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EXISTENCE OF BOUNDED SOLUTIONS OF NEUMANN PROBLEM FOR A NONLINEAR DEGENERATE ELLIPTIC EQUATION

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Abstract. We prove the existence of bounded solutions of Neumann problem for nonlinear degenerate elliptic equations of second order in divergence form. We also study some properties as the Phragmén-Lindelöf property and the asymptotic behavior of the solutions of Dirichlet problem associated to our equation in an unbounded domain.

1. Introduction

We consider the equation

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x,u,\nabla u) - c_0|u|^{p-2}u = f(x,u,\nabla u) \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^m$, $m \geq 2$, c_0 is a positive constant, ∇u is the gradient of unknown function u and f is a nonlinear function which has the growth of rate p, $1 < p < m$, respect to gradient ∇u. We assume that the following degenerate ellipticity condition is satisfied,

$$\lambda(|u|) \sum_{i=1}^{m} a_i(x,u,\eta)\eta_i \geq \nu(x)|\eta|^p, \tag{1.2}$$

where $\eta = (\eta_1, \eta_2, \dots, \eta_m)$, $|\eta|$ denotes the modulus of η, ν and λ are positive functions with properties to be specified later on.

We study the nonlinear Neumann boundary problem for (1.1) with the boundary condition

$$\sum_{i=1}^{m} a_i(x,u,\nabla u)\cos(\overrightarrow{n}, x_i) + c_2|u|^{p-2}u + F(x,u) = 0 \quad (c_2 > 0),\ x \in \partial\Omega, \tag{1.3}$$

where $\partial\Omega$ is locally Lipschitz boundary (see [1]) and $\overrightarrow{n} = \overrightarrow{n}(x)$ is the outwardly directed (relative to Ω) unit vector normal to $\partial\Omega$ at every point $x \in \partial\Omega$.

Many results for linear and quasilinear elliptic equations of second order have been established starting with pionering papers [13, 16], and arriving to the most

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recent [2, 7, 20, 21, 22]. For example, in the very recent paper [21] the existence of positive solutions for p-Laplacian, with nonlinear Neumann boundary conditions, is proved by a priori estimates and topological methods.

The Dirichlet problem for the equation of the type (1.1) in nondegenerate case on bounded domains was studied by Boccardo, Murat and Puel in [3, 4], using the method of sub and supersolutions. Afterwards, Drabek and Nicolosi in [8], assuming condition (1.2), studied the weak solvability of general boundary value problem for equation (1.1), obtaining more general results than [3, 4]. Let us also mention, on the related topic and in degenerate-case, [5, 6] and [10, 11].

In this article the basic idea of [8] is used: the question of the existence of solutions is handled by priori estimates, in the energy space corresponding to the given problem and in L^∞, together with the theory of equations with pseudomonotone operators.

This article is organized as follows. In Section 2 we formulate the hypotheses, we state our problem and the main existence theorem. Section 3 consists of preliminary assertions which are sufficient in the proof of our main results. In Section 4 we prove the existence theorem and we give an example where all our assumptions are satisfied. In Section 5 we study asymptotic behavior of the solution of the Dirichlet problem associated to equation (1.1) in an unbounded domain. Finally, in Section 6 we shall show that a theorem, like the Phragmén-Lindelöf one, holds for Dirichlet problem, in the case of p-Laplacian, in a cylindrical unbounded domain of $\mathbb{R}^m$; the analogous question for higher-order linear equations was first investigated by P.D. Lax in [14].

2. Hypotheses and formulation of the main results

We shall suppose that $\mathbb{R}^m$ $(m \geq 2)$ is the m-dimensional Euclidean space with elements $x = (x_1, x_2, \dots, x_m)$. Let Ω be an open bounded nonempty subset of $\mathbb{R}^m$, $\partial\Omega$ be locally lipschitzian. The symbols $\mathrm{meas}_m(\cdot)$ and $\mathrm{meas}(\cdot)$ will denote the m-dimensional Lebesguel measure and the $(m-1)$-dimensional Hausdorff measure, respectively.

We denote by $L^q(\partial\Omega)$, $(1 \leq q < \infty)$ the Lebesgue space of q-summable functions on $\partial\Omega$ with respect to the $(m-1)$-dimensional Hausdorff measure, with obvious modifications if $q = \infty$.

Let p be a real number such that $1 < p < m$. We use, on the weight function $\nu(x)$, the hypothesis

(H1) $\nu : \Omega \to (0, +\infty)$ is a measurable function such that

$$\nu(x) \in L^1_{\rm loc}(\Omega), \quad \Big(\frac{1}{\nu(x)}\Big)^{\frac{1}{p-1}} \in L^1_{\rm loc}(\Omega).$$

We shall denote by $W^{1,p}(\nu, \Omega)$ the set of all real functions $u \in L^p(\Omega)$ having the weak derivative $\frac{\partial u}{\partial x_i}$ with the property $\nu\big|\frac{\partial u}{\partial x_i}\big|^p \in L^1(\Omega)$, for $i = 1, \dots, m$. $W^{1,p}(\nu, \Omega)$ is a Banach space respect to the norm

$$\|u\|_{1,p} = \Big[\int_\Omega (|u|^p + \nu|\nabla u|^p)\, dx\Big]^{1/p}.$$

The space $\mathring{W}^{1,p}(\nu, \Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\nu, \Omega)$. Put $W = W^{1,p}(\nu, \Omega) \cap L^\infty(\Omega)$.

Remark 2.1. There exists a positive number K_0 such that for every $u \in W^{1,p}(\nu,\Omega)$ it is also $\min_\Omega(u,K) \in W^{1,p}(\nu,\Omega)$ for every $K \geq K_0$. Details concerning this assertion can be found in Nicolosi [19].

Remark 2.2. For every $u \in W$ and for every $\gamma > 0$ it is $u|u|^\gamma \in W$. Details concerning this assertion can be found in Guglielmino and Nicolosi [10].

We have alsothe following hypotheses

(H2) There exists $t > \frac{m}{p-1}$ such that

$$\frac{1}{\nu(x)} \in L^t(\Omega).$$

From (H1) and (H2) there is a continuous inclusion ξ of $W^{1,p}(\nu,\Omega)$ in $W^{1,p\tau}(\Omega)$, where $\tau = (1+\frac{1}{t})^{-1}$. So, from Sobolev embedding, if we set

$$p^\star = \frac{mp}{m-p+m/t},$$

then, we have $W^{1,p}(\nu,\Omega) \subset L^{p^\star}(\Omega)$ and there exists $\hat{c} > 0$ depending only on m,p,t,Ω and $\|1/\nu\|_{L^t(\Omega)}$ such that for every $u \in W^{1,p}(\nu,\Omega)$

$$\Big(\int_\Omega |u|^{p^\star}\,dx\Big)^{1/p^\star} \leq \hat{c}\|u\|_{1,p}.$$

In this connection see, for instance, [11], [12] and [17, Theorem 3.1].

Next, by the theorem of trace for Sobolev spaces (see for instance [18, Cap. 2, pag.77] or [13]), we know that for any $u \in W^{1,p\tau}(\Omega)$, there exists a unique element $\gamma_0 u \in L^{\tilde{p}}(\partial\Omega)$ where

$$\tilde{p} = p\tau(m-1)(m-p\tau)^{-1} = \frac{(m-1)p}{m-p+m/t}$$

and, the mapping γ_0 is continuous linear from $W^{1,p\tau}(\Omega)$ to $L^{\tilde{p}}(\partial\Omega)$. Obviously, $\gamma_0 \circ \xi$ is a continuous linear map of $W^{1,p}(\nu,\Omega)$ to $L^{\tilde{p}}(\partial\Omega)$ and for $u|_{\partial\Omega} = (\gamma_0\circ\xi)(u)$, the trace of u on $\partial\Omega$, the following inequality holds:

$$\Big(\int_{\partial\Omega} |u|_{\partial\Omega}|^{\tilde{p}}\,ds\Big)^{1/\tilde{p}} \leq c'\|u\|_{1,p}, \quad \text{for all } u \in W^{1,p}(\nu,\Omega),$$

where c' is a positive constant depending only on m,p,t,Ω and $\|1/\nu\|_{L^t(\Omega)}$.

When clear from the context, for $u \in W^{1,p}(\nu,\Omega)$, we shall write u instead of $u|_{\partial\Omega}$.

Remark 2.3. Hypotheses (H1) and (H2) imply that $W^{1,p}(\nu,\Omega)$ is compactly embedded in $L^p(\Omega)$. The proof of this assertion is the same as that for $p=2$ (see [11]). Furthermore, as the linear and continuous map γ_0 from $W^{1,p\tau}(\Omega)$ in $L^q(\partial\Omega)$ is compact for every q: $1 \leq q < \tilde{p}$ (see [18, Cap. 2, pag.103]), then, it is also compact the embedding $\gamma_0\circ\xi$ of $W^{1,p}(\nu,\Omega)$ in $L^q(\partial\Omega)$. It will be useful to note that $W^{1,p}(\nu,\Omega)$ is reflexive. For the proof of this fact it is possible to use the same procedure as in [1, pag.46].

We need the following structural hypotheses:

(H3) The functions $f(x,u,\eta)$, $a_i(x,u,\eta)$ $(i=1,2,\dots,m)$ are Caratheodory functions in $\Omega\times\mathbb{R}\times\mathbb{R}^m$, i.e. measurable with respect to x for every $(u,\eta) \in \mathbb{R}\times\mathbb{R}^m$ and continuous with respect to (u,η) for almost all $x \in \Omega$.

(H4) The function $F(x,u)$ is a Caratheodory function in $\partial\Omega\times\mathbb{R}$, i.e. measurable with respect to x for every $u\in\mathbb{R}$ and continuous with respect to u for almost all $x\in\partial\Omega$.

(H5) There exist a number σ and a function $f^*(x)$ such that

$$\max\big(0,\frac{2-p}{2}\big)<\sigma<1,\quad f^*\in L^1(\Omega),$$

$$|f(x,u,\eta)|\le\lambda(|u|)\Big[f^*(x)+|u|^{p-1+\sigma}+\big(\nu^{1/p}(x)|\eta|\big)^{p-1+\sigma}+\nu(x)|\eta|^p\Big] \tag{2.1}$$

holds for almost all $x\in\Omega$ and for all real numbers $u,\eta_1,\eta_2,\dots,\eta_m$

(H6) There exists a function $F^*\in L^\infty(\partial\Omega)$ such that

$$|F(x,u)|\le\lambda(|u|)+F^*(x) \tag{2.2}$$

holds for almost all $x\in\partial\Omega$ and for every $u\in\mathbb{R}$.

(H7) There exists a function $F_0\in L^\infty(\partial\Omega)$ such that

$$uF(x,u)+F_0(x)\ge 0 \tag{2.3}$$

holds for almost all $x\in\partial\Omega$ and for every $u\in\mathbb{R}$.

(H8) There exist a nonnegative number $c_1<c_0$ and a function $f_0\in L^\infty(\Omega)$ such that for almost all $x\in\Omega$ and for all real numbers $u,\eta_1,\eta_2,\dots,\eta_m$,

$$uf(x,u,\eta)+c_1|u|^p+\lambda(|u|)\nu(x)|\eta|^p+f_0(x)\ge 0\,. \tag{2.4}$$

(H9) There exists a function $a^*\in L^{p/(p-1)}(\Omega)$ such that for almost all $x\in\Omega$ and for real numbers $u,\eta_1,\eta_2,\dots,\eta_m$,

$$\frac{|a_i(x,u,\eta)|}{\nu^{1/p}(x)}\le\lambda(|u|)\big[a^*(x)+|u|^{p-1}+\nu^{(p-1)/p}(x)|\eta|^{p-1}\big]\,. \tag{2.5}$$

(H10) Condition (1.2) is satisfied for almost all $x\in\Omega$ and for all real numbers $u,\eta_1,\eta_2,\dots,\eta_m$; the function $\lambda:[0,+\infty)\to[1,+\infty)$ is monotone and nondecreasing.

(H11) For almost all $x\in\Omega$ and all real numbers u, η_1, η_2, $\dots$, η_m, τ_1,τ_2, $\dots$, τ_m, the inequality

$$\sum_{i=1}^m\big[a_i(x,u,\eta)-a_i(x,u,\tau)\big](\eta_i-\tau_i)\ge 0 \tag{2.6}$$

holds while the inequality holds if and only if $\eta\ne\tau$.

In this article we study the problem of finding a function $u\in W$ such that

$$\begin{aligned}&\int_\Omega\Big\{\sum_{i=1}^m a_i(x,u,\nabla u)\frac{\partial w}{\partial x_i}+c_0|u|^{p-2}uw+f(x,u,\nabla u)w\Big\}\,dx\\&+\int_{\partial\Omega}\{c_2|u|^{p-2}uw+F(x,u)w\}\,ds=0\end{aligned} \tag{2.7}$$

holds for every $w\in W$. Hypotheses (H1)–(H6)and (H10) provide the correctness for this problem. We shall prove the following result:

Theorem 2.4. *Let* (H1)–(H11) *be satisfied. Then* (2.7) *has at least one solution.*

3. Auxiliary results

The first result of this section is an a priori estimate in $L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ for every solution of (2.7).

Lemma 3.1. *Let* (H1)–(H10) *be satisfied and let* u *be a solution of* (2.7)*. Then*

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\partial\Omega)} \leq K \tag{3.1}$$

where

$$K = 2\Big\{\frac{2}{c_3}[\|f_0\|_{L^\infty(\Omega)} + \|F_0\|_{L^\infty(\partial\Omega)}]\Big\}^{1/p}, \quad c_3 = \min(c_2, c_0 - c_1).$$

Proof. Let us take $w = u|u|^\gamma$ as a test function in (2.7) (see Remark 2.2) where γ is a positive number. We deduce that

$$\begin{aligned} &\int_\Omega |u|^\gamma \Big\{(\gamma+1)\sum_{i=1}^m a_i(x,u,\nabla u)\frac{\partial u}{\partial x_i} + c_0|u|^p + \ f(x,u,\nabla u)u\Big\}\,dx \\ &+ \int_{\partial\Omega} \Big\{c_2|u|^{\gamma+p} + F(x,u)u|u|^\gamma\Big\}\,ds = 0. \end{aligned}$$

By using (H7), (H8) and (H10) we obtain

$$\begin{aligned} &\int_\Omega |u|^\gamma \Big\{\Big[\frac{\gamma+1}{\lambda(\|u\|_{L^\infty(\Omega)})} - \lambda(\|u\|_{L^\infty(\Omega)})\Big]\nu|\nabla u|^p + (c_0-c_1)|u|^p - f_0\Big\}\,dx \\ &+ \int_{\partial\Omega} \{c_2|u|^{\gamma+p} - F_0|u|^\gamma\}\,ds \leq 0. \end{aligned}$$

Set γ such that $\gamma > [\lambda(\|u\|_{L^\infty(\Omega)})]^2 - 1$, from the above inequality it follows that

$$c_3\Big[\int_\Omega |u|^{\gamma+p}\,dx + \int_{\partial\Omega} |u|^{\gamma+p}\,ds\Big] \leq \int_\Omega |f_0||u|^\gamma\,dx + \int_{\partial\Omega} |F_0||u|^\gamma\,ds.$$

Then, by Hölder's inequality

$$\begin{aligned} &c_3\Big[\int_\Omega |u|^{\gamma+p}\,dx + \int_{\partial\Omega} |u|^{\gamma+p}\,ds\Big] \\ &\leq \Big[\Big(\int_\Omega |u|^{\gamma+p}\,dx\Big)^{\frac{\gamma}{\gamma+p}} + \Big(\int_{\partial\Omega} |u|^{\gamma+p}\,ds\Big)^{\frac{\gamma}{\gamma+p}}\Big] \\ &\quad\times \Big[\Big(\int_\Omega |f_0|^{(\gamma+p)/p}\,dx\Big)^{\frac{p}{\gamma+p}} + \Big(\int_{\partial\Omega} |F_0|^{(\gamma+p)/p}\,ds\Big)^{\frac{p}{\gamma+p}}\Big]. \end{aligned}$$

The above inequality implies

$$\begin{aligned} &\Big(\int_\Omega |u|^{\gamma+p}\,dx\Big)^{\frac{p}{\gamma+p}} + \Big(\int_{\partial\Omega} |u|^{\gamma+p}\,ds\Big)^{\frac{p}{\gamma+p}} \\ &\leq \frac{2^{\frac{p}{p+\gamma}+1}}{c_3}\big\{\|f_0\|_{L^\infty(\Omega)}(\operatorname{meas}_m\Omega)^{\frac{p}{\gamma+p}} + \|F_0\|_{L^\infty(\partial\Omega)}(\operatorname{meas}\partial\Omega)^{\frac{p}{\gamma+p}}\big\} \end{aligned}$$

Letting $\gamma \to +\infty$ we obtain (3.1). The proof is complete. $\square$

The second result of this Section is an a priori estimate for every solution u of (2.7), in the norm of $W^{1,p}(\nu,\Omega)$.

Lemma 3.2. *Let* (H1)–(H10) *be satisfied and let* u *be a solution of* (2.7). *Then there exists a constant* $M > 0$ *such that*

$$\|u\|_{1,p} \le M,$$

where M *depends only on* c_0, c_1, c_2, σ, p, $\|f_0\|_{L^\infty(\Omega)}$, $\|f^*\|_{L^1(\Omega)}$, $\lambda(s)$, $\operatorname{meas}_m \Omega$, $\operatorname{meas} \partial\Omega$ *and* $\|F_0\|_{L^\infty(\partial\Omega)}$.

Proof. We have (see the proof of the Lemma 3.1):

$$\begin{aligned}
&\int_\Omega |u|^\gamma \Big\{ \big[\frac{\gamma+1}{\lambda(\|u\|_{L^\infty(\Omega)})} - \lambda(\|u\|_{L^\infty(\Omega)}) \big] \nu |\nabla u|^p + (c_0 - c_1)|u|^p \Big\}\, dx \\
&+ \int_{\partial\Omega} c_2 |u|^{\gamma+p}\, ds \\
&\le \int_{\partial\Omega} |F_0||u|^\gamma\, ds + \int_\Omega |f_0||u|^\gamma\, dx.
\end{aligned}$$

Set γ such that $\gamma > \lambda(K)[1+\lambda(K)] - 1$, where K is the constant defined in previous Lemma. Then, from the last inequality we obtain

$$\int_\Omega |u|^\gamma [\nu |\nabla u|^p + (c_0 - c_1)|u|^p]\, dx \le K^\gamma \Big(\int_\Omega |f_0|\, dx + \int_{\partial\Omega} |F_0|\, ds \Big). \tag{3.2}$$

On the other hand if we take $w(x) = u(x)$ as a test function in relation (2.7), we have

$$\int_\Omega \Big\{ \sum_{i=1}^m a_i(x,u,\nabla u) \frac{\partial u}{\partial x_i} + c_0|u|^p + f(x,u,\nabla u)u \Big\}\, dx + \int_{\partial\Omega} F(x,u)u\, ds \le 0.$$

Applying inequalities (1.2), (2.1), (2.3) and Lemma 3.1 we obtain

$$\begin{aligned}
&\min\Big(\frac{1}{\lambda(K)}, c_0\Big) \|u\|_{1,p}^p \\
&\le \lambda(K) \int_\Omega \big[f^*|u| + |u|^{p+\sigma} + |u|(\nu^{1/p}|\nabla u|)^{p-1+\sigma} + |u|\nu|\nabla u|^p \big]\, dx + \int_{\partial\Omega} |F_0|\, ds.
\end{aligned}$$

Then, there exists a constant K_1, depending only on c_0, c_1, c_2, σ, $\lambda(s)$, $\|f_0\|_{L^\infty(\Omega)}$ and $\|F_0\|_{L^\infty(\partial\Omega)}$, such that

$$\|u\|_{1,p}^p \le K_1 \int_\Omega [f^*|u| + |u|^{p+\sigma} + |u|^{\tau'} \nu |\nabla u|^p]\, dx + \|F_0\|_{L^\infty(\partial\Omega)} \operatorname{meas}\ \partial\Omega, \tag{3.3}$$

where $\tau' = \frac{\sigma}{2} \frac{p}{p-1+\sigma}$ (see also [8, (3.4)]).

We use (3.1), (3.2) to estimate the first term on the right-hand side of previous inequality:

$$\begin{gathered}
\int_\Omega f^*|u|\, dx \le \|u\|_{L^\infty(\Omega)} \|f^*\|_{L^1(\Omega)} \le K \|f^*\|_{L^1(\Omega)} \\
\int_\Omega |u|^{p+\sigma}\, dx \le \|u\|_{L^\infty(\Omega)}^{p+\sigma} \operatorname{meas}_m \Omega \le K^{p+\sigma} \operatorname{meas}_m \Omega, \\
\int_\Omega |u|^{\tau'} \nu |\nabla u|^p\, dx \le K^{\tau'} \Big(\int_\Omega |f_0|\, dx + \int_{\partial\Omega} |F_0|\, ds \Big) \quad \text{if } \tau' > \lambda(K)[1+\lambda(K)] - 1.
\end{gathered}$$

In the case $\tau' \le \lambda(K)[1+\lambda(K)] - 1$, we first apply Young's inequality to obtain

$$|u|^{\tau'} \le \epsilon + C(\epsilon, \tau', \gamma)|u|^\gamma, \quad \gamma > \lambda(K)[1+\lambda(K)] - 1;$$

hence,

$$\int_\Omega |u|^{\tau'}\nu|\nabla u|^p\,dx \le \epsilon\|u\|_{1,p}^p + C(\epsilon,\tau',\gamma)K^\gamma\Big(\int_\Omega |f_0|\,dx + \int_{\partial\Omega}|F_0|\,ds\Big).$$

The above inequalities and (3.3) give $\|u\|_{1,p} \le M$, where M depends only on c_0, c_1, c_2, p, σ, $\|f_0\|_{L^\infty(\Omega)}$, $\|F_0\|_{L^\infty(\partial\Omega)}$, $\operatorname{meas}_m\Omega$, meas $\partial\Omega$, $\|f^*\|_{L^1(\Omega)}$, $\lambda(s)$. The proof is complete. □

We want to emphasize that the constants K and M in previous Lemmas do not depend on u. Moreover, Hypothesis (H2) in such Lemmas is only used for defining the trace of u on $\partial\Omega$.

The following lemma will be useful in verifying the assumptions of the Leray-Lions Theorem in the proof of Lemma 3.4.

Lemma 3.3. *Let* (H1), (H3), (H9)–(H11) *be satisfied. Let* $u \in W^{1,p}(\nu,\Omega)$ *and* $\{u_n\}$ *be a sequence in* $W^{1,p}(\nu,\Omega)$ *such that there exists a constant* $\Lambda > 0$ *for which* $\|u_n\|_{1,p} \le \Lambda$ *and* $\lambda(|u_n(x)|) \le \Lambda$ *for almost all* $x \in \Omega$ *and for every* $n = 1, 2, \dots$. *Moreover, let us suppose* $\lim_{n\to+\infty}\|u_n - u\|_{L^p(\Omega)} = 0$ *and*

$$\lim_{n\to+\infty}\int_\Omega \sum_{i=1}^m [a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u)]\frac{\partial(u_n-u)}{\partial x_i}\,dx = 0. \tag{3.4}$$

Then

$$\lim_{n\to+\infty}\int_\Omega \nu|\nabla u_n - \nabla u|^p\,dx = 0.$$

The proof of the above lemma is an easy modification of the proof of [8, Lemma 3.3]. The following Lemma is a direct application of the Leray-Lions Theorem.

Lemma 3.4. *Assume that* $\lambda(s) \equiv \lambda$, *with* λ *a positive constant. Let us suppose that* (H1)–(H4), (H9)–(H11) *are satisfied. Let us suppose moreover that for every* $u \in \mathbb{R}$, $(\eta_1,\dots,\eta_m) \in \mathbb{R}^m$ *and for almost all* $x \in \Omega$, *it holds*

$$|f(x,u,\eta)| \le \lambda,$$

and for almost all $x \in \partial\Omega$ *and for all* $u \in \mathbb{R}$,

$$|F(x,u)| \le \lambda.$$

Then (2.7) *has at least one solution.*

Proof. Let us consider the operator

$$A(u,v) : W^{1,p}(\nu,\Omega)\times W^{1,p}(\nu,\Omega) \to (W^{1,p}(\nu,\Omega))^\star,$$

defined by

$$\begin{aligned}\langle A(u,v),w\rangle &= \int_\Omega\Big\{\sum_{i=1}^m a_i(x,u,\nabla v)\frac{\partial w}{\partial x_i} + c_0|u|^{p-2}uw + f(x,u,\nabla u)w\Big\}\,dx\\ &\quad + \int_{\partial\Omega}[c_2|u|^{p-2}uw + F(x,u)w]\,ds\end{aligned}$$

for every $w \in W^{1,p}(\nu,\Omega)$, and the operator $T : W^{1,p}(\nu,\Omega) \to (W^{1,p}(\nu,\Omega))^\star$ defined by

$$T(u) = A(u,u), \quad u \in W^{1,p}(\nu,\Omega).$$

Using (H9), it is easy to check that the operator $A(u,v)$ is a bounded operator. Moreover,

$$\big\langle A(v,v),v\big\rangle \geq \min(\frac{1}{\lambda},c_0)\|v\|_{1,p}^p - \lambda\|v\|_{1,p}\big[(\operatorname{meas}_m \Omega)^{(p-1)/p} + c'(\operatorname{meas}\partial\Omega)^{(\tilde{p}-1)/\tilde{p}}\big].$$

Hence

$$\lim_{\|v\|_{1,p}\to+\infty} \frac{\big\langle T(v),v\big\rangle}{\|v\|_{1,p}} = +\infty.$$

Now, we shall verify that the operator $A(u,v)$ satisfies the other assumptions of the Leray-Lions Theorem (see [15, Theorem 1]; see, also, [9]):

(i) Continuity and monotony in v: from (H11),

$$\langle A(u,u) - A(u,v), u-v\rangle \geq 0.$$

Moreover, we observe that

$$\lim_{n\to+\infty}\big\langle A(u_n,v_n),w\big\rangle = \big\langle A(u,v),w\big\rangle \quad \text{for every } w\in W^{1,p}(\nu,\Omega),$$

if

$$(u_n,v_n)\to(u,v) \quad \text{in } W^{1,p}(\nu,\Omega)\times W^{1,p}(\nu,\Omega).$$

For example, we prove that

$$\lim_{n\to+\infty}\int_{\partial\Omega}|v_n|^{p-2}v_n w\,ds = \int_{\partial\Omega}|v|^{p-2}vw\,ds. \tag{3.5}$$

Now, Hypothesis (H2) implies

$$\|v_n - v\|_{L^p(\partial\Omega)} \leq c'(\operatorname{meas}\partial\Omega)^{(\tilde{p}-p)/p\tilde{p}}\|v_n - v\|_{1,p}$$

then $v_n\to v$ in $L^p(\partial\Omega)$. Let E be an arbitrary measurable subset of $\partial\Omega$. It results

$$\int_E |v_n|^{p-1}|w|\,ds \leq \int_E |v_n|^p\,ds + \int_E |w|^p\,ds.$$

The strong convergence of v_n to v in $L^p(\partial\Omega)$ implies that $\{|v_n|^p\}$ are equiintegrable. Then the above inequality together with Hypothesis (H2) imply that $\{|v_n|^{p-1}|w|\}$ is also an equiintegrable sequence of functions. Hence (3.5) follows from Vitali's theorem.

(ii) Continuity of $A(u,v)$ with respect to v: let $u_n \rightharpoonup u$ in $W^{1,p}(\nu,\Omega)$ and $\lim_{n\to\infty}\langle A(u_n,u_n) - A(u_n,u), u_n - u\rangle = 0$, then, by Lemma 3.3, $u_n\to u$ in $W^{1,p}(\nu,\Omega)$; hence, by previous observation, we have that $A(u_n,v)\rightharpoonup A(u,v)$ in $(W^{1,p}(\nu,\Omega))^\star$, for every $v\in W^{1,p}(\nu,\Omega)$.

(iii) Continuity of $\big\langle A(u,v),u\big\rangle$ in u: we observe that if $v\in W^{1,p}(\nu,\Omega)$, $u_n\rightharpoonup u$ in $W^{1,p}(\nu,\Omega)$ and $A(u_n,v)\rightharpoonup v'$ in $(W^{1,p}(\nu,\Omega))^\star$, then $u_n\to u$ in $L^p(\Omega)$, $u_n\to u$ in $L^p(\partial\Omega)$, hence

$$\lim_{n\to\infty}\big\langle A(u_n,v),u_n - u\big\rangle = 0$$

and, therefore, $\langle A(u_n,v),u_n\rangle \to \big\langle v',u\big\rangle$ (see, also, [11, note (15)], where the special case $p=2$ is treated, but for Dirichlet problem, and, Remark 2.3).

Thus, all the assumptions of the Leray-Lions theorem (Hypothesis II) are satisfied. Hence the equation $Tu=0$ has at least one solution $u\in W^{1,p}(\nu,\Omega)$.

We shall prove that $u\in L^\infty(\Omega)\cap L^\infty(\partial\Omega)$. We set:

$$\Omega_k = \{x\in\Omega : u>k\}, \quad \partial\Omega_k = \{x\in\partial\Omega : u>k\}.$$

From (2.7), choosing $w = u - \min(u, k)$, $k > K_0$ (for K_0 see Remark 2.1), we have

$$\begin{aligned}&\int_{\Omega_k} \Big\{ \sum_{i=1}^m a_i(x, w+k, \nabla w) \frac{\partial w}{\partial x_i} + c_0 |w+k|^{p-1} w + f(x, w+k, \nabla w) w \Big\}\, dx \\ &+ \int_{\partial\Omega_k} \{c_2 |w+k|^{p-1} w + F(x, w+k) w\}\, ds = 0.\end{aligned}$$

Applying condition (1.2) we obtain

$$\min\big(\frac{1}{\lambda}, c_0\big) \|w\|_{1,p}^p \le \lambda \int_{\Omega_k} w\, dx + \lambda \int_{\partial\Omega_k} w\, ds.$$

The above inequality and (H4) imply

$$\|w\|_{1,p}^{p-1} \le \frac{\lambda[\hat{c}(\operatorname{meas}_m \Omega)^{(p^\star - \tilde{p})/p^\star \tilde{p}} + c']}{\min(\frac{1}{\lambda}, c_0)} [(\operatorname{meas}_m \Omega_k)^{(\tilde{p}-1)/\tilde{p}} + (\operatorname{meas} \partial\Omega_k)^{(\tilde{p}-1)/\tilde{p}}].$$

For $h > k$ we have

$$\begin{aligned}&\Big(\int_\Omega |w|^{\tilde{p}}\, dx\Big)^{\frac{p-1}{\tilde{p}}} + \Big(\int_{\partial\Omega} |w|^{\tilde{p}}\, ds\Big)^{\frac{p-1}{\tilde{p}}} \\ &\ge (h-k)^{p-1} \big\{(\operatorname{meas}_m \Omega_h)^{(p-1)/\tilde{p}} + (\operatorname{meas} \partial\Omega_h)^{(p-1)/\tilde{p}}\big\}.\end{aligned}$$

For $h > 0$, denote

$$\varphi(h) = \{\operatorname{meas}_m \Omega_h + \operatorname{meas} \partial\Omega_h\}.$$

We have

$$\varphi(h) \le \frac{\alpha}{(h-k)^{\tilde{p}}} [\varphi(k)]^{\frac{\tilde{p}-1}{p-1}}, \quad \text{if } h > k > K_0$$

where the positive constant α depends only on $\hat{c}$, c', c_0, λ, m, p, t, Ω.

Note that $\frac{\tilde{p}-1}{p-1} > 1$, then it follows from a lemma of Stampacchia [17, Lemma 3.11] that $\operatorname{ess\,sup}_\Omega u + \operatorname{ess\,sup}_{\partial\Omega} u < +\infty$. By this way also $\operatorname{ess\,sup}_\Omega(-u) + \operatorname{ess\,sup}_{\partial\Omega}(-u) < +\infty$. Hence $u \in L^\infty(\Omega) \cap L^\infty(\partial\Omega)$. □

4. Proof of Theorem 2.4

Proof. Let K be the constant defined in Lemma 3.1. We define

$$A_i(x, u, \eta) = \begin{cases} a_i(x, -K, \eta) & \text{if } u < -K \\ a_i(x, u, \eta) & \text{if } |u| \le K \\ a_i(x, K, \eta) & \text{if } u > K, \end{cases}$$

in $\Omega \times \mathbb{R} \times \mathbb{R}^m$. For every positive integer n we define:

$$f_n(x, u, \eta) = \begin{cases} f(x, u, \eta) & \text{if } |f| \le n \\ n \frac{f(x,u,\eta)}{|f(x,u,\eta)|} & \text{if } |f| > n \end{cases}$$

in $\Omega \times \mathbb{R} \times \mathbb{R}^m$,

$$F_n(x, u) = \begin{cases} F(x, u) & \text{if } |F| \le n \\ n \frac{F(x,u)}{|F(x,u)|} & \text{if } |F| > n \end{cases}$$

in $\partial\Omega \times \mathbb{R}$.

The functions $A_i(x, u, \eta)$, $f_n(x, u, \eta)$, $F_n(x, u)$, satisfy (H3)–(H11). It is sufficient to note, for example, that in $\Omega \times \mathbb{R} \times \mathbb{R}^m$,

$$|f_n(x, u, \eta)| \le |f(x, u, \eta)|,$$

and, that (H8) holds with $|f_0(x)|$ instead of $f_0(x)$. Analogous considerations verify the others assumptions. On the other hand, for every $u \in \mathbb{R}$, $(\eta_1, \dots, \eta_m) \in \mathbb{R}^m$ and for almost all $x \in \Omega$ it holds that

$$|f_n(x, u, \eta)| \le n\,,$$

and for almost all $x \in \partial\Omega$ and for all $u \in \mathbb{R}$,

$$|F_n(x, u)| \le n.$$

Then, it follows from Lemma 3.4 that, for every $n \in \mathbb{N}$, there exists $u_n \in W$ such that

$$\begin{aligned} &\int_\Omega \Big[\sum_{i=1}^m A_i(x, u_n, \nabla u_n)\frac{\partial w}{\partial x_i} + c_0|u_n|^{p-2}u_n w + f_n(x, u_n, \nabla u_n)w\Big]\,dx \\ &+ \int_{\partial\Omega} [c_2|u_n|^{p-2}u_n w + F_n(x, u_n)w]\,ds = 0 \end{aligned} \tag{4.1}$$

for every $w \in W$. An a priori estimate of Lemma 3.1 yields

$$\|u_n\|_{L^\infty(\Omega)} + \|u_n\|_{L^\infty(\partial\Omega)} \le K, \quad \text{for every } n \in \mathbb{N}, \tag{4.2}$$

and hence (4.1) can be written in the equivalent form

$$\begin{aligned} &\int_\Omega \Big[\sum_{i=1}^m a_i(x, u_n, \nabla u_n)\frac{\partial w}{\partial x_i} + c_0|u_n|^{p-2}u_n w + f_n(x, u_n, \nabla u_n)w\Big]\,dx \\ &+ \int_{\partial\Omega} [c_2|u_n|^{p-2}u_n w + F_n(x, u_n)w]\,ds = 0. \end{aligned} \tag{4.3}$$

It follows from Lemma 3.2 that for every $n \in \mathbb{N}$,

$$\|u_n\|_{1,p} \le M. \tag{4.4}$$

On the basis of (4.2) and (4.4) there exists a subsequence of $\{u_n\}$ (denoted again by $\{u_n\}$) such that $\{u_n\}$ converges weakly to u in $W^{1,p}(\nu, \Omega)$ and $\{u_n\}$ converges weakly* in $L^\infty(\Omega)$ and in $L^\infty(\partial\Omega)$ where $u \in W$ and $\|u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\partial\Omega)} \le K$. We shall prove that $u \in W$ is the solution of (2.7).

To pass to the limit in (4.3) for $n \to +\infty$ we have to prove that

$$\lim_{n\to+\infty} \int_\Omega \nu|\nabla u_n - \nabla u|^p\,dx = 0. \tag{4.5}$$

Now, the compact embedding of $W^{1,p}(\nu, \Omega)$ in $L^p(\Omega)$ implies the strong convergence of u_n to u in $L^p(\Omega)$ and hence also almost everywhere in $\partial\Omega$ (see Remark 2.3). Then, taking into account Lemma 3.3, to get (4.5) it will be sufficient to prove that (3.4) it holds.

Let us take $w = |u_n - u|^\gamma(u_n - u)$ as a test function in (4.3) where γ is a positive number. We deduce

$$\begin{aligned} &\int_\Omega \Big\{\sum_{i=1}^m a_i(x, u_n, \nabla u_n)(\gamma+1)|u_n - u|^\gamma \frac{\partial(u_n - u)}{\partial x_i} \\ &+ c_0|u_n|^{p-2}u_n|u_n - u|^\gamma(u_n - u) + f_n(x, u_n, \nabla u_n)|u_n - u|^\gamma(u_n - u)\Big\}\,dx \\ &+ \int_{\partial\Omega} \{c_2|u_n|^{p-2}u_n|u_n - u|^\gamma(u_n - u) + F_n(x, u_n)|u_n - u|^\gamma(u_n - u)\}\,ds \\ &= 0. \end{aligned}$$

From the above inequality, taking into account (1.2), (2.1), (2.2), (4.2), we obtain

$$\begin{aligned}
&\int_\Omega |u_n-u|^\gamma |\nabla u_n|^p \nu\,dx \\
&\le \int_\Omega \sum_{i=1}^m a_i(x,u_n,\nabla u_n)(\gamma+1)|u_n-u|^\gamma \frac{\partial u}{\partial x_i} \\
&\quad + c_0K^{p-1}\int_\Omega |u_n-u|^{\gamma+1}\,dx + 2K\lambda(K)\int_\Omega [|f^*|+K^{p-1+\sigma}+1]|u_n-u|^\gamma\,dx \\
&\quad + c_2\int_{\partial\Omega} |u_n|^{p-1}|u_n-u|^{\gamma+1}\,ds + \int_{\partial\Omega}[|F^*|+\lambda(K)]|u_n-u|^{\gamma+1}\,ds,
\end{aligned}$$

where γ is such that $\frac{\gamma+1}{\lambda(K)} - 4K\lambda(K) > 1$.

By Lebesgue theorem, the first three addends in the right hand side of previous inequality go to 0 as $n \to +\infty$ (see, [8, Lemma 3.4, pp. 229-230]). We prove, for example, that

$$\lim_{n\to\infty}\int_{\partial\Omega}[|F^*|+\lambda(K)]|u_n-u|^{\gamma+1}\,ds = 0,$$

this integral is absent in [8]. It results that a.e. $x \in \partial\Omega$,

$$[|F^*|+\lambda(K)]|u_n-u|^{\gamma+1} \le (2K)^{\gamma+1}[|F^*|+\lambda(K)] \in L^1(\partial\Omega).$$

As $u_n \to u$ a.e. in $\partial\Omega$, it will be enough to apply Lebesgue theorem again. Then, it follows

$$\lim_{n\to+\infty}\int_\Omega |u_n-u|^\gamma|\nabla u_n|^p\nu\,dx = 0,$$

and, so, applying Hölder inequality

$$\lim_{n\to+\infty}\int_\Omega |u_n-u||\nabla u_n|^p\nu\,dx = 0. \tag{4.6}$$

By (4.3) we obtain

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^m [a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u)]\frac{\partial(u_n-u)}{\partial x_i}\,dx \\
&= -\int_\Omega c_0|u_n|^{p-2}u_n(u_n-u)\,dx - \int_\Omega f_n(x,u_n,\nabla u_n)(u_n-u)\,dx \\
&\quad - \int_\Omega \sum_{i=1}^m a_i(x,u,\nabla u)\frac{\partial(u_n-u)}{\partial x_i}\,dx \\
&\quad + \int_\Omega \sum_{i=1}^m [a_i(x,u,\nabla u)-a_i(x,u_n,\nabla u)]\frac{\partial(u_n-u)}{\partial x_i}\,dx \\
&\quad - \int_{\partial\Omega} c_2|u_n|^{p-2}u_n(u_n-u)\,ds - \int_{\partial\Omega} F_n(x,u_n)(u_n-u)\,ds.
\end{aligned}$$

Now, all addends in the right-hand side of previous inequality go to 0 as $n \to +\infty$. For example, we shall estimate the second and the last addend. We have

$$\begin{aligned}
&\int_\Omega |f_n(x,u_n,\nabla u_n)||u_n-u|\,dx \\
&\le \lambda(K)\int_\Omega [K^{p-1+\sigma}+1+|f^*|]|u_n-u|\,dx + 2\lambda(K)\int_\Omega |u_n-u||\nabla u_n|^p\nu\,dx.
\end{aligned}$$

From the Lebesgue theorem and (4.6), the above inequality implies

$$\lim_{n\to+\infty}\int_\Omega f_n(x,u_n,\nabla u_n)(u_n-u)\,dx=0.$$

Next

$$\int_{\partial\Omega}|F_n(x,u_n)||u_n-u|\,ds\le[\lambda(K)+\|F^*\|_{L^\infty(\partial\Omega)}]\int_{\partial\Omega}|u_n-u|\,ds.$$

Taking into account that the imbedding of $W^{1,p}(\Omega)$ in $L^1(\partial\Omega)$ is compact (see Remark 2.3), the above inequality implies

$$\lim_{n\to+\infty}\int_{\partial\Omega}F_n(x,u_n)(u_n-u)\,dx=0.$$

For details concerning others passage to the limit see [8, pag. 228]. Consequently

$$\int_\Omega\sum_{i=1}^m[a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u)]\frac{\partial(u_n-u)}{\partial x_i}\,dx$$

tends to zero as $n\to+\infty$. So, $u_n\to u$ in $W^{1,p}(\nu,\Omega)$.

Now, to prove that the function $u\in W$ is the solution of (2.7) it is sufficient to pass to the limit as $n\to\infty$. For example, we prove that

$$\lim_{n\to+\infty}\int_{\partial\Omega}F_n(x,u_n)w\,ds=\int_{\partial\Omega}F(x,u)w\,ds \tag{4.7}$$

for every $w\in W$.

We fix $\epsilon>0$ and a point $x_0\in\partial\Omega$ such that $u_n(x_0)\to u(x_0)$ as $n\to+\infty$ and the function $F(x_0,u)$ is continuous with respect u. Then there is a number $n_\epsilon\in\mathbb{N}$ such that for any $n>n_\epsilon$,

$$-n<F(x_0,u(x_0))-\epsilon<F(x_0,u_n(x_0))<\epsilon+F(x_0,u(x_0))<n.$$

These inequalities and the definition of the function $F_n(x,u)$ imply that for any $n>n_\epsilon$, $F_n(x_0,u_n(x_0))=F(x_0,u_n(x_0))$ and

$$|F_n(x_0,u_n(x_0))-F(x_0,u(x_0))|<\epsilon.$$

In this way $F_n(x,u_n(x))\to F(x,u(x))$ a.e. on $\partial\Omega$. Next, from definition of $F_n(x,u)$ and (2.2) we have

$$|F_n(x,u_n(x))w(x)|\le[\lambda(K)+\|F^*\|_{L^\infty(\partial\Omega)}]|w(x)|$$

a.e. $x\in\partial\Omega$. Now, a new application of the Lebesgue theorem gives (4.7). The proof is complete. □

Now, we show an example where all assumptions are satisfied. Let Ω be a bounded open set of $\mathbb{R}^m$ such that $0\in\partial\Omega$. Put

$$\nu(x)=|x|^\gamma\quad\text{for }0<\gamma<p-1.$$

Then the function ν satisfies Hypotheses (H1) and (H2) with t such that

$$\frac{m}{p-1}<t<\frac{m}{\gamma}.$$

Consider the boundary-value problem

$$-\operatorname{div}\Big(\frac{|x|^\gamma}{1+|u|^p}|\nabla u|^{p-2}\nabla u\Big)+e^u-|u|^p+|x|^\gamma|\nabla u|^p=g(x)\quad\text{in }\Omega, \tag{4.8}$$

$$\frac{|x|^\gamma}{1+|u|^p}|\nabla u|^{p-2}\sum_{i=1}^m \frac{\partial u}{\partial x_i}\cos(\overrightarrow{n},x_i)+\frac{1}{e}u|u|^{p-2}+\frac{e^{u-1}}{2}=0 \quad \text{on } \partial\Omega, \tag{4.9}$$

where $g(x)\in L^\infty(\Omega)$. In this case we have:

$$\begin{gathered}
a_i(x,u,\nabla u)=\frac{|x|^\gamma}{1+|u|^p}|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}, \quad i=1,2,\dots,m;\\
f(x,u,\nabla u)=e^u-|u|^p-u|u|^{p-2}+|x|^\gamma|\nabla u|^p-g(x), \quad c_0=1;\\
F(x,u)=\frac{1}{2e}u|u|^{p-2}+\frac{e^{u-1}}{2}; \quad c_2=\frac{1}{2e}\,.
\end{gathered}$$

If we put $\lambda(|u|)=e^{|u|^p}$, it is possible to verify all the Hypotheses (H3)–(H11). To verify (H3), for example, it will be sufficient to note that the function $(|u|^p+ue^u)$ has minimum (≤ 0) in $(-\infty,+\infty)$.

Hence, BVP (4.8), (4.9) has at least one weak solution in the sense (2.7), i.e. there exists at least one $u\in W$ such that

$$\begin{aligned}
&\int_\Omega \frac{|x|^\gamma}{1+|u|^p}|\nabla u|^{p-2}\nabla u\nabla w\,dx+\int_\Omega[e^u-|u|^p+|x|^\gamma|\nabla u|^p]w\,dx\\
&+\int_{\partial\Omega}\big\{\frac{1}{e}u|u|^{p-2}+\frac{e^{u-1}}{2}\big\}w\,ds\\
&=\int_\Omega gw\,dx
\end{aligned}$$

holds for every $w\in W$.

Examples concerning the Dirichlet problem related to (1.1) can be found in [8, Section 6].

5. Asymptotic behavior near infinity of solutions to the Dirichlet problem for (1.1)

Let $\Omega=\{x\in\mathbb{R}^m:|x|>r\}$, r be a positive constant. For $n\in\mathbb{N}$, we denote

$$\Omega_n=\Omega\cap\{x\in\mathbb{R}^m:|x|<n\}.$$

We introduce the hypothesis

(H12) The function $\nu=\nu(x):\Omega\to(0,+\infty)$ is a measurable function such that $\nu\in L^\infty(\Omega)$. For every $n\in\mathbb{N}$, there exists a real number $\delta_n>\max(\frac{m}{p},\frac{1}{p-1})$ such that $1/\nu\in L^{\delta_n}(\Omega_n)$.

We set

$$L^1(\Omega)+L^{p/(p-1)}(\Omega)=\{f_1(x)+f_2(x):f_1\in L^1(\Omega),f_2\in L^{p/(p-1)}(\Omega)\}.$$

Let (H3), (H5), (H8)–(H12) be satisfied with $f_0\in L^1(\Omega)\cap L^\infty(\Omega)$, $f^*\in L^1(\Omega)+L^{p/(p-1)}(\Omega)$ and let $u\in \mathring{W}^{1,p}(\nu,\Omega)\cap L^\infty(\Omega)$ such that

$$\int_\Omega\Big\{\sum_{i=1}^m a_i(x,u,\nabla u)\frac{\partial w}{\partial x_i}+c_0|u|^{p-2}uw+\ f(x,u,\nabla u)w\Big\}\,dx=0 \tag{5.1}$$

for every $w\in \mathring{W}^{1,p}(\nu,\Omega)\cap L^\infty(\Omega)$. The function u exists because of [8, Theorem 2.2].

Theorem 5.1. *Let* (H3), (H5), (H8)–(H12) *be satisfied, with the function* f^* *in* $L^1(\Omega) + L^{p/(p-1)}(\Omega)$, *and*

$$|f_0(x)| + |a^*(x)| \le \tilde{c}e^{-\delta_1|x|}, \quad x \in \Omega, \tag{5.2}$$

with $\tilde{c}$ *and* δ_1 *positive constants. Let us consider* $u \in \mathring{W}^{1,p}(\nu,\Omega) \cap L^\infty(\Omega)$ *that satisfies* (5.1) *for every* $w \in \mathring{W}^{1,p}(\nu,\Omega) \cap L^\infty(\Omega)$. *Then*

$$\int_{|x|>\lambda} |u|^p \, dx \le Ce^{-\delta_3\lambda} \tag{5.3}$$

for every $\lambda \ge r$, *where* δ_3 *and* C *are positive constants depending on known parameters.*

Proof. Let us define in $\mathbb{R}^m$ a Lipschitzian function $\theta(x)$, $0 \le \theta(x) \le 1$, such that $\theta(x) = 0$ if $0 < |x| < r+1$, $\theta(x) = 1$ if $|x| > r+2$. Define in $\mathbb{R}^m$ the function $\theta_R(x)$, $0 \le \theta_R(x) \le 1$, such that $\theta_R(x) = 1$ if $|x| < R$, $\theta_R(x) = 0$ if $|x| > R+1$, and let $\theta_R(x)$ be a Lipschitzian function.

Take in (5.1) as a test function $w = u|u|^\gamma e^{\gamma\tau(x)}\theta\theta_R$ where $\tau(x) = \beta|x|$ if $|x| < L$, $\tau(x) = \beta L$ for $|x| > L$ and the positive constants γ, β will be stated later on. Moreover, let us suppose that real numbers L, R are such that $r+2 < L < R$.

After easy computations, by (1.2) and (2.4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m} e^{\gamma\tau(x)}|u|^\gamma\theta\theta_R\Big\{\Big[\frac{\gamma+1}{\lambda(\|u\|_{L^\infty(\Omega)})} - \lambda(\|u\|_{L^\infty(\Omega)})\Big]\nu|\nabla u|^p \\
&+ (c_0 - c_1)|u|^p\Big\} \, dx \\
&\le \gamma\int_{\mathbb{R}^m}\sum_{i=1}^m |a_i(x,u,\nabla u)|\Big|\frac{\partial\tau(x)}{\partial x_i}\Big||u|^{\gamma+1}e^{\gamma\tau(x)}\theta\theta_R \, dx \\
&\quad + \int_{\mathbb{R}^m}\sum_{i=1}^m |a_i(x,u,\nabla u)||u|^{\gamma+1}e^{\gamma\tau(x)}|\nabla\theta|\theta_R \, dx \\
&\quad + \int_{\mathbb{R}^m}\sum_{i=1}^m |a_i(x,u,\nabla u)||u|^{\gamma+1}e^{\gamma\tau(x)}|\nabla\theta_R|\theta \, dx \\
&\quad + \int_{\mathbb{R}^m} e^{\gamma\tau(x)}|f_0||u|^\gamma\theta\theta_R \, dx.
\end{aligned} \tag{5.4}$$

Now, we choose γ in such that

$$\frac{\gamma+1}{\lambda(\|u\|_{L^\infty(\Omega)})} - \lambda(\|u\|_{L^\infty(\Omega)}) = 2.$$

Then, we can estimate from below the left-hand side of (5.4) by

$$2\int_{r+2<|x|<L} e^{\gamma\beta|x|}|u|^\gamma\nu|\nabla u|^p \, dx + (c_0 - c_1)\int_{r+2<|x|<L} e^{\gamma\beta|x|}|u|^{\gamma+p} \, dx. \tag{5.5}$$

Next, we shall estimate every addend of right hand side of (5.4). By (2.5), (5.2) and the definitions of $\tau(x), \theta(x), \theta_R(x)$, it results that

$$
\begin{aligned}
&\gamma \int_{\mathbb{R}^m} \sum_{i=1}^m |a_i(x,u,\nabla u)| \Big|\frac{\partial \tau(x)}{\partial x_i}\Big| |u|^{\gamma+1} e^{\gamma\tau(x)} \theta\theta_R \,dx \\
&\le \gamma\beta \int_{|x|<L} e^{\gamma\beta|x|} \sum_{i=1}^m |a_i(x,u,\nabla u)||u|^{\gamma+1}\,dx \\
&\le \gamma\beta d_1 \Big[\int_{\mathbb{R}^m} |a^*(x)| e^{\gamma\beta|x|}\,dx + e^{\gamma\beta(r+2)}\Big] \\
&\quad + 2m\gamma\beta\lambda(\|u\|_{L^\infty(\Omega)})\|\nu\|_{L^\infty(\Omega)}^{1/p} \int_{r+2<|x|<L} e^{\gamma\beta|x|}|u|^{\gamma+p}\,dx \\
&\quad + m\gamma\beta\lambda(\|u\|_{L^\infty(\Omega)})\|\nu\|_{L^\infty(\Omega)}^{1/p} \int_{r+2<|x|<L} e^{\gamma\beta|x|}|u|^{\gamma}\nu|\nabla u|^p\,dx;
\end{aligned} \tag{5.6}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \sum_{i=1}^m |a_i(x,u,\nabla u)||u|^{\gamma+1} e^{\gamma\tau(x)} |\nabla\theta|\theta_R\,dx \\
&\le e^{\gamma\beta(r+2)} \int_{|x|<r+2} \sum_{i=1}^m |a_i(x,u,\nabla u)||u|^{\gamma+1}|\nabla\theta|\,dx \\
&\le d_2 e^{\gamma\beta(r+2)} \lambda(\|u\|_{L^\infty(\Omega)})\|u\|_{L^\infty(\Omega)}^{\gamma+1} \\
&\quad \times \int_{|x|<r+2} [a^*(x)\nu^{1/p} + |u|^{p-1}\nu^{1/p} + \nu|\nabla u|^{p-1}]\,dx \\
&\le d_3 e^{\gamma\beta(r+2)};
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \sum_{i=1}^m |a_i(x,u,\nabla u)||u|^{\gamma+1} e^{\gamma\tau(x)} |\nabla\theta_R|\theta\,dx \\
&\le e^{\gamma\beta L} \int_{R<|x|<R+2} \sum_{i=1}^m |a_i(x,u,\nabla u)||u|^{\gamma+1}|\nabla\theta_R|\,dx \\
&\le d_4 e^{\gamma\beta L} \lambda(\|u\|_{L^\infty(\Omega)})\|\nu\|_{L^\infty(\Omega)}^{1/p} \Big(\|u\|_{L^\infty(\Omega)}^{\gamma+1} + 1\Big) \\
&\quad \times \Big[\int_{R<|x|<R+2} |a^*(x)|\,dx + \int_{R<|x|<R+2} (|u|^p + \nu|\nabla u|^p)\,dx\Big];
\end{aligned} \tag{5.8}
$$

$$
\int_{\mathbb{R}^m} e^{\gamma\tau(x)}|f_0||u|^\gamma \theta\theta_R\,dx \le \tilde{c}\|u\|_{L^\infty(\Omega)}^\gamma \int_{\mathbb{R}^m} e^{(\gamma\beta-\delta_1)|x|}\,dx, \tag{5.9}
$$

where the constants d_i $(i = 1,2,3,4)$ are positive and depend only on m, p, $\lambda(s)$, $\|u\|_{L^\infty(\Omega)}$, $\|u\|_{1,p}$, $\|\nu\|_{L^\infty(\Omega)}$, $\|a^*\|_{L^1(\Omega)}$ and r.

From (5.4), estimates (5.5)–(5.9), letting $R \to +\infty$, we obtain

$$
\begin{aligned}
&2\int_{r+2<|x|<L} e^{\gamma\beta|x|}|u|^\gamma\nu|\nabla u|^p\,dx + (c_0 - c_1)\int_{r+2<|x|<L} e^{\gamma\beta|x|}|u|^{\gamma+p}\,dx \\
&\le d_3 e^{\gamma\beta(r+2)} + \gamma\beta d_1\Big[\tilde{c}\int_{\mathbb{R}^m} e^{(\gamma\beta-\delta_1)|x|}\,dx + e^{\gamma\beta(r+2)}\Big] \\
&\quad + 2m\gamma\beta\lambda(\|u\|_{L^\infty(\Omega)})\|\nu\|_{L^\infty(\Omega)}^{1/p}\int_{r+2<|x|<L} e^{\gamma\beta|x|}|u|^{\gamma+p}\,dx
\end{aligned}
$$

$$+ m\gamma\beta\lambda(\|u\|_{L^\infty(\Omega)})\|\nu\|^{1/p}_{L^\infty(\Omega)} \int_{r+2<|x|<L} e^{\gamma\beta|x|}|u|^\gamma \nu|\nabla u|^p\,dx$$
$$+ \tilde{c}\|u\|^\gamma_{L^\infty(\Omega)} \int_{\mathbb{R}^m} e^{(\gamma\beta-\delta_1)|x|}\,dx,$$

for every real numbers $L > r+2$, $\beta > 0$; where γ is a fixed real number, $\gamma > 2$.

Fix β such that

$$0 < \beta < \min\Big(\frac{\delta_1}{\gamma}, \frac{c_0 - c_1}{2m\gamma\lambda(\|u\|_{L^\infty(\Omega)})\|\nu\|^{1/p}_{L^\infty(\Omega)}}, \frac{2}{m\gamma\lambda(\|u\|_{L^\infty(\Omega)})\|\nu\|^{1/p}_{L^\infty(\Omega)}}\Big).$$

Then, for every $L > r+2$, we obtain

$$\int_{r+2<|x|<L} e^{\gamma\beta|x|}|u|^{\gamma+p}\,dx \le M$$

where M depends only on m, p, r, β, γ, c_0, c_1, $\tilde{c}$, $\lambda(s)$, $\|u\|_{L^\infty(\Omega)}$, $\|u\|_{1,p}$, $\|\nu\|_{L^\infty(\Omega)}$ and δ_1. Letting $L \to +\infty$, the above inequality implies

$$\int_{|x|>r} e^{\delta_2|x|}|u|^{\gamma+p}\,dx \le M_1 \tag{5.10}$$

where $\delta_2 = \gamma\beta$ and $M_1 = e^{\gamma\beta(r+2)}\|u\|^{\gamma+p}_{L^\infty(\Omega)} \operatorname{meas}_m(r < |x| < r+2) + M$. Hence (5.3) follows from (5.10). The proof is complete. $\square$

We give an example where Hypothesis (H12) is satisfied. Let $\Omega = \{x \in \mathbb{R}^m : |x| > 1\}$. We consider the function $\nu : \Omega \to (0, +\infty)$ defined by

$$\nu(x) = \big[(|x|-1)e^{-(|x|-1)}\big]^\gamma, \quad \gamma \in (0, (p-1)/m).$$

Then

$$\nu(x) \le \big(\frac{1}{e}\big)^\gamma, \quad x \in \Omega.$$

For every integer $n \ge 2$, we set $\Omega_n = \{x \in \mathbb{R}^m : 1 < |x| < n\}$. Then, the function $1/\nu(x) \in L^{\delta_n}(\Omega_n)$ for every δ_n satisfying $m/(p-1) < \delta_n < 1/\gamma$.

6. Phragmén-Lindelöf theorem

Now, we shall consider weak solutions of (1.1) for the Dirichlet problem, with p-Laplacian, in a cylindrical unbounded domain.

Let $0 \le a < b \le +\infty$ and define the set

$$\pi_{a,b} = \{x \in \mathbb{R}^m : x' \in \Omega', a < x_m < b\},$$

where $x' = (x_1, \dots, x_{m-1})$, Ω' is a bounded domain in $\mathbb{R}^{m-1}$, $m \ge 3$, with a smooth boundary $\partial\Omega'$; $\pi_a = \pi_{a,\infty}$. Let p be a real number such that $1 < p < m-1$.

For the next theorem we need the following hypotheses:

(H13) Let $\hat{\nu} = \hat{\nu}(x') : \Omega' \to (0, +\infty)$ be a measurable such that

$$\hat{\nu} \in L^\infty(\Omega'), \quad \big(\frac{1}{\hat{\nu}}\big) \in L^t(\Omega'),$$

with $t > \max(\frac{m}{p}, \frac{1}{p-1})$;

(H14) Let $f(x,u,\eta)$ be a Caratheodory function in $\pi_0 \times \mathbb{R} \times \mathbb{R}^m$ such that for almost all $x=(x',x_m) \in \pi_0$ and for all $(u,\eta) \in \mathbb{R} \times \mathbb{R}^m$,

$$|f(x,u,\eta)| \le \lambda(|u|)[f^*(x)+\hat{\nu}(x')|\eta|^p], \quad f^* \in L^1(\pi_0)+L^{p/(p-1)}(\pi_0),$$

$$c_1|u|^p + uf(x,u,\eta) \ge -f_0(x), \;\; f_0 \in L^1(\pi_0) \cap L^\infty(\pi_0),$$

where $\lambda : [0,+\infty) \to [1,+\infty)$ is a monotone nondecreasing function and c_1 is a positive constant.

Theorem 6.1. *Let* (H13), (H14) *be satisfied. Let $\tilde{\lambda} : [0,+\infty) \to [1,+\infty)$ be a nondecreasing function such that $\tilde{\lambda}(s) \le \lambda(s)$ for all $s \ge 0$. Let c_0 be a positive constant such that $c_0 > c_1$. Let $u \in \mathring{W}^{1,p}(\hat{\nu},\pi_0) \cap L^\infty(\pi_0)$ satisfy*

$$\int_{\pi_0} \Big\{ \frac{\hat{\nu}}{\tilde{\lambda}(|u|)} \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_i} + c_0|u|^{p-2}uw + f(x,u,\nabla u)w \Big\}\, dx = 0 \tag{6.1}$$

for an arbitrary function $w \in \mathring{W}^{1,p}(\hat{\nu},\pi_0) \cap L^\infty(\pi_0)$ (the function u exists by [8, Theorem 2.2]*). Let us assume that for some $a \ge 0$,*

$$c_1|u|^p + uf(x,u,\eta) \ge 0$$

for almost all $x \in \pi_a$ and for all $(u,\eta) \in \mathbb{R} \times \mathbb{R}^m$.
Then there exists a positive constant α, depending on m, p, t, Ω', $\|u\|_{L^\infty(\pi_0)}$, $\|u\|_{1,p}$, $\lambda(s)$, $\|\hat{\nu}\|_{L^\infty(\Omega')}$ and $\|1/\hat{\nu}\|_{L^t(\Omega')}$, such that

$$\int_{\pi_0} e^{\alpha x_m} \hat{\nu} \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^p \, dx \le D\,,$$

where D is a positive number depending only on known parameters.

Proof. For the sake of simplicity, we will assume throughout that

$$c_1|u|^p + uf(x,u,\eta) \ge 0 \tag{6.2}$$

for almost all $x \in \pi_0$ and for all $(u,\eta) \in \mathbb{R} \times \mathbb{R}^m$. Let $\theta(x) \in C^1(\mathbb{R})$ be a function such that $\theta(x) = 1$ if $x < \frac{1}{2}$, $\theta(x) = 0$ if $x > 1$, $0 \le \theta(x) \le 1$, $|\theta'(x)| \le \beta$.

For every $b \ge 0$, we consider $\theta_b(x_m) = \theta(x_m - b)$. It results $0 \le \theta_b(x_m) \le 1$ and $|\theta_b'(x_m)| \le \beta$ for all $b \ge 0$. Let b be a real number, $b > 0$. Let us prove that

$$\begin{aligned} &\int_{\pi_0} \frac{\hat{\nu}}{\tilde{\lambda}(|u|)} \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^p dx + \int_{\pi_0} \{c_0|u|^p + f(x,u,\nabla u)u\}\, dx \\ &= \int_{\pi_0} \frac{\hat{\nu}}{\tilde{\lambda}(|u|)} \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}(\theta_b u)\, dx \\ &\quad + \int_{\pi_0} \{c_0|u|^p + f(x,u,\nabla u)u\}\theta_b\, dx. \end{aligned} \tag{6.3}$$

The function $w = (\theta_c(x_m) - \theta_b(x_m))u \in \mathring{W}^{1,p}(\hat{\nu},\pi_0) \cap L^\infty(\pi_0)$, $c > b > 0$, so by (6.1), we have

$$\begin{aligned} &\int_{\pi_0} \frac{\hat{\nu}}{\tilde{\lambda}(|u|)} \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}[(\theta_c - \theta_b)u] + c_0|u|^p(\theta_c-\theta_b) \\ &+ f(x,u,\nabla u)(\theta_c - \theta_b)u\, dx = 0, \end{aligned}$$

hence, in (6.3) the right hand side does not depend on b. It results

$$\begin{aligned}&\int_{\pi_0} \frac{\hat\nu}{\tilde\lambda(|u|)} \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i}(\theta_b u)\,dx + \int_{\pi_0} c_0|u|^p\theta_b\,dx + \int_{\pi_0} f(x,u,\nabla u)u\theta_b\,dx \\ &= \int_{\pi_0} \frac{\hat\nu}{\tilde\lambda(|u|)} \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^p \theta_b\,dx + \int_{\pi_0} \frac{\hat\nu}{\tilde\lambda(|u|)}\Big|\frac{\partial u}{\partial x_m}\Big|^{p-2}\frac{\partial u}{\partial x_m} u\theta_b'\,dx \\ &\quad + \int_{\pi_0} c_0|u|^p\theta_b\,dx + \int_{\pi_0} f(x,u,\nabla u)u\theta_b\,dx.\end{aligned} \tag{6.4}$$

By (H13) and (6.2), Hölder's inequality and the definition of function θ_b it follows that

$$\begin{aligned}&\Big|\int_{\pi_0} \frac{\hat\nu}{\tilde\lambda(|u|)}\Big|\frac{\partial u}{\partial x_m}\Big|^{p-2}\frac{\partial u}{\partial x_m} u\theta_b'\,dx\Big| \\ &\le \beta(\sup_{\Omega'}\hat\nu)^{1/p}\Big(\int_{\pi_{b+\frac12,b+1}} \hat\nu \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^p\,dx\Big)^{(p-1)/p}\Big(\int_{\pi_{b+\frac12,b+1}} |u|^p\,dx\Big)^{1/p}.\end{aligned} \tag{6.5}$$

Next, from the weighted Friedrichs inequality (see, [17, Corollary 3.3]), we have

$$\int_{\Omega'} |u|^p\,dx' \le \alpha_1 \int_{\Omega'} \hat\nu(x') \sum_{i=1}^{m-1}\Big|\frac{\partial u}{\partial x_i}\Big|^p\,dx', \tag{6.6}$$

where the positive constant α_1 depends only on m, p, Ω' and $\|1/\hat\nu\|_{L^t(\Omega')}$.

From (6.5) and (6.6) we obtain

$$\begin{aligned}\Big|\int_{\pi_0} \frac{\hat\nu}{\tilde\lambda(|u|)}\Big|\frac{\partial u}{\partial x_m}\Big|^{p-2}\frac{\partial u}{\partial x_m} u\theta_b'\,dx\Big| &\le \int_{\pi_0}\hat\nu\Big|\frac{\partial u}{\partial x_m}\Big|^{p-1}|u|\,|\theta_b'|\,dx \\ &\le \alpha_2(\sup_{\Omega'}\hat\nu)^{1/p}\int_{\pi_{b+\frac12,b+1}} \hat\nu\sum_{i=1}^m\Big|\frac{\partial u}{\partial x_i}\Big|^p\,dx,\end{aligned} \tag{6.7}$$

where the positive constant α_2 depends only on m, p, β, Ω' and $\|1/\hat\nu\|_{L^t(\Omega')}$. Hence

$$\lim_{b\to+\infty}\int_{\pi_0} \frac{\hat\nu}{\tilde\lambda(|u|)}\Big|\frac{\partial u}{\partial x_m}\Big|^{p-2}\frac{\partial u}{\partial x_m} u\theta_b'\,dx = 0. \tag{6.8}$$

From (6.4), letting $b\to+\infty$, taking into account that the left hand side does not depend on b, by Lebesgue theorem and (6.8) we obtain (6.3).

Next, by (6.2), (6.3), $c_0 > c_1$, an easy computation gives

$$\int_{\pi_0} \frac{\hat\nu}{\tilde\lambda(|u|)} \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^p\,dx \le \int_{\pi_0} \frac{\hat\nu}{\tilde\lambda(|u|)} \sum_{i=1}^m \Big|\frac{\partial u}{\partial x_i}\Big|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}(\theta_b u)\,dx, \tag{6.9}$$

for every $b > 0$.

From (6.9) and (6.7) we obtain

$$\begin{aligned}\int_{\pi_{b+\frac12}} \hat\nu\sum_{i=1}^m\Big|\frac{\partial u}{\partial x_i}\Big|^p\,dx &\le \lambda(\|u\|_{L^\infty(\pi_0)})[\alpha_2(\sup_{\Omega'}\hat\nu)^{1/p}+1]\int_{\pi_{b+\frac12,b+1}} \hat\nu\sum_{i=1}^m\Big|\frac{\partial u}{\partial x_i}\Big|^p\,dx \\ &= (\alpha_3+1)\int_{\pi_{b+\frac12,b+1}} \hat\nu\sum_{i=1}^m\Big|\frac{\partial u}{\partial x_i}\Big|^p\,dx,\end{aligned}$$

for every $b > 0$, where the positive constant α_3 depends on m, p, β, Ω', $\|\hat{\nu}\|_{L^\infty(\Omega')}$, $\|u\|_{L^\infty(\pi_0)}$, $\lambda(s)$ and $\|1/\hat{\nu}\|_{L^t(\Omega')}$ Consequently,

$$I_{b+1}(u) \leq \frac{\alpha_3}{\alpha_3 + 1} I_b(u), \quad \forall b > 0,$$

where, for every $a \geq 0$,

$$I_a(u) = \int_{\pi_a} \hat{\nu} \sum_{i=1}^m \Big| \frac{\partial u}{\partial x_i} \Big|^p \, dx, \quad A = I_0(u) < \infty.$$

This formula, by induction, gives

$$I_{b+n}(u) \leq s^n I_b(u) \leq A s^n,$$

for $n \in \mathbb{N}$, $b > 0$ and $s = \frac{\alpha_3}{\alpha_3+1}$. We can write last relation in this way

$$I_{b+n}(u) \leq A e^{n \log s}, \quad \text{for every } b > 0,\ n \in \mathbb{N} \cup \{0\}.$$

It is simple to verify that above inequality gives

$$I_\lambda(u) \leq \alpha_4 e^{-\lambda \tilde{\alpha}}, \quad \text{for all } \lambda > 0,$$

where $\alpha_4 = A e^{\tilde{\alpha}}$ and $\tilde{\alpha} = -\log s > 0$.

Now, fixing α such that $0 < \alpha < \tilde{\alpha}$, we have

$$\begin{aligned}
\int_{\pi_0} e^{\alpha x_m} \hat{\nu} \sum_{i=1}^m \Big| \frac{\partial u}{\partial x_i} \Big|^p \, dx &= \sum_{j=0}^{+\infty} \int_{\pi_{j,j+1}} e^{\alpha x_m} \hat{\nu} \sum_{i=1}^m \Big| \frac{\partial u}{\partial x_i} \Big|^p \, dx \\
&\leq \sum_{j=0}^{+\infty} e^{\alpha(j+1)} \int_{\pi_{j,j+1}} \hat{\nu} \sum_{i=1}^m \Big| \frac{\partial u}{\partial x_i} \Big|^p \, dx \\
&\leq \sum_{j=0}^{+\infty} e^{\alpha(j+1)} I_j(u) \\
&\leq \alpha_4 \sum_{j=0}^{+\infty} e^{\alpha(j+1)} e^{-j\tilde{\alpha}} < +\infty.
\end{aligned}$$

The proof is complete. $\square$

As in Section 4, we will show an example where all assumptions are fulfilled. Let $\Omega' = \{x' = (x_1, x_2, \dots, x_{m-1}) \in \mathbb{R}^{m-1} : |x'| < 1\}$. Put

$$\hat{\nu}(x') = [d(x', \partial\Omega')]^\rho = (1 - |x'|)^\rho$$

for $\rho : 0 < \rho < \min\big(\frac{p}{m}, (p-1)\big)$. Then the function $\hat{\nu}$ satisfies (H13) with t arbitrarily taken as follows:

$$\max\Big(\frac{m}{p}, \frac{1}{p-1}\Big) < t < \frac{1}{\rho}.$$

Let us define in $\pi_0 \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ the function $f(x, u, \eta)$ by

$$f(x, u, \eta) = u e^u (1 - |x'|)^\rho |\eta|^p - g_1(x),$$

where $g_1(x) \in L^\infty(\pi_0)$ has compact support. It is possible to verify (H14) by setting $\lambda(|u|) = e^{2|u|}$, and, taking into account that

$$\frac{1}{2}|u|^p + u f(x, u, \eta) \geq -2^{\frac{1}{p-1}} |g_1(x)|^{\frac{p}{p-1}} \tag{6.10}$$

for almost all $x \in \pi_0$ and for all $(u, \eta) \in \mathbb{R} \times \mathbb{R}^m$. Then, from [8, Theorem 2.2], there exists a function $u \in \mathring{W}^{1,p}(\hat{\nu}, \pi_0) \cap L^\infty(\pi_0)$ such that

$$\begin{aligned} &\int_{\pi_0} \Big\{ \frac{(1-|x'|)^\rho}{e^{2|u|}} \sum_{i=1}^m \Big| \frac{\partial u}{\partial x_i} \Big|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i} + |u|^{p-2} uw + u e^u (1-|x'|)^\rho |\nabla u|^p w \Big\} \, dx \\ &= \int_{\pi_0} g_1 w \, dx \end{aligned}$$

for every arbitrary function $w \in \mathring{W}^{1,p}(\hat{\nu}, \pi_0) \cap L^\infty(\pi_0)$. In this case $c_0 = 1$.

From (6.10) because of the support of g_1, there exists a positive number a such that

$$\frac{1}{2} |u|^p + u f(x, u, \eta) \geq 0$$

for almost all $x \in \pi_a$ and for all $(u, \eta) \in \mathbb{R} \times \mathbb{R}^m$. So, it is possible to apply Theorem 6.1 to the function u.

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