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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO 3D KELVIN-VOIGT-BRINKMAN-FORCHHEIMER EQUATIONS WITH UNBOUNDED DELAYS

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Abstract. In this article we consider a 3D Kelvin Voigt Brinkman Forchheimer equations involving unbounded delays in a bounded domain $\Omega \subset \mathbb{R}^3$. First, we show the existence and uniqueness of weak solutions by using the Galerkin approximations method and the energy method. Second, we prove the existence and uniqueness of stationary solutions by employing the Brouwer fixed point theorem. Finally, we study the stability of stationary solutions via the direct classical approach and the construction of a Lyapunov function. We also give a sufficient condition for the polynomial stability of the stationary solution for a special case of unbounded variable delay.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. We consider the 3D Kelvin-Voigt-Brinkman-Forchheimer equations with delays in Ω,

$$
\begin{gathered}
\partial_t(u-\alpha^2\nabla u)-\nu\Delta u+(u\cdot\nabla)u+\nabla p+f(u)=g(t,u_t)+h(t)\\
\text{in } (0,T)\times\Omega,\\
\operatorname{div} u=0 \quad \text{in } (0,T)\times\Omega,\\
u(x,t)=0 \quad \text{in } (0,T)\times\partial\Omega,\\
u(\theta,x)=\phi(\theta,x), \quad \text{in } (-\infty,0]\times\Omega,
\end{gathered} \tag{1.1}
$$

where $\nu>0$ is the kinematic viscosity, $\alpha>0$ is a scale parameter with dimension of length, $u=u(x,t)=(u_1,u_2,u_3)$ is the velocity field of the fluid, p is the pressure, h is a nondelayed external force field, g is another external force term and contains hereditary characteristic u_t, where u_t is the function defined on $(-\infty,0]$ by $u_t(\theta)=u(t+\theta)$, $\theta\in(-\infty,0]$, ϕ is the initial datum on the interval.

The nonlinearity $f\in C^1(\mathbb{R}^3,\mathbb{R}^3)$ satisfies the following conditions:

$$
\begin{gathered}
f'(u)v\cdot v\ge(-K+\kappa|u|^{\beta-1})|v|^2, \quad \forall u,v\in\mathbb{R}^3,\\
|f'(u)|\le C_f(1+|u|^{\beta-1}), \quad \forall u\in\mathbb{R}^3,
\end{gathered} \tag{1.2}
$$

where K,κ,C_f, are some positive constants, $\beta\ge 1$ (in the case of $\beta>3$ to show the uniqueness of solutions) and $u\cdot v$ stands for standard inner product in $\mathbb{R}^3$. A

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typical example for f is

$$f(u) = au + b|u|^{\beta-1}u, \quad \beta \in [1; +\infty), \tag{1.3}$$

where $a \in \mathbb{R}$ and $b > 0$ are the Darcy and Forchheimer coefficients respectively.

The case $\alpha \equiv 0$ and $G \equiv 0$ has been studied by Zelik and Kalantarov [15]. The case of the so-called subcritical growth rate of the nonlinearity f (for $\beta \le 3$ in (1.3)) has been considered in the literature. The aim in [15] is to remove this growth restriction and verify the global existence, uniqueness and dissipativity of smooth solutions of the Brinkman-Forchheimer equations for a large class of nonlinearity f with an arbitrary growth exponent $\beta > 3$.

Note that the case $f \equiv 0$ and $G \equiv 0$ corresponds to the classical Navier-Stokes-Voigt problem. The existence, long-time behavior and regularity of solutions to the 3D Navier-Stokes-Voigt equations without delays in bounded domains and unbounded domains satisfying the Poincare's inequality have been studied by many mathematicians [2, 3, 6, 9, 10, 13, 14, 22, 24, 25]. There are many results involving PDEs in fluid mechanics with delays [1, 7, 8, 18, 19, 20] and many results about asymptotic behavior to PDEs [21, 23]. However, all the results with finite delay (constant delays, bounded variable delay or bounded distributed delay) has been studied in the phase spaces $C([-h, 0); X)$ and $L^2(-h, 0; X)$ with a suitable Banach space X, or infinite distributed delay in $C_\gamma(X)$, where

$$C_\gamma(X) = \big\{\varphi \in C((-\infty, 0]; X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } X\big\} \quad (\gamma > 0)$$

is the Banach space endowed with the norm

$$\|\varphi\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\|\varphi(\theta)\|_X.$$

In this paper, following [16] we continue studying the system (1.1) with unbounded variable delays in the space

$$BCL_{-\infty}(X) = \big\{\varphi \in C((-\infty, 0]; X) : \lim_{\theta\to-\infty} \varphi(\theta) \text{ exists in } X\big\}$$

which is a Banach space equipped with the norm

$$\|\varphi\|_{BCL_{-\infty}(X)} = \sup_{\theta\in(-\infty,0]} \|\varphi(\theta)\|_X.$$

The main novelty of this article is that we are interested in the problem with unbounded delays. The stability of stationary solutions to the 3D Kelvin-Voigt-Brinkman-Forchheimer equations with unbounded delays, has apparently not been studied previously.

We will discuss the existence and uniqueness of the weak solution and stationary solution. Moreover, stability will be established for the stationary solution. The existence and the uniqueness of solution is proved by using the classic Galerkin approximation and the energy method. The existence of stationary solution is established by employing a corollary of the Brouwer fixed point theorem. The stability of stationary solution is shown by using the direct classical method, Lyapunov functions and giving a sufficient condition for a special case of unbounded variable delays.

The rest of this paper is organized as follows. In section 2, we will set up some spaces and lemmas which will be used in the later sections. Section 3 will be devoted to the existence and uniqueness of solutions of the model. In section 4, we will study the existence, uniqueness and stability of stationary solutions.

2. Preliminaries

We consider the space

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\}.$$

Let H be the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$ defined by

$$(u, v) = \sum_{j=1}^{3} \int_\Omega u_j(x) v_j(x) dx \quad \text{for } u, v \in (L^2(\Omega))^3.$$

We also denote V the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with norm $\|\cdot\|$, and the associated scalar product $((\cdot,\cdot))$ defined by

$$((u, v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_j} \frac{\partial v_j}{\partial x_i} dx \quad \text{for } u, v \in (H_0^1(\Omega))^3.$$

We use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot,\cdot \rangle_{V,V'}$ for the dual pairing between V and V'. We recall the Stokes operator $A : V \to V'$ by $\langle Au, v\rangle = ((u, v))$. Denote by P the Helmholtz-Leray orthogonal projection in $(H_0^1(\Omega))^3$ onto the space V. Then $Au = -P\Delta u$, for all $u \in D(A) = (H^2(\Omega))^3 \cap V$. The Stokes operator A is a positive self-adjoint operator with compact inverse. Hence there exists a complete orthonormal set of eigenfunctions $\{w_j\}_{j=1}^\infty \subset H$ such that $Aw_j = \lambda_j w_j$ and

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \le \lambda_j \to +\infty \quad \text{as } t \to \infty.$$

We have the following Poincaré inequalities

$$\begin{aligned} \|u\|^2 &\ge \lambda_1 |u|^2 \quad \forall u \in V, \\ |u|^2 &\ge \lambda_1 \|u\|_*^2 \quad \forall u \in H. \end{aligned} \tag{2.1}$$

From (2.1), we have

$$|u|^2 \ge d_0(|u|^2 + \alpha^2\|u\|^2) \quad \forall u \in V,$$

where $d_0 = \lambda_1/(1 + \alpha^2\lambda_1)$. Furthermore, for $\alpha > 0$, the operator $I + \alpha^2 A$ has a compact inverse $(I + \alpha^2 A)^{-1} : D(A)' \to H$ with the estimate

$$\|(I + \alpha^2 A)^{-1} u\| \le \alpha^{-2}\|u\|_* \quad \forall u \in V'.$$

We define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall u, v, w \in V,$$

and $B : V\times V \to V'$ by $\langle B(u, v), w\rangle = b(u, v, w)$. We can write $B(u, v) = P[(u\cdot\nabla)v]$. It is easy to check that if $u, v, w \in V$, then $b(u, v, w) = -b(u, w, v)$, and in particular,

$$b(u, v, v) = 0, \quad \forall u, v \in V. \tag{2.2}$$

Using Hölder's inequality and Ladyzhenskaya's inequality, we can choose the best positive constant c_0 such that

$$|b(u, v, w)| \le c_0\|u\|\|v\||w|^{1/2}\|w\|^{1/2}, \ \ \forall u, v, w \in V. \tag{2.3}$$

From (2.3) and using Poincaré's inequality (2.1), we obtain

$$|b(u, v, w)| \le c_0\lambda_1^{-1/4}\|u\|\|v\|\|w\|, \quad \forall u, v, w \in V. \tag{2.4}$$

We will assume that $f \in L^2(0,T;V')$. For the function $g : [0,T] \times BCL_{-\infty}(H) \to (L^2(\Omega))^3$,we have the following assumptions:

(A1) For any $\xi \in BCL_{-\infty}(H)$, the mapping $[0,T] \ni t \mapsto g(t,\xi) \in (L^2(\Omega))^3$ is measurable.

(A2) $g(\cdot,0) = 0$.

(A3) There exists a constant $L_g > 0$ such that, for any $t \in [0,T]$ and all $\xi, \eta \in BCL_{-\infty}(H)$,
$$|g(t,\xi) - g(t,\eta)| \le L_g \|\xi - \eta\|_{BCL_{-\infty}(H)}.$$

Some examples of g satisfying (A1)–(A3) can be seen in [16].

We can rewrite the 3D Kelvin-Voigt-Brinkman-Forchheimer equations (1.1) in the functional form
$$\begin{gathered}\partial_t(u + \alpha^2 Au) + \nu Au + B(u,u) + Pf(u) = Pg(t,u_t) + Ph(t),\\ \text{in } (0,T)\times\Omega,\\ u(\theta) = \phi(\theta), \quad \theta \in (-\infty,0].\end{gathered} \tag{2.5}$$

3. Existence and uniqueness of weak solutions

We first give the definition of a weak solution.

Definition 3.1. Given an initial datum $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$, a weak solution u to (1.1) in the interval $(-\infty,T]$, $T>0$, is a function $u \in C((-\infty,T];H) \cap L^2(0,T;V) \cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$ with $u(\theta) = \phi(\theta)$, $\theta \le 0$ and $\frac{du}{dt} \in L^2(0,T;V') + L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega))$ such that, for all $v \in V$, and a.e. $t \in (0,T)$,
$$\begin{aligned}&\frac{d}{dt}((u(t),v) + \alpha^2((u(t),v))) + \nu((u(t),v)) + b(u(t),u(t),v) + \langle f(u),v\rangle\\ &= \langle h(t),v\rangle + (g(t,u_t),v).\end{aligned}$$

Now we show the existence of weak solutions.

Theorem 3.2. *Consider $h \in L^2(0,T;V')$, $g : [0,T] \times BCL_{-\infty}(H) \to H$ satisfying* (A1)–(A3) *and $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$ are given. Then there exists a unique weak solution to* (1.1)*.*

Proof. **Existence.** We split the proof of the existence into several steps.

Step 1: A Galerkin scheme. Let $\{v_j\}_{j=1}^\infty$ be the basis consisting of eigenfunctions of the Stokes operator A, which is orthonormal in H and orthogonal in V. Denote $V_m = \operatorname{span}\{v_1,\dots,v_m\}$ and consider the projector $P_m u = \sum_{j=1}^m (u,v_j)v_j$. Define also
$$u^m(t) = \sum_{j=1}^m \gamma_{m,j}(t) v_j,$$
where the coefficients $\gamma_{m,j}$ are required to satisfy
$$\begin{aligned}&\frac{d}{dt}((u^m(t),v_j) + \alpha^2((u^m(t,v_j))) + \nu((u^m(t),v_j))\\ &+ b(u^m(t),u^m(t),v_j) + \langle f(u^m),v_j\rangle\\ &= \langle h(t),v_j\rangle + (g(u_t^m),v_j),\end{aligned} \tag{3.1}$$
for $j = 1,\dots,m$, and the initial condition $u^m(\theta) = P_m\phi(\theta)$ for $\theta \in (-\infty,0]$.

This system of ordinary functional differential equations with infinite delay in the unknowns $(\gamma_{m,1}(t),\dots,\gamma_{m,m}(t))$ fulfills the conditions for the existence and uniqueness of local solutions (see [11], [12]). Hence, we conclude that the approximate solutions u^m to (3.1) exists uniquely and locally on $[0,t^*)$ with $0 \le t^* \le T$. Next, we will obtain a priori estimates and ensure that the solutions u^m exist in whole interval $[0,T]$.

Step 2: A priori estimates. Multiplying (3.1) by $\gamma_{m,j}(t), j = 1,\dots,m$, summing up and using (2.2), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2) + \nu\|u^m(t)\|^2 + \int_\Omega f(u^m)u^m dx\\ &= \langle h(t), u^m(t)\rangle + (g(u_t^m), u^m(t)).\end{aligned}$$

Using the inequality $f(u)\cdot u \ge -K + \kappa|u|^{\beta+1}$, the Cauchy inequality and noting that $|u^m(t)| \le \|u_t^m\|_{BCL_{-\infty}(H)}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2) + \nu\|u^m(t)\|^2 + \kappa\int_\Omega |u^m|^{\beta+1}dx\\ &\le K|\Omega| + \|h(t)\|_*\|u^m(t)\| + L_g\|u_t^m\|_{BCL_{-\infty}(H)}|u^m(t)|\\ &\le K|\Omega| + \frac{\nu}{2}\|u^m(t)\|^2 + \frac{\|h(t)\|_*^2}{2\nu} + L_g\|u_t^m\|^2_{BCL_{-\infty}(H)},\end{aligned}$$

and hence

$$\begin{aligned}&\frac{d}{dt}(|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2) + \nu\|u^m(t)\|^2 + 2\kappa\int_\Omega |u^m|^{\beta+1}dx\\ &\le 2K|\Omega| + \frac{\|h(t)\|_*^2}{\nu} + 2L_g\|u_t^m\|^2_{BCL_{-\infty}(H)}.\end{aligned}$$

Integrating from 0 to t, we obtain

$$\begin{aligned}&|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2 + \nu\int_0^t \|u^m(s)\|^2 ds + 2\kappa\int_0^t\int_\Omega |u^m|^{\beta+1}dxds\\ &\le 2K|\Omega|t + |u^m(0)|^2 + \alpha^2\|u^m(0)\|^2 + \frac{1}{\nu}\int_0^t \|h(s)\|_*^2 ds\\ &\quad + 2L_g\int_0^t \|u_s^m\|^2_{BCL_{-\infty}(H)}ds.\end{aligned}\tag{3.2}$$

In particular, for any $t > 0$,

$$\begin{aligned}&\sup_{-t<\theta\le 0}|u^m(t+\theta)|^2 + \alpha^2\|u^m(t)\|^2\\ &\le 2K|\Omega|t + \|\phi\|^2_{BCL_{-\infty}(H)} + \alpha^2\|\phi(0)\|^2 + \frac{1}{\nu}\int_0^t \|h(s)\|_*^2 ds\\ &\quad + 2L_g\int_0^t \big(\|u_s^m\|^2_{BCL_{-\infty}(H)} + \alpha^2\|u^m(s)\|^2\big)ds.\end{aligned}$$

Since

$$\begin{aligned}&\|u_t^m\|^2_{BCL_{-\infty}(H)} + \alpha^2\|u^m(t)\|^2\\ &= \max\Big\{\sup_{-t<\theta\le 0}|u^m(t+\theta)|^2 + \alpha^2\|u^m(t)\|^2; \sup_{\theta\le -t}|u^m(t+\theta)|^2 + \alpha^2\|u^m(t)\|^2\Big\}\\ &\le \max\Big\{\sup_{-t<\theta\le 0}|u^m(t+\theta)|^2 + \alpha^2\|u^m(t)\|^2; \|\phi\|^2_{BCL_{-\infty}(H)} + \alpha^2\|u^m(t)\|^2\Big\},\end{aligned}$$

we obtain

$$\begin{aligned}
&\|u^m_t\|^2_{BCL_{-\infty}(H)} + \alpha^2\|u^m(t)\|^2 \\
&\le 2K|\Omega|t + 2\|\phi\|^2_{BCL_{-\infty}(H)} + \alpha^2\|\phi(0)\|^2 + \frac{1}{\nu}\int_0^t \|h(s)\|^2_* ds \\
&\quad + 2L_g\int_0^t \big(\|u^m_s\|^2_{BCL_{-\infty}(H)} + \alpha^2\|u^m(s)\|^2\big)ds.
\end{aligned}$$

By the Gronwall inequality we have

$$\begin{aligned}
&\|u^m_t\|^2_{BCL_{-\infty}H)} + \alpha^2\|u^m(t)\|^2 \\
&\le e^{2L_g t}\Big(2K|\Omega|T + \|\phi\|^2_{BCL_{-\infty}(H)} + \alpha^2\|\phi(0)\|^2 + \frac{1}{\nu}\int_0^t (\|h(s)\|^2_*)ds\Big).
\end{aligned}$$

Then we obtain the following estimates: for any $R > 0$ such that $\|\phi\|_{BCL_{-\infty}(H)} \le R$, there exists a constant C depending on ν, L_g, f, such that

$$\|u^m_t\|^2_{BCL_{-\infty}(H)} + \alpha^2\|u^m(t)\|^2 \le C(T,R), \quad \forall t \in [0,T],\ \forall m \ge 1. \tag{3.3}$$

In particular,

$$\{u^m\} \text{ is uniformly bounded in } L^\infty(0,T;BCL_{-\infty}(H)) \cap L^\infty(0,T;V).$$

From (3.2) and the uniform estimates above, we obtain

$$\begin{aligned}
&\nu\int_0^t\int_\Omega \|u^m(s)\|^2 dxds + 2\kappa\int_0^t\int_\Omega |u^m(s)|^{\beta+1}dxds \\
&\le |u^m(0)|^2 + \alpha^2\|u^m(0)\|^2 + K|\Omega|\int_0^t ds + \frac{1}{\nu}\int_0^t \|h(s)\|^2_* ds \\
&\quad + 2L_g\int_0^t \|u^m_s\|^2_{BCL_{-\infty}(H)}ds \\
&\le |u^m(0)|^2 + \alpha^2\|u^m(0)\|^2 + \int_0^t \left(\frac{1}{\nu}\|h(s)\|^2_* + 2L_g C(T,R) + K|\Omega|\right)ds.
\end{aligned}$$

Then $\{u^m\}$ is uniformly bounded in $L^2(0,T;V) \cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$. Using (1.2), we obtain that $|f(u)| \le C(1+|u|^\beta)$ with C depending on C_f. Hence,

$$\begin{aligned}
\int_0^t\int_\Omega |f(u)|^{\frac{\beta+1}{\beta}}dxdt &\le C\int_0^t\int_\Omega (1+|u|^\beta)^{\frac{\beta+1}{\beta}}dxdt \\
&\le C\int_0^t\int_\Omega (1+|u|^{\beta+1})dxdt.
\end{aligned}$$

From the boundedness of $\{u^m\}$ in $L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$, we obtain $\{f(x,u^m)\}$ is bounded in $L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega))$. Now, we prove the boundedness of $\{\frac{du^m}{dt}\}$. We have

$$\begin{aligned}
\frac{d}{dt}(u^m(t) + \alpha^2 Au^m(t)) = &-\nu Au^m(t) - P_mB(u^m,u^m) - Pf(u^m) \\
&+ P_mh(t) + P_mg(t,u^m_t).
\end{aligned} \tag{3.4}$$

From (2.4), (3.3) and (3.4), we obtain

$$
\begin{aligned}
&\Big\|\frac{d}{dt}(u^m+\alpha^2 Au^m)\Big\|_* \\
&\le \nu\|Au^m\|_* + \|B(u^m,u^m)\|_* + \|f(u^m)\|_{L^{(\beta+1)/\beta}(\Omega)} + \|h(t)\|_* + \|g(t,u_t^m)\|_* \\
&\le \nu\|u^m\| + c_0\lambda_1^{-1/4}\|u^m\| + \|f(u^m)\|_{L^{(\beta+1)/\beta}(\Omega)} + \|h(t)\|_* + \lambda_1^{-1/2}|g(t,u_t^m)| \\
&\le \nu\|u^m\| + c_0\lambda_1^{-1/4}\|u^m\| + \|f(u^m)\|_{L^{(\beta+1)/\beta}(\Omega)} \\
&\quad + \|h(t)\|_* + L_g\lambda_1^{-1/2}\|u^m\|_{BCL_{-\infty}(H)} \\
&\le C(T,R), \quad \forall m\ge 1.
\end{aligned}
$$

This implies that $\frac{d}{dt}(u^m+\alpha^2Au^m)$ is uniformly bounded in

$$
L^2(0,T;V') + L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega)).
$$

Then $\{\frac{du^m}{dt}\}$ is uniformly bounded in $L^2(0,T;V') + L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega))$.

Step 3. Approximation in $BCL_{-\infty}(H)$ of the initial datum. We will show that

$$
P_m\phi \to \phi \quad \text{in } BCL_{-\infty}(H). \tag{3.5}
$$

Assume on the contrary that (3.5) is not true. Then there exists $\epsilon>0$ and a subsequence, relabeled similarly, such that

$$
\|P_m\phi(\theta_m) - \phi(\theta_m)\| > \epsilon, \quad \forall m. \tag{3.6}
$$

One can assume that $\theta_m\to-\infty$, otherwise if $\theta_m\to\theta$, then $P_m\phi(\theta_m)\to\phi(\theta)$, since $\|P_m\phi(\theta_m)-\phi(\theta\| \le \|P_m\phi(\theta_m)-P_m\phi(\theta)\| + \|P_m\phi(\theta)-\phi(\theta)\| \to 0$ as $m\to+\infty$. But with $\theta_m\to-\infty$ as $m\to+\infty$, we obtain that

$$
\|P_m\phi(\theta_m)-\phi(\theta_m)\| = \|P_m\phi(\theta_m)) - P_mx\| + \|P_mx - x\| + \|x-\phi(\theta_m)\| \to 0,
$$

where $x=\lim_{\theta\to-\infty}\phi(\theta)$. This contradicts (3.6), so (3.5) holds.

Step 4: Compactness results. We obtain

$$
\begin{gathered}
u^m \rightharpoonup^* u \text{ weakly in } L^\infty(0,T;V), \\
u^m \rightharpoonup u \text{ in } L^2(0,T;V)\cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega)), \\
f(u^m) \rightharpoonup \chi \text{ in } L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega)), \\
\frac{du^m}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;V') + L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega)).
\end{gathered}
$$

Since $\{u^m\}$ is uniformly bounded in $L^2(0,T;V)$ and $\{\frac{du^m}{dt}\}$ is uniformly bounded in $L^2(0,T;V') + L^{(\beta+1)/\beta}(0,T;L^{(\beta+1)/\beta}(\Omega))$ and by using the Aubin-Lions compactness lemma, we deduce that $u^m\to u$ strongly in $L^2(0,T;(L^2(\Omega))^3)$. Thus, we have (up to a subsequence)

$$
u^m \to u \quad \text{a.e. in } \Omega_T.
$$

From the continuity of f, we obtain that $f(u^m)\to f(u)$ a.e. in Ω_T. Because of the uniqueness of the limit, we have $f(u)\equiv\chi$.

Next, we show that

$$
u^m \to u \quad \text{in } C([0,T];H), \tag{3.7}
$$

by applying the Arzelà-Ascoli lemma (following the same method in [17]). We have

$$
\sup_{\theta\le 0}|u^m(t+\theta) - u(t+\theta)|
$$

$$
\begin{aligned}
&\le \max\Big\{\sup_{\theta\le -t}|P_m\phi(\theta+t)-\phi(\theta+t)|, \sup_{-t\le\theta\le 0}|u^m(t+\theta)-u(t+\theta)|\Big\}\\
&\le \max\Big\{|P_m\phi-\phi|_{BCL_{-\infty}(H)}, \sup_{-t\le\theta\le 0}|u^m(t+\theta)-u(t+\theta)|\Big\}\to 0.
\end{aligned}
$$

Then (3.5) and (3.7) imply that

$$
u_t^m \to u_t \quad \text{in } BCL_{-\infty}(H), \quad \forall t\in[0,T].
$$

Therefore, taking into account (A3), we have

$$
g(\cdot,u^m_\cdot)\to g(\cdot,u_\cdot) \text{ in } L^2(0,T;H).
$$

Finally, we can pass to the limit in (3.1) and conclude that u solves (1.1).

Uniqueness. Let u, v be two weak solutions of problem (2.5) with the same initial condition. Setting $w=u-v$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|w|^2+\alpha^2\|w\|^2)+\nu\|w\|^2+b(u,u,w)-b(v,v,w)\\
&+\int_\Omega (f(u_1)-f(u_2))(u_1-u_2)dx\\
&=(g(u_t)-g(v_t),w).
\end{aligned}
\tag{3.8}
$$

It is known (see [5]) that there exists two nonnegative constants $\alpha=\alpha(\beta)$ and C_f such that

$$
\int_\Omega (f(u)-f(v))(u-v)dx \ge -C_f|u-v|^2+\alpha\int_\Omega(|u|^{\beta-1}+|v|^{\beta-1})|u-v|^2dx. \tag{3.9}
$$

Using that f satisfies (3.9), then (3.8) becomes

$$
\begin{aligned}
&\frac{d}{dt}(|w|^2+\alpha^2\|w\|^2)+2\nu\|w\|^2+\alpha\int_\Omega(|u|^{\beta-1}+|v|^{\beta-1})|u-v|^2dx\\
&\le C_f|w|^2+2\int_\Omega|((w\cdot\nabla)u)\cdot w|dx+2\int_\Omega|g(u_t)-g(v_t)|\cdot|w(t)|dx.
\end{aligned}
$$

By Holder's inequality and Young's inequality, we have

$$
2\int_\Omega|((w\cdot\nabla)u)\cdot w|dx\le 2|u||w||\nabla w|\le \nu|\nabla w|^2+C|u|^2|w|^2,
$$

where $C=C(\nu)$. Assuming that $\beta-1>2$ and using Young's inequality again, we obtain

$$
2\int_\Omega|((w\cdot\nabla)u)\cdot w|dx\le \nu|\nabla w|^2+\alpha(|u|^{\beta-1}+|v|^{\beta-1})|w|^2+C|w|^2,
$$

where $C=C(\nu,\alpha)$.

Taking (A3) into account and using Young's inequality, we obtain

$$
\begin{aligned}
\frac{d}{dt}(|w(t)|^2+\alpha^2\|w\|^2)+\nu\|w\|^2 &\le C_f|w|^2+C|w|^2+2L_g\|w_t\|_{BCL_{-\infty}(H)}|w(t)|\\
&\le (C_f+C)|w|^2+\frac{\nu\lambda_1}{2}|w|^2+\frac{L_g}{\lambda_1}\|w\|^2_{BCL_{-\infty}(H)}\\
&\le (C_f+C)|w|^2+\frac{\nu}{2}\|w\|^2+\frac{L_g}{\lambda_1}\|w\|^2_{BCL_{-\infty}(H)}.
\end{aligned}
$$

Thus, we have

$$|w(t)|^2 + \alpha\|w\|^2 \le (C_f + C)\int_0^t |w(s)|^2 ds + \frac{L_g}{\lambda_1}\int_0^t \|w_s\|^2_{BCL_{-\infty}(H)} ds.$$

Since $w(\theta) = 0$ for $\theta \le 0$, then taking the maximum in $[0,t]$ for any $t \in [0,T]$, we have

$$\|w_t\|^2_{BCL_{-\infty}(H)} \le (C_f + C + \frac{L_g}{\lambda_1})\int_0^t \|w_s\|^2_{BCL_{-\infty}(H)} ds.$$

We complete the proof of uniqueness after applying the Gronwall inequality. □

4. Existence and stability of stationary solutions

4.1. **Existence of weak stationary solutions.** To study the existence and the other properties of stationary solutions, we need to impose some extra assumptions. Firstly, we assume that h is independent of time, i.e., $h(t) \equiv h \in V'$. Denote by i the trivial immersion $i : H \to BCL_{-\infty}(H)$ given by $i(u) = \widetilde{u}$ with $\widetilde{u}(t) = u$ for all $t \le 0$. We now require that g satisfy

(A4) $g(s,\xi) = g(t,\xi)$ for any $s,t \in \mathbb{R}_+$ and $\xi \in i(H)$.

If (A2)–(A4) hold, we trivially have that $\tilde{g} : H \to (L^2(\Omega))^3$ defined by $\tilde{g}(u) = g(0, i(u))$, i.e., $\tilde{g} = g_{|\mathbb{R}_+\times i(H)}$, is of course autonomous, Lipschitz (with the same Lipschitz constant L_g) and $\tilde{g}(0) = 0$.

Hence, the stationary equation to (2.5) is the following form which does not contain a delay term:

$$\nu Au + B(u,u) + Pf(u) = Ph + P\tilde{g}(u). \tag{4.1}$$

Let us consider the definition of stationary solutions to problem (1.1).

Definition 4.1. A weak stationary solution to (1.1) is an element $u^* \in V$ such that

$$\nu((u^*, v)) + b(u^*, u^*, v) + \langle f(u^*), v\rangle = \langle h, v\rangle + (\tilde{g}(u^*), v),\ \forall v \in V.$$

Theorem 4.2. *Assume* (A2)–(A4) *hold and* $h \in V'$. *If* $2L_g < \nu\lambda_1$ *then problem* (1.1) *admits at least one stationary solution* u^* *satisfying*

$$\|u^*\| \le \Big(\frac{\lambda_1(2\nu K|\Omega| + \|h\|_*^2)}{\nu(\lambda_1\nu - 2L_g)}\Big)^{1/2}, \tag{4.2}$$

and

$$\nu > C_f\lambda_1^{-1} + L_g\lambda_1^{-1} + 2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega| + \|h\|_*^2)}{\nu(\lambda_1\nu - 2L_g)}\Big)^{1/2}, \tag{4.3}$$

then the stationary solution of (1.1) *is unique.*

Proof. **Existence.** Estimate (4.2) can be obtained by taking into account that, any stationary solution u^*, if it exists, should satisfy

$$\nu((u^*, u^*)) + \langle f(u^*), v\rangle = \langle h, u^*\rangle + (\tilde{g}(u^*), u^*).$$

Using inequality $f(u)\cdot u \ge -K + \kappa|u|^{\beta+1}$ again, we have

$$\nu\|u^*\|^2 + \kappa\int_\Omega |u^*|^{\beta+1} dx \le K|\Omega| + \|h\|_*\|u^*\| + L_g\lambda_1^{-1}\|u^*\|^2.$$

Hence we obtain the desired estimate.

To show the existence, let $\{v_j\}_{j=1}^{\infty}$ be the basis of V consisting of eigenfunctions of the operator A. For each $m \geq 1$, let us define $V_m = \operatorname{span}\{v_1, \dots, v_m\}$ and an approximate stationary solution u^m of (1.1) by

$$\begin{gathered}
u^m = \sum_{i=1}^{m} \gamma_{mi} v_i, \\
\nu((u^m, v_i)) + b(u^m, u^m, v_i) + \langle f(u^m), v_i\rangle = \langle h, v_i\rangle + (\tilde{g}(u^m), v_i), \\
i = 1, \dots, m.
\end{gathered} \tag{4.4}$$

To prove the existence of u^m, we define the operators $R_m : V_m \to V_m$ by

$$[R_m u, v] := \nu((u, v)) + b(u, u, v) + \langle f(u), v\rangle - \langle h, v\rangle - (\tilde{g}(u), v), \quad \forall u, v \in V_m.$$

For all $u \in V_m$, we have

$$\begin{aligned}
[R_m u, u] &= \nu((u, u)) + \int_{\Omega} f(u) u dx - \langle h, u\rangle - (\tilde{g}(u), u) \\
&\geq \nu\|u\|^2 + \kappa \int_{\Omega} |u|^{\beta+1} dx - K|\Omega| - \|h\|_* \|u\| - L_g \lambda_1^{-1} \|u\|^2 \\
&= \big(\frac{\nu}{2} - L_g \lambda_1^{-1}\big)\|u\|^2 + \kappa \int_{\Omega} |u|^{\beta+1} dx - K|\Omega| - \frac{1}{2\nu}\|h\|_*^2.
\end{aligned}$$

It follows that $((R_m u, u)) \geq 0$ for $\|u\|_X = \|u\| + \|u\|_{L^{\beta+1}} = k$ sufficiently large, we obtain

$$k = \Big(\frac{2K|\Omega|\nu + \|h\|_*^2}{\nu(\lambda_1 \nu - 2L_G)}\Big)^{1/2} + \Big(\frac{2K|\Omega|\nu + \|h\|_*^2}{2\nu\kappa}\Big)^{1/(\beta+1)},$$

where $2L_g < \nu\lambda_1$. So, there exists a solution $u^m \in V_m$ satisfying $R_m(u^m) = 0$. Replacing v_i by u^m in (4.4) and taking (2.2) into account, we obtain

$$\nu\|u^m\|^2 + \langle f(u^m), u^m\rangle = \langle h, u^m\rangle + (\tilde{g}(u^m), u^m).$$

This implies

$$\nu\|u^m\|^2 + \kappa \int_{\Omega} |u^m|^{\beta+1} dx \leq K|\Omega| + \|h\|_* \|u^m\| + L_g \lambda_1^{-1} \|u^m\|^2.$$

Hence

$$\Big(\frac{\nu}{2} - \frac{L_g}{\lambda_1}\Big)\|u^m\|^2 + \kappa\|u^m\|_{L^{\beta+1}}^{\beta+1} \leq K|\Omega| + \frac{1}{2\nu}\|h\|_*^2.$$

We extract from $\{u^m\}$ a subsequence $\{u^{m'}\}$, which converges weakly in $V \cap L^{\beta+1}$ to some limit u. Since the domain Ω is bounded, the injection of V into H is compact. Thus,

$$u^{m'} \to u \quad \text{weakly in } V \text{ and strongly in } H,$$

up to a subsequence. Using the same method in the step compactness results in the Theorem 3.2, we pass to the limit in (4.4) using the sequence m' and find that u is a stationary solution of (1.1).

Uniqueness. Suppose that u^* and v^* are two stationary solutions of (4.1). Then

$$\begin{aligned}
&\nu((u^* - v^*, v)) + b(u^*, u^*, v) - b(v^*, v^*, v) + \int_{\Omega} (f(u^*) - f(v^*)) v dx \\
&= (\tilde{g}(u^*) - \tilde{g}(v^*), v)
\end{aligned}$$

for all $v \in V$. Choosing $v = u^* - v^*$, we have

$$\begin{aligned}&\nu((u^* - v^*, u^* - v^*)) + \int_\Omega (f(u^*) - f(v^*))(u^* - v^*)dx \\ &= b(u^*, u^*, u^* - v^*) - b(v^*, v^*, u^* - v^*) + (\tilde{g}(u^*) - \tilde{g}(v^*), u^* - v^*).\end{aligned}$$

Using estimate (2.4), we obtain

$$|b(u^*, u^*, u^* - v^*) - b(v^*, v^*, u^* - v^*)| \le 2c_0\lambda_1^{-1/4}\|u^*\|\|u^* - v^*\|^2.$$

From $\int_\Omega (f(u) - f(v))(u - v)dx \ge -C_f|u - v|^2$, we have

$$\nu\|u^* - v^*\|^2 \le C_f|u^* - v^*|^2 + 2c_0\lambda_1^{-1/4}\|u^*\|\|u^* - v^*\|^2 + L_g\lambda_1^{-1}\|u^* - v^*\|^2.$$

Using estimate (4.2) we deduce that

$$(\nu - C_f\lambda_1^{-1} - L_g\lambda_1^{-1} - 2c_0\lambda_1^{-1/4}\|v\|)\|u - v\|^2 \le 0,$$

and hence the uniqueness follows from condition (4.3). □

4.2. **Stability results.**

Definition 4.3. A stationary u^* to (1.1) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\phi \in BCL_{-\infty}(H)$ satisfies $\|\phi - i(u^*)\|_{BCL_{-\infty}(H)} \le \delta$, then the solution $u(\cdot;\phi)$ to (1.1) exists for all $t \ge 0$ and satisfies $|u(t;\phi) - u^*| < \varepsilon$ for any $t \ge 0$.

We consider the case of $g(t, u_t) = G(u(t - \rho(t)))$, where $G : H \to (L^2(\Omega))^3$ is a measurable function satisfying $G(0) = 0$, and assume that there exists $L_g > 0$ such that

$$|G(u) - G(v)| \le L_g|u - v|, \quad \forall u, v \in H. \tag{4.5}$$

Consider a function $\rho(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\rho_* = \max_{t\in[0,T]} \rho'(t) < 1$. The system (2.5) becomes

$$\frac{d}{dt}(u + \alpha^2 Au) = -\nu Au - B(u, u) - Pf(u) + Ph + PG(u(t - \rho(t))), \tag{4.6}$$

with initial condition $u(\theta) = \phi(\theta), \theta \in (-\infty, 0]$. We then have the following stability results.

4.2.1. *Local stability via a direct approach.*

Theorem 4.4. *Assume that $h \in V'$ and (4.5) hold. If $\nu\lambda_1 > 2L_g$ then there exists at least one stationary solution u^* to (4.1) satisfying (4.2). Moreover, if*

$$\nu > C_f\lambda_1^{-1} + 2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega| + \|h\|_*^2)}{\nu(\lambda_1\nu - 2L_g)}\Big)^{1/2} + \frac{(2 - \rho_*)L_g\lambda_1^{-1}}{2(1 - \rho_*)}, \tag{4.7}$$

then the stationary solution u^* is unique, and there exists $C = C(\rho_*, L_g)$ such that the solution u to (4.6) satisfies

$$\begin{aligned}&|u(t) - u^*|^2 + \alpha^2\|u(t) - u^*\|^2 \\ &\le C\Big(|\phi(0) - u^*|^2 + \alpha^2\|\phi(0) - u^*\|^2 + \|\phi - u^*\|^2_{L^2(-\rho(0),0;H)}\Big),\end{aligned}$$

for all $\phi \in BCL_{-\infty}(H)$.

Proof. We first see that all assumptions of Theorem 4.2 are satisfied (note that if ν satisfies (4.7) then condition (4.3) is obtained). Hence, the existence and uniqueness of stationary solution u^* satisfying (4.2) are established. If we write $u(t)$ in the form $u(t) = u^* + v(t)$, then from (2.5), $v(t)$ satisfies

$$\frac{d}{dt}(v+\alpha^2 Av)+\nu Av+B(u,u)-B(u^*,u^*)+f(u)-f(u^*)=G(u(t)-\rho(t))-G(u^*).$$

Multiplying this equation by v, integrating over Ω and using (2.2), we obtain

$$\begin{aligned}
&\frac{d}{dt}(|v(t)|^2+\alpha^2\|v(t)\|^2)+2\int_\Omega (f(u)-f(u^*))(u-u^*)dx\\
&= -2\nu\|v(t)\|^2+2(b(u,u,v(t))-b(u^*,u^*,v(t)))+2((G(u(t-\rho(t)))-G(u^*),v(t)).
\end{aligned}$$

Using Cauchy's inequality, the Lipschitz condition on G and the following estimate which is obtained as in (2.2),

$$|b(u,u,v(t))-b(u^*,u^*,v(t))| \le 2c_0\lambda_1^{-1/4}\|u^*\|\|v(t)\|^2,$$

Using $\int_\Omega (f(u)-f(u^*))(u-u^*)dx \ge -C_f|u-u^*|^2$, we obtain

$$\begin{aligned}
&\frac{d}{dt}(|v(t)|^2+\alpha^2\|v(t)\|^2)\\
&\le -2\nu\|v(t)\|^2+4c_0\lambda_1^{-1/4}\|u^*\|\|v(t)\|^2+2C_f|v(t)|^2+L_g|v(t-\rho(t))|^2\\
&\quad +L_g|v(t)|^2\\
&\le L_g|v(t-\rho(t))|^2+\Big(4c_0\lambda_1^{-1/4}\|u^*\|+2C_f\lambda_1^{-1}+L_g\lambda_1^{-1}-2\nu\Big)\|v(t)\|^2.
\end{aligned} \tag{4.8}$$

Using (4.2) and integrating the above inequality from 0 to t, we obtain

$$\begin{aligned}
&|v(t)|^2+\alpha^2\|v(t)\|^2\\
&\le |v(0)|^2+\alpha^2\|v(0)\|^2+L_g\int_0^t |v(s-\rho(s))|^2ds\\
&+\Big(L_g\lambda_1^{-1}+2C_f\lambda_1^{-1}+4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}-2\nu\Big)\int_0^t\|v(s)\|^2ds.
\end{aligned}$$

Letting $\tau(s)=s-\rho(s)$, we obtain

$$\begin{aligned}
\int_0^t |v(s-\rho(s))|^2ds &= \frac{1}{1-\rho'}\int_{-\rho(0)}^t |v(\tau)|^2d\tau \le \frac{1}{1-\rho_*}\int_{-\rho(0)}^t |v(\tau)|^2d\tau\\
&= \frac{1}{1-\rho_*}\int_{-\rho(0)}^0 |v(\tau)|^2d\tau+\frac{1}{1-\rho_*}\int_0^t |v(\tau)|^2d\tau.
\end{aligned}$$

We then have

$$\begin{aligned}
&|v(t)|^2+\alpha^2\|v(t)\|^2\\
&\le |v(0)|^2+\alpha^2\|v(0)\|^2+\frac{L_g}{1-\rho_*}\int_{-\rho(0)}^0 |v(\tau)|^2d\tau+\frac{L_g}{1-\rho_*}\int_0^t |v(\tau)|^2d\tau\\
&\quad+\Big(L_g\lambda_1^{-1}+2C_f\lambda_1^{-1}+4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}-2\nu\Big)\int_0^t\|v(s)\|^2ds\\
&\le |v(0)|^2+\alpha^2\|v(0)\|^2+\frac{L_g}{1-\rho_*}\int_{-\rho(0)}^0 |v(\tau)|^2d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\Big(L_g\lambda_1^{-1}+2C_f\lambda_1^{-1}+4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+\frac{L_g\lambda_1^{-1}}{(1-\rho_*)}-2\nu\Big)\\
&\times\int_0^t\|v(s)\|^2ds\\
&\le|v(0)|^2+\alpha^2\|v(0)\|^2+\frac{L_g}{1-\rho_*}\int_{-\rho(0)}^0|v(\tau)|^2d\tau\\
&+\Big(2C_f\lambda_1^{-1}+4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+\frac{(2-\rho_*)L_g\lambda_1^{-1}}{(1-\rho_*)}-2\nu\Big)\\
&\times\int_0^t\|v(s)\|^2ds.
\end{aligned}
$$

If

$$
\nu>C_f\lambda_1^{-1}+2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+\frac{(2-\rho_*)L_g\lambda_1^{-1}}{2(1-\rho_*)},
$$

then

$$
|v(t)|^2\le|v(0)|^2+\frac{L_g}{1-\rho_*}\int_{-\rho(0)}^0|v(\tau)|^2d\tau.
$$

We can choose $C=\max\{1,\frac{L_g}{1-\rho_*}\}$, and the proof is complete. □

4.2.2. *Asymptotical stability via the construction of Lyapunov functionals.*

Theorem 4.5. *Suppose that $f\in V'$ and (4.3) hold. If $\nu\lambda_1>2L_g$ then there exist at least one weak stationary solution u^* to (4.1) satisfying (4.2). In addition, if*

$$
\nu\ge C_f\lambda_1^{-1}+2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+\frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}
$$

then the stationary solution u^* is unique, stable, and satisfies

$$
\begin{aligned}
&\int_0^\infty(|u(s)-u^*|^2+\alpha^2\|u(s)-u^*\|^2)ds\\
&\le|\phi(0)-u^*|^2+\alpha^2\|\phi(0)-u^*\|^2+\frac{L_g}{\sqrt{1-\rho_*}}\|\phi-u^*\|^2_{L^2(-\rho(0),0;H)}.
\end{aligned}
\tag{4.9}
$$

for any solution u to (4.6) with $\phi\in BCL_{-\infty}(H)$. Furthermore, if

$$
\nu>C_f\lambda_1^{-1}+2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+\frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}
$$

then u^* is asymptotically stable.

Proof. Since all the assumptions of Theorem 4.2 are satisfied, there exists a unique stationary solution u^* to (4.1) satisfying (4.2). Let us set $w(t)=u(t)-u^*$. Then it satisfies

$$
\begin{aligned}
\frac{d}{dt}(w(t)+\alpha^2Aw(t))&=-\nu Aw(t)-B(u(t),u(t))+B(u^*,u^*)\\
&\quad-f(u)+f(u^*)+P(G(u(t-\rho(t)))-G(u^*)),
\end{aligned}
\tag{4.10}
$$

with initial condition $w(\theta)=\phi(\theta)-u^*$, $\theta\in(-\infty,0]$. For any $\phi\in BCL_{-\infty}(H)$, and any $t>0$ we define

$$
U(t,\phi)=|\phi(0)-u^*|^2+\alpha^2\|\phi(0)-u^*\|^2+\frac{c}{1-\rho_*}\int_{t-\rho(t)}^t|u(s)-u^*|^2ds,
$$

where the constant $c > 0$ is to be chosen later. Then for any $u(.;\phi)$ of (4.6) with initial data $\phi \in BCL_{-\infty}(H)$, we have

$$U(t,u_t) = |u(t)-u^*|^2 + \alpha^2\|u(t)-u^*\|^2 + \frac{c}{1-\rho_*}\int_{t-\rho(t)}^t |u(s)-u^*|^2 ds. \tag{4.11}$$

From (4.10) and using an estimate similar to (2.4) we obtain

$$\begin{aligned}
&\frac{d}{dt}U(t,w_t)\\
&= 2\Big\langle \frac{d}{dt}(w(t)+\alpha^2 Aw(t)), w(t)\Big\rangle + \frac{c}{1-\rho_*}|w(t)|^2 - \frac{c(1-\rho'(t))}{1-\rho_*}|w(t-\rho(t))|^2\\
&\le -2\nu\|w(t)\|^2 + 4c_0\lambda_1^{-1/4}\|u^*\|\|w(t)\|^2 + 2C_f|w(t)|^2\\
&\quad + 2L_g|w(t-\rho(t))||w(t)| + \frac{c}{1-\rho_*}|w(t)|^2 - \frac{c(1-\rho'(t))}{1-\rho_*}|w(t-\rho(t))|^2.
\end{aligned}$$

By Cauchy's inequality, Poincaré's inequality (2.1) and (4.2), we obtain

$$\begin{aligned}
&\frac{d}{dt}U(t,w_t)\\
&\le -2\nu\|w(t)\|^2 + 4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega| + \|h\|_*^2)}{\nu(\lambda_1\nu - 2L_g)}\Big)^{1/2}\|w(t)\|^2 + 2C_f|w(t)|^2\\
&\quad + 2L_g|w(t-\rho(t))||w(t)| + \frac{c}{1-\rho_*}|w(t)|^2 - c|w(t-\rho(t))|^2\\
&\le -2\nu\|w(t)\|^2 + 4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega| + \|h\|_*^2)}{\nu(\lambda_1\nu - 2L_g)}\Big)^{1/2}\|w(t)\|^2 + 2C_f|w(t)|^2\\
&\quad + 2\Big(\frac{c}{2}|w(t-\rho(t))|^2 + \frac{L_g^2}{2c}|w(t)|^2\Big) + \frac{c}{1-\rho_*}|w(t)|^2 - c|w(t-\rho(t))|^2\\
&\le -2\Big(\nu - 2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega| + \|h\|_*^2)}{\nu(\lambda_1\nu - 2L_g)}\Big)^{1/2} - \frac{L_g^2\lambda_1^{-1}}{2c} - \frac{c\lambda_1^{-1}}{2(1-\rho_*)} - C_f\lambda_1^{-1}\Big)\\
&\quad \times \|w(t)\|^2.
\end{aligned}$$

If we choose $c = L_g\sqrt{1-\rho_*}$, then the coefficient in the right-hand side takes it minimum value. We conclude that

$$\begin{aligned}
&\frac{d}{dt}U(t,w_t)\\
&\le -2\Big(\nu - 2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega| + \|h\|_*^2)}{\nu(\lambda_1\nu - 2L_g)}\Big)^{1/2} - \frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}} - C_f\lambda_1^{-1}\Big)\|w(t)\|^2.
\end{aligned} \tag{4.12}$$

Integrating (4.12) from 0 to t, we obtain

$$\begin{aligned}
&U(t,w_t) + 2\Big(\nu - 2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega| + \|h\|_*^2)}{\nu(\lambda_1\nu - 2L_g)}\Big)^{1/2}\\
&- \frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}} - C_f\lambda_1^{-1}\Big)\int_0^t \|w(s)\|^2 ds\\
&\le U(0,u_0).
\end{aligned} \tag{4.13}$$

From (4.11), we have

$$U(t,w_t) \ge |u(t)-u^*|^2 + \alpha^2\|u(t)-u^*\|^2$$

and

$$U(0,u_0)=|\phi(0)-u^*|^2+\alpha^2\|\phi(0)-u^*\|^2+\frac{L_g}{\sqrt{1-\rho_*}}\|\phi-u^*\|^2_{L^2(-\rho(0),0;H)}.$$

Then using the Poincaré inequality (2.1), inequality (4.13) becomes

$$\begin{aligned}
&|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\\
&+2\lambda_1\Big(\nu-2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}-\frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}-C_f\lambda_1^{-1}\Big)\\
&\times\int_0^t|u(s)-u^*|^2ds\\
&\le|\phi(0)-u^*|^2+\alpha^2\|\phi(0)-u^*\|^2+\frac{L_g}{\sqrt{1-\rho_*}}\|\phi-u^*\|^2_{L^2(-\rho(0),0;H)}.
\end{aligned}\tag{4.14}$$

Therefore, if

$$\nu\ge 2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+\frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}+C_f\lambda_1^{-1},$$

then the stationary solution u^* is stable and satisfies (4.9). If

$$\nu> 2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+\frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}+C_f\lambda_1^{-1},$$

then from (4.14) we obtain

$$\begin{aligned}
&\int_0^\infty(|u(s)-u^*|^2+\alpha^2\|u(s)-u^*\|^2)ds\\
&\le|\phi(0)-u^*|^2+\alpha^2\|\phi(0)-u^*\|^2+\frac{L_g}{\sqrt{1-\rho_*}}\|\phi-u^*\|^2_{L^2(-\rho(0),0;H)}.
\end{aligned}$$

By the continuity in time of u in H, we deduce that $\lim_{t\to\infty}|u(t)-u^*|^2=0$, i.e. the stationary solution u^* is asymptotically stable. □

Since $\frac{(2-\rho_*)L_g\lambda_1^{-1}}{2(1-\rho_*)}>\frac{L_g\lambda_1^{-1}}{\sqrt{1-\rho_*}}$ for $\rho_*\in(0,1)$, Theorem 4.5 is an improvement of Theorem 4.4.

4.2.3. *Polynomial stability: the proportional delay case.* We now consider the 3D Kelvin-Voigt-Brinkman-Forchheimer equations with proportional delay, which is a particular case of unbounded variable delay. More precisely, we assume $\rho(t)=(1-q)t$ with $q\in(0,1)$. We will show the polynomial stability of the stationary solution.

First, we consider the pantograph equation

$$x'(t)=ax(t)+bx(qt),t>0,x(0)=x_0,q\in(0,1).\tag{4.15}$$

The following lemmas are key tools in the proof of polynomial stability results.

Lemma 4.6 ([4, Lemma 3.4]). *Let $a\in\mathbb{R},b>0$ and $q\in(0,1)$. Assume x is the solution to* (4.15) *with $x(0)>0$. Suppose $p\in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfies*

$$D^+p(t)\le ap(t)+bp(qt),\quad t\ge0,$$

with $0<p(0)<x(0)$ and where D^+ denotes the Dini derivative. Then $p(t)\le x(t)$ for all $t\ge0$.

Lemma 4.7 ([4, Lemma 3.5])**.** *Let x be a solution to* (4.15)*. If $a<0, b\in\mathbb{R}$, then there exists $C=C(a,b,q)>0$ such that*

$$\limsup_{t\to\infty}\frac{|x(t)|}{t^\mu}=C|x(0)|,$$

where $\mu\in\mathbb{R}$ satisfies $0=a+|b|q^\mu$. Then, for some (possibly new) $C=C(a,b,q)>0$, we have

$$|x(t)|\le C|x(0)|(1+t)^\mu,\ t\ge 0.$$

We are ready to state the main result in this subsection.

Theorem 4.8. *Assume that $f\in V'$ and consider* (4.6) *with $\rho(t)=(1-q)t$, for $q\in(0,1)$. If $\nu>L_g\lambda_1^{-1}$ then there exists at least one weak stationary solution u^* to* (4.1) *satisfying* (4.2)*. Furthermore, if*

$$\nu>2c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+\frac{1}{2}L_g\lambda_1^{-1}+C_f\lambda_1^{-1}, \tag{4.16}$$

then u^* is asymptotically stable with polynomial rate, that is, there exists $C=C(\nu,L_g,q)>0$ such that

$$|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\le C(|\phi(0)-u^*|^2+\alpha^2\|\phi(0)-u^*\|^2)(1+t)^\mu,$$

for all $t\ge 0$, where

$$\mu=\log_q\Big(\frac{2\nu-4c_0\lambda_1^{1/4}\big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\big)^{1/2}-L_g\lambda_1^{-1}-2C_f\lambda_1^{-1}}{L_g\lambda_1^{-1}}\Big)<0. \tag{4.17}$$

Proof. The existence and uniqueness of a stationary solution u^* to (4.1) follows from Theorem 4.2. Let $v(t)=u(t)-u^*$, then $v(t)$ satisfies

$$\begin{aligned}&\frac{d}{dt}v(t)+\nu Av(t)-B(u(t),u(t))+B(u^*,u^*)+f(u)-f(u^*)\\&=P(G(u(qt))-G(u^*)),\end{aligned}$$

with initial condition $v(\theta)=\phi(\theta)-u^*$, $\theta\in(-\infty,0]$. Then, using the same argument as in estimate (4.8), taking $\rho(t)=(1-q)t$, and using the bound (4.2), we have

$$\begin{aligned}&\frac{d}{dt}(|v(t)|^2+\alpha^2\|v(t)\|^2)\\&\le\Big(4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+L_g\lambda_1^{-1}+2C_f\lambda_1^{-1}-2\nu\Big)\\&\quad\times(\|v(t)\|^2+\alpha^2\|v(t)\|^2)+L_g(|v(qt)|^2+\alpha^2\|v(qt)\|^2).\end{aligned} \tag{4.18}$$

Denoting $x(t)=|v(t)|^2+\alpha^2\|v(t)\|^2$, noting that

$$4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+L_g\lambda_1^{-1}+2C_f\lambda_1^{-1}-2\nu<0,$$

and using the Poincaré inequality (2.1), we obtain from (4.18) that

$$x'(t)\le\lambda_1\Big(4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega|+\|h\|_*^2)}{\nu(\lambda_1\nu-2L_g)}\Big)^{1/2}+L_g\lambda_1^{-1}+2C_f\lambda_1^{-1}-2\nu\Big)x(t)+L_gx(qt).$$

Applying Lemmas 4.6 and 4.7, there exists $C=C(\nu,L_g,q)>0$ such that

$$x(t)\le Cx(0)(1+t)^\mu,\quad\forall t\ge 0,$$

where μ satisfies

$$\lambda_1\Big(4c_0\lambda_1^{1/4}\Big(\frac{(2\nu K|\Omega| + \|h\|_*^2)}{\nu(\lambda_1\nu - 2L_g)}\Big)^{1/2} + L_g\lambda_1^{-1} + 2C_f\lambda_1^{-1} - 2\nu\Big) + L_g q^{\mu} = 0.$$

that is, μ is given by (4.17). Note that if $\mu < 0$, we obtain the polynomial stability of the stationary solution u^*. The proof is complete. $\square$

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