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A Q-ANALOGUE OF KUMMER'S EQUATION

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Abstract. In this article we define a q-analogue of Kummer's equation. It has two singular points. Near the singular point at zero, using the Frobenius method, we obtain two linearly independent series solutions in any one of three cases according to the difference of roots of the characteristic equation. Near the singular point at infinity, given that the only formal series solution is divergent, we find two integral solutions which are convergent under some condition. Finally, using the q-analogue of Kummer's equation, we deduce six contiguous relations about the q-hypergeometric series ${}_1\Phi_1$.

1. Introduction

Kummer's equation can be written as [5]

$$zu''(z) + (b - z)u'(z) - au(z) = 0, \tag{1.1}$$

where $a, b, z \in \mathbb{C}$. It has a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. We know that there are two formal series solutions around $z = 0$, i.e.

$$\begin{aligned} u_1 &:= \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} = {}_1F_1(a; b; z), \\ u_2 &:= z^{1-b} \sum_{n=0}^{\infty} \frac{(a+1-b)_n z^n}{(2-b)_n n!} = z^{1-b}{}_1F_1(a+1-b; 2-b; z), \end{aligned} \tag{1.2}$$

where $(a)_n$ is the shifted factorial or Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)(a+2)\cdots(a+n-1), & \text{if } n \geq 1, \end{cases}$$

and ${}_1F_1(a; b; z)$ is a generalized hypergeometric series. The solutions u_1 and u_2 can be considered as functions of a, b, z with the other two variables held constant. Then u_1 defines an entire function of a or z except when $b = 0, -1, -2, \dots$. As a function of b it is analytic except for poles at the non-positive integers. u_2 defines an entire function of a or z except when $b = 2, 3, \dots$. As a function of b it is analytic except for poles at the positive integers greater than 1.

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The formal series solution at $z = \infty$ is

$$u_3 := z^{-a} \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n (a+1-b)_n}{n! z^n} = z^{-a} {}_2F_0\big(a, a+1-b; -; -\frac{1}{z}\big). \tag{1.3}$$

The series u_3 is divergent, but it has an integral representation.

$$u_3 = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{c-a-1} dt. \tag{1.4}$$

This integral converges for $\operatorname{Re} a > 0$ and $\operatorname{Re} z > 0$ and is a solution of the Kummer's equation.

Let $0 < q < 1$. For a function $u(z)$ of the complex variable z with $z \neq 0$, define its q-derivative as

$$D_q u(z) = \frac{u(z) - u(qz)}{(1-q)z}, \tag{1.5}$$

and its nth order q-derivative as

$$D_q^n u(z) = D_q(D_q^{n-1} u(z)). \tag{1.6}$$

An nth order q-difference equation is

$$P_n(D_q) u(z) = 0, \tag{1.7}$$

where $P_n(\cdot)$ is an nth order polynomial.

When q tends to 1, the q-difference operator D_q "tends" to the usual derivation. Hence every differential equation can be discretized by a q-difference equation.

The Euler's hypergeometric equation

$$z(1-z)u''(z) + [c - (a+b+1)z]u'(z) - abu(z) = 0 \tag{1.8}$$

has a q-analogue difference equation

$$z(q^c - q^{a+b+1} z) D_q^2 u(z) + [[c]_q - (q^b [a]_q + q^a [b+1]_q) z] D_q u(z) - [a]_q [b]_q u(z) = 0, \tag{1.9}$$

where $a, b, c, z \in \mathbb{C}$ and $[a]_q = \frac{1-q^a}{1-q}$. Some of series solutions of equation (1.9) are obtained in the form of basic hypergeometric series in [4]. However, it seems that nowadays very little has been known about the q-analogues of Kummer's equation.

In this study, we define a q-analogue of Kummer's equation. In trying to get formal series solutions, we find that the Frobenius method used in the classical ordinary differential equations is also applicable to the q-difference equations when the singular point is regular. At the regular singular point of zero, the characteristic equation has two roots. According to the difference of these two roots, there are three cases to consider. For each case, we obtain two linearly independent series solutions. Near the irregular singular point at infinity, given that the only formal series solution is divergent, we find two integral solutions which are convergent under certain condition. Finally, six contiguous relations about the q-hypergeometric series ${}_1\Phi_1$ are presented.

The rest of this article is organized as follows. In section 2, we obtain a q-analogue difference equation satisfied by the q-hypergeometric series ${}_r\Phi_s$. As a special case, we have a q-analogue of Kummer's equation. In section 3, we define the singular points for the second order q-difference equations. For the q-analogue of Kummer's equation it has two singular points at 0 and ∞. At the singular point 0, using the Frobenius method we obtain the series solutions of the q-analogue of Kummer's equation. At the singular point ∞, given the only series solution is divergent, we derive two integral solutions which are convergent under some condition. In section

4, by using the q-analogue of Kummer's equation, we find six contiguous relations about the q-hypergeometric series ${}_1\Phi_1$.

2. q-analogue of Kummer's equation

For the convenience of statement of q-difference equations, we still use the definition of the basic hypergeometric series defined in [6]. Let $0 < q < 1$, a basic hypergeometric series or q-hypergeometric series is

$$\begin{aligned} {}_r\Phi_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z\right] &= {}_r\Phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ &:= \sum_{n=0}^{\infty} \frac{(a_1;q)_n \dots (a_r;q)_n}{(b_1;q)_n \dots (b_s;q)_n (q;q)_n} z^n, \end{aligned} \tag{2.1}$$

where $r, s \in \mathbb{N} := \{0, 1, 2, \dots\}$, $a_1, \dots, a_r, b_1, \dots, b_s, z \in \mathbb{C}$ and $(a;q)_n$ is q-shifted factorial defined by

$$(a;q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1. \end{cases}$$

To avoid zeros in the denominator of series (2.1), we require that

$$b_1, \dots, b_s \neq 1, q^{-1}, q^{-2}, \dots .$$

For the series in (2.1),

$$\frac{(n+1)\text{th term}}{n\text{th term}} = \frac{(1-a_1q^n)\cdots(1-a_rq^n)z}{(1-b_1q^n)\cdots(1-b_sq^n)(1-q^{n+1})}.$$

The series (2.1) will terminate if an only if, for some $i = 1, \dots, r$, we have $a_i \in \{1, q^{-1}, q^{-2}, \dots\}$. If $a_i = q^{-k}$ $(k = 0, 1, 2, \dots)$, then all terms in the series with $n > k$ will vanish. In the non-vanishing case, by the ratio test, the convergence radius of (2.1) is 1.

Since

$$\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = \lim_{q\to 1} \frac{1-q^a}{1-q} \frac{1-q^{a+1}}{1-q} \cdots \frac{1-q^{a+n-1}}{1-q} = (a)_n,$$

we can view the q-shifted factorial as a q-analogue of the shifted factorial. Hence ${}_r\Phi_s$ is a q-analogue of ${}_rF_s$ by the formal (termwise) limit [4]

$$\lim_{q\to 1} {}_r\Phi_s\left[\begin{matrix} q^{a_1}, \dots, q^{a_r} \\ q^{b_1}, \dots, q^{b_s} \end{matrix}; q, (1-q)^{1+s-r}z\right] = {}_rF_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z\right]. \tag{2.2}$$

The hypergeometric series

$$u(z) = {}_rF_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z\right]$$

is a formal solution of the differential equation [1]:

$$\{\delta(\delta+b_1-1)\cdots(\delta+b_s-1) - z(\delta+a_1)\cdots(\delta+a_r)\}u(z) = 0, \tag{2.3}$$

where $\delta = z\frac{d}{dz}$.

It has a q-analogue difference equation

$$\begin{aligned} &\Big[\delta_q(q^{b_1-1}\delta_q+[b_1-1]_q)\cdots(q^{b_s-1}\delta_q+[b_s-1]_q) \\ &-z(q^{a_1}\delta_q+[a_1]_q)\cdots(q^{a_r}\delta_q+[a_r]_q)\Big]u(z)=0, \end{aligned} \tag{2.4}$$

where

$$\delta_q=zD_q,\quad [a]_q=\frac{1-q^a}{1-q}.$$

For solutions of the equation (2.4) we have the following result.

Theorem 2.1. *The equation* (2.4) *has a series solution as*

$$_r\Phi_s(q^{a_1},\dots,q^{a_r};q^{b_1},\dots,q^{b_s};q,(1-q)^{1+s-r}z)$$

which converges when $|(1-q)^{1+s-r}z|<1$.

Proof. Let $_r\Phi_s:={}_r\Phi_s(q^{a_1},\dots,q^{a_r};q^{b_1},\dots,q^{b_s};q,(1-q)^{1+s-r}z)$. By a straightforward calculation, it holds

$$[q^{b_s-1}\delta_q+[b_s-1]_q]_r\Phi_s=[b_s-1]_q+\sum_{n=1}^{\infty}\frac{(q^{a_1};q)_n\cdots(q^{a_r};q)_n(1-q)^{n(1+s-r)}z^n}{(1-q)(q^{b_1};q)_n\cdots(q^{b_s};q)_{n-1}(q;q)_n}.$$

Then, it gives

$$\begin{aligned} &(q^{b_1-1}\delta_q+[b_1-1]_q)\cdots(q^{b_s-1}\delta_q+[b_s-1]_q)_r\Phi_s \\ &=[b_1-1]_q\cdots[b_s-1]_q+\sum_{n=1}^{\infty}\frac{(q^{a_1};q)_n\cdots(q^{a_r};q)_n(1-q)^{n(1+s-r)}z^n}{(1-q)^s(q^{b_1};q)_{n-1}\cdots(q^{b_s};q)_{n-1}(q;q)_n} \\ &=[b_1-1]_q\cdots[b_s-1]_q+\sum_{n=0}^{\infty}\frac{(q^{a_1};q)_{n+1}\cdots(q^{a_r};q)_{n+1}(1-q)^{(n+1)(1+s-r)}z^{n+1}}{(1-q)^s(q^{b_1};q)_n\cdots(q^{b_s};q)_n(q;q)_{n+1}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\delta_q(q^{b_1-1}\delta_q+[b_1-1]_q)\cdots(q^{b_s-1}\delta_q+[b_s-1]_q)_r\Phi_s \\ &=\sum_{n=0}^{\infty}\frac{(q^{a_1};q)_{n+1}\cdots(q^{a_r};q)_{n+1}(1-q)^{(n+1)(1+s-r)}z^n}{(1-q)^{s+1}(q^{b_1};q)_n\cdots(q^{b_s};q)_n(q;q)_n}, \end{aligned} \tag{2.5}$$

and

$$(q^{a_r}\delta_q+[a_r]_q)_r\Phi_s=\sum_{n=0}^{\infty}\frac{(q^{a_1};q)_n\cdots(q^{a_r};q)_{n+1}(1-q)^{n(1+s-r)}z^n}{(1-q)(q^{b_1};q)_n\cdots(q^{b_s};q)_n(q;q)_n}.$$

That is,

$$\begin{aligned} &z(q^{a_1}\delta_q+[a_1]_q)\cdots(q^{a_r}\delta_q+[a_r]_q)_r\Phi_s \\ &=\sum_{n=0}^{\infty}\frac{(q^{a_1};q)_{n+1}\cdots(q^{a_r};q)_{n+1}(1-q)^{n(1+s-r)}z^{n+1}}{(1-q)^r(q^{b_1};q)_n\cdots(q^{b_s};q)_n(q;q)_n}. \end{aligned} \tag{2.6}$$

Since the right-hand sides of (2.5) and (2.6) are the same, we arrive at our conclusion. ∎

We consider the case where $r=s=1$ in (2.4) and the resulting equation is called the q-analogue of Kummer's equation. Let $a_1=a$ and $b_1=c$, it can be re-written as

$$q^czD_q^2u(z)+([c]_q-q^az)D_qu(z)-[a]_qu(z)=0. \tag{2.7}$$

In the next section, we will study other solutions of (2.7) and discuss the properties of these solutions.

3. Solutions of the q-analogue of Kummer's equation

Just as in the second-order ordinary differential equations, we try to find the formal series solutions of (2.7). First of all, we give the definition of singular points of the second order q-difference equations.

Consider the q-difference equation:

$$p_2(z)D_q^2u(z) + p_1(z)D_qu(z) + p_0(z)u(z) = 0 \tag{3.1}$$

where $p_2(z)$, $p_1(z)$ and $p_0(z)$ are analytic in the neighborhood of $z = z_0$. When $p_2(z_0) \neq 0$, the point z_0 is called an *ordinary point* of the equation (3.1). When $p_2(z_0) = 0$ and $p_1(z_0)$ and/or $p_0(z_0)$ is not zero, then $z = z_0$ is called a *singular point* of (3.1). The point $z = z_0$ is called a *regular singular point* of (3.1) if

$$\lim_{z\to z_0} \frac{(z-z_0)p_1(z)}{p_2(z)} \quad \text{and} \quad \lim_{z\to z_0} \frac{(z-z_0)^2p_0(z)}{p_2(z)}$$

both exist. If one of these limits does not exist, the singular point is *irregular.*

It is easy to see that the equation (2.7) has singular points at $z = 0$ and $z = \infty$. So we have two cases to consider.

3.1. Solutions at $z = 0$.

Equation (2.7) has the form as in (3.1), where

$$p_2(z) = q^cz, \quad p_1(z) = [c]_q - q^az, \quad p_0(z) = -[a]_q.$$

Both of them are analytic near zero. Moreover, it holds

$$\lim_{z\to 0} \frac{zp_1(z)}{p_2(z)} = \frac{z([c]_q - q^az)}{q^cz} = [c]_qq^{-c} \quad \text{and} \quad \lim_{z\to 0} \frac{z^2p_0(z)}{p_2(z)} = 0.$$

Thus, the singular point $z = 0$ is regular. We know that by using the Frobenius method [7] one can find the formal series solutions of the second order ordinary differential equations. In fact, as we can see below, the Frobenius method is also applicable to the second order q-difference equations. Note that the detailed procedure to find the formal series solutions of the q-difference equations is described in [4].

Let

$$L[u] = q^czD_q^2u(z) + ([c]_q - q^az)D_qu(z) - [a]_qu(z). \tag{3.2}$$

Then equation (2.7) becomes $L[u] = 0$. Since both difference operators D_q^2 and D_q are linear, the operator $L[\cdot]$ is also linear, i.e.,

$$L[\alpha u + \beta] = \alpha L[u] + \beta.$$

for any complex numbers α and β.

We assume the solution of the form

$$u = \sum_{n=0}^{\infty} d_nz^{\lambda+n} \text{ with } d_0 \neq 0. \tag{3.3}$$

It holds

$$\begin{aligned} D_q u(z) &= \sum_{n=0}^{\infty} d_n [\lambda+n]_q z^{\lambda+n-1}, \\ D_q^2 u(z) &= \sum_{n=0}^{\infty} d_n [\lambda+n]_q [\lambda+n-1]_q z^{\lambda+n-2}. \end{aligned} \tag{3.4}$$

Substituting (3.3), (3.4) in (3.2), we obtain

$$\begin{aligned} L[u] =& \sum_{n=0}^{\infty} d_n [\lambda+n]_q (q^c[\lambda+n-1]_q + [c]_q) z^{\lambda+n-1} \\ & - \sum_{n=0}^{\infty} d_n (q^a[\lambda+n]_q + [a]_q) z^{\lambda+n}. \end{aligned} \tag{3.5}$$

Changing the indices of n in the second term of (3.5) and then isolating the terms with $n = 0$, we have

$$\begin{aligned} L[u] =& d_0[\lambda]_q (q^c[\lambda-1]_q + [c]_q) z^{\lambda-1} \\ & + \sum_{n=1}^{\infty} d_n [\lambda+n]_q (q^c[\lambda+n-1]_q + [c]_q) z^{\lambda+n-1} \\ & - \sum_{n=1}^{\infty} d_{n-1} (q^a[\lambda+n-1]_q + [a]_q) z^{\lambda+n-1}. \end{aligned} \tag{3.6}$$

Choosing the c_n's to satisfy the recurrence relations:

$$d_n[\lambda+n]_q (q^c[\lambda+n-1]_q + [c]_q) = d_{n-1}(q^a[\lambda+n-1]_q + [a]_q). \tag{3.7}$$

From the equation (3.7)

$$d_n = \frac{(q^{\lambda+a};q)_n (1-q)^n}{(q^{\lambda+c};q)_n (q^{\lambda+1};q)_n} d_0, \tag{3.8}$$

(3.6) becomes

$$L[u] = d_0[\lambda]_q (q^c[\lambda-1]_q + [c]_q) z^{\lambda-1}. \tag{3.9}$$

The equation

$$[\lambda]_q (q^c[\lambda-1]_q + [c]_q) = 0 \tag{3.10}$$

is called the *indicial equation.*

The solutions of the above indicial equation are

$$\lambda = 0 \text{ or } \lambda = 1-c.$$

According to the values of the above roots, there are three cases to consider about the solutions of (2.7).

Case 1. If c is not an integer, by (3.8) we have

$$\begin{gathered} u_1 := u|_{\lambda=0} = d_0\, {}_1\Phi_1(q^a; q^c; q, (1-q)z), \\ u_2 := u|_{\lambda=1-c} = d_0 z^{1-c}\, {}_1\Phi_1(q^{a+1-c}; q^{2-c}; q, (1-q)z). \end{gathered}$$

Since c is not an integer, the solutions u_1 and u_2 are linearly independent. Thus, any linear combination of u_1 and u_2 is a solution of the equation (2.7).

Case 2. If $c = 1$, then

$$u_1 := u|_{\lambda=0} = u|_{\lambda=1-c} = d_0\, {}_1\Phi_1(q^a; q; q, (1-q)z).$$

To obtain the other solution, we note that (3.9) becomes

$$L[u] = d_0[\lambda]_q^2 z^{\lambda-1}. \tag{3.11}$$

Then differentiating with respect to λ in (3.11), we have

$$L[\frac{\partial u}{\partial \lambda}] = \frac{\partial L[u]}{\partial \lambda} = 2d_0 z^{\lambda-1}[\lambda]_q \frac{q^\lambda \ln q}{q-1} + d_0[\lambda]_q^2 z^{\lambda-1}\ln z. \tag{3.12}$$

If $\lambda = 0$, then the right-hand side of (3.12) is zero. Thus, the other solution of (2.7) is

$$\begin{aligned} u_2 :=& \frac{\partial u}{\partial \lambda}\Big|_{\lambda=0} \\ =& d_0 \ln z + d_0 \sum_{n=1}^{\infty} \frac{(q^a;q)_n (1-q)^n z^n}{(q;q)_n (q;q)_n} \\ &\times \Big[\ln z + \ln q \sum_{j=0}^{n-1}\Big(\frac{2}{1-q^{1+j}} - \frac{1}{1-q^{a+j}} - 1\Big)\Big]. \end{aligned}$$

Case 3. If c is an integer and $c \neq 1$, there are two cases to consider.

(i) If $c < 1$, then in the equation (3.8), when $\lambda = 0$ and $n = 1-c$, the denominator $(q^{\lambda+c};q)_n = 0$. Thus, we take $d_0 = g_0(1-q^\lambda)$. Since

$$(q^{\lambda+c};q)_{1-c} = (1-q^{\lambda+c})\cdots(1-q^\lambda),$$

our assumed solution (3.3) has the form

$$\begin{aligned} u_g :=& g_0 z^\lambda \sum_{n=0}^{\infty} \frac{(1-q^\lambda)(q^{\lambda+a};q)_n (1-q)^n z^n}{(q^{\lambda+c};q)_n (q^{\lambda+1};q)_n} \\ =& g_0 z^\lambda \sum_{n=0}^{-c} \frac{(1-q^\lambda)(q^{\lambda+a};q)_n (1-q)^n z^n}{(q^{\lambda+c};q)_n (q^{\lambda+1};q)_n} \\ &+ \frac{g_0 z^\lambda}{(q^{\lambda+c};q)_{-c}} \sum_{n=1-c}^{\infty} \frac{(q^{\lambda+a};q)_n (1-q)^n z^n}{(q^{\lambda+1};q)_{n-1+c}(q^{\lambda+1};q)_n}. \end{aligned}$$

Hence, the first $1-c$ terms of u_g vanish when $\lambda = 0$. Then the first solution takes the form

$$u_1 := u_g|_{\lambda=0} = \frac{g_0}{(q^c;q)_{-c}} \sum_{n=1-c}^{\infty} \frac{(q^a;q)_n (1-q)^n z^n}{(q;q)_{n-1+c}(q;q)_n}.$$

For the solution $u_g|_{\lambda=1-c}$, let $m = 1-c > 0$. Then we find

$$\begin{aligned} u_g|_{\lambda=1-c} &= g_0 \sum_{n=0}^{\infty} \frac{(1-q^m)(q^{a+m};q)_n (1-q)^n z^{n+m}}{(q;q)_n (q^{1+m};q)_n} \\ &= g_0 \sum_{n=m}^{\infty} \frac{(1-q^m)(q^{a+m};q)_{n-m}(1-q)^{n-m} z^n}{(q;q)_{n-m}(q^{1+m};q)_{n-m}} \\ &= \frac{(q^c;q)_{-c}(1-q^m)(q;q)_m}{(q^a;q)_m (1-q)^m} u_1 \end{aligned}$$

Thus, u_1 and $u|_{\lambda=1-c}$ are not linearly independent. To obtain the other linearly independent solution, we note that

$$L[u_g] = g_0 z^{\lambda-1}\frac{(1-q^{\lambda})^2(1-q^{\lambda+c-1})}{(1-q)^2},$$

whose derivative with respect to λ is

$$\begin{aligned}
L[\frac{\partial u_g}{\partial\lambda}] =& \frac{\partial L[u_g]}{\partial\lambda}\\
=& g_0 z^{\lambda-1}\ln z\frac{(1-q^{\lambda})^2(1-q^{\lambda+c-1})}{(1-q)^2}\\
&- 2g_0 z^{\lambda-1}q^{\lambda}\ln q\frac{(1-q^{\lambda})(1-q^{\lambda+c-1})}{(1-q)^2}\\
&- g_0 z^{\lambda-1}\ln q\frac{(q^{\lambda}-1)^2 q^{\lambda+c-1}}{(q-1)^2}.
\end{aligned}\tag{3.13}$$

If $\lambda = 0$, the right-hand side of (3.13) is zero. Thus, the other solution is

$$\begin{aligned}
u_2 :=& \frac{\partial u_g}{\partial\lambda}\Big|_{\lambda=0}\\
=& - g_0\ln q\sum_{n=0}^{-c}\frac{(q^a;q)_n(1-q)^n z^n}{(q^c;q)_n(q;q)_n}\\
&+ \frac{g_0}{(q^c;q)_{-c}}\sum_{n=1-c}^{\infty}\Big\{\frac{(q^a;q)_n(1-q)^n z^n}{(q;q)_{n-1+c}(q;q)_n}\Big[\ln z + \ln q\sum_{j=0}^{n-1}\Big(\frac{1}{1-q^{1+j}} - \frac{1}{1-q^{a+j}}\Big)\\
&+ \ln q\sum_{j=0,j\neq -c}^{n-1}\frac{q^{c+j}}{1-q^{c+j}}\Big]\Big\}.
\end{aligned}$$

(ii) If $c > 1$, then in the equation (3.8), when $\lambda = 1-c$ and $n = c-1$, the denominator $(q^{\lambda+1};q)_n = 0$. Hence, we take $d_0 = h_0(1-q^{\lambda+c-1})$. Since

$$(q^{\lambda+1};q)_{c-1} = (1-q^{\lambda+1})\cdots(1-q^{\lambda+c-1}),$$

our assumed solution (3.3) takes a new form

$$\begin{aligned}
u_h :=& h_0\sum_{n=0}^{\infty}\frac{(1-q^{\lambda+c-1})(q^{\lambda+a};q)_n}{(q^{\lambda+c};q)_n(q^{\lambda+1};q)_n}(1-q)^n z^{\lambda+n}\\
=& h_0\sum_{n=0}^{c-2}\frac{(1-q^{\lambda+c-1})(q^{\lambda+a};q)_n}{(q^{\lambda+c};q)_n(q^{\lambda+1};q)_n}(1-q)^n z^{\lambda+n}\\
&+ \frac{h_0}{(q^{\lambda+1};q)_{c-2}}\sum_{n=c-1}^{\infty}\frac{(q^{\lambda+a};q)_n}{(q^{\lambda+c};q)_{n-c+1}(q^{\lambda+c};q)_n}(1-q)^n z^{\lambda+n}.
\end{aligned}$$

So the first $c-1$ terms of u_h vanish when $\lambda = 1-c$.

Then the first solution takes the form

$$u_1 := u_h|_{\lambda=1-c} = \frac{h_0 z^{1-c}}{(q^{2-c};q)_{c-2}}\sum_{n=c-1}^{\infty}\frac{(q^{a-c+1};q)_n}{(q;q)_{n-c+1}(q;q)_n}(1-q)^n z^n.$$

Let $k = c - 1 > 0$. Then it has

$$\begin{aligned} u_h|_{\lambda=0} &= h_0 \sum_{n=0}^{\infty} \frac{(1-q^{c-1})(q^a;q)_n}{(q^c;q)_n(q;q)_n}(1-q)^n z^n \\ &= h_0 \sum_{n=k}^{\infty} \frac{(1-q^{c-1})(q^a;q)_{n-k}}{(q^c;q)_{n-k}(q;q)_{n-k}}(1-q)^{n-k} z^{n-k} \\ &= \frac{(1-q^{c-1})(q^{2-c};q)_{c-2}(q;q)_{c-1}}{(q^{a-c+1};q)_{c-1}(1-q)^{c-1}} u_1. \end{aligned}$$

Thus, u_1 and $u_h|_{\lambda=0}$ are not linearly independent.

For the other solution, by (3.9), it has

$$L[u_h] = h_0 \frac{(1-q^\lambda)(1-q^{\lambda+c-1})^2}{(1-q)^2} z^{\lambda-1}$$

whose derivative with respect to λ is

$$\begin{aligned} L\big[\frac{\partial u_h}{\partial \lambda}\big] =& \frac{\partial L[u_h]}{\partial \lambda} \\ =& - h_0 \frac{q^\lambda \ln q (1-q^{\lambda+c-1})^2}{(1-q)^2} z^{\lambda-1} \\ & - 2h_0 \frac{(1-q^\lambda)(1-q^{\lambda+c-1}) q^{\lambda+c-1}\ln q}{(1-q)^2} z^{\lambda-1} \\ & + h_0 \frac{(1-q^\lambda)(1-q^{\lambda+c-1})^2}{(1-q)^2} z^{\lambda-1}\ln z. \end{aligned} \tag{3.14}$$

If $\lambda = 1 - c$, the right-hand side of (3.14) is zero. Thus the other solution is

$$\begin{aligned} u_2 :=& \frac{\partial u_h}{\partial \lambda}\Big|_{\lambda=1-c} \\ =& - h_0 z^{1-c} \ln q \sum_{n=0}^{c-2} \frac{(q^{a-c+1};q)_n}{(q;q)_n (q^{2-c};q)_n}(1-q)^n z^n \\ & + \frac{h_0 z^{1-c}}{(q^{2-c};q)_{c-2}} \sum_{n=c-1}^{\infty} \Big\{ \frac{(q^{a-c+1};q)_n}{(q;q)_{n+1-c}(q;q)_n}(1-q)^n z^n \Big[\ln z \\ & + \ln q \sum_{j=0}^{n-1} \Big(\frac{1}{1-q^{1+j}} - \frac{1}{1-q^{a-c+1+j}} \Big) + \ln q \sum_{j=0, j\neq c-2}^{n-1} \frac{q^{2-c+j}}{1-q^{2-c+j}} \Big] \Big\}. \end{aligned}$$

Consequently, we obtain the following results.

Theorem 3.1. *Equation (2.7) has a regular singular point zero. At the singular point zero:*

(i) *if c is not an integer, then there are two linearly independent series solutions*

$$u_1 = d_{01}\Phi_1(q^a; q^c; q, (1-q)z),$$
$$u_2 = d_0 z^{1-c}{}_1\Phi_1(q^{a-c+1}; q^{2-c}; q, (1-q)z).$$

(ii) *If $c = 1$, then there are two linearly independent series solutions*

$$u_1 = d_{01}\Phi_1(q^a; q; q, (1-q)z),$$

$$
\begin{aligned}
u_2 = d_0 \ln z + d_0 \sum_{n=1}^{\infty} & \frac{(q^a;q)_n(1-q)^n z^n}{(q;q)_n(q;q)_n} \\
& \times \Big[\ln z + \ln q \sum_{j=0}^{n-1}\Big(\frac{2}{1-q^{1+j}} - \frac{1}{1-q^{a+j}} - 1\Big)\Big].
\end{aligned}
$$

(iii) *If c is a non-positive integer, then there are two linearly independent series solutions*

$$
u_1 = \frac{g_0}{(q^c;q)_{-c}} \sum_{n=1-c}^{\infty} \frac{(q^a;q)_n(1-q)^n z^n}{(q;q)_{n-1+c}(q;q)_n},
$$

$$
\begin{aligned}
u_2 =& - g_0 \ln q \sum_{n=0}^{-c} \frac{(q^a;q)_n(1-q)^n z^n}{(q^c;q)_n(q;q)_n} \\
&+ \frac{g_0}{(q^c;q)_{-c}} \sum_{n=1-c}^{\infty} \Big\{ \frac{(q^a;q)_n(1-q)^n z^n}{(q;q)_{n-1+c}(q;q)_n} \Big[\ln z \\
&+ \ln q \sum_{j=0}^{n-1}\Big(\frac{1}{1-q^{1+j}} - \frac{1}{1-q^{a+j}}\Big) + \ln q \sum_{j=0, j\neq -c}^{n-1} \frac{q^{c+j}}{1-q^{c+j}}\Big]\Big\}.
\end{aligned}
$$

(iv) *If c is a positive integer such that $c > 1$, then there are two linearly independent series solutions*

$$
u_1 = \frac{h_0 z^{1-c}}{(q^{2-c};q)_{c-2}} \sum_{n=c-1}^{\infty} \frac{(q^{a-c+1};q)_n}{(q;q)_{n-c+1}(q;q)_n}(1-q)^n z^n
$$

$$
\begin{aligned}
u_2 =& - h_0 z^{1-c} \ln q \sum_{n=0}^{c-2} \frac{(q^{a-c+1};q)_n}{(q;q)_n(q^{2-c};q)_n}(1-q)^n z^n \\
&+ \frac{h_0 z^{1-c}}{(q^{2-c};q)_{c-2}} \sum_{n=c-1}^{\infty} \Big\{ \frac{(q^{a-c+1};q)_n}{(q;q)_{n+1-c}(q;q)_n}(1-q)^n z^n \Big[\ln z \\
&+ \ln q \sum_{j=0}^{n-1}\Big(\frac{1}{1-q^{1+j}} - \frac{1}{1-q^{a-c+1+j}}\Big) + \ln q \sum_{j=0, j\neq c-2}^{n-1} \frac{q^{2-c+j}}{1-q^{2-c+j}}\Big]\Big\}.
\end{aligned}
$$

As a solution of (2.7), we now seek other representation forms of ${}_1\Phi_1$, especially in its integral representation forms.

Theorem 3.2. *For $0 < q < 1$, $|z| < 1$ and $\mathrm{Re}(a) > 0$, it holds*

$$
\begin{aligned}
{}_1\Phi_1(q^a;q^c;q,z) &= \frac{(q^a;q)_\infty}{(q^c;q)_\infty (z;q)_\infty} {}_2\Phi_0(q^{c-a}, z; -; q, q^a) \\
&= \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-a)} \int_0^1 \frac{t^{a-1}(qt;q)_\infty}{(q^{c-a}t;q)_\infty (zt;q)_\infty} d_q t.
\end{aligned} \tag{3.15}
$$

Proof. By a direct calculation, we have

$$
\begin{aligned}
{}_1\Phi_1(q^a;q^c;q,z) &= \frac{(q^a;q)_\infty}{(q^c;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{c+n};q)_\infty}{(q;q)_n (q^{a+n};q)_\infty} z^n \\
&= \frac{(q^a;q)_\infty}{(q^c;q)_\infty} \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(q^{c-a};q)_m}{(q;q)_m} q^{m(a+n)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(q^a;q)_\infty}{(q^c;q)_\infty} \sum_{m=0}^{\infty} \frac{(q^{c-a};q)_m}{(q;q)_m (zq^m;q)_\infty} q^{am} \\
&= \frac{(q^a;q)_\infty}{(q^c;q)_\infty} \sum_{m=0}^{\infty} \frac{(q^{c-a};q)_m (z;q)_m}{(q;q)_m (z;q)_\infty} q^{am} \\
&= \frac{(q^a;q)_\infty}{(q^c;q)_\infty (z;q)_\infty} {}_2\Phi_0(q^{c-a}, z; -; q, q^a).
\end{aligned}
$$

To prove the second equality, we note that

$$
\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty} (1-q)^{1-a}.
$$

Then, it yields

$$
\begin{aligned}
&\frac{(q^a;q)_\infty}{(q^c;q)_\infty (z;q)_\infty} {}_2\Phi_0(q^{c-a}, z; -; q^a) \\
&= \frac{(q^a;q)_\infty}{(q^c;q)_\infty} \sum_{m=0}^{\infty} \frac{(q^{c-a};q)_\infty (q^{m+1};q)_\infty}{(q^{c-a+m};q)_\infty (zq^m;q)_\infty (q;q)_\infty} q^{am} \\
&= \frac{(q^a;q)_\infty (q^{c-a};q)_\infty}{(q^c;q)_\infty (q;q)_\infty} \frac{1}{1-q} \int_0^1 \frac{(qt;q)_\infty t^{a-1}}{(q^{c-a}t;q)_\infty (zt;q)_\infty} d_q t \\
&= \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-a)} \int_0^1 \frac{t^{a-1}(qt;q)_\infty}{(q^{c-a}t;q)_\infty (zt;q)_\infty} d_q t.
\end{aligned}
$$

□

The q-hypergeometric series ${}_1\Phi_1$ also has a q-analogue of Barnes' contour integral. The proof is very similar to [2, (4.2.2)].

Theorem 3.3.

$$
\begin{aligned}
&{}_1\Phi_1(a;c;q,z) \\
&= \frac{(a;q)_\infty}{(c;q)_\infty (q;q)_\infty} \Big(\frac{-1}{2\pi i}\Big) \int_{-i\infty}^{i\infty} \frac{(q^{1+s};q)_\infty (cq^s;q)_\infty}{(aq^s;q)_\infty} \frac{\pi(-z)^s}{\sin \pi s} ds \\
&= \frac{\Gamma_q(c)}{\Gamma_q(a)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma_q(a+s)\Gamma(-s)\Gamma(1+s)}{\Gamma_q(c+s)\Gamma_q(1+s)} (-z)^s ds.
\end{aligned} \tag{3.16}
$$

3.2. **Solutions at ∞.** For the solutions near infinity, we rewrite the equation (2.7) as

$$
q^{c-1}u(q^2 z) - [1 + q^{c-1} + q^a(q-1)z]u(qz) + [1 + (q-1)z]u(z) = 0. \tag{3.17}
$$

Then let $t = z^{-1}$, $p = q^{-1}$ and $w(t) = u(t^{-1})$. Then (3.17) becomes

$$
tw(p^2 t) - [p^{c-a-2}(1-p) + (1+p^{c-1})t]w(pt) + [p^{c-2}(1-p) + p^{c-1}t]w(t) = 0. \tag{3.18}
$$

Equation (3.18) has a q-derivative form

$$
t^3 D_p^2 w(t) + [p^{c-a-3}t - [c-2]_p t^2] D_p w(t) + p^{c-3}[-a]_p w(t) = 0, \tag{3.19}
$$

where

$$
\begin{aligned}
D_p w(t) &= \frac{w(t) - w(pt)}{(1-p)t}, \\
D_p^2 w(t) &= \frac{w(p^2 t) - (1+p)w(pt) + pw(t)}{(1-p)^2 p t^2},
\end{aligned} \tag{3.20}
$$

and $[a]_p = \frac{1-p^a}{1-p}$.

Equation (3.19) has the form as the one given in (3.1), where

$$p_2(t) = t^3,\ p_1(t) = p^{c-a-3}t - [c-2]_p t^2,\ p_0(t) = p^{c-3}[-a]_p.$$

All of them are analytic near zero. Since

$$\lim_{t\to 0} \frac{tp_1(t)}{p_2(t)} = \lim_{t\to 0} \frac{t[p^{c-a-3}t - [c-2]_p t^2]}{t^3}$$

does not exist, the singular point $t = 0$ of the equation (3.19) is irregular. Thus the singular point $z = \infty$ of the equation (2.7) is irregular.

If we still assume that the form of the solution is

$$u = \sum_{n=0}^{\infty} f_n z^{\lambda-n} \quad \text{with } f_0 \neq 0, \tag{3.21}$$

then we find

$$\begin{gathered} D_q u = \sum_{n=0}^{\infty} f_n[\lambda-n]_q z^{\lambda-n-1}, \\ D_q^2 u = \sum_{n=0}^{\infty} f_n[\lambda-n]_q[\lambda-n-1]_q z^{\lambda-n-2}. \end{gathered} \tag{3.22}$$

Substituting (3.21) and (3.22) into the equation (2.7) leads to

$$\begin{aligned} &\sum_{n=0}^{\infty} f_n[\lambda-n]_q\big(q^c[\lambda-n-1]_q + [c]_q\big) z^{\lambda-n-1} \\ &- \sum_{n=0}^{\infty} f_n\big(q^a[\lambda-n]_q + [a]_q\big) z^{\lambda-n} = 0. \end{aligned} \tag{3.23}$$

Changing the indices of the first term of (3.23) and then isolating terms with $n = 0$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} f_{n-1}[\lambda-n+1]_q\big(q^c[\lambda-n]_q + [c]_q\big) z^{\lambda-n} \\ &- \sum_{n=1}^{\infty} f_n\big(q^a[\lambda-n]_q + [a]_q\big) z^{\lambda-n} - f_0(q^a[\lambda]_q + [a]_q) z^{\lambda} = 0 \end{aligned}$$

From the third term of the last equation, we have the indicial equation

$$f_0(q^a[\lambda]_q + [a]_q) = 0.$$

In view of $f_0 \neq 0$, there is one solution $\lambda = -a$.

From the rest terms, we obtain a recurrence relation between f_n and f_{n-1} for any $n \geq 1$:

$$f_n = -\frac{q^{\lambda+c-a}(1-q^{-\lambda+n-1})(1-q^{-c-\lambda+n})}{(1-q^{-a-\lambda+n})(1-q)q^{n-1}} f_{n-1}.$$

From the recurrence relation, when $\lambda = -a$, we deduce that

$$f_n = (-1)^n \frac{q^{n(c-2a)}(q^a;q)_n(q^{a-c+1};q)_n}{(q;q)_n(1-q)^n q^{n(n-1)/2}} f_0.$$

Thus, we obtain a formal series solution as

$$u_3 = f_0 z^{-a} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(c-2a)} (q^a;q)_n (q^{a-c+1};q)_n}{(q;q)_n (1-q)^n q^{n(n-1)/2}} z^{-n}. \tag{3.24}$$

Unfortunately, the series in (3.24) is divergent, so it does not directly give a solution of the equation (2.7). However, it is possible to find integral solutions which are convergent under some conditions. Before that we need some properties about the definite q-integral.

Suppose $t, z \in \mathbb{C}$. The *definite q-integral* of a function $f(t)$ is

$$\int_0^z f(t) d_q t = (1-q) \sum_{j=0}^{\infty} z q^j f(z q^j). \tag{3.25}$$

From this definition we can deduce a more general formula:

$$\int_0^z f(t) d_q g(t) := \int_0^z f(t) D_q g(t) d_q t = \sum_{j=0}^{\infty} f(zq^j)(g(zq^j) - g(zq^{j+1})), \tag{3.26}$$

and a q-analogue of integration by parts [3]:

$$\int_0^z g(qt) d_q f(t) = g(z) f(z) - g(0) f(0) - \int_0^z f(t) d_q g(t). \tag{3.27}$$

The *improper q-integral* of $f(t)$ on $[0, +\infty)$ is defined by

$$\int_0^{\infty} f(t) d_q t = (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j). \tag{3.28}$$

Now, we need a formula about the q-derivative of a definite q-integral.

Proposition 3.4. *If $\alpha, z \in \mathbb{C} \backslash \{0\}$, then for any positive integer k, it holds*

$$D_q \Big(\int_0^{\frac{1}{\alpha z^k}} f(z,t) d_q t \Big) = \int_0^{\frac{1}{\alpha z^k}} D_q f(z,t) d_q t - \sum_{j=0}^{k-1} \frac{q^{j-k}}{\alpha z^{k+1}} f\Big(qz, \frac{q^{j-k}}{\alpha z^k}\Big), \tag{3.29}$$

where D_q is the q-derivative with respect to z.

Proof. By the definition (3.25)

$$\int_0^{\frac{1}{\alpha z^k}} f(z,t) d_q t = (1-q) \sum_{j=0}^{\infty} \frac{q^j}{\alpha z^k} f\big(z, \frac{q^j}{\alpha z^k}\big).$$

Then the q-derivative with respect to z of the above definite q-integral is

$$\begin{aligned}
& D_q \Big(\int_0^{\frac{1}{\alpha z^k}} f(z,t) d_q t \Big) \\
&= \frac{1}{(1-q)z} \Big[(1-q) \sum_{j=0}^{\infty} \frac{q^j}{\alpha z^k} f(z, \frac{q^j}{\alpha z^k}) - (1-q) \sum_{j=0}^{\infty} \frac{q^j}{\alpha q^k z^k} f(qz, \frac{q^j}{\alpha q^k z^k}) \Big] \\
&= \frac{1}{(1-q)z} \Big[(1-q) \sum_{j=0}^{\infty} \frac{q^j}{\alpha z^k} f(z, \frac{q^j}{\alpha z^k}) - (1-q) \sum_{j=0}^{\infty} \frac{q^j}{\alpha z^k} f(qz, \frac{q^j}{\alpha z^k}) \\
&\quad + (1-q) \sum_{j=0}^{\infty} \frac{q^j}{\alpha z^k} f(qz, \frac{q^j}{\alpha z^k}) - (1-q) \sum_{j=0}^{\infty} \frac{q^j}{\alpha q^k z^k} f(qz, \frac{q^j}{\alpha q^k z^k}) \Big]
\end{aligned}$$

$$
\begin{aligned}
&= (1-q)\sum_{j=0}^{\infty}\frac{q^j}{\alpha z^k}\frac{f(z,\frac{q^j}{\alpha z^k})-f(qz,\frac{q^j}{\alpha z^k})}{(1-q)z}\\
&\quad+\sum_{j=0}^{\infty}\frac{q^j}{\alpha z^{k+1}}f(qz,\frac{q^j}{\alpha z^k})-\sum_{j=0}^{\infty}\frac{q^{j-k}}{\alpha z^{k+1}}f\big(qz,\frac{q^{j-k}}{\alpha z^k}\big)\\
&=\int_0^{\frac{1}{\alpha z^k}} D_q f(z,t)d_qt-\sum_{j=0}^{k-1}\frac{q^{j-k}}{\alpha z^{k+1}}f\big(qz,\frac{q^{j-k}}{\alpha z^k}\big).
\end{aligned}
$$

□

Remark 3.5. When $q\to 1$, the formula (3.29) becomes

$$
\frac{d}{dz}\Big(\int_0^{\frac{1}{\alpha z^k}} f(z,t)d_qt\Big)=\int_0^{\frac{1}{\alpha z^k}}\frac{\partial}{\partial z}f(z,t)d_qt-\frac{k}{\alpha z^{k+1}}f\big(z,\frac{1}{\alpha z^k}\big),
$$

which is fundamental in calculus.

Next, we try to find an integral solution of (2.7) which has the form

$$
u_1(z)=\int_0^{\frac{1}{(1-q)q^2z}} E_q^{-qzt}g(qt)d_qt, \tag{3.30}
$$

where the q-analogue of exponential function is

$$
E_q^{-qzt}=((1-q)qzt;q)_\infty .
$$

Recall that the q-derivative with respect to z of E_q^{-qzt} is

$$
D_qE_q^{-qzt}=-qtE_q^{-q^2zt}.
$$

By Proposition 3.4 we have

$$
\begin{aligned}
D_qu_1(z)&=\int_0^{\frac{1}{(1-q)q^2z}} -qtE_q^{-q^2zt}g(qt)d_qt\\
&\quad-\frac{1}{(1-q)q^3z^2}E_q^{-q^2zt}g(qt)\Big|_{t=1/(1-q)q^3z}\\
&=-\int_0^{\frac{1}{(1-q)q^2z}} qtE_q^{-q^2zt}g(qt)d_qt,
\end{aligned} \tag{3.31}
$$

and

$$
\begin{aligned}
D_q^2u_1(z)&=-D_q\Big(\int_0^{\frac{1}{(1-q)q^2z}} qtE_q^{-q^2zt}g(qt)d_qt\Big)\\
&=\int_0^{\frac{1}{(1-q)q^2z}} q^3t^2E_q^{-q^3zt}g(qt)d_qt\\
&\quad+\frac{1}{(1-q)q^3z^2}qtE_q^{-q^3zt}g(qt)\big|_{t=1/(1-q)q^3z}\\
&=\int_0^{\frac{1}{(1-q)q^2z}} q^3t^2E_q^{-q^3zt}g(qt)d_qt.
\end{aligned} \tag{3.32}
$$

Substituting (3.30), (3.31), (3.32) in (2.7), we deduce that

$$
\int_0^{\frac{1}{(1-q)q^2z}} g(qt)[-D_{q,t}(E_q^{-q^2zt})q^{c+1}t^2-D_{q,t}(E_q^{-qzt})q^at
$$

$$
\begin{aligned}
&- qt[c]_q E_q^{-q^2zt} - [a]_q E_q^{-qzt}]d_q t \\
&= \int_0^{\frac{1}{(1-q)q^2z}} E_q^{-q^2zt} D_{q,t}(g(t)q^{c-1}t^2) + E_q^{-qzt} D_{q,t}(g(t)q^{a-1}t) \\
&- qt[c]_q E_q^{-q^2zt} g(qt) - [a]_q E_q^{-qzt} g(qt) d_q t \\
&= \int_0^{\frac{1}{(1-q)q^2z}} E_q^{-q^2zt}\{D_{q,t}(g(t)q^{c-1}t^2) + [1-(1-q)qzt] D_{q,t}(g(t)q^{a-1}t) \\
&- g(qt)[qt[c]_q + [a]_q - (1-q)q[a]_q zt]\} d_q t = 0.
\end{aligned}
$$

where $D_{q,t}$ is the q-derivative with respect to t. We then get a q-difference equation about $g(t)$ as

$$\{1 + [q-(1-q)qz]t\}g(qt) - \{q^{a-1} + [q^{c-1} - q^{a-1}(1-q)qz]t\}g(t) = 0. \tag{3.33}$$

From the recurrence relation (3.33), one can obtain the following result immediately.

Lemma 3.6. *The solution of the q-difference equation* (3.33) *is*

$$g(t) = g_0 t^{a-1} \frac{([(1-q)qz-q]t;q)_\infty}{([(1-q)qz-q^{c-a}]t;q)_\infty},$$

where g_0 is a nonzero constant.

Thus, we can re-express $u_1(z)$ as

$$u_1(z) = g_0 q^{a-1} \int_0^{\frac{1}{(1-q)q^2z}} E_q^{-qzt} t^{a-1} \frac{([(1-q)qz-q]qt;q)_\infty}{([(1-q)qz-q^{c-a}]qt;q)_\infty} d_q t. \tag{3.34}$$

To consider the convergence of the q-integral in (3.34), in view of the definition of the definite q-integral, we obtain

$$
\begin{aligned}
&\int_0^{\frac{1}{(1-q)q^2z}} E_q^{-qzt} t^{a-1} \frac{([(1-q)qz-q]qt;q)_\infty}{([(1-q)qz-q^{c-a}]qt;q)_\infty} d_q t \\
&= \sum_{j=0}^{\infty} \frac{q^{ja}(q^{j-1};q)_\infty ([(1-q)qz-q]\frac{q^{j-1}}{(1-q)z};q)_\infty}{(1-q)^{a-1}q^{2a}z^a ([(1-q)qz-q^{c-a}]\frac{q^{j-1}}{(1-q)z};q)_\infty}.
\end{aligned} \tag{3.35}
$$

Denote the j-th term of the infinite series in (3.35) as a_j. Then, we find

$$\lim_{j\to\infty} |\frac{a_{j+1}}{a_j}| = \lim_{j\to\infty} \Big|\frac{q^a}{1-q^{j-1}} \frac{1-\frac{q^{j-1}[(1-q)qz-q^{c-a}]}{(1-q)z}}{1-\frac{q^{j-1}[(1-q)qz-q]}{(1-q)z}}\Big| = |q^a|.$$

If $\mathrm{Re}(a) > 0$, then $|q^a| < 1$. By the ratio test, the infinite series in (3.35) converges absolutely.

Since there is another q-analogue of the exponential function, we also need to try another integral solution of the equation (2.7):

$$u_2(z) = \int_0^{\infty} e_q^{-zt} g(qt) d_q t, \tag{3.36}$$

where

$$e_q^{-zt} = \frac{1}{(-(1-q)zt;q)_\infty}.$$

The q-derivative of e_q^{-zt} with respect to z is

$$D_q e_q^{-zt} = -te_q^{-zt}.$$

So we further have

$$\begin{aligned} D_q u(z) &= -\int_0^\infty t e_q^{-zt} g(qt) d_q t, \\ D_q^2 u(z) &= \int_0^\infty t^2 e_q^{-zt} g(qt) d_q t. \end{aligned} \tag{3.37}$$

Substituting (3.36), (3.37) in (2.7) gives

$$\begin{aligned} &\int_0^\infty g(qt)\Big[-(D_{q,t}e_q^{-zt})q^c t^2 - (D_{q,t}e_q^{-zt})q^a t - ([a]_q + [c]_q t)e_q^{-zt}\Big] d_q t \\ &= \int_0^\infty e_q^{-zt}\Big[D_{q,t}\Big(g(t)q^{c-2}t^2\Big) + D_{q,t}\Big(g(t)q^{a-1}t\Big) - g(qt)([a]_q + [c]_q t)\Big] d_q t = 0, \end{aligned}$$

where $D_{q,t}$ is the q-derivative with respect to t. From the above equation, we obtain a q-difference equation about $g(t)$:

$$(1+t)g(qt) - \big(q^{a-1} + q^{c-2}t\big)\, g(t) = 0. \tag{3.38}$$

For the solution of (3.38), it is not difficult to obtain the following lemma.

Lemma 3.7. *The solution of the q-difference equation* (3.38) *is*

$$g(t) = g_0 t^{a-1} \frac{(-t;q)_\infty}{(-q^{c-a-1}t;q)_\infty},$$

where g_0 is a nonzero constant.

Thus, we can re-express $u_2(z)$ as

$$u_2(z) = g_0 q^{a-1} \int_0^\infty e_q^{-zt} t^{a-1} \frac{(-qt;q)_\infty}{(-q^{c-a}t;q)_\infty} d_q t. \tag{3.39}$$

For the convergence of the q-integral in (3.39), by the definition of the improper q-integral (3.28) we have

$$\begin{aligned} &\int_0^\infty e_q^{-zt} t^{a-1} \frac{(-qt;q)_\infty}{(-q^{c-a}t;q)_\infty} d_q t \\ &= \sum_{j=-\infty}^{\infty} \frac{q^{ja}(-q^{j+1};q)_\infty}{(-(1-q)q^j z;q)_\infty (-q^{c-a+j};q)_\infty} \\ &=: \sum_{j=0}^{\infty} a_j + \sum_{j=-\infty}^{j=-1} a_j =: I_1 + I_2. \end{aligned} \tag{3.40}$$

Then, we obtain

$$\lim_{j\to+\infty} \Big|\frac{a_{j+1}}{a_j}\Big| = \lim_{j\to+\infty} \Big|\frac{q^a[1+(1-q)q^j z](1+q^{c-a+j})}{1+q^{j+1}}\Big| = |q^a|.$$

If $\mathrm{Re}(a) > 0$, then $|q^a| < 1$. By the ratio test, the infinite series I_1 converges absolutely.

For the infinite series

$$I_2 = \sum_{j=1}^{\infty} \frac{q^{-ja}(-q^{-j+1};q)_\infty}{(-(1-q)q^{-j}z;q)_\infty (-q^{c-a-j};q)_\infty} \tag{3.41}$$

we denote the jth term of (3.40) by b_j, and find that

$$\lim_{j\to+\infty}\Big|\frac{b_{j+1}}{b_j}\Big| = \lim_{j\to+\infty}|q^{-a}|\Big|\frac{1+q^{-j}}{[1+(1-q)q^{-j-1}z](1+q^{c-a-j-1})}\Big| = 0<1.$$

Again by the ratio test, the infinite series I_2 converges absolutely. Thus, if $\mathrm{Re}(a) > 0$, the q-integral in (3.39) converges absolutely.

Consequently, for the equation (2.7) we have the following result.

Theorem 3.8. *When* $\mathrm{Re}(a) > 0$*, the equation* (2.7) *has two convergent integral solutions:*

$$u_1 = g_0 q^{a-1}\int_0^{\frac{1}{(1-q)q^2 z}} E_q^{-qzt}t^{a-1}\frac{([(1-q)qz-q]qt;q)_\infty}{([(1-q)qz-q^{c-a}]qt;q)_\infty}d_qt,$$
$$u_2 = g_0 q^{a-1}\int_0^{\infty} e_q^{-zt}t^{a-1}\frac{(-qt;q)_\infty}{(-q^{c-a}t;q)_\infty}d_qt.$$

4. Contiguous relations

Since ${}_1\Phi_1(q^a;q^c;q,(1-q)z)$ is a solution of (2.7), we now consider contiguous relations about ${}_1\Phi_1(q^a;q^c;q,(1-q)z)$. It is easily verified that

$$D_q\left({}_1\Phi_1(q^{a-1};q^{c-1};q,(1-q)z)\right) = \frac{1-q^{a-1}}{1-q^{c-1}}{}_1\Phi_1(q^a;q^c;q,(1-q)z).$$

By (2.7), the function ${}_1\Phi_1(q^{a-1};q^{c-1};q,(1-q)z)$ also satisfies

$$q^{c-1}zD_q^2u(z) + ([c-1]_q - q^{a-1}z)D_qu(z) - [a-1]_qu(z) = 0.$$

From the above two equations, we obtain a contiguous relation:

Proposition 4.1. *When* $|z| < 1/(1-q)$*, we have*

$$\begin{aligned}&[c]_q([c-1]_q - q^{a-1}z){}_1\Phi_1(q^a;q^c;q,(1-q)z)\\ &-[c]_q[c-1]_q{}_1\Phi_1(q^{a-1};q^{c-1};q,(1-q)z)\\ &+q^{c-1}[a]_qz{}_1\Phi_1(q^{a+1};q^{c+1};q,(1-q)z)=0.\end{aligned}\tag{4.1}$$

Using (2.7), we deduce a set of four relations from which six contiguous relations can be derived by equating the $\binom{2}{4}$ pairs of them. The first two relations are as follows.

Lemma 4.2. *When* $|z| < 1/(1-q)$*, it holds*

$$\delta_q(\Phi) = -[-a]_q(\Phi(a+) - \Phi), \tag{4.2}$$
$$\delta_q(\Phi) = -[1-c]_q(\Phi(c-) - \Phi), \tag{4.3}$$

where

$$\delta_q = zD_q, \quad \Phi = {}_1\Phi_1(q^a;q^c;q,(1-q)z),$$
$$\Phi(a+) = {}_1\Phi_1(q^{a+1};q^c;q,(1-q)z),$$
$$\Phi(c-) = {}_1\Phi_1(q^a;q^{c-1};q,(1-q)z).$$

Proof. Since

$${}_1\Phi_1(q^a;q^c;q,(1-q)z) = \sum_{n=0}^{\infty}\frac{(q^a;q)_n}{(q^c;q)_n}\frac{z^n}{[n]_q^!},$$

where $[n]_q^! = [1]_q[2]_q \cdots [n]_q$, we have

$$\delta_q(\Phi) = \sum_{n=1}^{\infty} \frac{(q^a;q)_n}{(q^c;q)_n} \frac{z^n}{[n-1]_q^!}. \tag{4.4}$$

On the other hand, we know that

$$\begin{aligned} \Phi(a+) - \Phi &= \sum_{n=1}^{\infty} \Big(\frac{(q^{a+1};q)_n}{(q^c;q)_n(q;q)_n} - \frac{(q^a;q)_n}{(q^c;q)_n(q;q)_n} \Big) (1-q)^n z^n \\ &= \sum_{n=1}^{\infty} \frac{q^a(q^{a+1};q)_{n-1}}{(q^c;q)_n} \frac{(1-q)z^n}{[n-1]_q^!}, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \Phi(c-) - \Phi &= \sum_{n=1}^{\infty} \Big(\frac{(q^a;q)_n}{(q^{c-1};q)_n(q;q)_n} - \frac{(q^a;q)_n}{(q^c;q)_n(q;q)_n} \Big) (1-q)^n z^n \\ &= \sum_{n=1}^{\infty} \frac{q^{c-1}(q^a;q)_n}{[c-1]_q(q^c;q)_n} \frac{z^n}{[n-1]_q^!}. \end{aligned} \tag{4.6}$$

Combining the equations (4.4) and (4.5), we arrive at relation (4.2). Similarly, relation (4.3) is proved by combining (4.4) and (4.6). □

To obtain the other two relations, we rewrite (2.7) as

$$\{\delta_q(q^{c-1}\delta_q + [c-1]_q) - z(q^a\delta_q + [a]_q)\}u(z) = 0. \tag{4.7}$$

By reducing the order of δ_q in the equation (4.7), we have the following lemma.

Lemma 4.3. *When $|z| < 1/(1-q)$, it holds*

$$\delta_q(\Phi) = (q^{a-c}z + q^{1-a}[a-c]_q)\Phi - q^{1-a}[a-c]_q\Phi(a-), \tag{4.8}$$

$$\delta_q(\Phi) = zq^{a-c}\Phi + z\frac{[a-c]_q}{[c]_q}\Phi(c+), \tag{4.9}$$

where

$$\begin{gathered} \delta_q = zD_q, \quad \Phi = {}_1\Phi_1(q^a;q^c;q,(1-q)z), \\ \Phi(a-) = {}_1\Phi_1(q^{a-1};q^c;q,(1-q)z), \\ \Phi(c+) = {}_1\Phi_1(q^a;q^{c+1};q,(1-q)z). \end{gathered}$$

Proof. To prove (4.8), by (4.7) we have

$$\{\delta_q(q^{c-1}\delta_q + [c-1]_q) - z(q^{a-1}\delta_q + [a-1]_q)\}\Phi(a-) = 0. \tag{4.10}$$

The operator $\delta_q(q^{c-1}\delta_q + [c-1]_q)$ can be factored as

$$\delta_q(q^{c-1}\delta_q + [c-1]_q) = (q^{a-1}\delta_q + [a-1]_q)\Big(q^{c-a}\delta_q + \frac{[c-a]_q}{q^{a-1}}\Big) + [1-a]_q[c-a]_q.$$

A direct calculation gives

$$\begin{aligned}
&(q^{a-1}\delta_q + [a-1]_q)\Phi(a-) \\
&= \sum_{n=1}^{\infty} \frac{(q^{a-1};q)_n}{(q^c;q)_n} \frac{q^{a-1}z^n}{[n-1]_q^!} + \sum_{n=0}^{\infty} \frac{(q^{a-1};q)_n}{(q^c;q)_n} \frac{[a-1]_q z^n}{[n]_q^!} \\
&= \sum_{n=1}^{\infty} \frac{(q^{a-1};q)_n z^n}{(q^c;q)_n [n-1]_q^!} \Big(q^{a-1} + \frac{[a-1]_q}{[n]_q}\Big) + [a-1]_q \\
&= [a-1]_q \Phi.
\end{aligned} \tag{4.11}$$

Note that the operators $q^{a-1}\delta_q + [a-1]_q$ and $q^{c-a}\delta_q + \frac{[c-a]_q}{q^{a-1}}$ are commutative. Then (4.10) becomes

$$\Big(q^{c-a}\delta_q + \frac{[c-a]}{q^{a-1}} - z\Big)[a-1]_q\Phi + [1-a]_q[c-a]_q\Phi(a-) = 0.$$

This implies the relation (4.8).

The proof of (4.9) is similar. By (4.7), we have

$$\{\delta_q(q^c\delta_q + [c]_q) - z(q^a\delta_q + [a]_q)\}\Phi(c+) = 0. \tag{4.12}$$

Factor the operator $z(q^a\delta_q + [a]_q)$ as

$$z(q^a\delta_q + [a]_q) = zq^{a-c}(q^c\delta_q + [c]_q) + z[a-c]_q.$$

Note that

$$\begin{aligned}
(q^c\delta_q + [c]_q)\Phi(c+) &= \sum_{n=1}^{\infty} \frac{(q^a;q)_n}{(q^{c+1};q)_n} \frac{q^c z^n}{[n-1]_q^!} + \sum_{n=0}^{\infty} \frac{(q^a;q)_n}{(q^{c+1};q)_n} \frac{[c]_q z^n}{[n]_q^!} \\
&= \sum_{n=1}^{\infty} \frac{(q^a;q)_n z^n}{(q^{c+1};q)_n [n-1]_q^!} \Big(q^c + \frac{[c]_q}{[n]_q}\Big) + [c]_q \\
&= [c]_q \Phi.
\end{aligned} \tag{4.13}$$

Then (4.12) becomes

$$(\delta_q - zq^{a-c})[c]_q\Phi - z[a-c]_q\Phi(c+) = 0.$$

So, relation (4.9) is established. □

Consequently, by Lemmas 4.2 and 4.3, we can derive six contiguous relations as follows.

Theorem 4.4 (contiguous relations)**.** *When $|z| < 1/(1-q)$, it holds*

$$\begin{gathered}
[-a]_q\Phi(a+) - [1-c]_q\Phi(c-) = ([-a]_q - [1-c]_q)\Phi, \\
[-a]_q\Phi(a+) - q^{1-a}[a-c]_q\Phi(a-) = ([-a]_q - q^{1-a}[a-c]_q - q^{a-c}z)\Phi, \\
[-a]_q[c]_q\Phi(a+) + z[a-c]_q\Phi(c+) = [c]_q([-a]_q - q^{a-c}z)\Phi, \\
q^{1-a}[a-c]_q\Phi(a-) - [1-c]_q\Phi(c-) = (q^{a-c}z - [1-a]_q)\Phi, \\
[c]_q[1-c]_q\Phi(c-) + z[a-c]_q\Phi(c+) = [c]_q([1-c]_q - q^{a-c}z)\Phi, \\
q^{1-a}[c]_q\Phi(a-) + z\Phi(c+) = q^{1-a}[c]_q\Phi,
\end{gathered}$$

where

$$\Phi = {}_1\Phi_1(q^a; q^c; q, (1-q)z),$$

$$\Phi(a\pm) = {}_1\Phi_1(q^{a\pm}; q^c; q, (1-q)z),$$
$$\Phi(c\pm) = {}_1\Phi_1(q^a; q^{c\pm}; q, (1-q)z).$$

References

[1] G. E. Andrews, R. Askey, R. Roy; *Special Functions*, Cambridge University Press, Cambridge, 1999.
[2] G. George, M. Rahman; *Basic Hypergeometric Series*, 2^{nd} edition, Cambridge University Press, 2004.
[3] V. Kac, P. Cheung; *Quantum Calculus*, Springer-Verlag, New York, 2002.
[4] T. H. Koornwinder; *q-special Functions, A Tutorial*, arXiv math/9403216.
[5] L. J. Slater; *Confluent Hypergeometric Functions*, Cambridge University Press, 1960.
[6] L. J. Slater; *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.
[7] Z. X. Wang, D. R. Guo; *Special Functions*, World Scientific Publishing Co Pte Ltd, 1989.

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