + w_1^{(3)}(l_2)\tilde{w}_2(l_2) + w_1''(l_3)\tilde{w}_2'(l_3) - w_1''(l_2)\tilde{w}_2'(l_2) \\
&\quad + \rho_2 w_2'(l_3)\tilde{w}_2(l_3) - \rho_2 w_2'(l_2)\tilde{w}_2(l_2).
\end{aligned}$$

Since $\tilde{U} \in \mathbb{H}\times\mathbb{H}$, we have that $\tilde{u}_2(l_0) = \tilde{u}_2'(l_0) = \tilde{w}_2(l_3) = \tilde{w}_2'(l_3) = 0$, $\tilde{v}_2(l_1) = \tilde{u}(l_1)$, and $\tilde{v}_2(l_2) = \tilde{w}_2(l_2)$. Then, it follows that

$$\begin{aligned}
&b(U, (\tilde{u}_2, \tilde{v}_2, \tilde{w}_2)^\top) \\
&= \big[-u_1^{(3)}(l_1) + \rho_1 u_2'(l_1) - v_1'(l_1)\big]\tilde{u}_2(l_1) + \big[w_1^{(3)}(l_2) - \rho_2 w_2'(l_2) + v_1'(l_2)\big]\tilde{w}_2(l_2) \\
&\quad + u_1''(l_1)\tilde{u}_2'(l_1) - w_1''(l_2)\tilde{w}_2'(l_2) = 0,
\end{aligned}$$

if and only if

$$\begin{gathered}
u_1^{(3)}(l_1) - \rho_1 u_2'(l_1) + v_1'(l_1) = 0, \\
w_1^{(3)}(l_2) - \rho_2 w_2'(l_2) + v_1'(l_2) = 0, \\
u_1''(l_1) = 0, \\
w_1''(l_2) = 0,
\end{gathered}$$

which are the transmission conditions (1.6)-(1.9) that we have in the description of the problem in Section 1 if $u_2 = \partial_t u_1$ and $w_2 = \partial_t w_1$. This motivates the following definition.

Definition 2.1. We say that $U \in \mathscr{H}$ satisfies the transmission conditions (1.6)-(1.9) in the weak sense if

$$\langle AU, \tilde{U} \rangle_{\mathscr{H}} = a(U, \tilde{U}) \quad \text{for all } \tilde{U} \in \mathbb{H} \times \mathbb{H}. \tag{2.6}$$

Now, we define the operator

$$\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}, \quad \mathscr{A}U := AU, \tag{2.7}$$

where

$$\begin{aligned} D(\mathscr{A}) := \big\{ U \in \mathbb{H} \times \mathbb{H} : u_1^{(4)} \in L^2(I_1),\, v_1'', \in L^2(I_2), w_1^{(4)} \in L^2(I_3), \\ U \text{ satisfies (1.6)-(1.9) in the weak sense} \big\}. \end{aligned}$$

In this way, problem (1.1)-(1.12) can be written in abstract form as the Cauchy problem

$$\frac{dU}{dt}(t) = \mathscr{A}U(t) \quad (t > 0), \quad U(0) = U_0. \tag{2.8}$$

3. Well posedness

Now, we show that problem (2.8) is well posed, which means, that for every $U \in D(\mathscr{A})$, (2.8) has one and only one classical solution which depends continuously on U_0. For this, we prove that the operator $\mathscr{A}$ defined in (2.7) generates a C_0-semigroup of contractions on $\mathscr{H}$. To achieve that, we use the Lumer-Phillips theorem.

Proposition 3.1. *The following assertions hold:*

(a) *$\mathscr{A}$ is dissipative.*
(b) *$I - \mathscr{A}$ is surjective.*
(c) *$D(\mathscr{A})$ is dense in $\mathscr{H}$.*

Proof. Let $U \in D(\mathscr{A})$. From (2.6) we have

$$\langle \mathscr{A}U, U \rangle_{\mathscr{H}} = -\rho_1 \|u_2'\|^2_{L^2(I_1)} - \beta \|v_2\|^2_{L^2(I_2)} - \rho_2 \|w_2'\|^2_{L^2(I_3)} \le 0, \tag{3.1}$$

from which (a) follows. For the surjectivity we will use the Lax-Milgram theorem. Let $F = (f_1, g_1, h_1, f_2, g_2, h_2)^\top \in \mathscr{H}$. We need to show that there exists a $U = (u_1, v_1, w_1, u_2, v_2, w_2)^\top \in D(\mathscr{A})$ such that $(I - \mathscr{A})U = F$, i.e.

$$u_1 - u_2 = f_1 \in H^2_{l_0}, \tag{3.2}$$
$$v_1 - v_2 = g_1 \in H^1(I_2), \tag{3.3}$$
$$w_1 - w_2 = h_1 \in H^2_{l_3}, \tag{3.4}$$
$$u_1^{(4)} + u_2 - \rho_1 u_2'' = f_2 \in L^2(I_1), \tag{3.5}$$
$$-v_1'' + (1+\beta) v_2 = g_2 \in L^2(I_2), \tag{3.6}$$
$$w_1^{(4)} + w_2 - \rho_2 w_2'' = h_2 \in L^2(I_3). \tag{3.7}$$

Plugging (3.2)–(3.4) in (3.5)–(3.7) respectively, we have to solve

$$u_1^{(4)} + u_1 - \rho_1 u_1'' = f_1 + f_2 - \rho_1 f_1'', \tag{3.8}$$
$$-v_1'' + (1+\beta) v_1 = g_2 + (1+\beta) g_1, \tag{3.9}$$
$$w_1^{(4)} + w_1 - \rho_2 w_1'' = h_1 + h_2 - \rho_2 h_1''. \tag{3.10}$$

We define the sesquilinear form $\mathscr{B} : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ by

$$\begin{aligned}\mathscr{B}(Y, \Phi) := \langle y_1, \phi_1\rangle_{H^2_{l_0}} + \rho_1\langle y_1', \phi_1'\rangle_{L^2(I_1)} + \langle y_1, \phi_1\rangle_{L^2(I_1)} + \langle y_2', \phi_2'\rangle_{L^2(I_2)} \\ + (1+\beta)\langle y_2, \phi_2\rangle_{L^2(I_2)} + \langle y_3, \phi_3\rangle_{H^2_{l_3}} + \rho_2\langle y_3', \phi_3'\rangle_{L^2(I_3)} + \langle y_3, \phi_3\rangle_{L^2(I_3)},\end{aligned}$$

for $Y := (y_1, y_2, y_3)^\top$, $\Phi := (\phi_1, \phi_2, \phi_3)^\top \in \mathbb{H}$. It is easy to see that $\mathscr{B} : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is continuous and coercive.

Now, for $(f_1, g_1, h_1, f_2, g_2, h_2)^\top \in \mathscr{H}$, we define $\Lambda : \mathbb{H} \to \mathbb{C}$ by

$$\begin{aligned}\Lambda(\Phi) := \langle f_1 + f_2, \phi_1\rangle_{L^2(I_1)} + \rho_1\langle f_1', \phi_1'\rangle_{L^2(I_1)} + \langle g_2 + (1+\beta)g_1, \phi_2\rangle_{L^2(I_2)} \\ + \langle h_1 + h_2, \phi_3\rangle_{L^2(I_3)} + \rho_2\langle h_1', \phi_3'\rangle_{L^2(I_3)},\end{aligned}$$

for all $\Phi \in \mathbb{H}$. It is also easy to see that $\Lambda : \mathbb{H} \to \mathbb{C}$ is an antilinear continuous functional. By the Lax-Milgram theorem exists a unique $Y = (y_1, y_2, y_3)^\top \in \mathbb{H}$ such that

$$\mathscr{B}(Y, \Phi) = \Lambda(\Phi) \quad \text{for all } \Phi \in \mathbb{H}. \tag{3.11}$$

In particular, if $\phi_1 \in C_c^\infty(I_1)$ and $\phi_2 = \phi_3 = 0$, we have in (3.11) that

$$\begin{aligned}&\langle y_1, \phi_1\rangle_{H^2_{l_0}} + \rho_1\langle y_1', \phi_1'\rangle_{L^2(I_1)} + \langle y_1, \phi_1\rangle_{L^2(I_1)} \\ &= \langle f_1 + f_2, \phi_1\rangle_{L^2(I_1)} + \rho_1\langle f_1', \phi_1'\rangle_{L^2(I_1)},\end{aligned}$$

which can be written, in distributional sense, as

$$\langle y_1^{(4)} - \rho_1 y_1'' + y_1, \phi_1\rangle = \langle f_1 + f_2 - \rho_1 f_1'', \phi_1\rangle,$$

for all $\phi_1 \in C_c^\infty(I_1)$. This implies that

$$y_1^{(4)} - \rho_1 y_1'' + y_1 = f_1 + f_2 - \rho_1 f_1'' \tag{3.12}$$

in sense of distributions. Since $y_1, y_1'', f_1, f_2, f_1'' \in L^2(I_1)$, we conclude that $y_1^{(4)} \in L^2(I_1)$. Similarly, taking $\phi_1 = \phi_2 = 0$ and $\phi_3 \in C_c^\infty(I_3)$, and then $\phi_1 = \phi_3 = 0$ and $\phi_2 \in C_c^\infty(I_2)$, we obtain

$$y_3^{(4)} - \rho_2 y_3'' + y_3 = h_1 + h_2 - \rho_2 h_1'' \tag{3.13}$$

and

$$-y_2'' + (1+\beta)y_2 = g_2 + (1+\beta)g_1 \tag{3.14}$$

in distributional sense, respectively. These equations imply that $y_3^{(4)} \in L^2(I_3)$ and $y_2'' \in L^2(I_2)$.

Let $(u_1, v_1, w_1) := (y_1, y_2, y_3)$ and $(u_2, v_2, w_2) := (u_1 - f_1, v_1 - g_1, w_1 - h_1)$. Hence $U := (u_1, v_1, w_1, u_2, v_2, w_2)^\top \in \mathbb{H} \times \mathbb{H}$, $u_1^{(4)} \in L^2(I_1)$, $v_1'' \in L^2(I_2)$ and $w_1^{(4)} \in L^2(I_3)$. From (3.11), for $\tilde{U} = (\tilde{u}_1, \tilde{v}_1, \tilde{w}_1, \tilde{u}_2, \tilde{v}_2, \tilde{w}_2)^\top \in \mathbb{H} \times \mathbb{H}$ arbitrary we have

$$\begin{aligned}&\langle u_1, \tilde{u}_2\rangle_{H^2_{l_0}} + \rho_1\langle u_1', \tilde{u}_2'\rangle_{L^2(I_1)} + \langle u_1, \tilde{u}_2\rangle_{L^2(I_1)} + \langle v_1', \tilde{v}_2'\rangle_{L^2(I_2)} \\ &+ (1+\beta)\langle v_1, \tilde{v}_2\rangle_{L^2(I_2)} + \langle w_1, \tilde{w}_2\rangle_{H^2_{l_3}} + \rho_2\langle w_1', \tilde{w}_2'\rangle_{L^2(I_3)} + \langle w_1, \tilde{w}_2\rangle_{L^2(I_3)} \\ &= \langle f_1 + f_2, \tilde{u}_2\rangle_{L^2(I_1)} + \rho_1\langle f_1', \tilde{u}_2'\rangle_{L^2(I_1)} + \langle g_2 + (1+\beta)g_1, \tilde{v}_2\rangle_{L^2(I_2)} \\ &\quad + \langle h_1 + h_2, \tilde{w}_2\rangle_{L^2(I_3)} + \rho_2\langle h_1', \tilde{w}_2'\rangle_{L^2(I_3)},\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\langle u_1-(f_1+f_2),\tilde u_2\rangle_{L^2(I_1)}+\langle v_1-(g_1+g_2),\tilde v_2\rangle_{L^2(I_2)}\\
&+\langle w_1-(h_1+h_2),\tilde w_2\rangle_{L^2(I_3)}\\
&=-\langle u_1,\tilde u_2\rangle_{H^2_{l_0}}-\rho_1\langle (u_1-f_1)',\tilde u_2'\rangle_{L^2(I_1)}-\langle v_1',\tilde v_2'\rangle_{L^2(I_2)}\\
&\quad-\beta\langle v_1-g_1,\tilde v_2'\rangle_{L^2(I_2)}-\langle w_1,\tilde w_2\rangle_{H^2_{l_3}}-\rho_2\langle (w_1-h_1)',\tilde w_2'\rangle_{L^2(I_3)}.
\end{aligned}\tag{3.15}$$

From (3.12)–(3.14) and $(u_2,v_2,w_2)=(u_1-f_1,v_1-g_1,w_1-h_1)$ it follows that

$$\begin{gathered}
\langle -u_1^{(4)}+\rho_1u_2'',\tilde u_2\rangle_{L^2(I_1)}=\langle u_1-(f_1+f_2),\tilde u_2\rangle_{L^2(I_1)},\\
\langle v_1''-\beta v_2,\tilde v_2\rangle_{L^2(I_2)}=\langle v_1-(g_1+g_2),\tilde v_2\rangle_{L^2(I_2)},\\
\langle -w_1^{(4)}+\rho_2w_2'',\tilde w_2\rangle_{L^2(I_3)}=\langle w_1-(h_1+h_2),\tilde w_2\rangle_{L^2(I_3)}.
\end{gathered}$$

Adding the three equations above and using (3.15) we obtain

$$\begin{aligned}
&\langle -u_1^{(4)}+\rho_1u_2'',\tilde u_2\rangle_{L^2(I_1)}+\langle v_1''-\beta v_2,\tilde v_2\rangle_{L^2(I_2)}+\langle -w_1^{(4)}+\rho_2w_2'',\tilde w_2\rangle_{L^2(I_3)}\\
&=\langle u_1-(f_1+f_2),\tilde u_2\rangle_{L^2(I_1)}+\langle v_1-(g_1+g_2),\tilde v_2\rangle_{L^2(I_2)}\\
&\quad+\langle w_1-(h_1+h_2),\tilde w_2\rangle_{L^2(I_3)}\\
&=-\langle u_1,\tilde u_2\rangle_{H^2_{l_0}}-\rho_1\langle (u_1-f_1)',\tilde u_2'\rangle_{L^2(I_1)}-\langle v_1',\tilde v_2'\rangle_{L^2(I_2)}-\beta\langle (v_1-g_1),\tilde v_2\rangle_{L^2(I_2)}\\
&\quad-\langle w_1,\tilde w_2\rangle_{H^2_{l_3}}-\rho_2\langle (w_1-h_1)',\tilde w_2'\rangle_{L^2(I_3)}\\
&=-\langle u_1,\tilde u_2\rangle_{H^2_{l_0}}-\rho_1\langle u_2',\tilde u_2'\rangle_{L^2(I_1)}-\langle v_1',\tilde v_2'\rangle_{L^2(I_2)}-\beta\langle v_2,\tilde v_2\rangle_{L^2(I_2)}\\
&\quad-\langle w_1,\tilde w_2\rangle_{H^2_{l_3}}-\rho_2\langle w_2',\tilde w_2'\rangle_{L^2(I_3)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle AU,\tilde U\rangle_{\mathscr H}&=\langle u_2,\tilde u_1\rangle_{H^2_{l_0}}+\langle v_2',\tilde v_1'\rangle_{L^2(I_2)}+\langle w_2,\tilde w_1\rangle_{H^2_{l_3}}+\langle -u_1^{(4)}+\rho_1u_2'',\tilde u_2\rangle_{L^2(I_1)}\\
&\quad+\langle v_1''-\beta v_2,\tilde v_2\rangle_{L^2(I_2)}+\langle -w_1^{(4)}+\rho_2w_2'',\tilde w_2\rangle_{L^2(I_3)}\\
&=\langle u_2,\tilde u_1\rangle_{H^2_{l_0}}+\langle v_2',\tilde v_1'\rangle_{L^2(I_2)}+\langle w_2,\tilde w_1\rangle_{H^2_{l_3}}-\langle u_1,\tilde u_2\rangle_{H^2_{l_0}}\\
&\quad-\rho_1\langle u_2',\tilde u_2'\rangle_{L^2(I_1)}-\langle v_1',\tilde v_2'\rangle_{L^2(I_2)}-\beta\langle v_2,\tilde v_2\rangle_{L^2(I_2)}-\langle w_1,\tilde w_2\rangle_{H^2_{l_3}}\\
&\quad-\rho_2\langle w_2',\tilde w_2'\rangle_{L^2(I_3)}.
\end{aligned}$$

Hence U satisfies the transmission conditions (1.6)-(1.9) in the weak sense. From this we conclude that $U\in D(\mathscr A)$, i.e. (b) holds. Since $\mathscr H$ is a Hilbert space, from (a) and (b) it follows that $D(\mathscr A)$ is dense in $\mathscr H$, see [20, Thm. 4.6, Chapter 1]. □

Theorem 3.2. *The operator $\mathscr A$ is the generator of a C_0-semigroup $(S(t))_{t\ge 0}$ of contractions on the Hilbert space $\mathscr H$. In consequence, for each $U_0\in D(\mathscr A)$ the Cauchy problem* (2.8) *has a unique classical solution $U\in C^1([0,\infty),\mathscr H)$ which depends continuously on the initial data.*

Proof. From Proposition 3.1 and the Lumer-Phillips theorem we have that $\mathscr A$ is the generator of a contraction C_0-semigroup over $\mathscr H$. Hence, for each $U_0\in D(\mathscr A)$, the Cauchy problem (2.8) has a unique classical solution $U\in C^1([0,\infty),\mathscr H)$ which depends continuously on the initial data, i.e. the problem is well posed. □

4. Exponential stability

In this section, we prove the main results of this article. First we prove the exponential stability of the semigroup $(S(t))_{t\geq 0}$ generated by $\mathscr{A}$ if we have damping in the three subdomains, i.e., if ρ_1, ρ_2, and β are positive. We recall that we use the convention $z(t)'$ to indicate the derivative of the function $z(t) := z(\cdot, t)$ with respect to the spatial variable x, where $z = z(x,t)$, whereas that the derivative of $z(t)$ with respect to t will be denoted by $\dot{z}(t)$ (see last paragraph in the introduction).

For $U_0 \in D(\mathscr{A})$, we have from Theorem 3.2 that $U(t) := S(t)U_0$ $(t \geq 0)$ is the classical solution of (2.8). In this case the energy $E(t)$ of the system is defined by

$$\begin{aligned} E(t) &:= \frac{1}{2}\|U(t)\|^2_{\mathscr{H}} \\ &= \frac{1}{2}\Big(\|u_1(t)\|^2_{H^2_{l_0}} + \|v_1(t)'\|^2_{L^2(I_2)} + \|w_1(t)\|^2_{H^2_{l_3}} + \|u_2(t)\|^2_{L^2(I_1)} \\ &\quad + \|v_2(t)\|^2_{L^2(I_2)} + \|w_2(t)\|^2_{L^2(I_3)}\Big). \end{aligned}$$

Note that

$$\begin{aligned} \frac{d}{dt}E(t) &= \operatorname{Re}\langle \mathscr{A}U(t), U(t)\rangle_{\mathscr{H}} \\ &= -\rho_1\|u_2(t)'\|^2_{L^2(I_1)} - \beta\|v_2(t)\|^2_{L^2(I_2)} - \rho_2\|w_2(t)'\|^2_{L^2(I_3)}, \end{aligned} \tag{4.1}$$

which shows that the system is dissipative if at least one damping is active ($\rho_1 + \rho_2 + \beta > 0$) and conservative if there is no damping at all ($\rho_1 = \rho_2 = \beta = 0$). Now, we prove the first main result of this article.

Theorem 4.1. *Let $\rho_1 > 0$, $\rho_2 > 0$ and $\beta > 0$. Then, the semigroup $(S(t))_{t\geq 0}$ is exponentially stable, i.e., for any $U_0 \in D(\mathscr{A})$ and $U(t) := S(t)U_0$ $(t \geq 0)$ we have*

$$E(t) \leq Ce^{-\alpha t}E(0)$$

with positive constants C and α.

Proof. For $U_0 \in D(\mathscr{A})$ and $t \geq 0$, let

$$\begin{aligned} U(t) &:= (u_1(t), v_1(t), w_1(t), u_2(t), v_2(t), w_2(t))^\top := S(t)U_0, \\ F(t) &:= \langle u_1(t), u_2(t)\rangle_{L^2(I_1)} + \langle v_1(t), v_2(t)\rangle_{L^2(I_2)} + \langle w_1(t), w_2(t)\rangle_{L^2(I_3)}. \end{aligned}$$

Then, using Cauchy-Schwarz and Young's inequalities we have

$$\begin{aligned} |F(t)| &\leq \frac{1}{2}\Big(\|u_1(t)\|^2_{L^2(I_1)} + \|u_2(t)\|^2_{L^2(I_1)} + \|v_1(t)\|^2_{L^2(I_2)} + \|v_2(t)\|^2_{L^2(I_2)} \\ &\quad + \|w_1(t)\|^2_{L^2(I_3)} + \|w_2(t)\|^2_{L^2(I_3)}\Big) \\ &\leq \frac{1}{2}\|U(t)\|^2_X, \end{aligned}$$

with $X := H^2(I_1) \times H^1(I_2) \times H^2(I_3) \times \mathbb{L}$. Because of the equivalence between the standard norm in X and the norm $\|\cdot\|_{\mathscr{H}}$, there exists $c_1 > 0$ such that

$$|F(t)| \leq c_1 E(t).$$

because $\frac{d}{dt}U(t) = \mathscr{A}U(t)$, i.e.,

$$\begin{pmatrix} \dot{u}_1(t) \\ \dot{v}_1(t) \\ \dot{w}_1(t) \\ \dot{u}_2(t) \\ \dot{v}_2(t) \\ \dot{w}_2(t) \end{pmatrix} = \begin{pmatrix} u_2(t) \\ v_2(t) \\ w_2(t) \\ -u_1(t)^{(4)} + \rho_1 u_2(t)'' \\ v_1(t)'' - \beta v_2(t) \\ -w_1(t)^{(4)} + \rho_2 w_2(t)'' \end{pmatrix},$$

we have

$$\begin{aligned} \frac{d}{dt}F(t) &= \langle \dot{u}_1(t), u_2(t)\rangle_{L^2(I_1)} + \langle u_1(t), \dot{u}_2(t)\rangle_{L^2(I_1)} + \langle \dot{v}_1(t), v_2(t)\rangle_{L^2(I_2)} \\ &\quad + \langle v_1(t), \dot{v}_2(t)\rangle_{L^2(I_2)} + \langle \dot{w}_1(t), w_2(t)\rangle_{L^2(I_3)} + \langle w_1(t), \dot{w}_2(t)\rangle_{L^2(I_3)} \\ &= \|u_2(t)\|^2_{L^2(I_1)} + \|v_2(t)\|^2_{L^2(I_2)} + \|w_2(t)\|^2_{L^2(I_3)} + \langle \Phi(t), \mathscr{A}U(t)\rangle_{\mathscr{H}} \\ &= \|u_2(t)\|^2_{L^2(I_1)} + \|v_2(t)\|^2_{L^2(I_2)} + \|w_2(t)\|^2_{L^2(I_3)} - \|u_1(t)\|^2_{H^2_{l_0}} \\ &\quad - \rho_1\langle u_1(t)', u_2(t)'\rangle_{L^2(I_1)} - \|v_1(t)'\|^2_{L^2(I_2)} - \beta\langle v_1(t), v_2(t)\rangle_{L^2(I_2)} \\ &\quad - \|w_1(t)\|^2_{H^2_{l_3}} - \rho_2\langle w_1(t)', w_2(t)'\rangle_{L^2(I_3)} \\ &= \|(u_2(t), v_2(t), w_2(t))\|^2_{\mathbb{L}} - \|(u_1(t), v_1(t), w_1(t))\|^2_{\mathbb{H}} \\ &\quad - \langle (\rho_1 u_1(t)', \beta v_1(t), \rho_2 w_1(t)'), (u_2(t)', v_2(t), w_2(t)')\rangle_{\mathbb{L}}. \end{aligned}$$

Where the weak transmission conditions were used with

$$\Phi(t) := (0,0,0,u_1(t), v_1(t), w_1(t))^\top.$$

Let $\delta > 0$. By Young's inequality and Poincaré's inequality, there exists $C_\delta > 0$ and $c_2 > 0$ such that

$$\begin{aligned} &- \langle (\rho_1 u_1(t)', \beta v_1(t), \rho_2 w_1(t)'), (u_2(t)', v_2(t), w_2(t)')\rangle_{\mathbb{L}} \\ &\le \delta\|(\rho_1 u_1(t)', \beta v_1(t), \rho_2 w_1(t)')\|^2_{\mathbb{L}} + C_\delta\|(u_2(t)', v_2(t), w_2(t)')\|^2_{\mathbb{L}} \\ &\le c_2\delta\|(u_1(t), v_1(t), w_1(t))\|^2_{\mathbb{H}} + C_\delta\|(u_2(t)', v_2(t), w_2(t)')\|^2_{\mathbb{L}}. \end{aligned}$$

Since $U(t) \in D(\mathscr{A})$ we have $u_2(t) \in H^2_{l_0}$ and $w_2(t) \in H^2_{l_3}$ which implies $u_2(t)(l_0) = 0$ and $w_2(t)(l_3) = 0$. Therefore from Poincaré's inequality we have

$$\begin{gathered} \|u_2(t)\|_{L^2(I_1)} \le \text{const.}\|u_2(t)'\|_{L^2(I_1)}, \\ \|w_2(t)\|_{L^2(I_3)} \le \text{const.}\|w_2(t)'\|_{L^2(I_3)}. \end{gathered}$$

From the three inequalities above, taking δ small enough such that $c_2\delta \le 1/2$, we obtain

$$\begin{aligned} \frac{d}{dt}F(t) &\le \|(u_2(t), v_2(t), w_2(t))\|^2_{\mathbb{L}} - \frac{1}{2}\|(u_1(t), v_1(t), w_1(t))\|^2_{\mathbb{H}} \\ &\quad + C_\delta\|(u_2(t)', v_2(t), w_2(t)')\|^2_{\mathbb{L}} \\ &\le c_3\|(u_2(t)', v_2(t), w_2(t)')\|^2_{\mathbb{L}} - \frac{1}{2}\|(u_1(t), v_1(t), w_1(t))\|^2_{\mathbb{H}} \end{aligned} \tag{4.2}$$

for a positive constant c_3. Now, let $L(t) := c_4E(t) + F(t)$ with c_4 a positive constant. Then, for c_4 large enough such that $2c_1 \le c_4$ and $-c_4\max\{\rho_1, \beta, \rho_2\} + c_3 \le -1/2$, it follows from (4.1), (4.2) and Poincaré's inequality applied to u_2 and w_2 that there are constants c_5 and c_6 such that

$$\begin{aligned}\frac{d}{dt}L(t) &\le -\frac{1}{2}\|(u_2(t)', v_2(t), w_2(t)')\|_{\mathbb{L}}^2 - \frac{1}{2}\|(u_1(t), v_1(t), w_1(t))\|_{\mathbb{H}}^2\\ &\le -\frac{c_5}{2}\|(u_2(t), v_2(t), w_2(t))\|_{\mathbb{L}}^2 - \frac{1}{2}\|(u_1(t), v_1(t), w_1(t))\|_{\mathbb{H}}^2\\ &\le -\min\{c_5, 1\}\frac{1}{2}\|U(t)\|_{\mathbb{H}\times\mathbb{L}}^2\\ &= -c_6 E(t).\end{aligned}\tag{4.3}$$

Since $|F(t)| \le c_1 E(t) \le c_4 E(t)/2$, we have

$$\frac{c_4}{2}E(t) \le L(t) \le \frac{3c_4}{2}E(t).$$

Therefore, (4.3) implies that $\frac{d}{dt}L(t) \le -\alpha L(t)$ for some positive constant α. By Gronwall's lemma, $L(t) \le Ce^{-\alpha t}L(0)$, which implies

$$E(t) \le \frac{2}{c_4}L(t) \le \frac{2}{c_4}Ce^{-\alpha t}L(0) \le 3Ce^{-\alpha t}E(0).$$

□

Now, we consider the case in which both beams are undamped. For this we need to show some regularity results.

Lemma 4.2. *Let I an open interval in the real line. For each $g \in L^2(I)$ there exists a unique $v \in H_0^1(I)$ such that*

$$\int_I v'\,\varphi' = -\int_I g\,\varphi \quad \forall\,\varphi \in H_0^1(I).\tag{4.4}$$

Furthermore, $v \in H^2(I)$.

Proof. It is easy to see that the bilinear form $(v,\varphi) \mapsto \int_I v'\,\varphi'$ is continuous in $H_0^1(I) \times H_0^1(I)$ and that, due to Poincaré's inequality, it is also coercive in $H_0^1(I)$. Then, Lax-Milgram theorem give the existence and uniqueness of the solution v of (4.4), since $\varphi \mapsto \int_I g\,\varphi$ is a continuous linear functional on $H_0^1(I)$, whenever $g \in L^2(I)$. Now, (4.4) implies that $(v')' = g$ in distributional sense. Since $g \in L^2(I)$ we have that $v' \in H^1(I)$ and therefore $v \in H^2(I)$. □

We will use the following lemma whose proof is similar to that of [8, Corollary 4.3].

Lemma 4.3. *Let $a < b$, $f \in L^2((a,b))$ and $z \in \mathbb{C}$. For sufficiently large $\lambda > 0$ there exists a unique $u \in H^4((a,b))$ such that*

$$\begin{gathered}u^{(4)} + \lambda u = f \quad \text{in } (a,b),\\ u(a) = u'(a) = 0,\\ u''(b) = 0,\\ u'''(b) = z.\end{gathered}$$

Theorem 4.4. *Let $U = (u_1, v_1, w_1, u_2, v_2, w_2)^\top \in D(\mathscr{A})$. Then $u_1 \in H^4(I_1)$, $v_1 \in H^2(I_2)$ and $w_1 \in H^4(I_3)$. In particular, the transmission conditions* (1.6)–(1.9) *hold in the classical sense.*

Proof. If $F = (f_1, g_1, h_1, f_2, g_2, h_2)^\top := \mathscr{A}U$, then

$$u_2 = f_1 \in H^2_{l_0}, \tag{4.5}$$

$$v_2 = g_1 \in H^1(I_2), \tag{4.6}$$

$$w_2 = h_1 \in H^2_{l_3}, \tag{4.7}$$

$$-u_1^{(4)} + \rho_1 u_2'' = f_2 \in L^2(I_1), \tag{4.8}$$

$$v_1'' - \beta v_2 = g_2 \in L^2(I_2), \tag{4.9}$$

$$-w_1^{(4)} + \rho_2 w_2'' = h_2 \in L^2(I_3). \tag{4.10}$$

From (4.6) and (4.9) we have

$$v_1'' = \beta g_1 + g_2. \tag{4.11}$$

Now, let v_0 be a smooth function on $\overline{I_2}$ such that $v_0(l_1) = u_1(l_1)$ and $v_0(l_2) = w_1(l_2)$ (we can take v_0 as an affine function for example) and set $\hat{v} := v_1 - v_0$. Then $\hat{v} \in H^1_0(I_2)$ and, by (4.11), we have

$$\hat{v}'' = \beta g_1 + g_2 - v_0''$$

in the distributional sense. Then $\hat{v}$ satisfies

$$\int_{I_2} \hat{v}'\,\phi' = -\int_{I_2} (\beta g_1 + g_2 - v_0'')\phi \quad \forall\, \phi \in H^1_0(I_2).$$

By Lemma 4.2 and $\beta g_1 + g_2 - v_0'' \in L^2(I_2)$, $\hat{v} \in H^2(I_2)$. Therefore, $v_1 = v_0 + \hat{v} \in H^2(I_2)$ and, in particular, Sobolev embedding theorem implies that $v_1 \in C^1(\overline{I_2})$.

On the other hand, we consider $\Phi := (0, 0, 0, \phi, \psi, 0)^\top$ with $\phi \in H^2_{l_0}$ arbitrary and $\psi \in H^1(I_2)$ such that $\psi(l_1) = \phi(l_1)$ and $\psi(l_2) = 0$. From the definition of $D(\mathscr{A})$ we obtain

$$\begin{aligned}\big\langle \mathscr{A}U, \Phi\big\rangle_{\mathscr{H}} &= \langle -u_1^{(4)} + \rho_1 u_2'', \phi\rangle_{L^2(I_1)} + \langle v_1'' - \beta v_2, \psi\rangle_{L^2(I_2)} \\ &= -\langle u_1'', \phi''\rangle_{L^2(I_1)} - \rho_1\langle u_2', \phi'\rangle_{L^2(I_1)} - \beta\langle v_2, \psi\rangle_{L^2(I_2)} - \langle v_1', \psi'\rangle_{L^2(I_2)}.\end{aligned}$$

Also, integrating by parts the terms $\langle u_2', \phi'\rangle_{L^2(I_1)}$ and $\langle v_1', \psi'\rangle_{L^2(I_2)}$ in the above equality, we have

$$\langle u_1^{(4)}, \phi\rangle_{L^2(I_1)} = \langle u_1'', \phi''\rangle_{L^2(I_1)} + \big[\rho_1 f_1'(l_1) - v_1'(l_1)\big]\overline{\phi}(l_1). \tag{4.12}$$

By Lemma 4.3, for sufficiently large $\lambda > 0$, there exists a unique $\tilde{u}_1 \in H^4(I_1)$ such that

$$\begin{gathered}\lambda\tilde{u}_1 + \tilde{u}_1^{(4)} = \lambda u_1 + \rho f_1'' - f_2 \quad \text{in } I_1, \\ \tilde{u}_1(l_0) = \tilde{u}_1'(l_0) = 0, \\ \tilde{u}_1''(l_1) = 0, \\ \tilde{u}_1'''(l_1) = \rho_1 f_1'(l_1) - v_1'(l_1).\end{gathered}$$

Then, for all $\phi \in H^2_{l_0}$, integration by parts twice and the boundary conditions above yield

$$\begin{aligned}&\langle \lambda\tilde{u}_1 + \tilde{u}_1^{(4)}, \phi\rangle_{L^2(I_1)} \\ &= \lambda\langle \tilde{u}_1, \phi\rangle_{L^2(I_1)} + \langle \tilde{u}_1'', \phi''\rangle_{L^2(I_1)} + \big[\rho_1 f_1'(l_1) - v_1'(l_1)\big]\overline{\phi}(l_1).\end{aligned} \tag{4.13}$$

For the same λ, adding up the term $\langle \lambda u_1, \phi\rangle_{L^2(I_1)}$ in (4.12) we obtain

$$\begin{aligned} &\langle \lambda u_1 + u_1^{(4)}, \phi\rangle_{L^2(I_1)} \\ &= \lambda\langle u_1, \phi\rangle_{L^2(I_1)} + \langle u_1'', \phi''\rangle_{L^2(I_1)} + \big[\rho_1 f_1'(l_1) - v_1'(l_1)\big]\overline{\phi}(l_1). \end{aligned} \tag{4.14}$$

From (4.6) and (4.8) we have

$$\lambda u_1 + u_1^{(4)} = \lambda u_1 + \rho_1 f_1'' - f_2 = \lambda \tilde{u}_1 + \tilde{u}_1^4.$$

Subtracting (4.14) from (4.13), with $\hat{u}_1 := \tilde{u}_1 - u_1$, we obtain

$$0 = \lambda\langle \hat{u}_1, \phi\rangle_{L^2(I_1)} + \langle \hat{u}_1'', \phi''\rangle_{L^2(I_1)}$$

for all $\phi \in H^2_{l_0}$. Since $\hat{u}_1 \in H^2_{l_0}$ we can set $\phi = \hat{u}_1$ in the last equality and we obtain $\hat{u}_1 = 0$, which implies $u_1 = \tilde{u}_1 \in H^4(I_1)$. In particular, the Sobolev embedding theorem guarantees that $u_1 \in C^3(\overline{I_1})$ and therefore the transmission conditions hold in the classical sense. The proof of $w_1 \in H^4(I_3)$ is similar, and therefore $w_1 \in C^3(\overline{I_3})$. $\square$

Now, we use the following frequency domain result, which gives us a necessary and sufficient condition for the exponential stability of a C_0-semigroup of contractions. For its proof see [11, 12, 21].

Proposition 4.5. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions in a Hilbert space H, generated by an operator A. Then the semigroup is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(A) \quad \text{and} \quad \|(i\lambda I - A)^{-1}\|_{\mathscr{L}(H)} \leq C \quad \forall \lambda \in \mathbb{R}. \tag{4.15}$$

The second main result of this article is the following.

Theorem 4.6. *If $\rho_1 = \rho_2 = 0$ and $\beta > 0$, then the semigroup $(S(t))_{t\geq 0}$ generated by $\mathscr{A}$ is exponentially stable.*

Proof. By Proposition 4.5 it is sufficient to show that $\mathscr{A}$ satisfies (4.15). First, we will show that $0 \in \rho(\mathscr{A})$. Let $F = (f_1, g_1, h_1, f_2, g_2, h_2)^\top \in \mathscr{H}$, then we have to find a $U = (u_1, v_1, w_1, u_2, v_2, w_2)^\top \in D(\mathscr{A})$ such that $-\mathscr{A}U = F$, which is equivalent to equations (4.5)–(4.10) with $-F$ replacing F. Then, plugging the first three equations in the last three, we obtain

$$u_1^{(4)} = f_2, \tag{4.16}$$

$$-v_1'' = g_2 + \beta g_1, \tag{4.17}$$

$$w_1^{(4)} = h_2. \tag{4.18}$$

Now, we define the sesquilinear form $\mathscr{B}_0 : \mathbb{H}\times\mathbb{H} \to \mathbb{C}$ by

$$\mathscr{B}_0(Y, \Phi) := \langle y_1, \phi_1\rangle_{H^2_{l_0}} + \langle y_2', \phi_2'\rangle_{L^2(I_2)} + \langle y_3, \phi_3\rangle_{H^2_{l_3}},$$

for $Y := (y_1, y_2, y_3)^\top$, $\Phi := (\phi_1, \phi_2, \phi_3)^\top \in \mathbb{H}$, and the antilinear functional $\Lambda : \mathbb{H} \to \mathbb{C}$ by

$$\Lambda(\Phi) := \langle f_2, \phi_1\rangle_{L^2(I_1)} + \langle g_2 + \beta g_1, \phi_2\rangle_{L^2(I_2)} + \langle h_2, \phi_3\rangle_{L^2(I_3)},$$

for all $\Phi \in \mathbb{H}$. Here antilinear means linear up to conjugated scalars, i.e. $\Lambda(\Phi_1 + \Phi_2) = \Lambda(\Phi_1) + \Lambda(\Phi_2)$ and $\Lambda(\alpha\Phi) = \overline{\alpha}\Lambda(\Phi)$. It is easy to see that $\mathscr{B}_0 : \mathbb{H}\times\mathbb{H} \to \mathbb{C}$

is continuous and coercive, and that $\Lambda : \mathbb{H} \to \mathbb{C}$ is continuous. Then, by the Lax-Milgram theorem, there exists a unique $Y := (y_1, y_2, y_3)^\top \in \mathbb{H}$ such that

$$\mathscr{B}_0(Y, \Phi) = \Lambda(\Phi) \quad \text{for all } \Phi \in \mathbb{H}. \tag{4.19}$$

As in the proof of Proposition 3.1, we obtain that $U := (y_1, y_2, y_3, -f_1, -g_1, -h_1)^\top \in D(\mathscr{A})$ and satisfies $-\mathscr{A}U = F$. On the other hand, if $\tilde{U} := (\tilde{u}_1, \tilde{v}_1, \tilde{w}_1, \tilde{u}_2, \tilde{v}_2, \tilde{w}_2)^\top \in D(\mathscr{A})$ solves $-\mathscr{A}\tilde{U} = F$, then $\mathscr{B}_0((\tilde{u}_1, \tilde{v}_1, \tilde{w}_1)^\top, \Phi) = \Lambda(\Phi)$ holds for all $\Phi \in \mathbb{H}$ by the definition of $D(\mathscr{A})$ and the weak transmission conditions. Therefore $U = \tilde{U}$ and $\mathscr{A}$ is a bijection. Since $\mathscr{A}$ is the generator of C_0-semigroup of contractions by Theorem 3.2, $\mathscr{A}$ is closed and hence $0 \in \rho(\mathscr{A})$.

By the Sobolev embedding theorem, we obtain that $\mathscr{A}^{-1}$ is a compact operator on $\mathscr{H}$, and therefore, the spectrum of $\mathscr{A}$ consists of eigenvalues only. Thus, we have to establish that there are no purely imaginary eigenvalues. Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $U \in D(\mathscr{A})$ with $\mathscr{A}U = i\lambda U$, i.e.

$$i\lambda u_1 = u_2, \tag{4.20}$$
$$i\lambda v_1 = v_2, \tag{4.21}$$
$$i\lambda w_1 = w_2, \tag{4.22}$$
$$i\lambda u_2 + u_1^{(4)} = 0, \tag{4.23}$$
$$i\lambda v_2 - v_1'' + \beta v_2 = 0, \tag{4.24}$$
$$i\lambda w_2 + w_1^{(4)} = 0. \tag{4.25}$$

From the dissipativity of $\mathscr{A}$,

$$0 = \operatorname{Re}\langle (i\lambda I - \mathscr{A})U, U\rangle_{\mathscr{H}} = \operatorname{Re}(i\lambda \|U\|_{\mathscr{H}}) - \operatorname{Re}\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = \beta \|v_2\|^2_{L^2(I_2)}.$$

Then, $v_1 = v_2 = 0$, i.e. $U = (u_1, 0, w_1, u_2, 0, w_2)^\top$. Since $v_1 \in H^1(I_2)$ the Sobolev inequality implies that $v_1(x) = 0$ for all $x \in \overline{I_2} = [l_1, l_2]$.

Multiplying (4.20) by $i\lambda$ and substituting in (4.23), we obtain

$$u_1^{(4)} - \lambda^2 u_1 = 0 \quad \text{in } (l_0, l_1). \tag{4.26}$$

Moreover u_1 satisfies the boundary conditions

$$u_1(l_0) = u_1'(l_0) = 0 \quad \text{and} \quad u_1''(l_1) = u_1'''(l_1) = 0. \tag{4.27}$$

Let $x := l_1 + (l_0 - l_1)\eta, \quad \eta \in [0, 1]$. Then

$$u_1(x) = u_1(l_1 + (l_0 - l_1)\eta) =: z(\eta) \quad \text{and} \quad \frac{d^k u_1}{dx^k} = \frac{1}{(l_0 - l_1)^k} \frac{d^k z}{d\eta^k}.$$

Therefore, problem (4.26)–(4.27) can be transformed into

$$\begin{gathered} z^{(4)} - a^2 z = 0 \quad \text{in } (0, 1), \\ z(0) = z''(0) = z'''(0) = 0, \\ z(1) = z'(1) = 0, \end{gathered} \tag{4.28}$$

where $a := (l_0 - l_1)^2 |\lambda| \neq 0$. The general solution of the ordinary differential equation in (4.28) is

$$z(\eta) = c_1 \cosh(\sqrt{a}\eta) + c_2 \sinh(\sqrt{a}\eta) + c_3 \cos(\sqrt{a}\eta) + c_4 \sin(\sqrt{a}\eta).$$

Now, we see that the boundary conditions in (4.28) imply that $c_1 = c_2 = c_3 = c_4 = 0$ and therefore $z \equiv 0$, i.e. $u_1 \equiv 0$. In similar way we obtain $w_1 \equiv 0$. From (4.20)

and (4.22) we have also $u_2 \equiv 0$ and $w_2 \equiv 0$. Thus, $U = 0$ and we conclude that $i\mathbb{R} \subset \rho(\mathscr{A})$.
Now, we will show that

$$\sup_{\lambda\in\mathbb{R}} \|(i\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} < \infty. \tag{4.29}$$

If this inequality is false, then there are sequences $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $(U_n)_{n\in\mathbb{N}} \subset D(\mathscr{A})$ such that $|\lambda_n| \xrightarrow[n\to\infty]{} \infty$, $\|U_n\|_{\mathscr{H}} = 1$ for all $n \in \mathbb{N}$ and

$$\|(i\lambda_n I - \mathscr{A})U_n\|_{\mathscr{H}} \to 0 \quad n \to \infty. \tag{4.30}$$

Let $F_n := (i\lambda_n I - \mathscr{A})U_n$. Because the standard norm on $H^2(I_1)\times H^1(I_2)\times H^2(I_3)\times L^2(I_1) \times L^2(I_2) \times L^2(I_3)$ is equivalent to the norm $\|\cdot\|_{\mathscr{H}}$ on $\mathscr{H}$, (4.30) implies

$$i\lambda_n u_{1,n} - u_{2,n} = f_{1,n} \to 0 \text{ in } H^2(I_1), \tag{4.31}$$
$$i\lambda_n v_{1,n} - v_{2,n} = g_{1,n} \to 0 \text{ in } H^1(I_2), \tag{4.32}$$
$$i\lambda_n w_{1,n} - w_{2,n} = h_{1,n} \to 0 \text{ in } H^2(I_3), \tag{4.33}$$
$$i\lambda_n u_{2,n} + u_{1,n}^{(4)} = f_{2,n} \to 0 \text{ in } L^2(I_1), \tag{4.34}$$
$$i\lambda_n v_{2,n} - v_{1,n}'' + \beta v_{2,n} = g_{2,n} \to 0 \text{ in } L^2(I_2), \tag{4.35}$$
$$i\lambda_n w_{2,n} + w_{1,n}^{(4)} = h_{2,n} \to 0 \text{ in } L^2(I_3), \tag{4.36}$$

where $\rho_1 = \rho_2 = 0$. From the dissipativity of $\mathscr{A}$, it follows that

$$0 \leftarrow \operatorname{Re}(\langle(i\lambda_n I - \mathscr{A})U_n, U_n\rangle_{\mathscr{H}}) = \operatorname{Re}[i\lambda_n\|U_n\|_{\mathscr{H}}^2 - \langle\mathscr{A}U_n, U_n\rangle_{\mathscr{H}}] = \beta\|v_{2,n}\|_{L^2(I_2)}^2,$$

i.e.

$$\|v_{2,n}\|_{L^2(I_2)} \to 0. \tag{4.37}$$

Now, (4.32), (4.35), and (4.37) imply

$$\|v_{1,n}\|_{L^2(I_2)}^2 \to 0, \quad |\lambda_n|\|v_{1,n}\|_{L^2(I_2)} \to 0, \quad \|\lambda_n^{-1} v_{1,n}''\|_{L^2(I_2)} \to 0. \tag{4.38}$$

From this and the Gagliardo-Nirenberg inequality, it follows that

$$\|v_{1,n}'\|_{L^2(I_2)} \le \|\lambda_n v_{1,n}\|_{L^2(I_2)}^{1/2} \|\lambda_n^{-1} v_{1,n}''\|_{L^2(I_2)}^{1/2} + \|v_{1,n}\|_{L^2(I_2)} \to 0. \tag{4.39}$$

Substituting $v_{2,n} = i\lambda_n v_{1,n} - g_{1,n}$ in (4.35), we have

$$g_{2,n} = -\lambda_n^2 v_{1,n} - i\lambda_n g_{1,n} - v_{1,n}'' + \beta v_{2,n}. \tag{4.40}$$

Now, taking L^2-product of (4.40) with $(l_2 - x)v_{1,n}'$ for $x \in I_2$, we obtain

$$\begin{aligned}
&\langle g_{2,n}, (l_2 - x)v_{1,n}'\rangle_{L^2(I_2)} \\
&= \lambda_n^2 \overline{\langle v_{1,n}, (l_2 - x)v_{1,n}'\rangle_{L^2(I_2)}} - \|\lambda_n v_{1,n}\|_{L^2(I_2)}^2 + (l_2 - l_1)|\lambda_n v_{1,n}(l_1)|^2 \\
&\quad - i\lambda_n\langle g_{1,n}, v_{1,n}\rangle_{L^2(I_2)} + \langle i\,(l_2 - x)g_{1,n}', \lambda_n v_{1,n}\rangle_{L^2(I_2)} \\
&\quad + i(l_2 - l_1)\lambda_n g_{1,n}(l_1)\overline{v_{1,n}}(l_1) - \|v_{1,n}'\|_{L^2(I_2)}^2 + \overline{\langle v_{1,n}'', (l_2 - x)v_{1,n}\rangle_{L^2(I_2)}} \\
&\quad + (l_2 - l_1)|v_{1,n}'(l_1)|^2 + \langle\beta v_{2,n}, (l_2 - x)v_{1,n}'\rangle_{L^2(I_2)},
\end{aligned}$$

or equivalently

$$
\begin{aligned}
&-\|\lambda_n v_{1,n}\|^2_{L^2(I_2)}+(l_2-l_1)|\lambda_n v_{1,n}(l_1)|^2-\|v'_{1,n}\|^2_{L^2(I_2)}\\
&+(l_2-l_1)|v'_{1,n}(l_1)|^2+2\mathrm{Re}\,\{\langle\beta v_{2,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}\}\\
&=\langle g_{2,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}+\overline{\langle\beta v_{2,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}}\\
&\quad-\overline{\lambda_n^2\langle v_{1,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}}+\langle ig_{1,n},\lambda_n v_{1,n}\rangle_{L^2(I_2)}\\
&\quad-\langle i(l_2-x)g'_{1,n},\lambda_n v_{1,n}\rangle_{L^2(I_2)}-i(l_2-l_1)\lambda_n g_{1,n}(l_1)\overline{v_{1,n}}(l_1)\\
&\quad-\overline{\langle v''_{1,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}}.
\end{aligned}\tag{4.41}
$$

From (4.40) we have that $\beta v_{2,n}=g_{2,n}+\lambda_n^2 v_{1,n}+i\lambda_n g_{1,n}+v''_{1,n}$ and therefore

$$
\begin{aligned}
&\langle\beta v_{2,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}\\
&=\langle g_{2,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}+\lambda_n^2\langle v_{1,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}\\
&\quad+\langle ig_{1,n},\lambda_n v_{1,n}\rangle_{L^2(I_2)}-\langle i(l_2-x)g'_{1,n},\lambda_n v_{1,n}\rangle_{L^2(I_2)}\\
&\quad-i(l_2-l_1)\lambda_n g_{1,n}(l_1)\overline{v_{1,n}}(l_1)+\langle v''_{1,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}.
\end{aligned}
$$

Substituting this in (4.41), we obtain

$$
\begin{aligned}
&(l_2-l_1)|\lambda_n v_{1,n}(l_1)|^2+(l_2-l_1)|v'_{1,n}(l_1)|^2\\
&=2\,\mathrm{Re}\Big\{\langle g_{2,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}-\langle i(l_2-x)g'_{1,n},\lambda_n v_{1,n}\rangle_{L^2(I_2)}\\
&\quad-i(l_2-l_1)g_{1,n}(l_1)\lambda_n\overline{v_1}(l_1)-\langle\beta v_{2,n},(l_2-x)v'_{1,n}\rangle_{L^2(I_2)}\\
&\quad+\langle ig_{1,n},\lambda_n v_{1,n}\rangle_{L^2(I_2)}\Big\}+\|\lambda_n v_{1,n}\|^2_{L^2(I_2)}+\|v'_{1,n}\|^2_{L^2(I_2)}.
\end{aligned}
$$

Now, by the Cauchy-Schwarz inequality and Young's inequality, for each $\epsilon>0$ there exists a $C_\epsilon>0$ such that

$$
\begin{aligned}
&(l_2-l_1)|\lambda_n v_{1,n}(l_1)|^2+(l_2-l_1)|v'_{1,n}(l_1)|^2\\
&\le 2\Big[(l_2-l_1)\|g_{2,n}\|_{L^2(I_2)}\|v'_{1,n}\|_{L^2(I_2)}+(l_2-l_1)\|g'_{1,n}\|_{L^2(I_2)}\|\lambda_n v_{1,n}\|_{L^2(I_2)}\\
&\quad+(l_2-l_1)\big(\varepsilon|\lambda_n v_{1,n}(l_1)|^2+C_\varepsilon|g_{1,n}(l_1)|^2\big)\\
&\quad+\beta(l_2-l_1)\|v_{2,n}\|_{L^2(I_2)}\|v'_{1,n}\|_{L^2(I_2)}+\|g_{1,n}\|_{L^2(I_2)}\|\lambda_n v_{1,n}\|_{L^2(I_2)}\Big]\\
&\quad+\|\lambda_n v_{1,n}\|^2_{L^2(I_2)}+\|v'_{1,n}\|^2_{L^2(I_2)}.
\end{aligned}
$$

From this inequality with $\varepsilon=1/4$, (4.32), (4.35), (4.37)–(4.39), and the trace theorem, it follows that $|\lambda_n v_{1,n}(l_1)|\to 0$ and $|v'_{1,n}(l_1)|\to 0$ as $n\to\infty$. Therefore,

$$
|\lambda_n u_{1,n}(l_1)|\to 0\quad\text{and}\quad|u'''_{1,n}(l_1)|\to 0\quad\text{as } n\to\infty]\tag{4.42}
$$

because the transmission conditions (1.5) and (1.6) with $\rho_1=0$.

Now, substituting $u_{2,n}=i\lambda_n u_{1,n}-f_{1,n}$ in (4.34), we obtain

$$
f_{2,n}=-\lambda_n^2 u_{1,n}-i\lambda_n f_{1,n}+u^{(4)}_{1,n}.\tag{4.43}
$$

Taking L^2-product of (4.43) with $(x-l_0)u'_{1,n}$ for $x\in I_1$, we obtain with integration by parts that

$$
\begin{aligned}
&\langle f_{2,n},(x-l_0)u'_{1,n}\rangle_{L^2(I_1)}\\
&=\|\lambda_n u_{1,n}\|^2_{L^2(I_1)}+\lambda_n^2\overline{\langle u_{1,n},(x-l_0)u'_{1,n}\rangle_{L^2(I_1)}}-(l_1-l_0)|\lambda_n u_{1,n}(l_1)|^2\\
&\quad+i\lambda_n\langle f_{1,n},u_{1,n}\rangle_{L^2(I_1)}+\langle i(x-l_0)f'_{1,n},\lambda_n u_{1,n}\rangle_{L^2(I_1)}\\
&\quad-i(l_1-l_0)f_{1,n}(l_1)\lambda_n\overline{u_{1,n}}(l_1)+\|u''_{1,n}\|^2_{L^2(I_1)}-\overline{\langle u''_{1,n},(x-l_0)u'''_{1,n}\rangle_{L^2(I_1)}}\\
&\quad+(l_1-l_0)u'''_{1,n}(l_1)\overline{u'_{1,n}}(l_1).
\end{aligned}\tag{4.44}
$$

From (4.43) we have $\lambda_n^2u_{1,n}=-f_{2,n}-i\lambda_nf_{1,n}+u^{(4)}_{1,n}$ and therefore

$$
\begin{aligned}
&\lambda_n^2\langle u_{1,n},(x-l_0)u'_{1,n}\rangle_{L^2(I_1)}\\
&=-\langle f_{2,n},(x-l_0)u'_{1,n}\rangle_{L^2(I_1)}+\langle if_{1,n},\lambda_nu_{1,n}\rangle_{L^2(I_1)}\\
&\quad+\langle i(x-l_0)f'_{1,n},\lambda_nu_{1,n}\rangle_{L^2(I_1)}-i(l_1-l_0)f_{1,n}(l_1)\lambda_n\overline{u_{1,n}}(l_1)\\
&\quad+2\|u''_{1,n}\|^2_{L^2(I_1)}+\langle u''_{1,n},(x-l_0)u'''_{1,n}\rangle_{L^2(I_1)}+(l_1-l_0)u'''_{1,n}(l_1)\overline{u'_{1,n}}(l_1).
\end{aligned}\tag{4.45}
$$

Now, plugging (4.45) in (4.44), we obtain

$$
\begin{aligned}
&\|\lambda_nu_{1,n}\|^2_{L^2(I_1)}+3\|u''_{1,n}\|^2_{L^2(I_1)}\\
&=2\operatorname{Re}\big\{\langle f_{2,n}\,,\,(x-l_0)u'_{1,n}\rangle_{L^2(I_1)}-\langle if_{1,n}\,,\,\lambda_nu_{1,n}\rangle_{L^2(I_1)}\\
&\quad-\langle i(x-l_0)f'_{1,n}\,,\,\lambda_nu_{1,n}\rangle_{L^2(I_1)}+i(l_1-l_0)f_{1,n}(l_1)\lambda_n\overline{u_{1,n}}(l_1)\\
&\quad-(l_1-l_0)u'''_{1,n}(l_1)\overline{u'_{1,n}}(l_1)\big\}+(l_1-l_0)|\lambda_nu_{1,n}(l_1)|^2.
\end{aligned}\tag{4.46}
$$

Note that the Gagliardo-Nirenberg inequality implies

$$
\|u'_{1,n}\|_{L^2(I_1)}\le\|u_{1,n}\|^{1/2}_{L^2(I_1)}\|u''_{1,n}\|^{1/2}_{L^2(I_1)}+\|u_{1,n}\|_{L^2(I_1)}
$$

and thus

$$
\|u'_{1,n}\|^2_{L^2(I_1)}\le3\|u_{1,n}\|^2_{L^2(I_1)}+\|u''_{1,n}\|_{L^2(I_1)}.\tag{4.47}
$$

Moreover, from the trace theorem that there exists a positive constant C such that

$$
|u'_{1,n}(l_1)|\le C\|u_{1,n}\|_{H^2(I_1)}\le C\|U_n\|_{\mathscr{H}}=C.\tag{4.48}
$$

Let ε_1, ε_2, and ε_3 be positive numbers. By Young's inequality in (4.46), there are positive constants C_{ε_1}, C_{ε_2}, and C_{ε_3} such that

$$
\begin{aligned}
&\|\lambda_nu_{1,n}\|^2_{L^2(I_1)}+3\|u''_{1,n}\|^2_{L^2(I_1)}\\
&\le\varepsilon_1\|u'_{1,n}\|^2_{L^2(I_1)}+C_{\varepsilon_1}\|f_{2,n}\|^2_{L^2(I_1)}+\varepsilon_2\|\lambda_nu_{1,n}\|^2_{L^2(I_1)}+C_{\varepsilon_2}\|f_{1,n}\|^2_{L^2(I_1)}\\
&\quad+\varepsilon_3\|\lambda_nu_{1,n}\|^2_{L^2(I_1)}+C_{\varepsilon_3}\|f'_{1,n}\|^2_{L^2(I_1)}+2(l_1-l_0)\big\{|f_{1,n}(l_1)||\lambda_nu_{1,n}(l_1)|\\
&\quad+|u'''_{1,n}(l_1)||u'_{1,n}(l_1)|+|\lambda_nu_{1,n}(l_1)|^2\big\}\\
&\le3\varepsilon_1\|\lambda_nu_{1,n}\|^2_{L^2(I_1)}+\varepsilon_1\|u''_{1,n}\|^2_{L^2(I_1)}+C_{\varepsilon_1}\|f_{2,n}\|^2_{L^2(I_1)}\\
&\quad+\varepsilon_2\|\lambda_nu_{1,n}\|^2_{L^2(I_1)}+C_{\varepsilon_2}\|f_{1,n}\|^2_{L^2(I_1)}+\varepsilon_3\|\lambda_nu_{1,n}\|^2_{L^2(I_1)}\\
&\quad+C_{\varepsilon_3}\|f'_{1,n}\|^2_{L^2(I_1)}+2(l_1-l_0)\big\{|f_{1,n}(l_1)||\lambda_nu_{1,n}(l_1)|+C|u'''_{1,n}(l_1)|\\
&\quad+|\lambda_nu_{1,n}(l_1)|^2\big\},
\end{aligned}\tag{4.49}
$$

where we used (4.47) and (4.48). Choosing ε_1, ε_2, and ε_3 small enough such that $3\varepsilon_1+\varepsilon_2+\varepsilon_3<1/2$, we obtain from (4.31), (4.34), (4.42), and (4.49) that

$$
\|\lambda_nu_{1,n}\|_{L^2(I_1)}\to0,\quad\text{and}\quad\|u''_{1,n}\|_{L^2(I_1)}\to0\quad\text{as }n\to\infty.\tag{4.50}
$$

Similarly we conclude that

$$\|\lambda_n w_{1,n}\|_{L^2(I_3)} \to 0 \quad \text{and} \quad \|w''_{1,n}\|_{L^2(I_3)} \to 0 \quad \text{as } n \to \infty. \tag{4.51}$$

Therefore, $\|U_n\|_{\mathscr{H}} \to 0$ as $n \to \infty$, because (4.31), (4.32), (4.33), (4.37), (4.39), (4.50), and (4.51), which is a contradiction. Thus we have proved that the $(S(t))_{t\geq 0}$ is exponentially stable. $\square$

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