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EXISTENCE OF PERIODIC SOLUTIONS FOR SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. By means of variational structure and critical point theory, we study the existence of periodic solutions for a second-order neutral differential equation

$$(p(t)x'(t-\tau))' + f(t, x(t), x(t-\tau), x(t-2\tau)) = g(t),$$
$$x(0) = x(2k\tau), x'(0) = x'(2k\tau).$$

where k is a given positive integer and τ is a positive number.

1. Results

In this paper we study the existence of periodic solutions of the second order problem

$$\begin{gathered}(p(t)x'(t-\tau))' + f(t, x(t), x(t-\tau), x(t-2\tau)) = g(t),\\ x(0) = x(2k\tau), x'(0) = x'(2k\tau).\end{gathered} \tag{1.1}$$

where $f \in C(\mathbb{R}^4, \mathbb{R}), p, g \in C(\mathbb{R}, \mathbb{R})$, k is a given positive integer and τ is a positive number.

The existence of periodic solutions to (1.1) will be studied under the hypotheses:

(H1) $f \in C(\mathbb{R}^4, \mathbb{R})$
(H2) There exists a continuously differentiable functional $F(t, u, v)$ in $C^1(\mathbb{R}^3, \mathbb{R})$ with

$$F'_u(t, x(t-\tau), x(t-2\tau)) + F'_v(t, x(t), x(t-\tau)) = f(t, x(t), x(t-\tau), x(t-2\tau))$$

(H3) $F(t, u, v)$ is τ-periodic in t
(H4) $p(t)$ is τ-periodic and $0 < m < p(t)$
(H5) $g(t)$ is τ-periodic and $\overline{g} = \frac{1}{2k\tau}\int_0^{2k\tau} |g(t)|^2 dt < m/2$.

In recent years, by using the continuation theorem of coincidence degree theory, the existence of periodic solutions to ordinary equation have been extensively studied. In articles [1, 2, 4, 5, 6], the following second-order scalar differential equations

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have been studied:

$$\begin{gathered}
x''(t) + ax'(t) + bx(t) + g(x(t-1)) = p(t), \\
x''(t) + m^2 x(t) + g(x(t-\tau)) = p(t), \\
x''(t) + f(t, x(t), x(t-\tau_0(t))x'(t) + \beta(t)g(x(t-\tau_1(t))) = p(t), \\
x''(t) + cx'(t) + g(t-\tau), x(t-\tau), x'(t-\tau)) = p(t).
\end{gathered}$$

However, the study of corresponding problem for second-order neutral differential system with variational structure and critical point theory, to the best of our knowledge, appeared rarely; see [3]. In this paper, we study the existence of periodic solutions to (1.1) by means of variational technique and critical point theory.,

For the reader's convenience, we recall some basic definitions. Let E be a real Banach space. A mapping I from E to $\mathbb{R}$ will be called a functional. A critical point of I is a point where $I'(x_0) = \theta$ and a critical value of I is a number c such that $I(x_0) = c$. In applications to differential equations, critical points correspond to weak solution of equations. Indeed this fact makes critical point theory an important existence tool in studying differential equations.

A functional I is weakly lower semi-continuous at $x \in C$ if

$$x_n \rightharpoonup x \Rightarrow \lim_{n\to\infty} \inf I(x_n) \geq I(x).$$

A functional I is coercive on C means that

$$I(x) \to +\infty \quad \text{as} \quad \|x\| \to \infty.$$

We will make use of a theorem in [7] to obtain the critical point of I. This theorem is crucial for arriving at our results.

Theorem 1.1 ([7]). *Let E be a reflexive Banach space, C be weakly closed subset of E, and $I : C \to \mathbb{R}$ be weakly lower semi-continuous and coercive. Then I has a minimum on C.*

The main result of this paper is as follows.

Theorem 1.2. *Under assumptions (H1)-(H5), problem* (1.1) *has at least one $2k\tau$-periodic solution.*

Proof. Let $H_0^1(0, 2k\tau) = \{x(t) \in L^2[0, 2k\tau] : x(0) = x(2k\tau), x'(0) = x'(2k\tau)\}$ denote the Hilbert space with norm and inner product

$$\|x\| = \Big(\int_0^{2k\tau} |x'(t)|^2 dt\Big)^{1/2}, \quad (x, y) = \int_0^{2k\tau} x'(t)y'(t)dt.$$

Since each $x \in E$ can be extended periodically to the whole line, we may do not distinguish x and its extension.

A variational method is used for the following functional defined on E,

$$I(x) = \int_0^{2k\tau} [\frac{p(t)}{2}|x'(t)|^2 - F(t, x(t), x(t-\tau)) + g(t)x(t)]dt\,.$$

For $x, y \in E$ and $\alpha \in \mathbb{R}$, we denote by $\varphi(\alpha)$ the function $I(x + \alpha y)$; i.e.,

$$\begin{aligned}
\varphi(\alpha) = \int_0^{2k\tau} & [\frac{p(t)}{2}(|x'(t) + \alpha y'(t)|^2) \\
& - F(t, x(t) + \alpha y(t), x(t-\tau) + \alpha y(t-\tau)) + g(t)[x(t) + \alpha y(t)\big]dt\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\varphi'(\alpha)\\
&=\int_0^{2k\tau}\big\{p(t)[x'(t)y'(t)+\alpha y'(t)^2]-[F_u'(t,x(t)+\alpha y(t),x(t-\tau)+\alpha y(t-\tau))y(t)\\
&\quad+F_v'(t,x(t)+\alpha y(t),x(t-\tau)+\alpha y(t-\tau))y(t-\tau)]+g(t)y(t)\big\}dt
\end{aligned}$$

So that

$$\begin{aligned}
\varphi'(0)&=\int_0^{2k\tau}\{p(t)x'(t)y'(t)-[F_u'(t,x(t),x(t-\tau))y(t)\\
&\quad+F_v'(t,x(t),x(t-\tau))y(t-\tau)]\}dt+\int_0^{2k\tau}g(t)y(t)dt\\
&=\int_0^{2k\tau}p(t)x'(t)dy(t)-\int_0^{2k\tau}[F_u'(t,x(t),x(t-\tau))y(t)\\
&\quad+F_v'(t,x(t),x(t-\tau))y(t-\tau)]dt+\int_0^{2k\tau}g(t)y(t)dt\\
&=p(t)x'(t)y(t)|_0^{2k\tau}-\int_0^{2k\tau}[p(t)x'(t)]'y(t)dt\\
&\quad-\int_0^{2k\tau}[F_u'(t,x(t),x(t-\tau))y(t)\\
&\quad+F_v'(t,x(t),x(t-\tau))y(t-\tau)]dt+\int_0^{2k\tau}g(t)y(t)dt\\
&=-\int_0^{2k\tau}[p(t)x'(t)]'y(t)dt-\int_0^{2k\tau}F_u'(t,x(t),x(t-\tau))y(t)dt\\
&\quad-\int_{-\tau}^{(2k-1)\tau}F_v'(t,x(t),x(t-\tau))y(t-\tau)dt+\int_0^{2k\tau}g(t)y(t)dt\\
&=-\int_0^{2k\tau}[p(t)x'(t)]'y(t)dt-\int_0^{2k\tau}F_u'(t,x(t),x(t-\tau))y(t)dt\\
&\quad-\int_0^{2k\tau}F_v'(t+\tau,x(t+\tau),x(t))y(t)dt+\int_0^{2k\tau}g(t)y(t)dt\\
&=-\int_0^{2k\tau}\{[(p(t)x'(t))'+F_u'(t,x(t),x(t-\tau))\\
&\quad+F_v'(t,x(t+\tau),x(t))-g(t)]y(t)\}dt
\end{aligned}$$

Therefore, the Euler equation corresponding to the functional $I(x)$ is

$$(p(t)x'(t))'+F_u'(t,x(t),x(t-\tau))+F_v'(t,x(t+\tau),x(t))-g(t)=0 \tag{1.2}$$

It is easy to see that this equation is is equivalent to (1.1), and that any critical point x of the functional I is a $2k\tau$-periodic solution of (1.1).

Since $F(t,u,v)\in C^1(\mathbb{R}^3,\mathbb{R})$, we have $\int_0^{2k\tau}F(t,x(t),x(t-\tau))dt\le c$; thus

$$
\begin{aligned}
I(x) &= \int_0^{2k\tau}[\frac{p(t)}{2}|x'(t)|^2 - F(t,x(t),x(t-\tau)) + g(t)x(t)]dt \\
&= \int_0^{2k\tau}\frac{p(t)}{2}|x'(t)|^2dt - \int_0^{2k\tau}F(t,x(t),x(t-\tau))dt + \int_0^{2k\tau}g(t)x(t)dt \\
&\ge \int_0^{2k\tau}\frac{p(t)}{2}|x'(t)|^2dt - \int_0^{2k\tau}F(t,x(t),x(t-\tau))dt - \int_0^{2k\tau}|g(t)x(t)|dt \\
&\ge \frac{m}{2}\|x\|^2 - c - (\int_0^{2k\tau}|g(t)|^2dt)(\int_0^{2k\tau}|x(t)|^2dt) \\
&= \frac{m}{2}\|x\|^2 - c - (2k\tau)\overline{g}(\int_0^{2k\tau}|x(t)|^2dt) \\
&\ge \frac{m}{2}\|x\|^2 - c - \overline{g}(\int_0^{2k\tau}|x'(t)|^2dt) \\
&\ge \frac{m-2\overline{g}}{2}\|x\|^2 - c.
\end{aligned}
$$

It is easy to see the functional I is coercive. If x_n weakly converges to x, then by the compact embedding of $H_0^1(0,2k\tau)$ into $C([0,2k\tau])$, we know the convergence is uniform in $[0,2k\tau]$. From the trivial inequality

$$
0\le \int_0^{2k\tau}p(t)[x_n'(t)-x'(t)]^2dt,
$$

we have

$$
\int_0^{2k\tau}p(t)x_n'^2(t)dt \ge 2\int_0^{2k\tau}p(t)x_n'(t)x'(t)dt - \int_0^{2k\tau}p(t)x'^2(t)dt
$$

Thus

$$
\begin{aligned}
I(x_n) &= \int_0^{2k\tau}[\frac{p(t)}{2}|x_n'(t)|^2 - F(t,x_n(t),x_n(t-\tau)) + g(t)x_n(t)]dt \\
&\ge \int_0^{2k\tau}p(t)x_n'(t)x'(t)dt - \frac{1}{2}\int_0^{2k\tau}p(t)x'^2(t)dt \\
&\quad - \int_0^{2k\tau}F(t,x_n(t),x_n(t-\tau))dt + \int_0^{2k\tau}g(t)x_n(t)dt\,,
\end{aligned}
$$

and hence

$$
\lim_{n\to\infty}\inf I(x_n)\ge I(x)\,.
$$

This implies that I is weakly lower semi-continuous on $H_0^1(0,2k\tau)$, and the existence of a minimum for I follows from Theorem 1.1. Thus (1.1) has at least one periodic solution. $\square$

Example. Let

$$\begin{aligned}
&f(t,x(t),x(t-\tau),x(t-2\tau))\\
&= -4(8+\sin^2\frac{2\pi t}{\tau})\{[\frac{1}{1+x^2(t-\tau)}+\frac{1}{1+x^2(t-2\tau)}]\frac{x(t-\tau)}{[1+x^2(t-\tau)]^2}\\
&\quad +[\frac{1}{1+x^2(t)}+\frac{1}{1+x^2(t-\tau)}]\frac{x(t)}{[1+x^2(t)]^2}\},
\end{aligned}$$

$p(t)=16+\cos^2\frac{\pi t}{\tau}$, and $g(t)=1+\sin^2\frac{2\pi t}{\tau}$. Then F can be chosen as

$$F(t,u,v)=(8+\sin^2\frac{2\pi t}{\tau})(\frac{1}{1+u^2}+\frac{1}{1+v^2})^2\,.$$

It is easy to see that $F(t,u,v)$ is τ-periodic in t, p(t) is τ-periodic with $0<15<p(t)$, $g(t)$ is τ-periodic and

$$\overline{g}=\frac{1}{2k\tau}\int_0^{2k\tau}|g(t)|^2dt\le\frac{1}{2k\tau}\int_0^{2k\tau}4dt<\frac{15}{2},$$

Since all the assumptions in Theorem 1.2 are satisfied, (1.1) has at least one periodic solution.

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References

[1] J. Belley, M. Virgilio; Periodic Duffing equations with delay. *Electron. J. Differential Equations* 2004 (2004), No. 30, 18 pp. (electronic).

[2] Gen Qiang Wang, Ju Rang Yan; Existence of Periodic Solutions for Second Order Nonlinear Neutral Delay Equations (in Chinese), *Acta Mathematica Sinica*, 47(2004), 379-384.

[3] Guo Zhi-ming, Xu Yuan-tong; Existence of periodic solutions to a class of second-order neutral differential difference equations. *Acta Anal. Funct. Appl.* 5 (2003), 13-19.

[4] W. Layton, Periodic solutions of nonlinear delay equations. *J. Math. Anal. Appl.* 77 (1980), 198-204.

[5] Lu Shiping, Ge Weigao; Existence of periodic solutions for a kind of second-order neutral functional differential equation. *Appl. Math. Comput.* 157 (2004), 433-448.

[6] Ma Shiwang, Wang Zhicheng, Yu Jianshe; Coincidence degree and periodic solutions of Duffing equations. *Nonlinear Anal.* 34 (1998), 443-460.

[7] J. Mawhin and M. Willem; Critical point theory and Hamiltonian systems. *Applied Mathematical Sciences,* 74. Springer-Verlag, New York, 1989.

[8] P. H. Rabinowitz; Minimax methods in critical point theory with applications to differential equations. *CBMS Regional Conference Series in Mathematics*, 65. by the American Mathematical Society, Providence, RI, 1986.

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