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A COMPARISON PRINCIPLE FOR A CLASS OF SUBPARABOLIC EQUATIONS IN GRUSHIN-TYPE SPACES

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Abstract. We define two notions of viscosity solutions to subparabolic equations in Grushin-type spaces, depending on whether the test functions concern only the past or both the past and the future. We then prove a comparison principle for a class of subparabolic equations and show the sufficiency of considering the test functions that concern only the past.

1. Background and Motivation

In [3], the author considered viscosity solutions to fully nonlinear subelliptic equations in Grushin-type spaces, which are sub-Riemannian metric spaces lacking a group structure. It is natural to consider viscosity solutions to subparabolic equations in this same environment. Our main theorem, found in Section 4, is a comparison principle for a class of subparabolic equations in Grushin-type spaces. We begin with a short review of the key geometric properties of Grushin-type spaces in Section 2 and in Section 3, we define two notions of viscosity solutions to subparabolic equations. Section 4 contains a parabolic comparison principle and the corollary showing the sufficiency of using test functions that concern only the past.

2. Grushin-type Spaces

We begin with $\mathbb{R}^n$, possessing coordinates $p = (x_1, x_2, \dots, x_n)$ and vector fields

$$X_i = \rho_i(x_1, x_2, \dots, x_{i-1}) \frac{\partial}{\partial x_i}$$

for $i = 2, 3, \dots, n$ where $\rho_i(x_1, x_2, \dots, x_{i-1})$ is a (possibly constant) polynomial. We decree that $\rho_1 \equiv 1$ so that

$$X_1 = \frac{\partial}{\partial x_1}.$$

A quick calculation shows that when $i < j$, the Lie bracket is given by

$$X_{ij} \equiv [X_i, X_j] = \rho_i(x_1, x_2, \dots, x_{i-1}) \frac{\partial \rho_j(x_1, x_2, \dots, x_{j-1})}{\partial x_i} \frac{\partial}{\partial x_j}.$$

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Because the ρ_i's are polynomials, at each point there is a finite number of iterations of the Lie bracket so that $\frac{\partial}{\partial x_i}$ has a non-zero coefficient. It follows that Hörmander's condition [6] is satisfied by these vector fields.

We may further endow $\mathbb{R}^N$ with an inner product (singular where the polynomials vanish) so that the span of the $\{X_i\}$ forms an orthonormal basis. This produces a sub-Riemannian manifold that we shall call g_n, which is also the tangent space to a generalized Grushin-type space G_n. Points in G_n will also be denoted by $p = (x_1, x_2, \dots, x_n)$. We observe that if $\rho_i \equiv 1$ for all i, then $g_n = G_n = \mathbb{R}^n$.

Given a smooth function f on G_n, we define the horizontal gradient of f as

$$\nabla_0 f(p) = (X_1 f(p), X_2 f(p), \dots, X_n f(p))$$

and the symmetrized second order (horizontal) derivative matrix by

$$((D^2 f(p))^\star)_{ij} = \frac{1}{2}(X_i X_j f(p) + X_j X_i f(p))$$

for $i, j = 1, 2, \dots n$.

Definition 2.1. The function $f : G_n \to \mathbb{R}$ is said to be C^1_{sub} if $X_i f$ is continuous for all $i = 1, 2, \dots, n$. Similarly, the function f is C^2_{sub} if $X_i X_j f(p)$ is continuous for all $i, j = 1, 2, \dots, n$.

Though G_n is not a Lie group, it is a metric space with the natural metric being the Carnot-Carathéodory distance, which is defined for points p and q as follows:

$$d_C(p, q) = \inf_\Gamma \int_0^1 \|\gamma'(t)\| dt.$$

Here Γ is the set of all curves γ such that $\gamma(0) = p$, $\gamma(1) = q$ and

$$\gamma'(t) \in \operatorname{span}\{\{X_i(\gamma(t))\}_{i=1}^n\}.$$

By Chow's theorem (see, for example, [1]) any two points can be joined by such a curve, which means $d_C(p, q)$ is an honest metric. Using this metric, we can define Carnot-Carathéodory balls and bounded domains in the usual way.

The Carnot-Carathéodory metric behaves differently at points where the polynomials ρ_i vanish. Fixing a point p_0, consider the n-tuple $r_{p_0} = (r^1_{p_0}, r^2_{p_0}, \dots, r^n_{p_0})$ where $r^i_{p_0}$ is the minimal number of Lie bracket iterations required to produce

$$[X_{j_1}, [X_{j_2}, [\cdots [X_{j_{r^i_{p_0}}}, X_i] \cdots](p_0) \neq 0.$$

Note that though the minimal length is unique, the iteration used to obtain that minimum is not. Note also that

$$\rho_i(p_0) \neq 0 \leftrightarrow r^i_{p_0} = 0.$$

Setting $R^i(p_0) = 1 + r^i_{p_0}$ we obtain the local estimate at p_0

$$d_C(p_0, p) \sim \sum_{i=1}^n |x_i - x_i^0|^{\frac{1}{R^i(p_0)}} \tag{2.1}$$

as a consequence of [1, Theorem 7.34]. Using this local estimate, we can construct a local smooth Grushin gauge at the point p_0, denoted $\mathcal{N}(p_0, p)$, that is comparable to the Carnot-Carathéodory metric. Namely,

$$(\mathcal{N}(p_0, p))^{2\mathcal{R}} = \sum_{i=1}^n (x_i - x_i^0)^{\frac{2\mathcal{R}}{R^i(p_0)}} \tag{2.2}$$

with

$$\mathcal{R}(p_0) = \prod_{i=1}^{n} R^i(p_0).$$

3. Subparabolic Jets and Solutions to Subparabolic Equations

In this section, we define and compare various notions of solutions to parabolic equations in Grushin-type spaces, in the spirit of [5, Section 8]. We begin by letting $u(p,t)$ be a function in $G_n \times [0,T]$ for some $T > 0$ and by denoting the set of $n \times n$ symmetric matrices by S^n. We consider parabolic equations of the form

$$u_t + F(t,p,u,\nabla_0 u,(D^2u)^\star) = 0 \tag{3.1}$$

for continuous and proper $F : [0,T] \times G_n \times \mathbb{R} \times g_n \times S^n \to \mathbb{R}$. Recall that F is proper means

$$F(t,p,r,\eta,X) \leq F(t,p,s,\eta,Y)$$

when $r \leq s$ and $Y \leq X$ in the usual ordering of symmetric matrices. [5] We note that the derivatives $\nabla_0 u$ and $(D^2u)^\star$ are taken in the space variable p. We call such equations *subparabolic*. Examples of subparabolic equations include the subparabolic P-Laplace equation for $2 \leq P < \infty$ given by

$$u_t + \Delta_P u = u_t - \operatorname{div}(\|\nabla_0 u\|^{P-2}\nabla_0 u) = 0$$

and the subparabolic infinite Laplace equation

$$u_t + \Delta_\infty u = u_t - \langle (D^2u)^\star \nabla_0 u, \nabla_0 u\rangle = 0.$$

Let $\mathcal{O} \subset G_n$ be an open set containing the point p_0. We define the parabolic set $\mathcal{O}_T \equiv \mathcal{O} \times (0,T)$. Following the definition of Grushin jets in [3], we can define the subparabolic superjet of $u(p,t)$ at the point $(p_0,t_0) \in \mathcal{O}_T$, denoted $P^{2,+}u(p_0,t_0)$, by using triples $(a,\eta,X) \in \mathbb{R} \times g_n \times S^n$ with $\eta = \sum_{i=1}^n \eta_j X_j$ and the ij-th entry of X denoted X_{ij}. We then have that $(a,\eta,X) \in P^{2,+}u(p_0,t_0)$ if

$$\begin{aligned}
u(p,t) \leq{}& u(p_0,t_0) + a(t-t_0) + \sum_{j\notin\mathcal{N}} \frac{1}{\rho_j(p_0)}(x_j - x_j^0)\eta_j \\
&+ \frac{1}{2}\sum_{j\notin\mathcal{N}} \frac{1}{(\rho_j(p_0))^2}(x_j - x_j^0)^2 X_{jj} \\
&+ \sum_{\substack{i,j\notin\mathcal{N}\\ i<j}} (x_i - x_i^0)(x_j - x_j^0)\big(\frac{1}{\rho_j(p_0)\rho_i(p_0)}X_{ij} - \frac{1}{2}\frac{1}{(\rho_j(p_0))^2}\frac{\partial \rho_j}{\partial x_i}(p_0)\eta_j\big) \\
&+ \sum_{k\in\mathcal{N}} \frac{1}{\beta}\sum_{j=1}^{n}(x_k - x_k^0)\frac{2}{\rho_j(p_0)}\big(\frac{\partial \rho_k}{\partial x_j}(p_0)\big)^{-1}X_{jk} + o(|t-t_0| + d_C(p_0,p)^2).
\end{aligned}$$

Here, as in [3], β is the number of non-zero terms in the final sum and we understand that if $\rho_j(p_0) = 0$ or $\frac{\partial \rho_{i_m}}{\partial x_j}(p_0) = 0$ then that term in the final sum is zero.

We define the subjet $P^{2,-}u(p_0,t_0)$ by

$$P^{2,-}u(p_0,t_0) = -P^{2,+}(-u)(p_0,t_0).$$

We also define the set theoretic closure of the superjet, denoted $\overline{P}^{2,+}u(p_0,t_0)$, by requiring $(a,\eta,X) \in \overline{P}^{2,+}u(p_0,t_0)$ exactly when there is a sequence

$$(a_n,p_n,t_n,u(p_n,t_n),\eta_n,X_n) \to (a,p_0,t_0,u(p_0,t_0),\eta,X)$$

with the triple $(a_n, \eta_n, X_n) \in P^{2,+}u(p_n, t_n)$. A similar definition holds for the closure of the subjet.

As in the subelliptic case, we may also define jets using the appropriate test functions. Namely, we consider the set $\mathcal{A}u(p_0, t_0)$ by

$$\mathcal{A}u(p_0, t_0) = \{\phi \in C^2_{\text{sub}}(\mathcal{O}_T) : u(p, t) - \phi(p, t) \leq u(p_0, t_0) - \phi(p_0, t_0) = 0\}$$

consisting of all test functions that touch from above. We define the set of all test functions that touch from below, denoted $\mathcal{B}u(p_0, t_0)$, by

$$\mathcal{B}u(p_0, t_0) = \{\phi \in C^2_{\text{sub}}(\mathcal{O}_T) : u(p, t) - \phi(p, t) \geq u(p_0, t_0) - \phi(p_0, t_0) = 0\}.$$

The following lemma is proved in the same way as the Euclidean version ([4] and [7]) except we replace the Euclidean distance $|p - p_0|$ with the local Grushin gauge $\mathcal{N}(p_0, p)$.

Lemma 3.1. *With the above notation, we have*

$$P^{2,+}u(p_0, t_0) = \{(\phi_t(p_0, t_0), \nabla_0\phi(p_0, t_0), (D^2\phi(p_0, t_0))^\star) : \phi \in \mathcal{A}u(p_0, t_0)\}$$

and

$$P^{2,-}u(p_0, t_0) = \{(\phi_t(p_0, t_0), \nabla_0\phi(p_0, t_0), (D^2\phi(p_0, t_0))^\star) : \phi \in \mathcal{B}u(p_0, t_0)\}.$$

We may now relate the traditional Euclidean parabolic jets found in [5] to the Grushin subparabolic jets via the following lemma.

Lemma 3.2. *Let the coordinates of the points $p, p_0 \in \mathbb{R}^n$ be $p = (x_1, x_2, \dots, x_n)$ and $p_0 = (x^0_1, x^0_2, \dots, x^0_n)$. Let $P^{2,+}_{\text{eucl}}u(p_0, t_0)$ be the traditional Euclidean parabolic superjet of u at the point (p_0, t_0) and let $(a, \eta, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n$ with $\eta = (\eta_1, \eta_2, \dots, \eta_n)$. Then*

$$(a, \eta, X) \in \overline{P}^{2,+}_{\text{eucl}}u(p_0, t_0)$$

gives the element

$$(a, \tilde{\eta}, \mathcal{X}) \in \overline{P}^{2,+}u(p_0, t_0)$$

where the vector $\tilde{\eta}$ is defined by

$$\tilde{\eta} = \sum_{i=1}^{n} \rho_i(p_0)\eta_i X_i$$

and the symmetric matrix $\mathcal{X}$ is defined by

$$\mathcal{X}_{ij} = \begin{cases} \rho_i(p_0)\rho_j(p_0)X_{ij} + \frac{1}{2}\frac{\partial \rho_j}{\partial x_i}(p_0)\rho_i(p_0)\eta_j & \text{if } i \leq j \\ \mathcal{X}_{ji} & \text{if } i > j. \end{cases}$$

The proof matches the subelliptic case in Grushin-type spaces as found in [3].

We then use these jets to define subsolutions and supersolutions to Equation (3.1).

Definition 3.3. Let $(p_0, t_0) \in \mathcal{O}_T$ be as above. The upper semicontinuous function u is a *viscosity subsolution* in $\mathcal{O}_T$ if for all $(p_0, t_0) \in \mathcal{O}_T$ we have $(a, \eta, X) \in P^{2,+}u(p_0, t_0)$ produces

$$a + F(t_0, p_0, u(p_0, t_0), \eta, X) \leq 0. \tag{3.2}$$

A lower semicontinuous function u is a *viscosity supersolution* in $\mathcal{O}_T$ if for all $(p_0, t_0) \in \mathcal{O}_T$ we have $(b, \nu, Y) \in P^{2,-}u(p_0, t_0)$ produces

$$b + F(t_0, p_0, u(p_0, t_0), \nu, Y) \geq 0. \tag{3.3}$$

A continuous function u is a *viscosity solution* in $\mathcal{O}_T$ if it is both a viscosity subsolution and viscosity supersolution.

We observe that the continuity of the function F allows Equations (3.2) and (3.3) to hold when $(a,\eta,X) \in \overline{P}^{2,+}u(p_0,t_0)$ and $(b,\nu,Y) \in \overline{P}^{2,-}u(p_0,t_0)$, respectively.

We also wish to define what [8] refers to as parabolic viscosity solutions. We first need to consider the sets

$$\mathcal{A}^- u(p_0,t_0) = \{\phi \in C^2_{\text{sub}}(\mathcal{O}_T) : u(p,t) - \phi(p,t) \le u(p_0,t_0) - \phi(p_0,t_0) = 0 \text{ for } t < t_0\}$$

consisting of all functions that touch from above only when $t < t_0$ and the set

$$\mathcal{B}^- u(p_0,t_0) = \{\phi \in C^2_{\text{sub}}(\mathcal{O}_T) : u(p,t) - \phi(p,t) \ge u(p_0,t_0) - \phi(p_0,t_0) = 0 \text{ for } t < t_0\}$$

consisting of all functions that touch from below only when $t < t_0$. Note that $\mathcal{A}^- u$ is larger than $\mathcal{A}u$ and $\mathcal{B}^- u$ is larger than $\mathcal{B}u$. These larger sets correspond physically to the past alone playing a role in determining the present.

We then have the following definition.

Definition 3.4. An upper semicontinuous function u on $\mathcal{O}_T$ is a *parabolic viscosity subsolution* in $\mathcal{O}_T$ if $\phi \in \mathcal{A}^- u(p_0,t_0)$ produces

$$\phi_t(p_0,t_0) + F(t_0,p_0,u(p_0,t_0),\nabla_0\phi(p_0,t_0),(D^2\phi(p_0,t_0))^\star) \le 0.$$

A lower semicontinuous function u on $\mathcal{O}_T$ is a *parabolic viscosity supersolution* in $\mathcal{O}_T$ if $\phi \in \mathcal{B}^- u(p_0,t_0)$ produces

$$\phi_t(p_0,t_0) + F(t_0,p_0,u(p_0,t_0),\nabla_0\phi(p_0,t_0),(D^2\phi(p_0,t_0))^\star) \ge 0.$$

A continuous function is a *parabolic viscosity solution* if it is both a parabolic viscosity supersolution and subsolution.

It is easily checked that parabolic viscosity sub(super-)solutions are viscosity sub(super-)solutions. The reverse implication will be a consequence of the comparison principle proved in the next section.

4. Comparison Principle

To prove our comparison principle, we will consider the function introduced in [3] given by $\varphi : G_n \times G_n \to \mathbb{R}$ given by

$$\varphi(p,q) = \sum_{i=1}^{n} \frac{1}{2^i}(x_i - y_i)^{2^i}$$

and show the existence of parabolic Grushin jet elements when considering subsolutions and supersolutions in G_n. This theorem is based on [5, Thm. 8.2], which details the Euclidean case.

Theorem 4.1. *Let u be a viscosity subsolution to Equation* (3.1) *and v be a viscosity supersolution to Equation* (3.1) *in the bounded parabolic set $\Omega \times (0,T)$ where Ω is a bounded domain. Let τ be a positive real parameter and let $\varphi(p,q)$ be as above. Suppose the local maximum of*

$$M_\tau(p,q,t) \equiv u(p,t) - v(q,t) - \tau\varphi(p,q)$$

occurs at the interior point (p_τ, q_τ, t_τ) *of the parabolic set* $\Omega\times\Omega\times(0,T)$*. Then, for each* $\tau > 0$*, there are elements* $(a, \tau\Upsilon_{p_\tau}, \mathcal{X}^\tau) \in \overline{P}^{2,+}u(p_\tau, t_\tau)$ *and* $(a, \tau\Upsilon q_\tau, \mathcal{Y}^\tau) \in \overline{P}^{2,-}v(q_\tau, t_\tau)$ *where*

$$(\Upsilon_{p_\tau})_i \equiv \rho_i(p_\tau)\frac{\partial\varphi(p_\tau, q_\tau)}{\partial x_i} = \rho_i(p_\tau)(x_i^\tau - y_i^\tau)^{2^i-1},$$
$$(\Upsilon_{q_\tau})_i \equiv -\rho_i(q_\tau)\frac{\partial\varphi(p_\tau, q_\tau)}{\partial y_i} = \rho_i(q_\tau)(x_i^\tau - y_i^\tau)^{2^i-1}$$

so that if

$$\lim_{\tau\to\infty} \tau\varphi(p_\tau, q_\tau) = 0,$$

then we have

$$|\,\|\Upsilon_{q_\tau}\|^2 - \|\Upsilon_{p_\tau}\|^2\,| = O(\varphi(p_\tau, q_\tau)^2), \tag{4.1}$$
$$\mathcal{X}^\tau \le \mathcal{Y}^\tau + \mathcal{R}^\tau \quad \textit{where } \lim_{\tau\to\infty}\mathcal{R}^\tau = 0. \tag{4.2}$$

We note that Equation (4.2) *uses the usual ordering of symmetric matrices.*

Proof. We first need to check that condition 8.5 of [5] is satisfied, namely that there exists an $r > 0$ so that for each M, there exists a C so that $b \le C$ when $(b, \eta, X) \in P^{2,+}_{\text{eucl}}u(p,t)$, $|p - p_\tau| + |t - t_\tau| < r$, and $|u(p,t)| + \|\eta\| + \|X\| \le M$ with a similar statement holding for $-v$. If this condition is not met, then for each $r > 0$, we have an M so that for all C, $b > C$ when $(b, \eta, X) \in P^{2,+}_{\text{eucl}}u(p,t)$. By Lemma 3.2 we would have

$$(b, \tilde{\eta}, \mathcal{X}) \in P^{2,+}u(p,t)$$

contradicting the fact that u is a subsolution. A similar conclusion is reached for $-v$ and so we conclude that this condition holds. We may then apply Theorem 8.3 of [5] and obtain, by our choice of φ,

$$(a, \tau D_p\varphi(p_\tau, q_\tau), X^\tau) \in \overline{P}^{2,+}_{\text{eucl}}u(p_\tau, t_\tau),$$
$$(a, -\tau D_q\varphi(p_\tau, q_\tau), Y^\tau) \in \overline{P}^{2,-}_{\text{eucl}}v(q_\tau, t_\tau).$$

Using Lemma 3.2 we define the vectors $\Upsilon_{p_\tau}(p_\tau, q_\tau)$ and $\Upsilon_{q_\tau}(p_\tau, q_\tau)$ by

$$\Upsilon_{p_\tau}(p_\tau, q_\tau) = \widetilde{D_p\varphi}(p_\tau, q_\tau),$$
$$\Upsilon_{q_\tau}(p_\tau, q_\tau) = -\widetilde{D_q\varphi}(p_\tau, q_\tau)$$

and we also define the matrices $\mathcal{X}$ and $\mathcal{Y}$ as in Lemma 3.2. Then by Lemma 3.2,

$$(a, \tau\Upsilon_{p_\tau}(p_\tau, q_\tau), \mathcal{X}^\tau) \in \overline{P}^{2,+}u(p_\tau, t_\tau),$$
$$(a, \tau\Upsilon_{q_\tau}(p_\tau, q_\tau), \mathcal{Y}^\tau) \in \overline{P}^{2,-}v(q_\tau, t_\tau).$$

Equations (4.1) and (4.2) are in [3, Lemma 4.2]. □

Using this theorem, we now define a class of parabolic equations to which we shall prove a comparison principle.

Definition 4.2. We say the continuous, proper function

$$F : [0,T]\times\overline{\Omega}\times\mathbb{R}\times g_n\times S^n \to \mathbb{R}$$

is *admissible* if for each $t \in [0,T]$, there is the same function $\omega : [0,\infty] \to [0,\infty]$ with $\omega(0+) = 0$ so that F satisfies

$$F(t,q,r,\nu,\mathcal{Y}) - F(t,p,r,\eta,\mathcal{X}) \le \omega\big(d_C(p,q) + \big|\, \|\nu\|^2 - \|\eta\|^2 \big| + \|\mathcal{Y}-\mathcal{X}\|\big). \quad (4.3)$$

We now formulate the comparison principle for the following problem.

$$u_t + F(t,p,u,\nabla_0 u,(D^2u)^\star) = 0 \quad \text{in } (0,T)\times\Omega \quad (4.4)$$

$$u(p,t) = h(p,t) \quad p \in \partial\Omega,\ t \in [0,T) \quad (4.5)$$

$$u(p,0) = \psi(p) \quad p \in \overline{\Omega} \quad (4.6)$$

Here, $\psi \in C(\overline{\Omega})$ and $h \in C(\overline{\Omega}\times[0,T))$. We also adopt the convention in [5] that a subsolution $u(p,t)$ to Problem (4.4)–(4.6) is a viscosity subsolution to (4.4), $u(p,t) \le h(p,t)$ on $\partial\Omega$ with $0 \le t < T$ and $u(p,0) \le \psi(p)$ on $\overline{\Omega}$. Supersolutions and solutions are defined in an analogous matter.

Theorem 4.3. *Let Ω be a bounded domain in G_n. Let F be admissible. If u is a viscosity subsolution and v a viscosity supersolution to Problem* (4.4)–(4.6) *then $u \le v$ on $[0,T)\times\Omega$.*

Proof. Our proof follows that of [5, Thm. 8.2] and so we discuss only the main parts.

For $\epsilon > 0$, we substitute $\tilde{u} = u - \frac{\varepsilon}{T-t}$ for u and prove the theorem for

$$u_t + F(t,p,u,\nabla_0 u,(D^2u)^\star) \le -\frac{\varepsilon}{T^2} < 0,$$
$$\lim_{t\uparrow T} u(p,t) = -\infty \quad \text{uniformly on } \overline{\Omega}$$

and take limits to obtain the desired result. Assume the maximum occurs at $(p_0,t_0) \in \Omega\times(0,T)$ with

$$u(p_0,t_0) - v(p_0,t_0) = \delta > 0.$$

Let

$$M_\tau = u(p_\tau,t_\tau) - v(q_\tau,t_\tau) - \tau\varphi(p_\tau,q_\tau)$$

with (p_τ,q_τ,t_τ) the maximum point in $\overline{\Omega}\times\overline{\Omega}\times[0,T)$ of $u(p,t)-v(q,t)-\tau\varphi(p,q)$. Using the same proof as [2, Lemma 5.2] we conclude that

$$\lim_{\tau\to\infty} \tau\varphi(p_\tau,q_\tau) = 0.$$

If $t_\tau = 0$, we have

$$0 < \delta \le M_\tau \le \sup_{\overline{\Omega}\times\overline{\Omega}} (\psi(p) - \psi(q) - \tau\varphi(p,q))$$

leading to a contradiction for large τ. We therefore conclude $t_\tau > 0$ for large τ. Since $u \le v$ on $\partial\Omega\times[0,T)$ by Equation (4.5), we conclude that for large τ, we have (p_τ,q_τ,t_τ) is an interior point. That is, $(p_\tau,q_\tau,t_\tau) \in \Omega\times\Omega\times(0,T)$. Using Lemma 3.2, we obtain

$$(a,\tau\Upsilon_{p_\tau}(p_\tau,q_\tau),\mathcal{X}^\tau) \in \overline{P}^{2,+}u(p_\tau,t_\tau),$$
$$(a,\tau\Upsilon_{q_\tau}(p_\tau,q_\tau),\mathcal{Y}^\tau) \in \overline{P}^{2,-}v(q_\tau,t_\tau)$$

satisfying the equations

$$a + F(t_\tau, p_\tau, u(p_\tau, t_\tau), \tau\Upsilon(p_\tau, q_\tau), \mathcal{X}^\tau) \le -\frac{\varepsilon}{T^2},$$
$$a + F(t_\tau, q_\tau, v(q_\tau, t_\tau), \tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}^\tau) \ge 0.$$

Using the fact that F is proper, the fact that $u(p_\tau, t_\tau) \ge v(q_\tau, t_\tau)$ (otherwise $M_\tau < 0$), and Equations (4.1) and (4.2), we have

$$\begin{aligned} 0 < \frac{\varepsilon}{T^2} &\le F(t_\tau, q_\tau, v(q_\tau, t_\tau), \tau\Upsilon_{q_\tau}(p_\tau, q_\tau), \mathcal{Y}^\tau) \\ &\quad - F(t_\tau, p_\tau, u(p_\tau, t_\tau), \tau\Upsilon_{p_\tau}(p_\tau, q_\tau), \mathcal{X}^\tau) \\ &\le \omega(d_C(p_\tau, q_\tau) + \tau|\, \|\Upsilon_q(p,q)\|^2 - \|\Upsilon_p(p,q)\|^2| + \|\mathcal{Y}^\tau - \mathcal{X}^\tau\|) \\ &= \omega(d_C(p_\tau, q_\tau) + C\tau\varphi(p_\tau, q_\tau) + \|\mathcal{R}_\tau\|). \end{aligned}$$

We arrive at a contradiction as $\tau \to \infty$. $\square$

We then have the following corollary, showing the equivalence of parabolic viscosity solutions and viscosity solutions.

Corollary 4.4. *For admissible F, we have the parabolic viscosity solutions are exactly the viscosity solutions.*

Proof. We showed above that parabolic viscosity sub(super-)solutions are viscosity sub(super-)solutions. To prove the converse, we will follow the proof of the subsolution case found in [8], highlighting the main details. Assume that u is not a parabolic viscosity subsolution. Let $\phi \in \mathcal{A}^- u(p_0, t_0)$ have the property that

$$\phi_t(p_0, t_0) + F(t_0, p_0, \phi(p_0, t_0), \nabla_0\phi(p_0, t_0), (D^2\phi(p_0, t_0))^\star) \ge \epsilon > 0$$

for a small parameter ϵ. Let $r > 0$ be sufficiently small so that the gauge $\mathcal{N}(p_0, p)$ is comparable to the distance $d_C(p_0, p)$. Define the gauge ball $B_{\mathcal{N}(p_0)}(r)$ by

$$B_{\mathcal{N}(p_0)}(r) = \{p \in G_n : \mathcal{N}(p_0, p) < r\}$$

and the parabolic gauge ball $S_r = B_{\mathcal{N}(p_0)}(r) \times (t_0 - r, t_0)$ and let ∂S_r be its parabolic boundary. Then the function

$$\tilde{\phi}_r(p, t) = \phi(p, t) + |t_0 - t|^{16R} - r^{16R} + (\mathcal{N}(p_0, p))^{16R}$$

is a classical supersolution for sufficiently small r. We then observe that $u \le \tilde{\phi}_r$ on ∂S_r but $u(p_0, t_0) > \tilde{\phi}(p_0, t_0)$. Thus, the comparison principle, Theorem 4.3, does not hold. Thus, u is not a viscosity subsolution. The supersolution case is identical and omitted. $\square$

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