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On a Class of Elliptic Systems in $\mathbb{R}^N$ *

David G. Costa

Abstract

We consider a class of variational systems in $\mathbb{R}^N$ of the form

$$\begin{cases} -\Delta u + a(x)u = F_u(x,u,v) \\ -\Delta v + b(x)v = F_v(x,u,v)\,, \end{cases}$$

where $a, b : \mathbb{R}^N \to \mathbb{R}$ are continuous functions which are coercive; i.e., $a(x)$ and $b(x)$ approach plus infinity as x approaches plus infinity. Under appropriate growth and regularity conditions on the nonlinearities $F_u(.)$ and $F_v(.)$, the (weak) solutions are precisely the critical points of a related functional defined on a Hilbert space of functions u, v in $H^1(\mathbb{R}^N)$.

By considering a class of potentials $F(x,u,v)$ which are nonquadratic at infinity, we show that a weak version of the Palais-Smale condition holds true and that a nontrivial solution can be obtained by the Generalized Mountain Pass Theorem.

Our approach allows situations in which $a(.)$ and $b(.)$ may assume negative values, and the potential $F(x,s)$ may grow either faster of slower than $|s|^2$

1 Introduction

In this paper we consider a class of semilinear elliptic systems in $\mathbb{R}^N$ of the form

$$(P) \qquad \begin{cases} -\Delta u + a(x)u &= f(x,u,v) \text{ in } \mathbb{R}^N \\ -\Delta v + b(x)v &= g(x,u,v) \text{ in } \mathbb{R}^N\,, \end{cases}$$

where $a, b : \mathbb{R}^N \to \mathbb{R}$ are continuous functions satisfying $a(x) \geq a_0\,,\ b(x) \geq b_0 \quad \forall x \in \mathbb{R}^N$ and such that $\lim_{|x|\to\infty} a(x) = \lim_{|x|\to\infty} b(x) = +\infty$. The nonlinearities $f, g : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ are also continuous with $f(x,0,0) = g(x,0,0) \equiv 0$, so that $(u,v) \equiv (0,0)$ solves (P) and we therefore must look for nontrivial solutions. We shall consider the variational situation in which $(f,g) = \nabla F$ for some

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C^1 function $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$, where ∇F stands for the gradient of F in the variables $U = (u, v) \in \mathbb{R}^2$.

In the scalar case $-\Delta u + a(x)u = f(x, u)$, among other results, P. Rabinowitz [14] showed existence of a nontrivial solution $u \in W^{1,2}(\mathbb{R}^N, \mathbb{R})$ under the assumption that $f(x, u)$ was *superlinear* with *subcritical growth*. This was done by a *mountain-pass type* argument [1] applied to the pertinent functional

$$I(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2}(|\nabla u|^2 + a(x)u^2) - F(x, u) \right) dx\,,$$

without the use of the Palais-Smale condition, which was not clear to hold true. On the other hand, Ding and Li showed in [8] existence of a nontrivial solution (u, v) for (P) by considering separate cases in which $f(x, u, v), g(x, u, v)$ were *superlinear* or *sublinear*.

Motivated by these results and using some recent ideas from [7, 6], our purpose in this paper is twofold. First we consider a class of *potentials* $F(x, u, v)$ which we call *nonquadratic at infinity* (cf. [7, 6]) and show that a weaker version of the Palais-Smale condition holds true so that a nontrivial solution of (P) can be obtained by a variant of the *Generalized Mountain-Pass Theorem* [12]. Such an existence result partially extends and, in fact, complements the above mentioned results of Rabinowitz and Ding-Li. Secondly we show that, under the hypotheses of *superlinearity* used in [14, 8], the Palais-Smale condition is indeed satisfied so that the standard Mountain-Pass Theorem can be used to prove those results. More precisely, we will prove Theorems 1.1 and 1.2 below, where the following hypotheses will be used:

(A_0) $a, b \in C(\mathbb{R}^N)$, $a(x) \ge a_0, b(x) \ge b_0$ for some positive constants a_0, b_0, and all $x \in \mathbb{R}^N$.

(A_1) $a(x) \to +\infty, b(x) \to +\infty$ as $|x| \to \infty$.

(F_0) $|\nabla f(x, U)| + |\nabla g(x, U)| \le c(1 + |U|^{p-1})$ for all $(x, U) \in \mathbb{R}^N \times \mathbb{R}^2$, where $f, g \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$, $c > 0$ and $1 \le p < (N + 2)/(N - 2)$ if $N \ge 3$ (or $1 \le p < \infty$ if $N = 1, 2$).

$(F_1)_\mu$ $U \cdot \nabla F(x, U) \ge \mu F(x, U) > 0$ for all $(x, U) \in \mathbb{R}^N \times \mathbb{R}^2 \backslash \{(0, 0)\}$.

$(F_2)_\nu$ $U \cdot \nabla F(x, U) - 2F(x, U) \ge a \mid U \mid^\nu > 0$ for all $(x, U) \in \mathbb{R}^N \times \mathbb{R}^2 \backslash \{(0, 0)\}$.

In what follows, we let $0 < \lambda_1 < \lambda_2 < \dots$ denote the distinct eigenvalues of the problem $-\vec{\Delta} U + A(x)U = \lambda U,\ x \in \mathbb{R}^N$, where $U = (u, v)$, $\vec{\Delta} = \mathrm{diag}(\Delta, \Delta)$ and $A(x) = \mathrm{diag}(a(x), b(x))$.

Theorem 1.1 *Suppose* $(A_0), (A_1)$ *and* $(F_0), (F_2)_\nu)$ *are satisfied with* $\nu > \frac{N}{2}(p - 1)$ *if* $N \ge 2$ *(or* $\nu > p - 1$ *if* $N = 1$*). If, in addition, we have*

(F_3) $\displaystyle \limsup_{|U|\to 0} \frac{2F(x, U)}{|U|^2} \le \alpha < \lambda_k < \beta \le \liminf_{|U|\to\infty} \frac{2F(x, U)}{|U|^2}$ *unif. for* $x \in \mathbb{R}^N$,

(F_4) $F(x,U) \geq \frac{1}{2}\lambda_{k-1}|U|^2$ *for all* $x \in \mathbb{R}^N$ *and* $U \in \mathbb{R}^2$*,*

then (P) *possesses a nonzero weak solution* $U \in C^1(\mathbb{R}^N, \mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^N, \mathbb{R}^2)$*.*

Theorem 1.2 *If* $(A_0), (A_1)$ *and* $(F_0), (F_1)_\mu$ *are satisfied with* $\mu > 2$*, then the functional* I *associated with problem* (P) *satisfies the Palais-Smale condition and* (P) *has a nonzero weak solution* $U \in C^1(\mathbb{R}^N, \mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^N, \mathbb{R}^2)$*.*

Remark 1.3 In the case that $a, b \in C^1(\mathbb{R}^N)$ and $f, g \in C^2(\mathbb{R}^N, \mathbb{R}^2)$ then, by standard bootstrap arguments, the weak C^1 solution U above is indeed a classical solution of (P).

Remark 1.4 Conditions $(F_3), (F_4)$ represent a *crossing* of the eigenvalue λ_k by the nonlinearity (f, g). On the other hand, when f and g are x-independent, a simple calculation shows that $(F_1)_\mu$ with $\mu > 2$ implies $\lim_{|U|\to 0} F(U)/|U|^2 = 0$ and $\lim_{|U|\to\infty} F(U)/|U|^2 = +\infty$, so that all eigenvalues are crossed in this case; in particular, $(F_3), (F_4)$ are automatically satisfied with $k = 1$ (and letting $\lambda_0 = 0$). Also, it is not hard to show (see Remark 2.5) that $(F_1)_\mu$ implies $(F_2)_\mu$ provided that we have $\liminf_{|U|\to 0} F(U)/|U|^\mu \geq a > 0$. In this case, when $p \leq 1 + 4/N$ and $N \geq 3$ in (F_0), Theorem 1.1 above extends Theorem 1.7 in [14].

Remark 1.5 It will be clear from the proof of Theorem 1.1 that a similar result holds with $(F_2)_\nu$ replaced by its "dual"

$(F_2)_\nu^-$

$$U \cdot \nabla F(x,U) - 2F(x,U) \leq -a \mid U \mid^\nu < 0$$

for all $x \in \mathbb{R}^N$, $U \in \mathbb{R}^2 \backslash \{(0,0)\}$.

2 Proofs of Theorems 1.1 and 1.2

Let $H^1 = H^1(\mathbb{R}^N, \mathbb{R}^2)$ denote the Sobolev space of pairs $U = (u, v)$ of L^2-functions $u, v : \mathbb{R}^N \to \mathbb{R}$ with weak derivatives $\partial u/\partial x_j$, $\partial v/\partial x_j$ $(j = 1, \ldots, N)$ also in $L^2(\mathbb{R}^N)$, endowed with its usual norm

$$\|U\|^2_{H^1} = \int (|\nabla U|^2 + |U|^2)\, dx = \int (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2)\, dx\,.$$

Throughout this paper, unless specified otherwise, all integrals are understood to be taken over all of $\mathbb{R}^N$. Given continuous functions $a, b : \mathbb{R}^N \to \mathbb{R}$ satisfying $a(x) \geq a_0 > 0$, $b(x) \geq b_0 > 0 \quad \forall x \in \mathbb{R}^N$, we consider the subspace $E \subset H^1$ defined by

$$E = \{U = (u, v) \in H^1 : \int (|\nabla u|^2 + |\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2)\, dx < \infty\}$$

and endowed with the norm

$$\|U\|^2 = \int (|\nabla u|^2 + |\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2)\, dx\,.$$

Since $a(x) \geq a_0 > 0$, $b(x) \geq b_0 > 0$, we clearly have the continuous embedding $E \hookrightarrow H^1$. We also recall that Sobolev's Theorem gives the continuous embeddings $H^1 \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R}^2)$ for all $2 \leq q \leq 2^* := 2N/(N-2)$, if $N \geq 3$ (respectively, $2 \leq q < \infty$ if $N = 1, 2$).

Now, let us consider the functional $I : E \to \mathbb{R}$ given by

$$\begin{aligned} I(u,v) &= \int \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2)\, dx - \int F(x,u,v)\, dx \\ &= \frac{1}{2}\|U\|^2 - N(U)\,. \end{aligned} \tag{1}$$

Assuming the growth condition (F_0), it can be shown (cf. Theorem A.VI in [4]) that the functional N is indeed well-defined and of class C^1 on H^1 and (hence) on the space E, with

$$\langle \nabla N(U), \Phi \rangle = \int (f(x,u,v)\varphi + g(x,u,v)\psi)\, dx \tag{2}$$

for all $U = (u,v)$, $\Phi = (\varphi, \psi) \in E$, where we are denoting by $\langle \cdot, \cdot \rangle$ the inner product on E. In fact, one can say more when both functions $a(x), b(x)$ are coercive, that is, when condition (A_1) is also satisfied.

Proposition 2.1 *(i) If (A_0) and (A_1) hold true, then the embedding $E \hookrightarrow L^2(\mathbb{R}^N, \mathbb{R}^2)$ is compact.*

(ii) Under conditions $(A_0), (A_1)$ and (F_0) the mapping $\nabla N : E \to E$ is compact.

Proof of (i) We will show that $U_m \to 0$ strongly in $L^2(\mathbb{R}^N, \mathbb{R}^2)$ whenever $U_m \rightharpoonup 0$ weakly in E. Indeed, let $C > 0$ be such that $\|U_m\| \leq C$. Given $\epsilon > 0$, pick $R > 0$ such that $a(x) \geq 2C^2/\epsilon$, $b(x) \geq 2C^2/\epsilon$ for all $|x| \geq R$ and denote by B_R the ball of radius R in $\mathbb{R}^N$. Then, since the restriction operator $U \mapsto U|_{B_R}$ is continuous from $H^1(\mathbb{R}^N, \mathbb{R}^2)$ into $H^1(B_R, \mathbb{R}^2)$, we also have that $U_m \rightharpoonup 0$ weakly in $H^1(B_R, \mathbb{R}^2)$. In particular, the compact embedding $H^1(B_R, \mathbb{R}^2) \hookrightarrow L^2(B_R, \mathbb{R}^2)$ implies that for some natural number m_0,

$$\int_{B_R} (|u_m|^2 + |v_m|^2)\ dx \leq \frac{\epsilon}{2} \quad \forall m \geq m_0\,. \tag{3}$$

On the other hand, by our choice of $R > 0$, we clearly have

$$\begin{aligned} \frac{2}{\epsilon}\int_{\mathbb{R}^N \setminus B_R} (|u_m|^2 + |v_m|^2)\, dx &\leq \frac{1}{C^2}\int_{\mathbb{R}^N \setminus B_R} (a(x)|u_m|^2 + b(x)|v_m|^2)\, dx \\ &\leq \frac{1}{C^2}\|U_m\|^2 \leq 1\,. \end{aligned} \tag{4}$$

Combining (3) and (4) we obtain that $|U_m|^2_{L^2} \leq \epsilon$ for all $m \geq m_0$.

Proof of (ii) We assume $N \geq 3$, the case $N = 1, 2$ being similar. Assumption (F_0) implies

$$|f(x,U) - f(x,\hat{U})| \leq \Big(a_1 + b_1(|U|^{p-1} + |\hat{U}|^{p-1})\Big)\,|U - \hat{U}|\,, \tag{5}$$

for all $x \in \mathbb{R}^N$, $U, \hat{U} \in \mathbb{R}^2$, with a similar estimate holding true for $g(x,U)$. Now, letting $2^* = 2N/(N-2)$, $p_1 = 2^*/(p-1)$, $p_2 = p_3 = 2p_1/(p_1 - 1)$ and recalling that $p < (N+2)/(N-2) = 2^* - 1$ in (F_0), we have that $p_1, p_2, p_3 > 1$ with $p_2, p_3 < 2^*$ and $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$. Therefore, (5) and Hölder's inequality give

$$\begin{aligned}&\int |(f(x,U) - f(x,\hat{U}))\varphi|\,dx \\ &\leq \quad A_1|U - \hat{U}|_{L^2}|\varphi|_{L^2} + B_1\left(|U|^{p-1}_{L^{2^*}} + |\hat{U}|^{p-1}_{L^{2^*}}\right)|U - \hat{U}|_{L^{p_2}}|\varphi|_{L^{p_3}}\,,\end{aligned} \tag{6}$$

for all $\varphi \in H^1(\mathbb{R}^N)$, with a similar estimate also holding for $g(x,U)$, namely,

$$\begin{aligned}&\int |(g(x,U) - g(x,\hat{U}))\psi|\,dx \\ &\leq \quad A_2|U - \hat{U}|_{L^2}|\psi|_{L^2} + B_2\left(|U|^{p-1}_{L^{2^*}} + |\hat{U}|^{p-1}_{L^{2^*}}\right)|U - \hat{U}|_{L^{p_2}}|\psi|_{L^{p_3}}\,,\end{aligned} \tag{7}$$

for all $\psi \in H^1(\mathbb{R}^N)$. From these, letting $(\varphi, \psi) = \nabla N(U) - \nabla N(\hat{U})$, we obtain

$$\|\nabla N(U) - \nabla N(\hat{U})\| \leq A|U - \hat{U}|_{L^2} + B\left(|U|^{p-1}_{L^{2^*}} + |\hat{U}|^{p-1}_{L^{2^*}}\right)|U - \hat{U}|_{L^{p_2}}\,. \tag{8}$$

On the other hand, using the continuous embedding $E \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R}^2)$, $2 \leq q \leq 2^*$, together with the *interpolation inequality* (where $1/q = \sigma/2 + (1-\sigma)/2^*$)

$$|U|_{L^q} \leq |U|^{\sigma}_{L^2}|U|^{1-\sigma}_{L^{2^*}} \quad \forall U \in L^2 \cap L^{2^*}$$

and the fact (proved in (i)) that the embedding $E \hookrightarrow L^2$ is compact, we infer that the embeddings $E \hookrightarrow L^q$ are also compact for $2 \leq q < 2^*$. Therefore, using (8) and recalling that $p_2 < 2^*$, we conclude that $\nabla N(U_m) \to \nabla N(\hat{U})$ *strongly* in E whenever $U_m \rightharpoonup \hat{U}$ *weakly* in E. The proof of Proposition 2.1 is complete.

Remark 2.2 Let $H = l^2(\mathbb{N})$ be the Hilbert space of square-summable sequences $a = (a_j)_{j \in N}$ with its usual norm $|a|^2_H = \sum a_j^2$. As is well-known, given a sequence $\{\epsilon_j\} \subset \mathbb{R}_+$ with $\lim_{j\to\infty} \epsilon_j = 0$, the operator $T : H \to H$ defined by $(Ta)_j = \epsilon_j a_j$ is a compact operator. This fact can also be stated by saying that, given a positive sequence $\{M_j\}$ with $\lim_{j\to\infty} M_j = +\infty$, the embedding $E \hookrightarrow H$ is compact, where $E = \{a = (a_j) \in H : \|a\|^2 := \sum M_j a_j^2 < \infty\}$. Proposition 2.1 (i) above is an expression of this fact to our present situation. We learned from P. Rabinowitz that similar versions of Proposition 2.1 (i) were also proved in [11, 8].

Next we recall a compactness condition of the Palais-Smale type which was introduced by Cerami in [5]. It was subsequently used by Bartolo-Benci-Fortunato [2] to prove a deformation theorem (Thm 1.3 in [2]) and, as a consequence, general minimax results as in Benci-Rabinowitz [3].

Definition 2.3 A functional $I \in C^1(E, \mathbb{R})$ is said to satisfy condition (C) if Any sequence $\{U_m\} \subset E$ such that $I(U_m)$ is bounded and $(1+\|U_m\|)\|\nabla I(U_m)\| \to 0$ possesses a convergent subsequence.

Note that (C) is implied by the usual Palais-Smale condition (PS): Any sequence $\{U_m\} \subset E$ such that $I(U_m)$ is bounded and $\|\nabla I(U_m)\| \to 0$ possesses a convergent subsequence.

In our case, where $I(U) = q(U) - N(U)$ is a perturbation of the quadratic form $q(U) = \frac{1}{2}\|U\|^2$, it turns out that if N is *superquadratic at infinity* in the sense of $(F_{1,\mu})$, then I satisfies the usual Palais-Smale condition (PS). In fact, we will show it suffices that I be *nonquadratic at infinity* in the sense of $(F_2)_\nu$ for condition (C) to be satisfied.

Proposition 2.4 *Assume that* $(A_0), (A_1)$ *and* (F_0) *hold true. Then:*

(i) Condition $(F_1)_\mu$ *implies* (PS) *whenever* $\mu > 2$*;*

(ii) Condition $(F_2)_\nu$ *implies* (C) *whenever* $\nu > \frac{N}{2}(p-1)$ *if* $N \geq 2$ *(or* $\nu > p-1$ *if* $N = 1, 2$*).*

Proof of (i) Let $\{U_m\} \subset E$ be such that $|I(U_m)| \leq K$ and $\|\nabla I(U_m)\| = \epsilon_m \to 0$. Then,

$$\begin{aligned} &(\frac{\mu}{2} - 1)\|U_m\|^2 \\ &= \mu I(U_m) - \langle \nabla I(U_m), U_m \rangle + \int [\mu F(x, U_m) - U_m \cdot \nabla F(x, U_m)]\, dx \\ &\leq \mu K + \epsilon_m \|U_m\| \end{aligned}$$

in view of $(F_1)_\mu)$, so that $\|U_m\|$ is bounded. Since $\nabla I(U) = U - \nabla N(U)$ and $\nabla N : E \to E$ is a compact mapping by Proposition 2.1 (ii), we conclude as usual that $\{U_m\}$ possesses a convergent subsequence.

Proof of (ii) We will assume $N \geq 3$ since the proof is similar for $N = 1, 2$. Recall that (F_0) gives

$$|F(x, U)| \leq C_1|U|^2 + C_2|U|^{p+1} \quad \forall x \in \mathbb{R}^N, \quad \forall U \in \mathbb{R}^2, \tag{9}$$

where $p+1 < 2^*$ and, without loss of generality, we may assume that $p+1 > \nu$. Thus, we have the *interpolation inequality*

$$|U|_{L^{p+1}} \leq |U|_{L^\nu}^{1-t} |U|_{L^{2^*}}^t \quad \forall U \in L^\nu \cap L^{2^*},$$

where $1/(p+1) = (1-t)/\nu + t/(2^*)$. Using the Sobolev embedding $E \hookrightarrow L^{2^*}$, we obtain

$$|U|_{L^{p+1}} \le C|U|_{L^\nu}^{1-t}\|U\|^t \quad \forall U \in L^\nu \cap E\,. \tag{10}$$

Now, let $\{U_m\} \subset E$ be such that $I(U_m)$ is bounded and $(1+\|U_m\|)\|\nabla I(U_m)\| \to 0$. Using $(F_2)_\nu$ we obtain

$$a|U_m|_{L^\nu} \le 2I(U_m) - \langle \nabla I(U_m), U_m\rangle \le K_1\,,$$

hence

$$|U_m|_{L^\nu} \le K_2 \quad \forall m \in \mathbb{N}\,. \tag{11}$$

In particular, writing $Q_m(x) = U_m(x)\cdot\nabla F(x, U_m(x)) - 2F(x, U_m(x))$, we have that

$$\limsup \int Q_m(x)\,dx \le K_1\,. \tag{12}$$

On the other hand, using (9) and (10), we obtain the estimate

$$\begin{aligned} \frac{1}{2}\|U_m\|^2 - I(U_m) &= \int F(x, U_m(x))\,dx \\ &\le C_1|U_m|_{L^2}^2 + C_2C^{p+1}|U_m|_{L^\nu}^{(1-t)(p+1)}\|U_m\|^{t(p+1)}\,, \end{aligned}$$

so that (11) implies

$$\|U_m\|^2 \le K_3 + K_4|U_m|_{L^2}^2 + K_5\|U_m\|^{t(p+1)}\,, \tag{13}$$

where a simple calculation shows that $t(p+1) < 2$ since $\nu > \frac{N}{2}(p-1)$. Finally, we prove the claim below, which implies that $\{U_m\}$ possesses a convergent subsequence as before.

Claim: $\{U_m\}$ has a bounded subsequence in E.

Suppose, by contradiction, that $\|U_m\| \to \infty$. Letting $W_m = U_m/\|U_m\|$ and using the compact embedding $E \hookrightarrow L^2$, we conclude that there exists $\hat{W} \in E$ such that $W_m \rightharpoonup \hat{W}$ *weakly* in E, $W_m \to \hat{W}$ *strongly* in L^2 and $W_m(x) \to \hat{W}(x)$ a. e. $x \in \mathbb{R}^N$. Now, dividing by $\|U_m\|^2$ in (13) and passing to the limit (recalling that $t(p+1) < 2$), we obtain

$$1 \le K_4|\hat{W}|_{L^2}^2\,,$$

so that $|\hat{W}| \ne 0$ and the set $S = \{x \in \mathbb{R}^N : |\hat{W}(x)| \ne 0\}$ has a positive measure. Thus, since $Q_m(x) \ge a|U_m(x)|^\nu \ge 0$ and $|U_m(x)| \to \infty$ for $x \in S$, an application of Fatou's Lemma gives

$$\lim \int Q_m(x)\,dx \ge \lim \int_S Q_m(x)\,dx = \infty\,,$$

which contradicts (12). The proof of Proposition 2.4 is complete. □

Remark 2.5 Consider the x-independent case. For simplicity, let $H(U) = F(U)/|U|^\mu$, and $K(U) = [U\cdot\nabla F(U) - 2F(U)]/|U|^\mu$. Then, it is easy to see that $(F_1)_\mu$ implies

$$r \mapsto H(rU) \text{ is nondecreasing in } r \in (0,+\infty) \text{ (for any } |U|=1) ,$$
$$K(U) \geq (\mu-2)\inf_{|V|=r} H(V) \quad \forall |U| \geq r > 0 .$$

In particular, since $H(U) > 0$ for $(0,0) \neq U \in \mathbb{R}^2$, the limits $a_+(U) = \lim_{r\to 0+} H(rU)$ will exist and $a_+(U) \geq 0$. Therefore, in the case that $a_+ = \inf_{|U|=1} a_+(U) > 0$, the above estimate shows that condition $(F_{2,\mu})$ holds with $a = (\mu-2)a_+ > 0$.

Now, before proving Theorems 1.1 and 1.2, we will make a small digression regarding a useful lower estimate for the functional $N(U) = \int_\Omega F(x,U)dx$ when the *potential* is a (continuous) function $F : \Omega\times\mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\liminf_{|U|\to\infty} \frac{F(x,U)}{|U|^2} \geq b > -\infty \quad \text{uniformly for } x \in \Omega , \tag{14}$$

with $\Omega \subset \mathbb{R}^N$ an arbitrary domain. Of course, we are also assuming that F satisfies

$$|F(x,U)| \leq C_1|U|^2 + C_2|U|^q , \tag{15}$$

for some $2 \leq q < \infty$, and that we have a continuous embedding $E \hookrightarrow L^2(\Omega)\cap L^q(\Omega)$, so that N is well-defined on the space E.

Let $\widehat{b} < b$ be given. Then, by (14), there exists $R > 0$ such that

$$F(x,U) \geq \widehat{b}|U|^2 \quad \forall x \in \Omega \text{ and } |U| \geq R , \tag{16}$$

hence

$$F(x,U) \geq \widehat{b}|U|^2 - \widehat{M} \quad \forall x \in \Omega \text{ and } U \in \mathbb{R}^2 ,$$

in view of (15). The above clearly gives the following lower estimate for the functional N,

$$N(U) \geq \widehat{b}|U|^2_{L^2} - \widehat{M}\mathrm{meas}(\Omega) \quad \forall U \in E ,$$

which is meaningful only when $\mathrm{meas}(\Omega) < \infty$, in which case it implies

$$\liminf_{\|U\|\to\infty} \frac{N(U) - \widehat{b}|U|^2_{L^2}}{\|U\|^2} \geq 0 . \tag{17}$$

We will show next that, even in the case of a general domain $\Omega \subset \mathbb{R}^N$, the above lower bound still holds provided E is compactly embedded in $L^2(\Omega)$.

Proposition 2.6 *Assume (14), (15) and that the embedding $E \hookrightarrow L^2(\Omega)$ is compact. Then (17) holds true.*

Proof In view of (16) and denoting $\Omega_R(U) = \{x \in \Omega : |U(x)| < R\}$, we can write

$$\begin{aligned} N(U) &\geq \widehat{b} \int_{\Omega \setminus \Omega_R(U)} |U|^2 \, dx + \int_{\Omega_R(U)} F(x,U) \, dx \\ &= \widehat{b}|U|_{L^2}^2 + \int_{\Omega_R(U)} [F(x,U) - \widehat{b}|U|^2] \, dx \, . \end{aligned}$$

Therefore, it suffices to show that $\liminf_{\|U\| \to \infty} N_R(U)/\|U\|^2 \geq 0$, where

$$N_R(U) = \int_{\Omega_R(U)} [F(x,U) - \widehat{b}|U|^2] \, dx \, .$$

We claim that $\lim_{\|U\| \to \infty} N_R/\|U\|^2 = 0$. Indeed, by contradiction, suppose that there exists $\delta_0 > 0$ and a sequence $\{U_m\} \subset E$ such that $\|U_m\| \to \infty$ and

$$\Big| \int_{0<|U_m|<R} [Q(x,U_m) - \widehat{b}]|U_m|^2 \, dx \Big| \geq \delta_0 \|U_m\|^2 \quad \forall m \in \mathbb{N} \, ,$$

where we are denoting $Q(x,U) = F(x,U)/|U|^2$, $U \neq (0,0)$. By taking a subsequence, if necessary, we may assume that the above holds without the absolute value (the case where $N_R(U_m) < 0$ is entirely similar). Now, let us define $W_m = U_m/\|U_m\|$. Then, since $\|W_m\| = 1$ and the embedding $E \hookrightarrow L^2$ is compact, there exists $\hat{W} \in E$ such that, for a suitable subsequence (which we still denote by $\{W_m\}$), we have

$$\begin{array}{ll} W_m \rightharpoonup \hat{W} & \text{weakly in } E \ , \\ W_m \to \hat{W} & \text{strongly in } L^2(\Omega) \ , \\ W_m(x) \to \hat{W}(x) & \text{a. e. } x \in \Omega \ , \\ |W_m(x)| \leq h(x) \in L^2(\Omega) \ . & \end{array}$$

Therefore, letting $H_m(x) = [Q_m(x, U_m(x)) - \widehat{b}]\chi_m(x)|W_m(x)^2|$ where χ_m is the characteristic function of the set $\Omega_R(U_m) = \{x \in \Omega \mid 0 < |U_m(x)| < R\}$, we have

$$\int_\Omega H_m(x) \, dx \geq \delta_0 > 0 \quad \forall m \in \mathbb{N} \, . \tag{18}$$

On the other hand, we observe that $|H_m(x)| \leq (|\widehat{b}| + M_R)h(x)^2 \in L^1(\Omega)$, where $M_R = \max_{|U| \leq R} |Q(x,U)| < \infty$ in view of (15). Moreover, $H_m(x) \to 0$ a. e. $x \in \Omega$ since, on $\widehat{\Omega} = \{x \in \Omega \mid |\widehat{W}(x)| = 0\}$ we clearly have $|W_m(x)| \to 0$, whereas, if $|\widehat{W}(x)| > 0$, we have $|U_m(x)| = \|U_m\||W_m(x)| \to +\infty$ so that $\chi_m(x) = 0$ for all m large. Therefore, by Lebesgue's theorem, we conclude that

$$\int_\Omega H_m(x) \, dx \to 0 \, ,$$

which is in contradiction with (18). The proof of Proposition 2.6 is complete.

Proof of Theorem 1.2 In view of Proposition 2.4 (i), it suffices to check that the conditions of the Mountain-Pass Theorem [1] are satisfied. Indeed, it is easy to see that the global assumption $(F_1)_\mu$ implies

$$\begin{array}{ll} (i) & F(x,U) \geq \min_{|V|=1} F(x,V)|U|^\mu > 0 \quad \forall x \in \mathbb{R}^N \text{ and } |U| \geq 1\,, \\ (ii) & 0 < F(x,U) \leq \max_{|V|=1} F(x,V)|U|^\mu \quad \forall x \in \mathbb{R}^N \text{ and } 0 < |U| \leq 1\,, \end{array} \tag{19}$$

where $\max_{|V|=1} |F(x,V)| \leq C$ in view of (F_0). In particular, (19)(ii) shows that

$$\lim_{|U|\to 0} \frac{F(x,U)}{|U|^2} = 0 \text{ uniformly for } x \in \mathbb{R}^N \;, \tag{20}$$

and (19)(i) shows that, given any bounded set $S \subset \mathbb{R}^N$, there exists $\widehat{C} = \widehat{C}(S)$, $\widehat{C} > 0$ with

$$F(x,U) \geq \widehat{C}|U|^\mu \quad \forall x \in S \text{ and } |U| \geq 1\,, \tag{21}$$

Now, using the embedding $E \hookrightarrow L^2$, it is clear from (20) that

$$\inf_{\|U\|=r} I(U) > 0$$

for all $r > 0$ sufficiently small. On the other hand, (21) shows that there exist many $e \in E$ such that $I(e) < 0$ (For instance, take $e = \rho\Phi$ with $0 \neq \Phi \in C^1(\mathbb{R}^N, \mathbb{R}^2)$ having compact support and $\rho > 0$ being sufficiently large). Therefore, the *geometry* of the mountain-pass theorem holds true and we can conclude the existence of a critical point $\widehat{U} \in E$ of the functional I with $I(\widehat{U}) > 0$. In other words, problem (P) has a nonzero weak solution $\widehat{U} \in H^1$ such that $b(x)^{1/2}\widehat{U} \in L^2$. Moreover, by the regularity theory, we also have $\widehat{U} \in C^1$. The proof of Theorem 1.2 is complete. □

Remark 2.7 It should be observed that, in our present case, we did not use the (system) analogue of assumption $f(x,0) = f_u(x,0) = 0$ made in [14], since the global condition $(F_1)_\mu$ already implies (2.20).

Proof of Theorem 1.1 Notice that, given $\gamma \in \mathbb{R}$, we can write (2.1) as

$$I(U) = \frac{1}{2}\langle U - \gamma TU, U\rangle - N_\gamma(U)\,, \tag{22}$$

where $N_\gamma(U) := N(U) - \frac{1}{2}\gamma|U|^2_{L^2}$ and $T : E \to E$ is defined by $\langle TU, \Phi\rangle = (U,\Phi)_{L^2} \quad \forall U, \Phi \in E$, so that T is a compact operator in view of Proposition 2.1 (i). In fact, it is easy to see that T is a *positive* operator and its eigenvalues $\{\tau_j\}_{j\in N}$ are the reciprocals of the eigenvalues of the eigenvalue problem $-\vec{\Delta}U + A(x)U = \lambda_j U \;,\; x \in \mathbb{R}^N$, that is, $\tau_j = 1/\lambda_j$. We denote by E^+_γ, E^0_γ and E^-_γ

the subspaces of E where $I-\gamma T$ is *positive definite*, *zero* and *negative definite*, respectively, and let $m_\gamma>0$ be such that

$$\begin{aligned}\frac{1}{2}\langle U-\gamma TU,U\rangle &\geq m_\gamma\|U\|^2 \quad \forall U\in E_\gamma^+\,,\\ \frac{1}{2}\langle U-\gamma TU,U\rangle &\leq -m_\gamma\|U\|^2 \quad \forall U\in E_\gamma^-\,.\end{aligned}$$

Also, we define the subspaces $E^+=E^+_{\lambda_{k-1}}$ and $E^-=E^-_{\lambda_{k-1}}\oplus E^0_{\lambda_{k-1}}$, so that $E=E^+\oplus E^-$.

Now, recalling the *crossing* condition (F_3), pick $\widehat{\alpha}<\widehat{\beta}$ so that $\alpha<\widehat{\alpha}<\lambda_k<\widehat{\beta}<\beta$. Then, there exists $\widehat{\delta}>0$ such that

$$F(x,U)\leq\frac{1}{2}\widehat{\alpha}|U|^2 \quad \forall\,|U|\leq\widehat{\delta}\,,$$

so that $F(x,U)\leq\frac{1}{2}\widehat{\alpha}|U|^2+M|U|^{p+1}\quad\forall x\in\mathbb{R}^N$ and $U\in\mathbb{R}^2$ and, hence,

$$I(U)\geq\frac{1}{2}(\|U\|^2-\widehat{\alpha}|U|^2_{L^2})-\widehat{M}\|U\|^{p+1}\quad\forall U\in E\,. \tag{23}$$

From (23), letting $\widehat{m}=m_{\widehat{\alpha}}$, it follows that

$$I(U)\geq\widehat{m}\|U\|^2-\hat{M}\|U\|^{p+1}=(\widehat{m}-\widehat{M}\|U\|^{p-1})\|U\|^2 \tag{24}$$

for all $U\in E^+$. Since we may assume $p>1$ in (F_0), we can find $\omega,\rho>0$ such that

$$I(U)\geq\omega\quad\forall U\in E^+\,,\quad |U\|=\rho\,. \tag{25}$$

On the other hand, we obtain from (F_4) that

$$I(U)\leq\frac{1}{2}(\|U\|^2-\lambda_{k-1}|U|^2_{L^2})\leq 0\quad\forall U\in E^-\,, \tag{26}$$

and, since (F_0) and (F_3) imply that (15) and (14) hold with $b=\frac{1}{2}\beta>\frac{1}{2}\widehat{\beta}$, we obtain from Proposition 2.6 that, given $\epsilon>0$, there exists $R_\epsilon>0$ such that

$$N(U)\geq\frac{1}{2}\widehat{\beta}|U|^2_{L^2}-\epsilon\|U\|^2\quad\forall\|U\|\geq R_\epsilon\,,$$

hence

$$I(U)\leq\frac{1}{2}(\|U\|^2-\widehat{\beta}|U|^2_{L^2})+\epsilon\|U\|^2\quad\forall\|U\|\geq R_\epsilon\,.$$

Therefore, as $\frac{1}{2}(\|U\|^2-\widehat{\beta}|U|^2_{L^2})\leq -m_{\widehat{\beta}}\|U\|^2\ \forall U\in E^-\oplus E^0_{\lambda_k}$, we can pick $0<\epsilon<m_{\widehat{\beta}}$ to get

$$I(U)\leq(-m_{\widehat{\beta}}+\epsilon)\|U\|^2<0\quad\forall\|U\|\geq R_\epsilon\,,\ U\in E^-\oplus E^0_{\lambda_k}\,. \tag{27}$$

Estimates (25)-(27) show that the functional I exhibits the *geometry* required by the Generalized Mountain-Pass Theorem (Thm 5.3 in [12]). Moreover, as shown in [2], a deformation theorem can be proved with condition (C) replacing the Palais-Smale condition (PS) and it turns out that the Generalized Mountain-Pass Theorem holds true under condition (C) (see [10] for details). Thus, in view of Proposition 2.4 (ii), we may conclude from (25)-(27) that I possesses a critical point $\hat{U} \in E$ with $I(\hat{U}) \geq \omega > 0$. In particular, $\hat{U} \neq 0$ since $I(0) = 0$ by (24) and (26). The proof of Theorem 1.1 is now complete.

3 Final Comments

In this section we make some comments regarding extensions of problem (P), the global assumptions $(F_1)_\mu$, $(F_2)_\nu$, and we present a simple example which illustrates the difference between these assumptions.

1) Using the method of [7], we could extend our results to include noncooperative systems of the form

$$(\hat{P}) \qquad \left\{ \begin{array}{rcl} -\Delta u + a(x)u + \delta v &=& f(x,u,v) \text{ in } \mathbb{R}^N \\ -\Delta v - \delta u + b(x)v &=& -g(x,u,v) \text{ in } \mathbb{R}^N \end{array} \right. ,$$

where $\delta > 0$ is given and $(f, g) = \nabla F$. In this case the corresponding functional $I : E \to \mathbb{R}$ is *strongly indefinite* and care should be taken in proving the required *linking* condition of the Generalized Mountain-Pass Theorem.

2) In the *scalar* case, it is well known that problem (P) arises naturally in connection with *standing wave* solutions of nonlinear Schrödinger Equations (see [4, 15])

$$i\frac{\partial \phi}{\partial t} = -\Delta\phi + V(x)\phi + g(|\phi|^2)\phi\,,\ x \in \mathbb{R}^N\,,\ t > 0\,,$$

that is, when one seeks time-periodic solutions of the form $\phi(x,t) = e^{-i\omega t}u(x)$ for some $\omega \in \mathbb{R}$. Indeed, in this case the function $u(x)$ must satisfy $-\Delta u + a(x)u = f(u)$ with $a(x) = V(x) - \omega$ and $f(u) = -g(|u|^2)u$. The corresponding functional is then given by

$$I(u) = \int \frac{1}{2}[|\nabla u|^2 + a(x)u^2 + G(|u|^2)]\ dx\ ,$$

where $G(s) = \int_0^s g(\sigma)\,d\sigma$.

3) As already noted in Remark 2.5, condition $(F_1)_\mu$ with $\mu > 2$ implies $(F_2)_\mu$ provided that

$$\liminf_{|U|\to 0} \frac{F(x,U)}{|U|^\mu} \geq a_+ > 0\,,$$

where we recall that the above limit is always nonnegative. One basic difference between these two *global* hypotheses is that, unlike $(F_1)_\mu$, condition $(F_2)_\nu$ is *insensitive* to quadratic terms. In particular, the coercive weight functions $a(x), b(x)$ in problem (P) do not have to be uniformly bounded away from zero.

4) Aside from showing the possibility of trading the *superquadraticity* condition $(F_1)_\mu$ for the *nonquadraticity* condition $(F_2)_\nu$, our approach shows that, in the "coercive" case, problem (P) behaves as if it were posed in a bounded domain $\Omega \subset \mathbb{R}^N$. We should mention that the more general case, in which $a(x)$ and $b(x)$ satisfy (A_0) but are not necessarily coercive, may indeed lack the "compactness" needed in our approach. In the *scalar* situation, by using *comparison arguments*, such a case was also treated by Rabinowitz in [14] under additional assumptions on $f(x,u)$.

5) Finally, we present an example that illustrates the difference between $(F_1)_\mu$ and $(F_2)_\nu$. Let

$$F_1(u) = u^2(\log|u| - 1)\,, \quad \text{for } |u| \geq 1\,.$$

It is not hard to show that F_1 can be extended to all of $\mathbb{R}$ as a function $F : \mathbb{R} \to \mathbb{R}$ of class C^2 such that $F^{(j)}(0) = 0$ for all $j \in \mathbb{N}$ and, for suitable $m > 0$ and $a > 0$, the function $\widehat{F}(u) = F(u) - m$ satisfies

$$u\widehat{F}'(u) - 2\widehat{F}(u) \geq a|u| \quad \forall u \in \mathbb{R}\,.$$

(For instance, define $F(u) = -e^{1-(1/|u|)}$ for $0 < |u| \leq 1$.) Therefore, in this example $\widehat{F}$ satisfies $(F_2)_\nu$ with $\nu = 1$ but it is *not superquadratic* and $(F_1)_\mu$ cannot hold with $\mu > 2$.

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David G. Costa
Department of Mathematical sciences, University of Nevada, Las Vegas, NV 89154.
Dpto. Matemática Universidade de Brasilia, 70910 Brasilia, DF Brazil
E-mail address: costa@nevada.edu