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INVERSE SPECTRAL PROBLEMS FOR NONLINEAR STURM-LIOUVILLE PROBLEMS

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Abstract. This paper concerns the nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t)) = \lambda u(t), \quad u(t) > 0, \quad t \in I := (0,1), \quad u(0) = u(1) = 0,$$

where λ is a positive parameter. We try to determine the nonlinear term $f(u)$ by means of the global behavior of the bifurcation branch of the positive solutions in $\mathbb{R}_+ \times L^2(I)$.

1. Introduction

We consider the nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I := (0,1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(0) = u(1) = 0, \tag{1.3}$$

where λ is a positive parameter. We assume that $f(u)$ satisfies the following conditions:

(A1) $f(u)$ is a function of C^1 for $u \ge 0$ satisfying $f(0) = f'(0) = 0$,
(A2) $g(u) := f(u)/u$ is strictly increasing for $u \ge 0$ ($g(0) := 0$),
(A3) $g(u) \to \infty$ as $u \to \infty$.

Then for each given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda(\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{I})$ with $\|u_\alpha\|_2 = \alpha$. The set $\{(\lambda(\alpha), u_\alpha); \alpha > 0\}$ gives all solutions of (1.1)–(1.3) and is an unbounded curve of class C^1 in $\mathbf{R}_+ \times L^2(I)$ emanating from $(\pi^2, 0)$ (cf. [1, 7]).

Typical examples of $f(u)$ are as follows:

$$f(u) = u^p \quad (u \ge 0),$$

$$f(u) = u^p\big(1 - \frac{1}{1+u^q}\big) \quad (u \ge 0),$$

where $p > 1$ and $q > 0$ are constants.

The equation (1.1)–(1.3) is motivated by the logistic equation of population dynamics and vibration of string with self-interaction, and has been extensively investigated by many authors. We refer to [1, 7, 11, 12, 13] and the references therein

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for the works in L^∞-framework from a viewpoint of local and global bifurcation theory. On the other hand, since (1.1)–(1.3) is regarded as an eigenvalue problem, it seems important to study (1.1)–(1.3) in L^2-framework. We refer to [2, 3, 4, 5, 6, 8, 9, 10, 14] for other works in this direction. In particular, the asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to 0$ have been established in [4, 5]. Therefore, the problem we have to consider here is a global behavior of u_α as $\alpha \to \infty$, and it is known from [1] that

$$\frac{u_\alpha(t)}{g^{-1}(\lambda(\alpha))} \to 1 \tag{1.4}$$

locally uniformly on I as $\alpha \to \infty$. By this, it is easy to see that for $\alpha \gg 1$

$$\alpha = \|u_\alpha\|_2 = (1 + o(1))g^{-1}(\lambda(\alpha)).$$

It follows from this that, in many cases, as $\alpha \to \infty$

$$\lambda(\alpha) = g(\alpha) + o(g(\alpha)). \tag{1.5}$$

Note that if $f(u) = u^p$ $(p > 1)$, then $g(\alpha) = \alpha^{p-1}$. Motivated by (1.5), the following asymptotic formula for $\lambda(\alpha)$ as $\alpha \to \infty$ has been given in [14].

Theorem 1.1 ([14]). *Let $f(u) = u^p$ $(p > 1)$. Let n be an arbitrary, fixed, number in $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Then the following asymptotic formula holds as $\alpha \to \infty$:*

$$\lambda(\alpha) = \alpha^{p-1} + C_0\alpha^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}} C_0^{k+2} \alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}), \tag{1.6}$$

where

$$C_0 = (p+3)\int_0^1 \sqrt{\frac{p-1}{p+1} - \xi^2 + \frac{2}{p+1}\xi^{p+1}}\,d\xi$$

and $a_k(p)$ is the polynomial ($\deg a_k(p) \le k+1$) which is determined by $a_0 = 1, a_1, \dots, a_{k-1}$.

Consider now the implication of Theorem 1.1 and (1.6) from the standpoint of inverse spectral problems.

Problem A. Assume that (1.6) holds for any $n \in \mathbb{N}_0$. Then does $f(u) = u^p$ $(p > 1)$ hold?

For the first step to solve this problem, we simplified the problem as follows in [15]:

Problem B. Assume that the following asymptotic formula is valid as $\alpha \to \infty$.

$$\lambda(\alpha) = \alpha^{p-1} + C_0\alpha^{(p-1)/2} + o(\alpha^{(p-1)/2}).$$

Then does $f(u) = u^p$ hold?

The answer to Problem B was given in [15].

Theorem 1.2 ([15]). *Let $f(u) = u^p(1 - 1/(1 + u^2))$. Furthermore, assume that $1 < p < 5$. Then as $\alpha \to \infty$*

$$\lambda(\alpha) = \alpha^{p-1} + C_0\alpha^{(p-1)/2} + o(\alpha^{(p-1)/2}). \tag{1.7}$$

Therefore, we regret to say that the answer to the Problem B is negative. However, it seems worth considering the following problem on which it is imposed stronger conditions than those in Problem B.

Problem C. Assume that the following asymptotic formula is valid as $\alpha \to \infty$.

$$\lambda(\alpha) = \alpha^{p-1} + C_0\alpha^{(p-1)/2} + \frac{1}{p-1}C_0^2 + o(1). \tag{1.8}$$

Then does $f(u) = u^p$ hold?

The purpose of this paper is to answer this question.

Theorem 1.3. *Let $f(u) = u^p(1 - 1/(1 + u^q))$, where $p > 1$.*

(i) *Assume that $(p-1)/2 < q < p + 1$. Then (1.7) holds as $\alpha \to \infty$.*
(ii) *Assume that $p - 1 < q < p + 1$. Then (1.8) holds as $\alpha \to \infty$.*

Therefore, unfortunately, the answer to the Problem C is not valid, either. It seems that the assumption in Problem C is still weak to solve the inverse spectral problem. However, it seems that Theorem 1.3 certainly is the meaningful step to give the answer to Problem A.

Our arguments here to prove Theorem 1.3 are quite straightforward and are different from those of Theorem 1.2, which are variant of the proof of Theorem 1.1. Therefore, in [15], it seems that the restriction $1 < p < 5$ for the case $q = 2$ is technical. However, it follows from Theorem 1.3 (i) that this restriction is not technical but optimal.

2. Proof of Theorem 1.3

In this section, C denotes various positive constants independent of $\lambda \gg 1$. We begin with the fundamental tools which play important roles in what follows. We denote by (λ, u_λ) the solution pair of (1.1)–(1.3) for $\lambda > \pi^2$. We know from [1] that there exists a unique solution $u_\lambda \in C^2(\bar{I})$ of (1.1)–(1.3) for a given $\lambda > \pi^2$. We use this notation in what follows. Therefore, $\alpha = \|u_\lambda\|_2$. It is well known that

$$u_\lambda(t) = u_\lambda(1-t), \quad t \in I, \tag{2.1}$$

$$u_\lambda'(t) > 0, \quad 0 \le t < \frac{1}{2}, \tag{2.2}$$

$$\|u_\lambda\|_\infty = u_\lambda(\frac{1}{2}). \tag{2.3}$$

We know by [1] that

$$\lambda = \|u_\lambda\|_\infty^{p-1}\frac{\|u_\lambda\|_\infty^q}{1 + \|u_\lambda\|_\infty^q} + \lambda_1, \tag{2.4}$$

where $\lambda_1 > 0$ is the remainder term of λ with respect to $\|u_\lambda\|_\infty$ and depends on λ. For $\lambda \gg 1$, we have

$$\lambda_1 \le C\lambda e^{-\sqrt{\kappa\lambda}/2}. \tag{2.5}$$

Here $\kappa > 0$ is a constant. For completeness, we give the proof of (2.5) in Appendix. Now multiply (1.1) by $u_\lambda'(t)$. Then

$$\Big(u_\lambda''(t) + \lambda u_\lambda(t) - u_\lambda(t)^p + \frac{u_\lambda^p(t)}{1 + u_\lambda^q(t)}\Big)u_\lambda'(t) = 0.$$

This along with (2.3) implies that

$$\begin{aligned} &\frac{1}{2}u_\lambda'(t)^2 + \frac{1}{2}\lambda u_\lambda(t)^2 - \frac{1}{p+1}u_\lambda(t)^{p+1} + \int_0^{u_\lambda(t)} \frac{\xi^p}{1+\xi^q}d\xi \equiv \text{constant} \\ &= \frac{1}{2}\lambda\|u_\lambda\|_\infty^2 - \frac{1}{p+1}\|u_\lambda\|_\infty^{p+1} + \int_0^{\|u_\lambda\|_\infty} \frac{\xi^p}{1+\xi^q}d\xi \quad (\text{put } t = 1/2). \end{aligned} \tag{2.6}$$

Let

$$L_\lambda(\theta) = \lambda(\|u_\lambda\|_\infty^2 - \theta^2) - \frac{2}{p+1}(\|u_\lambda\|_\infty^{p+1} - \theta^{p+1}) + 2\int_\theta^{\|u_\lambda\|_\infty} \frac{\xi^p}{1+\xi^q} d\xi. \tag{2.7}$$

This along with (2.2) and (2.6) implies that $u_\lambda'(t) = \sqrt{L_\lambda(u_\lambda(t))}$ for $0 \le t \le 1/2$. By this and (2.1), we obtain

$$\begin{aligned} \|u_\lambda\|_\infty^2 - \alpha^2 &= 2\int_0^{1/2} \frac{(\|u_\lambda\|_\infty^2 - u_\lambda^2(t))u_\lambda'(t)}{\sqrt{L_\lambda(u_\lambda(t))}} dt \\ &= 2\int_0^{\|u_\lambda\|_\infty} \frac{(\|u_\lambda\|_\infty^2 - \theta^2)}{\sqrt{L_\lambda(\theta)}} d\theta \\ &= \frac{2\|u_\lambda\|_\infty^2}{\sqrt{\lambda}} \int_0^1 \frac{1-s^2}{\sqrt{B_\lambda(s)}} ds \\ &= \frac{2\|u_\lambda\|_\infty^2}{\sqrt{\lambda}} \Big\{ \int_0^1 \frac{1-s^2}{\sqrt{A(s)}} ds + \int_0^1 \Big(\frac{1-s^2}{\sqrt{B_\lambda(s)}} - \frac{1-s^2}{\sqrt{A(s)}} \Big) ds \Big\} \\ &= \frac{2\|u_\lambda\|_\infty^2}{\sqrt{\lambda}} (C_1 + M_\lambda), \end{aligned} \tag{2.8}$$

where

$$A(s) := 1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}), \tag{2.9}$$

$$B_\lambda(s) := 1 - s^2 - \frac{2}{p+1}\frac{\|u_\lambda\|_\infty^{p-1}}{\lambda}(1 - s^{p+1}) + \frac{2}{\lambda\|u_\lambda\|_\infty^2}\int_{\|u_\lambda\|_\infty s}^{\|u_\lambda\|_\infty} \frac{\xi^p}{1+\xi^q} d\xi, \tag{2.10}$$

$$C_1 := \int_0^1 \frac{1-s^2}{\sqrt{A(s)}} ds, \tag{2.11}$$

$$M_\lambda := \int_0^1 \Big(\frac{1-s^2}{\sqrt{B_\lambda(s)}} - \frac{1-s^2}{\sqrt{A(s)}} \Big) ds. \tag{2.12}$$

By (2.8), we prove Theorem 1.3. Therefore, it is important to obtain the asymptotic formula for M_λ as $\lambda \to \infty$. To this end, we first prove the following lemma.

Lemma 2.1. *Let $0 < \epsilon \ll 1$ be fixed. Then there exists a constant $0 < \delta \ll 1$ such that for $1 - \epsilon \le s \le 1$ and $\lambda \gg 1$,*

$$A(s) = K_0(s)(1-s)^2, \tag{2.13}$$

$$B_\lambda(s) = K_1(\lambda)(1-s) + K_2(\lambda, s)(1-s)^2, \tag{2.14}$$

where

$$(p-1)(1-\delta) \le K_0(s) \le p - 1\,, \tag{2.15}$$

$$K_1(\lambda) = \frac{2\lambda_1}{\lambda}\,, \tag{2.16}$$

$$(p-1)(1-\delta) \le K_2(\lambda, s) \le p - 1\,. \tag{2.17}$$

Proof. We have $A(1) = 0$. Furthermore,

$$A'(s) = -2s + 2s^p, \quad A''(s) = -2 + 2ps^{p-1}. \tag{2.18}$$

By this and Taylor expansion, for $1-\epsilon \le s \le 1$ and $\lambda \gg 1$, we obtain

$$\begin{aligned} A(s) &= A(1) + A'(1)(s-1) + \frac{1}{2}A''(s_1)(s-1)^2 \\ &= \frac{1}{2}A''(s_1)(s-1)^2, \end{aligned}$$

where $1-\epsilon \le s < s_1 < 1$. This along with (2.18) implies (2.13). Next, we have $B(1) = 0$. Furthermore,

$$B'_\lambda(s) = -2s + 2s^p\frac{\|u_\lambda\|_\infty^{p-1}}{\lambda} - \frac{2}{\lambda\|u_\lambda\|_\infty^2}\frac{\|u_\lambda\|_\infty^{p+1}s^p}{1+\|u_\lambda\|_\infty^q s^q}.$$

Then by (2.4),

$$\begin{aligned} B'_\lambda(1) &= -2 + 2\frac{\|u_\lambda\|_\infty^{p-1}}{\lambda} - \frac{2}{\lambda\|u_\lambda\|_\infty^2}\frac{\|u_\lambda\|_\infty^{p+1}}{1+\|u_\lambda\|_\infty^q} \\ &= -2\Big(1 - \frac{\|u_\lambda\|_\infty^{p+q-1}}{\lambda(1+\|u_\lambda\|_\infty^q)}\Big) \\ &= -\frac{2\lambda_1}{\lambda}. \end{aligned}$$

Furthermore,

$$B''_\lambda(s) = \frac{2(p\|u_\lambda\|_\infty^{p-1}s^{p-1}-\lambda)}{\lambda} - \frac{2(p-q)s^{p+q-1}\|u_\lambda\|_\infty^{p+q-1} + ps^{p-1}\|u_\lambda\|_\infty^{p-1}}{\lambda(1+\|u_\lambda\|_\infty^q s^q)^2}.$$

Therefore, by (2.4), $2(p-1)(1-\delta) \le B''_\lambda(s) \le 2(p-1)$ for $1-\epsilon \le s \le 1$ and $\lambda \gg 1$. By this and Taylor expansion, we obtain

$$\begin{aligned} B_\lambda(s) &= B_\lambda(1) + B'_\lambda(s)(s-1) + \frac{1}{2}B''_\lambda(s_2)(s-1)^2 \\ &= K_1(\lambda)(1-s) + K_2(\lambda,s)(1-s)^2, \end{aligned}$$

where $1-\epsilon < s < s_2 < 1$. Thus the proof is complete. □

Lemma 2.2. *For $\lambda \gg 1$,*

$$M_\lambda = C_2(q)\|u_\lambda\|_\infty^{-q}(1+o(1)). \tag{2.19}$$

Here

$$C_2(q) = \int_0^1 \frac{(1-s^2)\{(1-s^{p+1})/(p+1) - (1-s^{p-q+1})/(p-q+1)\}}{A(s)^{3/2}}ds.$$

Proof. It is easy to see that

$$M_\lambda = \int_0^1 \frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)}+\sqrt{B_\lambda(s)})}ds.$$

Then by (2.4), (2.9) and (2.10), for $0 < s \le 1$ and $\lambda \gg 1$,

$$
\begin{aligned}
&A(s) - B_\lambda(s)\\
&= \frac{2}{p+1}\Big(\frac{\|u_\lambda\|_\infty^{p-1}}{\lambda} - 1\Big)(1-s^{p+1}) - \frac{2}{\lambda\|u_\lambda\|_\infty^2}\int_{\|u_\lambda\|_\infty s}^{\|u_\lambda\|_\infty} \frac{\xi^p}{1+\xi^q}d\xi\\
&= \frac{2}{p+1}(1+o(1))\|u_\lambda\|_\infty^{-q}(1-s^{p+1}) - \frac{2}{p-q+1}(1+o(1))\frac{\|u_\lambda\|_\infty^{p-q-1}}{\lambda}(1-s^{p-q+1})\\
&= 2\|u_\lambda\|_\infty^{-q}(1+o(1))\Big\{\frac{1}{p+1}(1-s^{p+1}) - \frac{1}{p-q+1}(1-s^{p-q+1})\Big\}.
\end{aligned} \tag{2.20}
$$

Furthermore, since $q < p+1$, as $\lambda \to \infty$,

$$
|A(0) - B_\lambda(0)| \le \Big|\frac{2}{p+1}\Big(\frac{\|u_\lambda\|_\infty^{p-1}}{\lambda} - 1\Big) - \frac{2}{\lambda\|u_\lambda\|_\infty^2}\int_0^{\|u_\lambda\|_\infty} \frac{\xi^p}{1+\xi^q}d\xi\Big| \le C\|u_\lambda\|_\infty^{-q}. \tag{2.21}
$$

By this and (2.20), for $0 \le s \le 1$, as $\lambda \to \infty$, $B_\lambda(s) \to A(s)$. We apply Lebesgue's convergence theorem to our situation. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. Then

$$
\begin{aligned}
M_\lambda &= \int_0^{1-\epsilon} \frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)}+\sqrt{B_\lambda(s)})}ds\\
&\quad + \int_{1-\epsilon}^1 \frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)}+\sqrt{B_\lambda(s)})}ds\\
&:= M_{1,\lambda} + M_{2,\lambda}.
\end{aligned} \tag{2.22}
$$

We know that $A(s)$ and $B_\lambda(s)$ is strictly decreasing for $0 \le s \le 1$ and $B_\lambda(1) = 0$ (cf. Appendix). So we see from (2.14) that for $0 \le s \le 1-\epsilon$ and $\lambda \gg 1$,

$$
\begin{gathered}
A(s) \ge A(1-\epsilon) \ge (p-1)(1-\delta)\epsilon^2,\\
B_\lambda(s) \ge B_\lambda(1-\epsilon) \ge (p-1)(1-\delta)\epsilon^2 > 0.
\end{gathered}
$$

By this, there exists a constant $C_\epsilon > 0$ such that for $0 \le s \le 1-\epsilon$ and $\lambda \gg 1$,

$$
\Big|\frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}\big(\sqrt{A(s)}+\sqrt{B_\lambda(s)}\big)}\Big| \le C_\epsilon.
$$

Therefore, by (2.4), (2.20), (2.21), (2.22) and Lebesgue's convergence theorem, we obtain

$$
\begin{aligned}
M_{1,\lambda} &= \int_0^{1-\epsilon} \frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)}+\sqrt{B_\lambda(s)})}ds\\
&\to \int_0^{1-\epsilon} \frac{(1-s^2)\{(1-s^{p+1})/(p+1) - (1-s^{p-q+1})/(p-q+1)\}}{A(s)^{3/2}}ds.
\end{aligned} \tag{2.23}
$$

By Lemma 2.1, for $1-\epsilon \le s \le 1$ and $\lambda \gg 1$,

$$\begin{aligned}
&\Big|\frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)}+\sqrt{B_\lambda(s)})}\Big| \\
&\le \frac{(1-s^2)\{K_1(\lambda)(1-s)+|K_0(s)-K_2(\lambda,s)|(1-s)^2\}}{(K_1(\lambda)(1-s)+K_2(\lambda,s)(1-s)^2)\sqrt{K_0(s)(1-s)^2}} \\
&\le 2\frac{(K_1(\lambda)(1-s)+|K_0(s)-K_2(\lambda,s)|(1-s)^2)}{\{K_1(\lambda)(1-s)+K_2(\lambda,s)(1-s)^2\}\sqrt{K_0(s)}} \\
&\le 2\frac{K_1(\lambda)+|K_0(s)-K_2(\lambda,s)|(1-s)}{\{K_1(\lambda)+K_2(\lambda,s)(1-s)\}\sqrt{K_0(s)}} \le C.
\end{aligned}$$

By this, we apply Lebesgue's convergence theorem to $M_{2,\lambda}$ to obtain

$$\begin{aligned}
M_{2,\lambda} &= \int_{1-\epsilon}^1 \frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)}+\sqrt{B_\lambda(s)})}ds \\
&\to \int_{1-\epsilon}^1 \frac{(1-s^2)\{(1-s^{p+1})/(p+1)-(1-s^{p-q+1})/(p-q+1)\}}{A(s)^{3/2}}ds.
\end{aligned}$$

By this and (2.23), we obtain (2.19). Thus the proof is complete. $\square$

Proof of Theorem 1.3. By (2.8) and Lemma 2.2, we obtain

$$\|u_\lambda\|_\infty^2 - \alpha^2 = \frac{2\|u_\lambda\|_\infty^2}{\sqrt{\lambda}}(C_1 + C_2\|u_\lambda\|_\infty^{-q}(1+o(1))).$$

By this, (2.4) and the Taylor expansion, for $\lambda \gg 1$,

$$\begin{aligned}
&\|u_\lambda\|_\infty^2 - \alpha^2 \\
&= 2\|u_\lambda\|_\infty^2\Big(\|u_\lambda\|_\infty^{p-1} - \|u_\lambda\|_\infty^{p-q-1}(1+o(1))\Big)^{-1/2}\big(C_1 + C_2\|u_\lambda\|_\infty^{-q}(1+o(1))\big) \\
&= 2\|u_\lambda\|_\infty^{(5-p)/2}(C_1 + (C_1/2 + C_2)\|u_\lambda\|_\infty^{-q}(1+o(1))).
\end{aligned} \tag{2.24}$$

By (2.4) and direct calculation, we obtain

$$\|u_\lambda\|_\infty = \lambda^{1/(p-1)}\Big(1 + \frac{1}{p-1}\lambda^{-q/(p-1)} + o\big(\lambda^{-q/(p-1)}\big)\Big). \tag{2.25}$$

By this, (2.24) and Taylor expansion,

$$\begin{aligned}
\alpha^2 &= \|u_\lambda\|_\infty^2 - 2\|u_\lambda\|_\infty^{(5-p)/2}(C_1 + (C_1/2 + C_2)\|u_\lambda\|_\infty^{-q}(1+o(1))) \\
&= \lambda^{2/(p-1)} + \frac{2}{p-1}\lambda^{(2-q)/(p-1)} + o\big(\lambda^{(2-q)/(p-1)}\big) \\
&\quad - 2\lambda^{(5-p)/(2(p-1))}\big\{C_1 + \big(\frac{5-p}{2(p-1)}C_1 + \frac{1}{2}C_1 + C_2\big)\lambda^{-q/(p-1)}(1+o(1))\big\}.
\end{aligned} \tag{2.26}$$

(i) Assume that $q > (p-1)/2$. Then for $\lambda \gg 1$,

$$\lambda^{(5-p)/(2(p-1))-q/(p-1)} \ll \lambda^{(2-q)/(p-1)} \ll \lambda^{(5-p)/(2(p-1))}.$$

By this and (2.26), we obtain

$$\alpha^2 = \lambda^{2/(p-1)} - 2C_1\lambda^{(5-p)/(2(p-1))} + \frac{2}{p-1}\lambda^{(2-q)/(p-1)}(1+o(1)).$$

Now we put

$$\lambda = \alpha^{p-1}(1 + C_3\alpha^{-\eta_1}(1 + o(1))).$$

Then by direct calculation, we obtain

$$C_3 = (p-1)C_1, \quad \eta_1 = \frac{p-1}{2}.$$

Since $(p-1)C_1 = C_0$, this implies Theorem 1.3 (i).
(ii) Furthermore, assume that $q > p-1$. We put

$$\lambda = \alpha^{p-1}(1 + (p-1)C_1\alpha^{-(p-1)/2} + C_4\alpha^{-\eta_2}(1 + o(1))).$$

Then by a straightforward calculation, we obtain

$$C_4 = (p-1)C_1^2, \quad \eta_2 = p-1.$$

This implies that for $\lambda \gg 1$

$$\lambda = \alpha^{p-1} + (p-1)C_1\alpha^{(p-1)/2} + (p-1)C_1^2 + o(1).$$

Since $(p-1)C_1 = C_0$, we obtain Theorem 1.3 (ii). Thus the proof is complete. □

3. Appendix

We first prove (2.5). We consider (1.1)–(1.3) with $f(u) = u^{p+q}/(1+u^q)$ for $p > 1$ and $q > 0$. We put $F(u) := \int_0^u f(s)ds$. Furthermore, let

$$Q_\lambda(\theta) = \lambda(\|u_\lambda\|_\infty^2 - \theta^2) - 2(F(\|u_\lambda\|_\infty) - F(\theta)). \tag{3.1}$$

For $0 \le t \le 1$, (3.1) is equivalent to (2.7). Then for $0 \le t \le 1/2$, we obtain

$$u_\lambda'(t) = \sqrt{Q(u_\lambda(t))}. \tag{3.2}$$

By this, we obtain

$$\frac{1}{2} = \int_0^{1/2} \frac{u_\lambda'(t)}{\sqrt{Q(u_\lambda(t)}}dt = \int_0^{\|u_\lambda\|_\infty} \frac{1}{\sqrt{Q_\lambda(\theta)}}d\theta = \frac{1}{\sqrt{\lambda}}\int_0^1 \frac{1}{\sqrt{R_\lambda(s)}}ds, \tag{3.3}$$

where

$$R_\lambda(s) := 1 - s^2 - \frac{2}{\lambda\|u_\lambda\|_\infty^2}(F(\|u_\lambda\|_\infty) - F(\|u_\lambda\|_\infty s)) \tag{3.4}$$

Let an arbitrary $0 < \epsilon \ll 1$ be fixed. Then for $1 - \epsilon \le s \le 1$, by Taylor expansion,

$$F(\|u_\lambda\|_\infty s) = F(\|u_\lambda\|_\infty) + f(\|u_\lambda\|_\infty)\|u_\lambda\|_\infty(s-1) + \frac{1}{2}f'(\|u_\lambda\|_\infty s_1)\|u_\lambda\|_\infty^2(s-1)^2,$$

where $s < s_1 < 1$ and s_1 depends on s. Since $f'(u) = pu^{p-1}(1 + o(1))$ for $u \gg 1$, there exists a constant $\delta > 0$ such that for $1 - \epsilon \le s \le 1$

$$\begin{aligned} R_\lambda(s) &= 2\big(1 - \frac{f(\|u_\lambda\|_\infty)}{\lambda\|u_\lambda\|_\infty}\big)(1-s) + \big(\frac{f'(\|u_\lambda\|_\infty s_1)}{\lambda} - 1\big)(1-s)^2 \\ &\ge 2\xi(1-s) + \delta(1-s)^2, \end{aligned} \tag{3.5}$$

where

$$\xi := 1 - \frac{f(\|u_\lambda\|_\infty)}{\lambda\|u_\lambda\|_\infty} > 0. \tag{3.6}$$

We know that $\xi \to 0$ as $\lambda \to \infty$. By this, (3.3) and (3.5), we obtain

$$\begin{aligned}\frac{\sqrt{\lambda}}{2} &\le \int_0^{1-\epsilon} \frac{1}{\sqrt{R_\lambda(s)}} ds + \int_{1-\epsilon}^1 \frac{1}{\sqrt{2\xi(1-s)+\delta(1-s)^2}} ds \\ &\le C + \int_0^\epsilon \frac{1}{\sqrt{2\xi v + \delta v^2}} dv \\ &= C + \delta^{-1}\Big[\log|2\delta v + 2\xi + 2\sqrt{\delta(\delta v^2 + 2\xi v)}|\Big]_0^\epsilon \\ &= \delta^{-1}(\log C - \log 2\xi).\end{aligned}$$

By this, we obtain

$$2\xi \le Ce^{-\delta\sqrt{\lambda}/2},$$

which along with (3.6) implies

$$\lambda \le \frac{f(\|u_\lambda\|_\infty)}{\|u_\lambda\|_\infty} + \frac{C}{2}\lambda e^{-\delta\sqrt{\lambda}/2}.$$

Thus the proof of (2.5) is complete. We next prove that $B_\lambda(s)$ is decreasing for $0 \le s \le 1$. Indeed, since $B_\lambda(s) = R_\lambda(s)$ in (3.4), by (A2) and (2.4),

$$\begin{aligned}B'_\lambda(s) &= -2s + \frac{2s}{\lambda}\frac{f(\|u_\lambda\|_\infty s)}{\|u_\lambda\|_\infty s} \\ &\le -2s + \frac{2s}{\lambda}\frac{f(\|u_\lambda\|_\infty)}{\|u_\lambda\|_\infty} < 0.\end{aligned}$$

Thus the proof is complete.

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