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ASYMPTOTIC EXPANSIONS FOR LINEAR SYMMETRIC HYPERBOLIC SYSTEMS WITH SMALL PARAMETER

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Abstract. The boundary layer functions method of Lyusternik-Vishik is used to obtain asymptotic expansions of the solutions to the Cauchy problem for linear symmetric hyperbolic systems with constant coefficients as the small parameter ε tends to zero.

1. Introduction

We consider the following Cauchy problem, which will be called (P_ϵ),

$$(P_0+\varepsilon P_1)U = F(x,t), \quad x \in \mathbb{R}^d, t>0, \tag{1.1}$$

$$U(\varepsilon,x,0)=U_0(x), \quad x\in\mathbb{R}^d \tag{1.2}$$

where $P_i = A_i\partial_t + B_i(\partial_x)+G_i$, $B_i(\partial_x)=\sum_{j=1}^d B_{ij}\ \partial_{x_j}$, $i=0,1, B_i, G_i$ are real constant $n\times n$ matrices, $d\geq 1, \varepsilon>0$ is a small parameter, $U,F:\mathbb{R}^d\times[0,\infty)\to\mathbb{R}^n$,

$$A_0=\begin{pmatrix} I_m & 0\\ 0 & 0\end{pmatrix}, \quad A_1=\begin{pmatrix} 0 & 0\\ 0 & I_{n-m}\end{pmatrix}, \quad 0\leq m\leq n,$$

and I_k is a identity matrix.

We shall investigate the behavior of the solution $U(\varepsilon,x,t)$ to the perturbed system (P_ε) as $\varepsilon\to 0$. The main question of perturbation theory is if the solution $U(\varepsilon,x,t)$ to the perturbed system tends to the solution $U(0,x,t)$ of the unperturbed system as $\varepsilon\to 0$. The answer depends on the structure of the operator $P=P_0+\varepsilon P_1$ and also on the norm which determines the convergence. If the smooth solution $U(\varepsilon,x,t)\to U(0,x,t)$ uniformly on its domain of definition $\mathcal{D}$, then (P_0) is called a *regularly perturbed system*. In the opposite case, the system (P_0) is called *singularly perturbed*. In this case, there arises a subset of $\mathcal{D}$ in which the solution $U(\varepsilon,x,t)$ has a singular behavior relative to ε. This subset is called *the boundary layer*. The function which defines the singular behavior of $U(\varepsilon,x,t)$ relative to ε within the boundary layer is called *the boundary layer function*. At present the investigations of the singularly perturbed problems are very much advanced. We refer the reader to sources [1] - [8], which contain a very large bibliography and also a survey of the results in the perturbation theory connected with the partial differential equations.

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Here we develop the results of the paper [9] in the d-dimensional case. We obtain the asymptotic expansions for the solutions $U(\varepsilon, x, t)$ on the positive power of the small parameter ε when the matrices B_i are symmetric, i.e. the operator P_ε is the hyperbolic one.

Below we use the following notations. For $s \in \mathbb{R}$ we denote by H^s the usual Sobolev spaces with the scalar product $(u,v)_s = \int_{\mathbb{R}^n}(1+\xi^2)^s \hat{u}(\xi)\bar{\hat{v}}(\xi)\,d\xi$, where $\hat{u} = F[u]$ and $F^{-1}[u]$ are the direct and the inverse Fourier transforms of u in S'. $H_n^s = (H^s)^n$ is the Hilbert space equipped with the scalar product $(f_1, f_2)_{s,n} = \sum_{j=1}^n (f_{1j}, f_{2j})_s$, $f_i = (f_{i1}, ..., f_{in}), i = 1,2$ and with the norm $\|\cdot\|_{s,n}$ generated by this scalar product. Let $\mathcal{D}'((a,b), X)$ be the space of vectorial distributions on (a,b) with values in Banach space X. Then for $k \in \mathbb{N}^*$ and $1 \le p \le \infty$ we set $W^{k,p}(a,b;X) = \{u \in \mathcal{D}'((a,b);X);\ u^{(j)} \in L^p(a,b;X), j = 0,1,\dots,k\}$, where $u^{(j)}$ is the distributional derivative of order j. If $k=0$ we set $W^{0,p}(a,b;X) = L^p(a,b;X)$. Let us denote $A = A_0 + \varepsilon A_1$, $B = B_0 + \varepsilon B_1$, $G = G_0 + \varepsilon G_1$, $L_j = B_j(\partial_x) + G_j$, $j = 0,1$, where $\partial_x = (\partial/\partial_{x_1}, \dots, \partial/\partial_{x_d})$. The special forms of matrices A_0 and A_1 involve the natural representations of matrices B_i, G_i by blocks

$$B_j = \begin{pmatrix} B_{j1} & B_{j2} \\ B_{j2}^* & B_{j3} \end{pmatrix}, \quad G_j = \begin{pmatrix} G_{j1} & G_{j2} \\ G_{j2}^* & G_{j3} \end{pmatrix}, \quad j = 0,1,$$

and $B_{j1}(\xi), G_{j1} \in M^m(\mathbb{R})$, $B_{j2}(\xi), G_{j2} \in M^{m\times(n-m)}(\mathbb{R})$, $B_{j3}(\xi), G_{j3} \in M^{n-m}(\mathbb{R})$, and "*" means transposition. Denote $L_{ij}(\partial_x) = B_{ij}(\partial_x) + G_{ij}$, $i = 0,1$, $j = 1,2,3$, and $F = \mathrm{col}(f,g)$, $U_0 = \mathrm{col}(u_0, u_1)$, where $f, u_0 \in M^{m\times 1}(\mathbb{R})$, $g, u_1 \in M^{(n-m)\times 1}(\mathbb{R})$.

Let us formulate the main assumptions to be used in the sequel.

(H1) $B_i(\xi)$, G_i, $i = 0,1$, are real symmetric matrices for $\xi \in \mathbb{R}^n$;

(H2) $(G\xi, \xi)_{\mathbb{R}^n} \ge (G_{03}\eta, \eta)_{\mathbb{R}^{n-m}} \ge q_0|\eta|^2$, with $q_0 > 0$, for all $\xi = (\xi', \eta) \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^{n-m}$.

Under the hypothesis (H1), the system (P_ε) is symmetric of the hyperbolic type. According to [7], the analysis of systems (P_0) and (P_ε) shows that:

a) If $m = n$, then the system (P_0) is of the hyperbolic type, regularly perturbed because in this case the boundary layer function is zero;

b) If $m = 0$, then the system (P_0) is of the elliptic type, singularly perturbed;

c) If $0 < m < n$, then the system (P_0) is well-posed in the sense of Petrovskii, singularly perturbed. In particular, if $\det B_{03} \ne 0$ and $B_{02} = 0$, then the system (P_0) is of the elliptic- parabolic type.

In the following section we shall give the formal asymptotic expansions of the solutions to the problem (P_ε) on the positive powers of the small parameter ε. The last two sections contain the validity of these formal expansions which lead to the main result theorem 3.5.

2. Formal asymptotic expansions

According to the method of Lyusternik-Vishik [2], for the solution $U(\varepsilon, x, t)$ to the problem (P_ε) we postulate the following asymptotic expansion

$$U(\varepsilon, x, t) = \sum_{k=0}^{N} \varepsilon^k (V_k(x,t) + Z_k(x,\tau)) + R_N(\varepsilon, x, t), \quad \tau = \frac{t}{\varepsilon}, \tag{2.1}$$

where $Z(x,\tau) = Z_0(x,\tau) + \cdots + \varepsilon^N Z_N(x,\tau)$ is the boundary layer function. It describes the singular behavior of solution $U(\varepsilon,x,t)$ relative to ε within a neighborhood of the set $\{(x,0), x \in \mathbb{R}^d\}$ which is the boundary layer. The function $V(x,t) = V_0(x,t) + \cdots + \varepsilon^N V_N(x,t)$ is the regular part of expansion (2.1). Usually function $Z(x,\tau)$ is considered small in some sense for large τ, i.e. $Z \to 0$ as $\tau \to \infty$. On the other hand, because $U(\varepsilon,x,t) \not\to U(0,x,t)$ as $\varepsilon \to 0$ within the boundary layer, then the function $Z(x,\tau)$ has to reduce the discrepancy between $U(\varepsilon,x,0)$ and $U(0,x,0)$.

Now, we formally substitute expansion (2.1) into (1.1) and identify the coefficients of the same powers of ε which contain the same variables. Then we get the following equations:

$$P_0 V_k = F_k(x,t), \quad x \in \mathbb{R}^d,\ t > 0, \tag{2.2}$$

where $F_0 = F$, $F_k = -P_1 V_{k-1}$, $k = 1,\dots,N$,

$$\begin{gathered} A_0 \partial_\tau Z_k = \mathcal{F}_k(x,\tau), \quad k = 0,1,\dots,N, \\ A_1(L_0 Z_N + L_1 Z_{N-1} + \partial_\tau Z_N) = 0, \quad x \in \mathbb{R}^d, \tau > 0, \end{gathered} \tag{2.3}$$

where $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = -L_0 Z_0 - A_1 \partial_\tau Z_0$, $\mathcal{F}_k = -L_0 Z_{k-1} - L_1 Z_{k-2} - A_1 \partial_\tau Z_{k-1}$, $k = 2,\dots,N$, and

$$(P_0 + \varepsilon P_1) R_N = \mathcal{F}(x,t,\varepsilon), \quad x \in \mathbb{R}^d,\ t > 0, \tag{2.4}$$

where $\mathcal{F} = -\varepsilon^{N+1}(P_1 V_N + L_1 Z_N) - \varepsilon^N A_0(L_0 Z_N + L_1 Z_{N-1})$.

Similarly, substituting (2.1) into initial condition (1.2) we obtain

$$R_N(\varepsilon,x,0) = 0, \quad x \in \mathbb{R}^d, \tag{2.5}$$

$$V_0(x,0) + Z_0(x,0) = U_0(x), \quad x \in \mathbb{R}^d, \tag{2.6}$$

$$V_k(x,0) + Z_k(x,0) = 0, \quad x \in \mathbb{R}^d,\ k = 1,\dots,N. \tag{2.7}$$

Let

$$Z_k = \begin{pmatrix} X_k \\ Y_k \end{pmatrix}, \quad V_k = \begin{pmatrix} v_k \\ w_k \end{pmatrix}, \quad F_k = \begin{pmatrix} f_k \\ g_k \end{pmatrix}, \quad \mathcal{F}_k = \begin{pmatrix} \mathcal{F}_{k1} \\ \mathcal{F}_{k2} \end{pmatrix},$$

where $X_k, v_k, f_k, \mathcal{F}_{k1} \in M^{m\times 1}(\mathbb{R})$, $Y_k, w_k, g_k, \mathcal{F}_{k2} \in M^{(n-m)\times 1}(\mathbb{R})$. Then from (2.3), (2.6), and (2.7) for X_k and Y_k, we get

$$\partial_\tau X_k = \mathcal{F}_{k1}, \quad X_k \to 0, \quad \tau \to +\infty, \tag{2.8}$$

and

$$\begin{gathered} \partial_\tau Y_k + L_{03} Y_k = \mathcal{F}_{k2}(x,\tau), \quad x \in \mathbb{R}^d, \tau > 0 \\ Y_k(x,0) = \begin{cases} u_1(x) - w_0(x,0), & \text{for } k = 0, \\ -w_k(x,0) & \text{for } k = 1,\dots,N,\ x \in \mathbb{R}^d, \end{cases} \end{gathered} \tag{2.9}$$

where

$$\begin{gathered} \mathcal{F}_{01} = 0, \quad \mathcal{F}_{11} = -L_{01} X_0 - L_{02} Y_0, \quad \mathcal{F}_{k1} = -L_{01} X_{k-1} - L_{02} Y_{k-1} \\ -L_{11} X_{k-2} - L_{12} Y_{k-2}, \quad k = 2,\dots,N, \\ \mathcal{F}_{02} = -L_{02}^* X_0, \quad \mathcal{F}_{k2} = -L_{02}^* X_k - L_{13} Y_{k-1} - L_{12}^* X_{k-1}, \\ L_{ij}^*(\xi) = B_{ij}^*(\xi) + G_{ij}^*, \quad k = 1,\dots,N. \end{gathered}$$

Similarly, from (2.2) and (2.6), (2.7) we obtain the problems for v_k and w_k,

$$\begin{gathered}\partial_t v_k + L_{01}v_k + L_{02}w_k = f_k(x,t),\\ L_{02}^* v_k + L_{03}w_k = g_k(x,t), \quad x\in\mathbb{R}^d,\ t>0,\\ v_k(x,0) = \begin{cases} u_0(x) - X_0(x,0), & \text{for } k=0,\\ -X_k(x,0), & \text{for } k=1,\dots,N, \quad x\in\mathbb{R}^d,\end{cases}\end{gathered} \tag{2.10}$$

Thus, we have obtained the problems for the functions X_k, Y_k, v_k, w_k and R_N. In the following sections we shall present the validity of the expansion (2.1).

3. Justification of expansion (2.1)

To study the problem (2.10) we examine the problem

$$\begin{gathered}\partial_t v + L_{01}v + L_{02}w = f(x,t),\\ L_{02}^* v + L_{03}w = g(x,t), \quad x\in\mathbb{R}^d,\ t>0,\\ v(x,0) = h(x), \quad x\in\mathbb{R}^d,\end{gathered} \tag{PV}$$

which is of the same type. To obtain the solvability of this problem and the regularity of their solutions we pass to the following problem for $\hat v$ and $\hat w$

$$\begin{gathered}\partial_t \hat v + (G_{01} + i|\xi| b_{01}(\xi))\hat v + (G_{02} + i|\xi| b_{02}(\xi))\hat w = \hat f(\xi,t),\\ (G_{02}^* + i|\xi| b_{02}^*(\xi))\hat v + (G_{03} + i|\xi| b_{03}(\xi))\hat w = \hat g(\xi,t),\\ \hat v(\xi,0) = \hat h(\xi).\end{gathered} \tag{$P\hat V$}$$

where $b_{ij}(\xi) = B_{ij}(\xi/|\xi|)$.

The following two lemmas will be proved in the following section.

Lemma 3.1. *Under the hypotheses (H1), (H2) the matrix $G_{03} + i|\xi| b_{03}(\xi)$ is invertible for $\xi\in\mathbb{R}^d$ and the function $\xi \to (G_{03} + i|\xi| b_{03}(\xi))^{-1}$ is bounded on $\mathbb{R}^d$.*

From Lemma 3.1 the problem $(P\hat V)$ receives the form

$$\begin{gathered}\frac{d}{dt}\hat v(\xi,t) + K(\xi)\hat v(\xi,t) = H(\xi,t),\\ \hat v(\xi,0) = \hat h(\xi),\end{gathered} \tag{3.1}$$

$$\hat w(\xi,t) = (G_{03} + i|\xi| b_{03}(\xi))^{-1}(\hat g(\xi,t) - (G_{02}^* + i|\xi| b_{02}^*(\xi))\hat v(\xi,t)), \tag{3.2}$$

where

$$\begin{aligned} K(\xi) =& G_{01} + i|\xi| b_{01}(\xi)\\ &- (G_{02} + i|\xi| b_{02}(\xi))(G_{03} + i|\xi| b_{03}(\xi))^{-1}(G_{02}^* + i|\xi| b_{02}^*(\xi))\\ H(\xi,t) =& \hat f(\xi,t) - (G_{02} + i|\xi| b_{02}(\xi))(G_{03} + i|\xi| b_{03}(\xi))^{-1}\hat g(\xi,t).\end{aligned} \tag{3.3}$$

Lemma 3.2. *Under the hypotheses (H1), (H2) the matrix $K(\xi)$ can be represented in the form*

$$K(\xi) = K_0(\xi) + i|\xi| K_1(\xi) + |\xi|^2 K_2(\xi), \quad \xi\in\mathbb{R}^d, \tag{3.4}$$

where the functions $\xi \to K_j(\xi)$, $j = 0,1,2$ are bounded on $\mathbb{R}^d$, K_1, K_2 are real symmetric and $K_2 \geq 0$.

These lemmas permit us to prove the following proposition.

Proposition 3.3. *Let the hypotheses (H1), (H2) be fulfilled and* $l \in \mathbb{N}^*$. *If* $h \in H_m^{s+2l+1}$, $F = \mathrm{col}(f,g) \in W^{l,1}(0,T;H_n^{s+2})$, *then there exists a unique strong solution* $V = \mathrm{col}(v,w) \in W^{l,\infty}(0,T;H_n^s)$ *of the problem (PV) and*

$$\|V\|_{W^{l,\infty}(0,T;H_n^s)} \le C(T)(\|h\|_{s+2l+1,m} + \|F\|_{W^{l,1}(0,T;H_n^{s+2})}). \tag{3.5}$$

Proof. Consider the Cauchy problem

$$\frac{d}{dt}\hat{v}(t) + K(\xi)\hat{v}(t) = 0, \quad \hat{v}(0) = \hat{h}, \quad 0 < t < T, \tag{3.6}$$

in the Hilbert space $H = \{f = (f_1,\dots,f_m);\ (1+|\xi|^2)^{\frac{s}{2}} f_k(\xi) \in L^2(\mathbb{R}^d),\ k = 1,\dots,m\}$, equipped with the scalar product $(f,g)_H = \int_{\mathbb{R}^d} (1+|\xi|^2)^s (f,\bar{g})_{\mathbb{R}^m}\, d\xi$. The representation (3.4) shows that the operator $-K(\xi) : H \to H$ satisfies the conditions

$$\mathrm{Re}(-Kf,f)_H \le \omega(f,f)_H, \quad \mathrm{Re}(-\bar{K}^* f,f)_H \le \omega(f,f)_H, \quad f \in H,$$

where $\omega = sup_{\xi\in\mathbb{R}^d}\|K_0(\xi)\|_{\mathbb{R}^m\to\mathbb{R}^m} + \delta$ with some $\delta > 0$. This means that the operator $-(K+\omega I)$ is maximal dissipative on H. According to [10] the Cauchy problem (3.6) generates a C_0 semigroup of operators $\{\hat{T}(t),\ t \ge 0\}$ on H. Since

$$\frac{d}{dt}\|\hat{v}(\cdot,t)\|_H^2 \le -(K_0\hat{v}(\cdot,t),\hat{v}(\cdot,t))_H - (\hat{v}(\cdot,t),K_0\hat{v}(\cdot,t))_H \le 2\omega\|\hat{v}(\cdot,t)\|_H^2,$$

we have $\|\hat{v}(\cdot,t)\|_H \le e^{\omega t}\|h\|_H$ for any $h \in H$, i.e. $\|\hat{T}(t)\| \le e^{\omega t}$. Due to Parseval's equality we get that the Cauchy problem

$$\frac{d}{dt}v(t) + \check{K}v(t) = 0, \quad v(0) = v_0, \quad 0 < t < T, \quad (F[\check{K}v] = K(\xi)\hat{v})$$

generates a C_0 semigroup of operators $\{T(t), t \ge 0\}$ on H_m^s, such that $v(\cdot,t) = T(t)v_0$ and $\|T(t)\| \le e^{\omega t}$. Then the semigroup $T_0(t) = T(t)e^{-\omega t}$ solves the Cauchy problem

$$\frac{d}{dt}z(t) + (\check{K}+\omega I)z(t) = f(t)e^{\omega t}, \quad z(0) = y_0, \quad 0 < t < T. \tag{3.7}$$

According to [11] for every $y_0 \in H_m^s$ and $f \in L^1(0,T;H_m^s)$ there exists a unique mild solution of this problem $z \in C([0,T];H_m^s)$, such that

$$z(t) = T_0(t)y_0 + \int_0^t T_0(t-s)f(s)e^{\omega s}\, ds$$

and hence

$$\|z\|_{C([0,T];H_m^s)} \le \|y_0\|_{s,m} + \|f\|_{L^1(0,T;H_m^s)}e^{\omega T}$$

Moreover, if $y_0 \in H_m^{s+2l}, f \in W^{l,1}(0,T;H_m^s)$ and $l \in \mathbb{N}^*$, then z is a strong solution of the problem (3.7), $z \in W^{l,\infty}(0,T;H_m^s)$ and

$$\|z\|_{W^{l,\infty}(0,T;H_m^s)} \le C(T)(\|y_0\|_{s+2l,m} + \|f\|_{W^{l,1}(0,T;H_m^s)}).$$

Note that the solution y to the Cauchy problem

$$\frac{d}{dt}y(t) + \check{K}y(t) = f, \quad y(0) = y_0, \quad 0 < t < T,$$

and the solution z to the problem (3.7) are connected by means of the equality $y(t) = e^{-\omega t}z(t)$. Consequently, for the same y_0, f and $l \in N^*$ we have

$$\|y\|_{W^{l,\infty}(0,T;H^s_m)} \le C(T)(\|y_0\|_{s+2l,m} + \|f\|_{W^{l,1}(0,T;H^s_m)}).$$

In view of (3.1), using the last estimate and boundedness of the matrix $(G_{03} + i|\xi|b(\xi))^{-1}$ we obtain the estimate

$$\|v\|_{W^{l,\infty}(0,T;H^s_m)} \le C(T)(\|h\|_{s+2l,m} + \|f\|_{W^{l,1}(0,T;H^s_m)} + \|g\|_{W^{l,1}(0,T;H^{s+1}_{n-m})}). \tag{3.8}$$

From (3.2) and (3.8) we get the estimate

$$\|w\|_{W^{l,\infty}(0,T;H^s_m)} \le C(T)(\|h\|_{s+2l+1,m} + \|f\|_{W^{l,1}(0,T;H^{s+1}_m)} + \|g\|_{W^{l,1}(0,T;H^{s+2}_{n-m})}). \tag{3.9}$$

Now, the estimates (3.8) and (3.9) imply the estimate (3.5). Proposition 3.3 is proved.

Consider the Cauchy problem

$$\begin{aligned} \partial_\tau Y + L_{03}Y &= \mathcal{F}(x,\tau), \quad x \in \mathbb{R}^d,\ \tau > 0, \\ Y(x,0) &= y_0(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{PY}$$

Proposition 3.4. *Let hypotheses (H1), (H2) be fulfilled and $l \in \mathbb{N}^*$. If $y_0 \in H^{s+l}_{n-m}$, $\mathcal{F} \in W^{l,1}_{\mathrm{loc}}(0,\infty;H^s_{n-m})$, then there exists a unique strong solution $Y \in W^{l,\infty}_{\mathrm{loc}}(0,\infty;H^s_{n-m})$ of the problem (PY). For this solution*

$$\begin{aligned} \|\partial^l_\tau Y(\cdot,\tau)\|_{s,n-m} \le{}& Ce^{-q_0\tau}(\|y_0\|_{s+l,n-m} + \sum_{\nu=0}^{l-1}\|\partial^\nu_\tau \mathcal{F}(\cdot,0)\|_{s+l-\nu-1,n-m} \\ &+ \int_0^\tau e^{q_0\theta}\|\partial^l_\tau \mathcal{F}(\cdot,\theta)\|_{s,n-m}\,d\theta) \end{aligned} \tag{3.10}$$

Proof. Under the hypotheses (H1), (H2) the operator $-L_{03}(\partial_x)$ is dissipative and generates the C_0 semigroup of contractions $S(\tau)$ on H^s_{n-m}. Then there exists a unique mild solution $Y \in C([0,\infty);H^s_{n-m})$ of Cauchy problem (PY). In the usual way it is not difficult to obtain the estimate $\|S(\tau)\| \le e^{-q_0\tau}, \tau \ge 0$. This estimate and formula

$$Y(\cdot,\tau) = S(\tau)y_0 + \int_0^\tau S(\theta)\mathcal{F}(\cdot,\tau-\theta)\,d\theta$$

involve the estimate (3.10) in the case $l = 0$. In the cases $l \ge 1$ the estimate (3.10) will be obtained by differentiating relative to τ the equation from (PY). Proposition 3.4 is proved.

Due to these propositions, we can determine the functions V_k and Z_k. Indeed, if $k = 0$, then from (2.8) it follows that $X_0 = 0$. Then from (2.10), due to Proposition

3.3, we find the main regular term $V_0 = \mathrm{col}(v_0, w_0)$ of expansion (2.1). Instantly, we have

$$w_0(x,0) = F^{-1}[(G_{03} + i|\xi|b_{03}(\xi))^{-1}(\hat{g}(\xi,0) - (G^*_{02} + i|\xi|b^*_{02}(\xi))\hat{u}_0(\xi))].$$

Moreover, Lemma 3.1 and the Parseval equality permit us to obtain the estimate

$$\|w_0(\cdot,0)\|_{s,n-m} \le C(\|g(\cdot,0)\|_{s,n-m} + \|u_0\|_{s+1,m}) \le C(\|U_0\|_{s+1,n} + \|F(\cdot,0)\|_{s,n}). \tag{3.11}$$

Due to proposition 3.4, this fact permits us to define the function Y_0 as a solution of Cauchy problem (2.9). Moreover, from (3.10) and (3.11) we have

$$\|\partial^l_\tau Y_0(\cdot,\tau)\|_{s,n-m} \le Ce^{-q_0\tau}(\|U_0\|_{s+l+1,n} + \|F(\cdot,0)\|_{s+l,n}). \tag{3.12}$$

Thus, we have defined the main singular term $Z_0 = \mathrm{col}(0, Y_0)$ of expansion (2.1).

Let us define the next terms of this expansion. Suppose that the terms $V_0, \dots, V_{k-1}$ and $Z_0, \dots, Z_{k-1}$ are already found. We shall find the terms V_k and Z_k and show that the estimates

$$\begin{aligned}\|V_k\|_{W^{l,\infty}(0,T;H^s_n)} \le{}& C(T)(\|U_0\|_{s+2l+3k+1,n} \\ &+ \|F(\cdot,0)\|_{s+2l+3k-2,n} + \|F\|_{W^{l,1}(0,T;H^{s+3k+2}_n)}),\end{aligned} \tag{3.13}$$

and

$$\|\partial^l_\tau Z_k(\cdot,\tau)\|_{s,n} \le Ce^{-q_0\tau}(1+\tau^k)(\|U_0\|_{s+l+k+1,n} + \|F(\cdot,0)\|_{s+l+k,n}) \tag{3.14}$$

hold, supposing that such estimates are true for previous terms. Note, that the estimates (3.13), (3.14) for V_0 and Z_0 follow from (3.5) and (3.12).

At first, solving the problem (2.8), we get $X_k(\cdot,\tau) = -\int_\tau^\infty \mathcal{F}_{k1}(\cdot,\theta)\,d\theta$, where the integral exists due to the estimate (3.14) for Z_{k-1}. From this formula using (3.14) for Z_{k-1} and for Z_{k-2} we obtain

$$\begin{aligned}\|\partial^l_\tau X_k(\cdot,\tau)\|_{s,m} ={}& \|\partial^{l-1}_\tau \mathcal{F}_{k1}(\cdot,\tau)\|_{s,m} \\ \le{}& C(\|\partial^{l-1}_\tau Z_{k-1}(\cdot,\tau)\|_{s+1,n} + \|\partial^{l-1}_\tau Z_{k-2}(\cdot,\tau)\|_{s+1,n}) \\ \le{}& Ce^{-q_0\tau}(1+\tau^{k-1})(\|U_0\|_{s+l+k,n} + \|F(\cdot,0)\|_{s+l+k-1,n})\end{aligned} \tag{3.15}$$

for $l \ge 1$. Similarly we get the estimate (3.15) in the case $l = 0$.

Because $v_k(\cdot,0) = -X_k(\cdot,0)$, due to Proposition 3.3 we solve the problem (2.10) and find V_k. Moreover, using (3.5), (3.13) for V_{k-1}, (3.15) for X_k and the estimate

$$\|V_k\|_{W^{l,\infty}(0,T;H^s_n)} \le C(T)(\|X_k(\cdot,0)\|_{s+2l+1,m} + \|V_{k-1}\|_{W^{l,\infty}(0,T;H^{s+3}_n)}),$$

we obtain the estimate (3.13) for V_k.

Instantly, we find

$$w_k(x,0) = F^{-1}[(G_{03} + i|\xi|b_{03}(\xi))^{-1}(\hat{g}_k(\xi,0) - (G^*_{02} + i|\xi|b^*_{02}(\xi))\hat{X}_k(\xi,0))]$$

and establish the estimate

$$\begin{aligned}\|w_k(\cdot,0)\|_{s,n-m} \le{}& C(\|g_k(\cdot,0)\|_{s,n-m} + \|X_k(\cdot,0)\|_{s+1,m}) \\ \le{}& C(\|X_{k-1}(\cdot,0)\|_{s+1,m} + \|X_k(\cdot,0)\|_{s+1,m} \\ &+ \|w_{k-1}(\cdot,0)\|_{s+1,n-m}) \\ \le{}& C(\|U_0\|_{s+k+1,n} + \|F(\cdot,0)\|_{s+k,n}).\end{aligned} \tag{3.16}$$

Also, using (3.14) for Z_{k-1} and (3.15) for X_k we have

$$\begin{aligned}\|\partial_\tau^l \mathcal{F}_{k2}(\cdot,\tau)\|_{s,n-m} \le& C(\|\partial_\tau^l X_k(\cdot,\tau)\|_{s+1,m} + \|\partial_\tau^l Z_{k-1}(\cdot,\tau)\|_{s+1,n}) \\ \le& Ce^{-q_0\tau}(1+\tau^{k-1})(\|U_0\|_{s+l+k+1,n} + \|F(\cdot,0)\|_{s+l+k,n}).\end{aligned} \tag{3.17}$$

From (3.10), (3.16) and (3.17) follows the estimate

$$\begin{aligned}\|\partial_\tau^l Y_k(\cdot,\tau)\|_{s,n-m} \le& Ce^{-q_0\tau}(\|w_k(\cdot,0)\|_{s+l,n-m} + \sum_{\nu=0}^{l-1} \|\partial_\tau^\nu \mathcal{F}_{k2}(\cdot,0)\|_{s+l-\nu-1,n-m} \\ &+ \int_0^\tau e^{q_0\theta}\|\partial_\tau^l \mathcal{F}_{k2}(\cdot,\theta)\|_{s,n-m}\,d\theta) \\ \le& Ce^{-q_0\tau}(1+\tau^k)(\|U_0\|_{s+l+k+1,n} + \|F(\cdot,0)\|_{s+l+k,n}).\end{aligned} \tag{3.18}$$

The estimates (3.15) and (3.18) imply the estimate (3.14) for Z_k.

Now we are ready to prove the main result.

Theorem 3.5. *Suppose that B and G satisfy conditions (H1), (H2) and $0 \le l < N+1$. If $U_0 \in H_n^{s+2l+3(N+1)}$, $F \in W^{l+1,1}(0,T;H_n^{s+2l+3(N+1)})$, then there exists a unique strong solution $U \in W^{l,\infty}(0,T;H_n^s)$ of the problem (P_ε). For this solution expansion (2.1) is true, where V_k and Z_k are determined by problems (2.10) and (2.8), (2.9) respectively and they satisfy the estimates (3.13), (3.14). For the remainder term $R_N = \mathrm{col}(R_{N1}, R_{N2})$ the estimate*

$$\|R_{N1}\|^2_{W^{l,\infty}(0,T;H_m^s)} + \varepsilon^{1/2}\|R_{N2}\|^2_{W^{l,\infty}(0,T;H_{n-m}^s)} \le C(T)\varepsilon^{N+1-l} \tag{3.19}$$

is true with $C(T)$ depending on T, $\|U_0\|_{s+2l+3(N+1),n}$, $\|F\|_{W^{l+1,1}(0,T;H_n^{s+2l+3(N+1)})}$ and q_0. In particular, if $N=0$, then

$$\|U - V_0 - Z_0\|_{C([0,T];H_n^s)} \le C(T)\varepsilon^{1/4}.$$

Proof. The solvability of the problem (P_ε) can be obtained using the theory of C_0 semigroup of operators [11]. Indeed, operator $-(B(\partial_x)+G)$ is closed and dissipative on H_n^s. This operator generates the C_0 semigroup of contractions on H_n^s, which solves the problem (P_ε). Moreover the conditions $U_0 \in H_n^{s+l}$, $F \in W^{l,1}(0,T;H_n^s)$, $\partial_t^\nu F(\cdot,0) \in H_n^{s+l-\nu-1}$, $\nu = 0,\dots,l-1$, $l \ge 1$ imply the regularity of solution $U \in W^{l,\infty}(0,T;H_n^s)$. It remains to prove the estimate (3.19). We shall prove this estimate using the method from [12]. Further all constants depending on the norms indicated in the Theorem 3.5 will be denoted by $C(T)$. Let us denote by $\mathcal{R}_l = \partial_t^l R_N$, $\mathcal{R}_{li} = \partial_t^l R_{Ni}$, $i=1,2$. From condition (H1) it follows that $(B\mathcal{R}_l, \mathcal{R}_l)_{s,n}$ is a pure imaginary value. Consequently,

$$\frac{d}{dt}(A\mathcal{R}_l(\cdot,t), \mathcal{R}_l(\cdot,t))_{s,n} + 2(G\mathcal{R}_l(\cdot,t), \mathcal{R}_l(\cdot,t))_{s,n} = 2Re(\partial_t^l\mathcal{F}(\cdot,t), \mathcal{R}_l(\cdot,t))_{s,n}$$

Then using $(H2)$, it is not difficult to get the inequality

$$\frac{d}{dt}(A\mathcal{R}_l(\cdot,t), \mathcal{R}_l(\cdot,t))_{s,n} + 2q_0(\mathcal{R}_{l2}(\cdot,t), \mathcal{R}_{l2}(\cdot,t))_{s,n-m} \le 2|(\partial_t^l\mathcal{F}(\cdot,t), \mathcal{R}_l(\cdot,t))_{s,n}|. \tag{3.20}$$

The estimates (3.13) and (3.14) yield

$$
\begin{aligned}
&|(\partial_t^l \mathcal{F}(\cdot,t), \mathcal{R}_l(\cdot,t))_{s,n}| \\
&\quad \le \varepsilon^{N+1} |(P_1(\partial_t^l V_N(\cdot,t)) + \varepsilon^{-l} L_1(\partial_\tau^l Z_N(\cdot,\tau)), \mathcal{R}_l(\cdot,t))_{s,n}| \\
&\qquad + \varepsilon^{N-l} |(L_0(\partial_\tau^l Z_N(\cdot,\tau)) + L_1(\partial_\tau^l Z_{N-1}(\cdot,\tau)), A_0 \mathcal{R}_l(\cdot,t))_{s,n}| \\
&\quad \le C(T)(\varepsilon^{N-l}\kappa(t)\|\mathcal{R}_{l1}(\cdot,t)\|_{s,m} + (\varepsilon^{N+1} + \kappa(t)\varepsilon^{N+1-l})\|\mathcal{R}_l(\cdot,t))\|_{s,n}),
\end{aligned} \tag{3.21}
$$

where $0 \le t \le T, \tau = t/\varepsilon$ and $\kappa(t) = e^{-q_0 t/\varepsilon}(1 + (t/\varepsilon)^N)$. Integrating (3.20) by t and using (3.21) we get

$$
\begin{aligned}
&\|\mathcal{R}_{l1}(\cdot,t))\|_{s,m}^2 + \varepsilon\|\mathcal{R}_{l2}(\cdot,t))\|_{s,n-m}^2 + 2q_0 \int_0^t \|\mathcal{R}_{l2}(\cdot,\theta)\|_{s,n-m}^2 \, d\theta \\
&\quad \le \|\mathcal{R}_{l1}(\cdot,0)\|_{s,m}^2 + \varepsilon\|\mathcal{R}_{l2}(\cdot,0))\|_{s,n-m}^2 + C(T)(\varepsilon^{N-l} \int_0^t \kappa(\theta)\|\mathcal{R}_{l1}(\cdot,\theta)\|_{s,m} \, d\theta \\
&\qquad + \int_0^t (\varepsilon^{N+1} + \kappa(\theta)\varepsilon^{N-l+1})\|\mathcal{R}_l(\cdot,\theta)\|_{s,n} \, d\theta), \quad 0 \le t \le T,
\end{aligned} \tag{3.22}
$$

Note that

$$
\mathcal{R}_l(\cdot,0) = \sum_{\nu=0}^{l-1} (-A^{-1}(B(\partial_x) + G))^{l-\nu-1} A^{-1} \partial_t^\nu \mathcal{F}(\cdot,0), \qquad l \ge 1,
$$

and according to (2.5) $\mathcal{R}_0(\cdot,0) = 0$. Therefore, using (3.14), (3.15) and the equality $A^{-1}A_0 = A_0$, we have

$$
\begin{aligned}
\|A^{-1}\partial_t^\nu \mathcal{F}(\cdot,0)\|_{s,n} &\le \varepsilon^{N+1}\|(A^{-1}P_1\partial_t^\nu V_N)(\cdot,0)\|_{s,n} + \varepsilon^{N+1-\nu}\|(A^{-1}L_1\partial_\tau^\nu Z_N)(\cdot,0)\|_{s,n} \\
&\quad + \varepsilon^{N-\nu}\|A_0(L_0\partial_\tau^\nu Z_N + L_1\partial_\tau^\nu Z_{N-1})(\cdot,0)\|_{s,n} \\
&\le C(T)(\varepsilon^N + \varepsilon^{N-\nu}) \le C(T)\varepsilon^{N-\nu}, \quad 0 < \varepsilon < 1, \ 0 \le \nu \le N,
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\|\mathcal{R}_l(\cdot,0)\|_{s,n} &\le \sum_{\nu=0}^{l-1} \|A^{-1}(B(\partial_x) + G))^{l-\nu-1} A^{-1}\partial_t^\nu \mathcal{F}(\cdot,0)\|_{s,n} \\
&\le C(T)\sum_{\nu=0}^{l-1} \varepsilon^{-(l-\nu-1)} \cdot \varepsilon^{N-\nu} \\
&\le C(T)\varepsilon^{N-l+1}.
\end{aligned} \tag{3.23}
$$

Further, if $l < N+1$ and ε is small, then for $0 \le t \le T$ we have the estimates

$$
\begin{aligned}
\int_0^t \kappa(\theta)\|\mathcal{R}_{l1}(\cdot,\theta)\|_{s,m} \, d\theta &\le \int_0^t \kappa(\theta)\, d\theta + \int_0^t \kappa(\theta)\|\mathcal{R}_{l1}(\cdot,\theta)\|_{s,m}^2 \, d\theta \\
&\le C(T)\varepsilon + \int_0^t \kappa(\theta)\|\mathcal{R}_{l1}(\cdot,\theta)\|_{s,m}^2 \, d\theta,
\end{aligned} \tag{3.24}
$$

and

$$\begin{aligned} C(T)&\int_0^t (\varepsilon^{N+1}+\kappa(\theta)\varepsilon^{N-l+1})\|\mathcal{R}_l(\cdot,\theta)\|_{s,n}\,d\theta \\ &\le C(T)\varepsilon^{N-l+1}+q_0\int_0^t \|\mathcal{R}_{l2}(\cdot,\theta)\|^2_{s,n-m}\,d\theta \\ &\quad + C(T)\int_0^t (\varepsilon^{N+1}+\kappa(\theta)\varepsilon^{N-l+1})\|\mathcal{R}_{l1}(\cdot,\theta)\|^2_{s,m}\,d\theta. \end{aligned} \tag{3.25}$$

Then due to estimates (3.23), (3.24) and (3.25) the inequality (3.22) receives the form

$$\begin{aligned} \|\mathcal{R}_{l1}(\cdot,t))\|^2_{s,m}&+\varepsilon\|\mathcal{R}_{l2}(\cdot,t))\|^2_{s,n-m}+q_0\int_0^t \|\mathcal{R}_{l2}(\cdot,\theta)\|^2_{s,n-m}\,d\theta \\ &\le C(T)(\varepsilon^{N-l+1}+\int_0^t (\varepsilon^{N+1}+\kappa(\theta)\varepsilon^{N-l})\|\mathcal{R}_{l1}(\cdot,\theta)\|^2_{s,m}\,d\theta),\ 0\le t\le T. \end{aligned}$$

Thanks to Gronwall's lemma, from the last inequality we get the estimates

$$\|\mathcal{R}_{l1}(\cdot,t)\|^2_{s,m}\le C(T)\varepsilon^{N-l+1},\quad 0\le t\le T, \tag{3.26}$$

and

$$\varepsilon\|\mathcal{R}_{l2}(\cdot,t)\|^2_{s,n-m}+q_0\int_0^t \|\mathcal{R}_{l2}(\cdot,\theta)\|^2_{s,n-m}\,d\theta\le C(T)\varepsilon^{N-l+1},\ 0\le t\le T. \tag{3.27}$$

From (3.27) and (3.23) follows the estimate

$$\begin{aligned} \|\mathcal{R}_{l2}(\cdot,t)\|^2_{s,n-m}&\le\|\mathcal{R}_{l2}(\cdot,0)\|^2_{s,n-m}+2\int_0^t \|\mathcal{R}_{l2}(\cdot,\theta)\|_{s,n-m}\|\mathcal{R}_{(l+1)2}(\cdot,\theta)\|_{s,n-m}\,d\theta \\ &\le C(T)\varepsilon^{2(N-l+1)}+2\left(\int_0^t \|\mathcal{R}_{l2}(\cdot,\theta)\|^2_{s,n-m}\,d\theta\right)^{1/2}\times \\ &\quad\left(\int_0^t \|\mathcal{R}_{(l+1)2}(\cdot,\theta)\|^2_{s,n-m}\,d\theta\right)^{1/2} \\ &\le C(T)\varepsilon^{N-l+1/2},\quad 0\le t\le T. \end{aligned} \tag{3.28}$$

The estimates (3.26) and (3.28) imply the estimate (3.19). Therefore, Theorem 3.5 is proved.

4. Proof of Lemmas

Proof of Lemma 3.1. To prove this lemma we shall use the method of simultaneous reduction of two matrices to the diagonal form [13]. As $G^*_{03}=G_{03}$ and $G_{03}>0$, then there exists an orthogonal matrix $T_1\in M^m(\mathbb{R})$, $T_1^*T_1=I_m$, such that $T_1^*G_{03}T_1=\Lambda_0^2=\operatorname{diag}(\lambda_1,\dots,\lambda_m)$, where $\lambda_k>0, k=1,\dots,m$, are the eigenvalues of matrix G_{03}. Let $C(\xi)=\Lambda_0^{-1}T_1^*b_{03}(\xi)T_1\Lambda_0^{-1}$. As the matrix $C(\xi)$ is real symmetric, then there exists an orthogonal matrix $T_2(\xi)\in M(\mathbb{R}^m)$, such that $T_2^*C(\xi)T_2=\Lambda(\xi)=\operatorname{diag}(\mu_1(\xi),\dots,\mu_m(\xi))$, where $\mu_1(\xi),\dots,\mu_m(\xi)$ are real eigenvalues of matrix $C(\xi)$. Thus we have

$$T^*(\xi)G_{03}T(\xi)=I_m,\qquad T^*(\xi)b_{03}(\xi)T(\xi)=\Lambda(\xi), \tag{4.1}$$

where $T(\xi) = T_1\Lambda_0^{-1}T_2(\xi)$. From (4.1) it follows

$$G_{03} + i|\xi|b_{03}(\xi) = {T^*}^{-1}(\xi)(I_m + i|\xi|\Lambda(\xi))T^{-1}(\xi).$$

It means that the matrix $G_{03} + i|\xi|b_{03}(\xi)$ is invertible and

$$(G_{03} + i|\xi|b_{03}(\xi))^{-1} = T(\xi)\Lambda_1(\xi)(I_m - i|\xi|\Lambda(\xi))T^*(\xi), \tag{4.2}$$

where $\Lambda_1(\xi) = \mathrm{diag}((1 + |\xi|^2\mu_1^2)^{-1}, \dots, (1 + |\xi|^2\mu_m^2)^{-1})$. The orthogonality of the matrix $T_2(\xi)$ implies the boundedness of the function $\xi \to T(\xi)$ on $\mathbb{R}^d$. Then the boundedness of matrix $(G_{03}+i|\xi|b_{03}(\xi))^{-1}$ follows from (4.2). Lemma 3.1 is proved.

Proof of Lemma 3.2. Let us substitute (4.2) into (3.3). Then we obtain the representation (3.4), where

$$\begin{aligned}
K_0(\xi) =& G_{01} - G_{02}T^*\Lambda_1T^*G_{02}^* - |\xi|^2(G_{02}T\Lambda_1\Lambda T^*b_{02}^* + b_{02}T\Lambda_1\Lambda T^*G_{02}^*),\\
K_1(\xi) =& b_{01} + G_{02}T\Lambda_1\Lambda T^*G_{02}^* - G_{02}T\Lambda_1T^*b_{02}^*\\
&- b_{02}T\Lambda_1T^*G_{02}^* - |\xi|^2b_{02}T\Lambda_1\Lambda T^*b_{02}^*,\\
K_2(\xi) =& b_{02}T\Lambda_1T^*b_{02}^*.
\end{aligned}$$

It is easy to see that $K_j(\xi)$, $j = 0, 1, 2$ are bounded on $\mathbb{R}^d$, and $K_1^* = K_1$, $K_2^* = K_2$. It remains to prove that $K_2 \geq 0$. According to Ostrowski's theorem [14, p.270], denoting by $\lambda_j(A)$, $j = 1, \dots, m$ the eigenvalues of real symmetric matrix A, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$, we have $\lambda_j(K_2(\xi)) = \lambda_j(b_{02}T\Lambda_1T^*b_{02}^*) = \theta_j\lambda_j(\Lambda_1) \geq 0$, where $0 \leq \lambda_1(b_{02}TT^*b_{02}^*) \leq \theta_j \leq \lambda_m(b_{02}TT^*b_{02}^*)$. It means that $K_2 \geq 0$. Therefore, Lemma 3.2 is proved.

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