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SHARPER ESTIMATES FOR THE EIGENVALUES OF THE DIRICHLET FRACTIONAL LAPLACIAN ON PLANAR DOMAINS

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Abstract. In this article, we study the eigenvalues of the Dirichlet fractional Laplacian operator $(-\Delta)^{\alpha/2}$, $0 < \alpha < 1$, restricted to a bounded planar domain $\Omega \subset \mathbb{R}^2$. We establish new sharper lower bounds in the sense of the Weyl law for the of sums of eigenvalues, which advance the recent results obtained in several articles even in a more general setting.

1. Introduction

Fractional Laplacian operators are usually considered as the prototype of non-local operators [9]. From an application standpoint, non-local operators recently attracted a great deal of attention as they appear in many studies such as graphene models [14], dislocation of crystals [11], obstacle problems [27], non-local minimal surfaces [8], and nonlinear & nonlocal evolution equations with anomalous diffusion in continuum mechanics [6, 16].

In this article, we establish estimates for the eigenvalues $\{\lambda_j^{(\alpha)}\}_{j=1}^{\infty}$ of the fractional Laplacian operators $(-\Delta)^{\alpha/2}$, $0 < \alpha < 1$, restricted to a planar domain. To this end, we consider the eigenvalue problem defined by

$$\begin{gathered} (-\Delta)^{\alpha/2} u_j = \lambda_j^{(\alpha)} u_j \quad \text{in } \Omega, \\ u_j = 0 \quad \text{in } \mathbb{R}^2 \backslash \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded connected domain with smooth boundary in $\mathbb{R}^2$. Since Ω is bounded, the spectrum of the fractional Laplacian is discrete and the eigenvalues $\{\lambda_j^{(\alpha)}\}_{j=1}^{\infty}$ (including multiplicities) can be sorted in an increasing order.

There are several equivalent ways to define the fractional Laplacian operator. For suitable test functions, including all functions $u \in C_0^{\infty}(\mathbb{R}^2)$, it can be defined as

$$(-\Delta)^{\alpha/2} u(x) = \mathcal{A}_\alpha \lim_{\epsilon \to 0^+} \int_{\{|y|>\epsilon\}} \frac{u(x+y) - u(x)}{|y|^{2+\alpha}} dy, \tag{1.2}$$

where $\mathcal{A}_\alpha$ is a well-known positive normalizing constant. In the course of proving analogous estimates involving eigenvalues of the fractional Laplacian, some of the

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known methods fail mainly due to the fractional power and non-locality of such operators. However, we can get around to this impediment by considering the fractional Laplacian operator on $\Omega \subset \mathbb{R}^2$ as a pseudo-differential operator with symbol $|\mu|^\alpha$ as

$$\widehat{(-\Delta)^{\alpha/2}}|_\Omega u(\mu) = |\mu|^\alpha \hat{u}(\mu), \quad 0 < \alpha \leq 2, \quad u \in H_0^{\alpha/2}(\Omega). \tag{1.3}$$

Here, $H_0^{\alpha/2}(\Omega)$ denotes the Sobolev space of order $\alpha/2$ and the Fourier transform is defined as

$$\hat{u}(\mu) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-i\mu\cdot x} u(x)\, dx.$$

When $\Omega = \mathbb{R}^2$, one can look at [12, Proposition 3.3] for the proof of the equivalence between the definitions in (1.2) and (1.3).

The fractional Laplacian can also be considered as the infinitesimal generator of the semigroup of the symmetric α-stable process, and therefore the two share the same set of eigenvalues. That allows one to use probabilistic machinery to prove estimates involving the fractional Laplacian operator. Let X_t denote the symmetric $\alpha-$stable process with the characteristic function

$$e^{-t|\mu|^\alpha} = E\big(e^{i\mu\cdot X_t}\big) = \int_{\mathbb{R}^2} e^{i\mu\cdot y} p_t^{(\alpha)}(y) dy, \quad t > 0,\ \mu \in \mathbb{R}^2, \tag{1.4}$$

where $p_t^{(\alpha)}(r,s) = p_t^{(\alpha)}(r-s)$ is called the transition density of the symmetric $\alpha-$stable process (or the heat kernel of the fractional Laplacian). Even though we do not know any particular process corresponding to $\alpha \in (0,1)$, it is worth mentioning that $\alpha = 1$ is called the Cauchy process. Another $\alpha-$stable process of importance is the Holtsmark distribution ($\alpha = 3/2$) that is used to model gravitational fields of stars (See e.g., [33]). Moreover, $\alpha-$stable processes share many of the basic properties of the Brownian motion. One of the most important features of the symmetric $\alpha-$stable processes ($0 < \alpha < 2$) is that they do not have continuous paths, which is related to non-locality of the fractional Laplacian operator [4, 7].

Although the case $\alpha = 2$ (i.e., Dirichlet Laplacian operator) is excluded in this paper, it is worthwhile to pause here to review some of the pertinent results involving the eigenvalues of the Dirichlet Laplacian operator. There is an extensive literature devoted to the inequalities involving the eigenvalues of the Dirichlet Laplacian. One may consult the articles [1, 2, 3, 13, 15, 18, 19, 21, 22, 29] and references therein for a through literature review. Note that these results are remarkably similar to what's already known for the fractional Laplacian, even though methods differ greatly mostly due to the non-locality of the fractional Laplacian. The legendary Weyl asymptotics result proved in [32] is the first such result that we report. Around 1912, Weyl [32] considered the eigenvalue problem

$$\begin{gathered} -\Delta u_j = \lambda_j u_j \quad \text{in } \Omega, \\ u_j = 0 \quad \text{in } , \mathbb{R}^2 \backslash \Omega, \end{gathered} \tag{1.5}$$

and proved that the eigenvalues λ_k of the Dirichlet Laplacian over the bounded domain $\Omega \subset \mathbb{R}^2$ satisfy the following asymptotic result

$$\lambda_k \sim \frac{4\pi k}{|\Omega|} \quad \text{as } k \to \infty, \tag{1.6}$$

where $|\Omega|$ designates the area of Ω. Almost 50 years later, Pólya [25] proved that if Ω is a plane covering domain (i.e, their congruent non-overlapping translations cover $\mathbb{R}^2$ without gaps), then the eigenvalues λ_k of the Dirichlet Laplacian satisfy

$$\lambda_k \geq \frac{4\pi k}{|\Omega|} \tag{1.7}$$

for any integer $k \geq 1$. Pólya also conjectured that his result in (1.7) could be generalized to an arbitrary bounded domain in $\mathbb{R}^d$ for $d \geq 2$. This question still remains open. The closest result to Pólya's inequality for an arbitrary bounded domain in $\mathbb{R}^d$ is due to Li and Yau [23]. It gives a lower bound for the sum of the eigenvalues of the Dirichlet Laplacian operator, which is sharp in the sense of Weyl asymptotics. In [30], the authors generalized this result by proving the following counterpart of the Li-Yau inequality for the fractional Laplacian operator $(-\Delta)^{\alpha/2}$, $0 < \alpha \leq 2$:

$$\sum_{j=1}^{k} \lambda_j^{(\alpha)} \geq \frac{2}{\alpha+2}(4\pi)^{\alpha/2}|\Omega|^{-\frac{\alpha}{2}} k^{1+\frac{\alpha}{2}}. \tag{1.8}$$

To look at this inequality from a different perspective, we can take the Legendre transform of the following result by Laptev [20] and obtain (1.8),

$$\sum_{j} \big(z - \lambda_j^{(\alpha)}\big)_+ \leq \frac{\alpha|\Omega|}{4\pi(\alpha+2)} z^{1+\frac{2}{\alpha}}. \tag{1.9}$$

Setting $\alpha = 2$ in (1.9) gives an earlier result by Berezin [5]. As before, we recover the original Li-Yau inequality after an application of the Legendre transform to Berezin's result. Thus, in what follows, we call (1.8) as the Berezin-Li-Yau inequality.

For $0 < \alpha \leq 2$, the following refinement of the Berezin-Li-Yau type result was also obtained in [30],

$$\sum_{j=1}^{k} \lambda_j^{(\alpha)} \geq \frac{2}{\alpha+2}(4\pi)^{\alpha/2}|\Omega|^{-\frac{\alpha}{2}} k^{1+\frac{\alpha}{2}} + \frac{\alpha}{48(\alpha+2)} \frac{|\Omega|^{2-\frac{\alpha}{2}}}{(4\pi)^{1-\frac{\alpha}{2}} I(\Omega)} k^{\alpha/2} \tag{1.10}$$

where $I(\Omega)$, the moment of inertia, is defined by

$$I(\Omega) = \min_{y \in \mathbb{R}^2} \int_{\Omega} |x-y|^2 \, dx.$$

By a translation of the origin and a rotation of axes if necessary, in the sequel, we assume that the origin is the center of mass of Ω and that

$$I(\Omega) = \int_{\Omega} |x|^2 \, dx. \tag{1.11}$$

Remark 1.1. (1) Equation (1.10) generalizes an earlier result obtained by Melas [24].

(2) Setting $\alpha = 2$ in (1.10) simplifies the right hand side, which allows one to take the Legendre transform and obtain a similar Berezin type bound with a shift.

The most recent result improving (1.10) appeared in [28] where they obtained that for $0 < \alpha \leq 2$,

$$\begin{aligned} \sum_{j=1}^{k} \lambda_j^{(\alpha)} \geq{}& \frac{2}{\alpha+2}(4\pi)^{\alpha/2}|\Omega|^{-\frac{\alpha}{2}} k^{1+\frac{\alpha}{2}} + \frac{\alpha}{48(\alpha+2)} \frac{|\Omega|^{2-\frac{\alpha}{2}}}{(4\pi)^{1-\frac{\alpha}{2}} I(\Omega)} k^{\alpha/2} \\ &+ \frac{\alpha^3}{12288(\alpha+2)^2} \frac{|\Omega|^{4-\frac{\alpha}{2}}}{(4\pi)^{2-\frac{\alpha}{2}} I(\Omega)^2} k^{\frac{\alpha-2}{2}} \end{aligned} \tag{1.12}$$

In [31] the authors sharpened (1.12) for $1 \leq \alpha \leq 2$ by showing that the eigenvalues $\{\lambda_j^{(\alpha)}\}_{j=1}^{\infty}$ of the fractional Laplacian operator in (1.1) defined on $\Omega \subset \mathbb{R}^2$ satisfy

$$\begin{aligned}\sum_{j=1}^{k} \lambda_j^{(\alpha)} \geq \frac{2}{\alpha+2}(4\pi)^{\alpha/2}|\Omega|^{-\frac{\alpha}{2}} k^{1+\frac{\alpha}{2}} &+ \frac{\alpha}{16(\alpha+2)} \frac{|\Omega|^{2-\frac{\alpha}{2}}}{(4\pi)^{1-\frac{\alpha}{2}} I(\Omega)} k^{\alpha/2} \\ &- \frac{\alpha}{640(\alpha+2)} \frac{|\Omega|^{4-\frac{\alpha}{2}}}{(4\pi)^{2-\frac{\alpha}{2}} I(\Omega)^2} k^{\frac{\alpha-2}{2}}.\end{aligned} \tag{1.13}$$

In view of [28, 30, 31], our aim in this article is to advance the results with a focus on obtaining a sharper lower bound for the Berezin-Li-Yau inequality in the case of a fractional Laplacian with $\alpha \in (0,1)$ defined on a planar domain. Precisely, we shall establish the following main result.

Theorem 1.2. *For $0 < \alpha < 1$, $k \geq 1$ the eigenvalues $\{\lambda_j^{(\alpha)}\}_{j=1}^{\infty}$ of the fractional Laplacian operator* (1.1) *defined on $\Omega \subset \mathbb{R}^2$ satisfy*

$$\begin{aligned}\sum_{j=1}^{k} \lambda_j^{(\alpha)} \geq \frac{2}{\alpha+2}(4\pi)^{\alpha/2}|\Omega|^{-\frac{\alpha}{2}} k^{1+\frac{\alpha}{2}} &+ \frac{\alpha}{48(\alpha+2)} \frac{|\Omega|^{2-\frac{\alpha}{2}}}{(4\pi)^{1-\frac{\alpha}{2}} I(\Omega)} k^{\alpha/2} \\ &+ \frac{\alpha^3}{(\alpha+1)(\alpha+2)^3 2^{2\alpha+1}} \frac{|\Omega|^{1+\frac{3\alpha}{2}}}{(4\pi)^{\frac{\alpha+1}{2}} I(\Omega)^{\alpha+\frac{1}{2}}} k^{\frac{1-\alpha}{2}}.\end{aligned} \tag{1.14}$$

It is worth noting that $1-\alpha > 0 > \alpha - 2$ for $0 < \alpha < 1$, thereby improving the earlier result (1.12).

Recently, Chen and Song [10] obtained that eigenvalues of the fractional Laplacian satisfy

$$\lambda_j^{(\alpha p)} \leq \left(\lambda_j^{(\alpha)}\right)^p \tag{1.15}$$

for each j and any constant $p \in (0,1]$. Since the eigenvalues are in the increasing order, $\lambda_j^{(\alpha p)} \leq \lambda_k^{(\alpha p)}$ for each $1 \leq j \leq k$. Thus, Theorem 1.2 along with an application of (1.15) leads to the following more general results.

Corollary 1.3. *For any $0 < \alpha, p < 1$, and each $k \geq 1$, the eigenvalues $\{\lambda_j^{(\alpha)}\}_{j=1}^{\infty}$ of the fractional Laplacian operator* (1.1) *defined on $\Omega \subset \mathbb{R}^2$ satisfy*

$$\begin{aligned}\sum_{j=1}^{k} \big(\lambda_j^{(\alpha)}\big)^p \geq \frac{2}{p\alpha+2}(4\pi)^{\frac{p\alpha}{2}}|\Omega|^{-\frac{p\alpha}{2}} k^{1+\frac{p\alpha}{2}} &+ \frac{p\alpha}{48(p\alpha+2)} \frac{|\Omega|^{2-\frac{p\alpha}{2}}}{(4\pi)^{1-\frac{p\alpha}{2}} I(\Omega)} k^{\frac{p\alpha}{2}} \\ &+ \frac{p^3\alpha^3}{(p\alpha+1)(p\alpha+2)^3 2^{2p\alpha+1}} \frac{|\Omega|^{1+\frac{3p\alpha}{2}}}{(4\pi)^{\frac{p\alpha+1}{2}} I(\Omega)^{p\alpha+\frac{1}{2}}} k^{\frac{1-p\alpha}{2}}.\end{aligned} \tag{1.16}$$

$$\begin{aligned}\big(\lambda_k^{(\alpha)}\big)^p \geq \frac{2}{p\alpha+2}(4\pi)^{\frac{p\alpha}{2}}|\Omega|^{-\frac{p\alpha}{2}} k^{\frac{p\alpha}{2}} &+ \frac{p\alpha}{48(p\alpha+2)} \frac{|\Omega|^{2-\frac{p\alpha}{2}}}{(4\pi)^{1-\frac{p\alpha}{2}} I(\Omega)} k^{\frac{p\alpha}{2}-1} \\ &+ \frac{p^3\alpha^3}{(p\alpha+1)(p\alpha+2)^3 2^{2p\alpha+1}} \frac{|\Omega|^{1+\frac{3p\alpha}{2}}}{(4\pi)^{\frac{p\alpha+1}{2}} I(\Omega)^{p\alpha+\frac{1}{2}}} k^{\frac{-1-p\alpha}{2}}.\end{aligned} \tag{1.17}$$

In view of the recent work [17, 26], it is worth noting that one can easily extend this for elliptic operators $\mathcal{E}_f$ defined by a kernel f on $\mathbb{R}^2$

$$\mathcal{E}_f u(x) = \lim_{\epsilon \to 0^+} \int_{\{|y|>\epsilon\}} (u(x+y) - u(x)) f(y)\, dy, \tag{1.18}$$

where f satisfies

$$f(y) \geq \sigma \frac{\mathcal{A}_\alpha}{|y|^{\alpha+2}}, \tag{1.19}$$

with the same normalizing constant $\mathcal{A}_\alpha$ in (1.2) and $\sigma > 0$. Notice that $f(y) = \mathcal{A}_\alpha |y|^{-\alpha-2}$ corresponds to the fractional Laplacian definition. This is particularly important for mixed stable processes. Now, let us consider the eigenvalue problem defined by

$$\begin{aligned} -\mathcal{E}_f u_j &= \lambda_j u_j \quad \text{in } \Omega, \\ u_j &= 0 \quad \text{in } \mathbb{R}^2 \backslash \Omega. \end{aligned} \tag{1.20}$$

It is shown in [26] that the spectrum of $\mathcal{E}_f$ is also discrete and the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ (including multiplicities) can be sorted in an increasing order. Also, the set of Fourier transforms $\{\hat{u}_j\}_{j=1}^{\infty}$ of $\{u_j\}_{j=1}^{\infty}$ forms an orthonormal set in $L^2(\mathbb{R}^2)$ since the set of eigenfunctions $\{u_j\}_{j=1}^{\infty}$ is an orthonormal set in $L^2(\Omega)$. Note that we use the same notation for eigenvalues and eigenfunctions to illuminate the striking similarities though they might be different for each $\mathcal{E}_f$.

Using an analogous approach, we can establish remarkable estimates for certain eigenvalue problems involving elliptic operators as follows:

Corollary 1.4. *For $0 < \alpha < 1$, $k \geq 1$ the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of problem* (1.20) *defined on $\Omega \subset \mathbb{R}^2$ satisfy*

$$\begin{aligned} \sum_{j=1}^{k} \lambda_j \geq \frac{2\sigma}{\alpha+2} (4\pi)^{\alpha/2} |\Omega|^{-\frac{\alpha}{2}} k^{1+\frac{\alpha}{2}} + \frac{\sigma\alpha}{48(\alpha+2)} \frac{|\Omega|^{2-\frac{\alpha}{2}}}{(4\pi)^{1-\frac{\alpha}{2}} I(\Omega)} k^{\alpha/2} \\ + \frac{\sigma\alpha^3}{(\alpha+1)(\alpha+2)^3 2^{2\alpha+1}} \frac{|\Omega|^{1+\frac{3\alpha}{2}}}{(4\pi)^{\frac{\alpha+1}{2}} I(\Omega)^{\alpha+\frac{1}{2}}} k^{\frac{1-\alpha}{2}}. \end{aligned} \tag{1.21}$$

and hence

$$\begin{aligned} \lambda_k \geq \frac{2\sigma}{\alpha+2} (4\pi)^{\alpha/2} |\Omega|^{-\frac{\alpha}{2}} k^{\frac{\alpha}{2}} + \frac{\sigma\alpha}{48(\alpha+2)} \frac{|\Omega|^{2-\frac{\alpha}{2}}}{(4\pi)^{1-\frac{\alpha}{2}} I(\Omega)} k^{\frac{\alpha}{2}-1} \\ + \frac{\sigma\alpha^3}{(\alpha+1)(\alpha+2)^3 2^{2\alpha+1}} \frac{|\Omega|^{1+\frac{3\alpha}{2}}}{(4\pi)^{\frac{\alpha+1}{2}} I(\Omega)^{\alpha+\frac{1}{2}}} k^{\frac{-1-\alpha}{2}}. \end{aligned} \tag{1.22}$$

Note that because of the additional third term on the right of (1.21) and (1.22), these estimates also improve the main result of [17], which is simply the multiple of the lower bound stated in (1.10) [30] by σ.

The outline of this article is as follows: In Section 2, we present relevant facts on the eigenvalues and eigenfunctions of the fractional Laplacian operator that are essential to prove our main results. In Section 3 we first report on some intermediate steps and provide the proof of our main results. Finally, we concisely discuss why our method works for certain elliptic operators in a more general setting.

2. Preliminaries

In this section, we review some of the relevant definitions and facts that play an important role in proving the estimates in (1.14). In spite of crucial differences, the machinery given can also be used to obtain analogous bounds for other operators, see for example [24, 30].

Throughout this article, we define the ball of radius r centered at x in $\mathbb{R}^2$ as $B_r(x) := \{y \in \mathbb{R}^2 : |x-y| \le r\}$ and the volume of the unit disk $B_1(x) \subset \mathbb{R}^2$ as $\omega_2 = \pi$.

We begin this section with a review of some well-known properties of the eigenfunctions of the fractional Laplacian operator. By using Plancherel's theorem, one can show that the set of eigenfunctions $\{u_j\}_{j=1}^{\infty}$ is an orthonormal set in $L^2(\Omega)$ because the set of Fourier transforms $\{\hat{u}_j\}_{j=1}^{\infty}$ of $\{u_j\}_{j=1}^{\infty}$ also forms an orthonormal set in $L^2(\mathbb{R}^2)$. To ease the notation in what follows we set

$$\mathcal{U}_k(\mu) := \sum_{j=1}^{k} |\hat{u}_j(\mu)|^2 = \frac{1}{4\pi^2} \sum_{j=1}^{k} \Big| \int_{\Omega} e^{-iz\cdot\mu} u_j(z)\, dz \Big|^2 \ge 0. \tag{2.1}$$

Notice that the integral is taken over Ω instead of $\mathbb{R}^2$ because the support of u_j is Ω. Interchanging the sum and integral and using $\|\hat{u}_j\|_2 = 1$, we obtain

$$\int_{\mathbb{R}^2} \mathcal{U}_k(\mu)\, d\mu = k. \tag{2.2}$$

The following upper bound for $\mathcal{U}_k$ is obtained by utilizing Bessel's inequality:

$$\mathcal{U}_k(\mu) \le \frac{1}{(2\pi)^2} \int_{\Omega} |e^{-iz\cdot\mu}|^2\, dz = \frac{|\Omega|}{4\pi^2}. \tag{2.3}$$

Furthermore, we observe that $\mathcal{U}_k$ defined by (2.1) also satisfies

$$\int_{\mathbb{R}^2} |\mu|^{\alpha} \mathcal{U}_k(\mu)\, d\mu = \sum_{j=1}^{k} \lambda_j^{(\alpha)} \tag{2.4}$$

because

$$\begin{aligned} \lambda_j^{(\alpha)} &= \langle u_j, \lambda_j^{(\alpha)} u_j \rangle \\ &= \Big\langle u_j, (-\Delta)^{\alpha/2}|_{\Omega} u_j \Big\rangle \\ &= \langle \hat{u}_j, \widehat{(-\Delta)^{\alpha/2}}|_{\Omega} u_j \rangle \\ &= \langle \hat{u}_j, |\mu|^{\alpha} \hat{u}_j \rangle \\ &= \int_{\mathbb{R}^2} |\mu|^{\alpha} |\hat{u}_j(\mu)|^2\, d\mu. \end{aligned} \tag{2.5}$$

Next, we find an estimate for $|\nabla \mathcal{U}_k|$. Notice that

$$\sum_{j=1}^{k} |\nabla \hat{u}_j(\mu)|^2 \le \frac{1}{4\pi^2} \int_{\Omega} \big| ize^{-iz\cdot\mu} \big|^2\, dz = \frac{I(\Omega)}{4\pi^2}. \tag{2.6}$$

For every μ, Hölder's inequality together with (2.3) and (2.6) yields

$$\begin{aligned} |\nabla \mathcal{U}_k(\mu)| &\le 2\Big(\sum_{j=1}^{k} |\hat{u}_j(\mu)|^2 \Big)^{1/2} \Big(\sum_{j=1}^{k} |\nabla \hat{u}_j(\mu)|^2 \Big)^{1/2} \\ &\le \Lambda := \frac{1}{2\pi^2} |\Omega|^{1/2} I(\Omega)^{1/2}. \end{aligned} \tag{2.7}$$

Now assume that $B_R(0)$ is the symmetric rearrangement of Ω so that $|\Omega| = \pi R^2$. Notice that

$$I(\Omega) \ge \int_{B_R(0)} |x|^2\, dx = \frac{1}{2}\pi R^4 = \frac{1}{2\pi} |\Omega|^2, \tag{2.8}$$

roughly leading to

$$\Lambda \geq \frac{|\Omega|^{3/2}}{2\sqrt{2}\pi^{5/2}}. \tag{2.9}$$

Let $\mathcal{U}_k^*(\mu)$ denote the decreasing radial rearrangement of $\mathcal{U}_k(\mu)$. Then, there exists a real valued absolutely continuous function $\varrho_k : [0,\infty) \to [0, |\Omega|/(4\pi^2)]$ such that

$$\mathcal{U}_k^*(\mu) = \varrho_k(|\mu|). \tag{2.10}$$

Also, we define the distribution function D_k by

$$D_k(t) := |\{\mu : \mathcal{U}_k(\mu) > t\}| = |\{\mu : \mathcal{U}_k^*(\mu) > t\}|.$$

Then, $D_k(\varrho_k(t)) = \pi t^2$. Indeed, since $\mathcal{U}_k^*(\mu)$ is decreasing, we obtain

$$D_k(\varrho_k(t)) = |\{\mu : \mathcal{U}_k^*(\mu) > \varrho_k(t)\}| = |\{\mu : |\mu| < t\}| = |B_t(0)| = \pi t^2.$$

Utilizing Federer's coarea formula in view of (2.3), we have

$$\begin{aligned} D_k(s) &= \int_s^\infty \int_{\{\mathcal{U}_k^{-1}(t)\}} \frac{1}{|\nabla \mathcal{U}_k|}\, dP\, dt \\ &= \int_s^{|\Omega|/(4\pi^2)} \int_{\{\mathcal{U}_k = t\}} \frac{1}{|\nabla \mathcal{U}_k|}\, dP\, dt, \end{aligned}$$

where P is the 1-dimensional Hausdorff measure. The isoperimetric inequality,

$$P(\partial\Omega) \geq 2\pi^{1/2}|\bar{\Omega}|^{1/2}, \quad \Omega \subset \mathbb{R}^2,$$

together with $\varrho_k'(t) \leq 0$, $t \geq 0$, yields the following inequalities

$$\begin{aligned} 2\pi t &= D_k'(\varrho_k(t))\varrho_k'(t) \\ &= -\varrho_k'(t) \int_{\{\mathcal{U}_k = \varrho_k(t)\}} \frac{1}{|\nabla \mathcal{U}_k|}\, dP \\ &\geq -\frac{1}{\Lambda}\varrho_k'(t) P(\{\mathcal{U}_k = \varrho_k(t)\}) \\ &\geq -\frac{1}{\Lambda}\varrho_k'(t) 2\pi^{1/2} D_k(\varrho_k(t))^{1/2} \\ &= -\frac{2\pi t}{\Lambda}\varrho_k'(t). \end{aligned}$$

In conclusion, all these lead to the estimate

$$0 \leq -\varrho_k'(t) \leq \Lambda. \tag{2.11}$$

The crux of the matter in this work is obtaining an elementary but new sharper inequality rather than using a Taylor series expansion in its previous counterparts. We give a short proof so that the exposition is clearly self-contained.

Lemma 2.1. *For $t \geq 0$, $s > 0$, and $0 < \alpha < 1$ we have the following inequality:*

$$t^{\alpha+2} \geq \frac{\alpha+2}{2} t^2 s^\alpha - \frac{\alpha}{2} s^{\alpha+2} + \frac{\alpha}{2} s^\alpha (t-s)^2 + \alpha t s^{1-\alpha}(t^\alpha - s^\alpha)^2 \tag{2.12}$$

Proof of Lemma 2.1. First, let us see that

$$h(z,\alpha) := 2z^{\alpha+2} - (\alpha+2)z^2 + \alpha - \alpha(z-1)^2 - 2\alpha z(z^\alpha - 1)^2 \geq 0. \tag{2.13}$$

h can be rewritten as

$$h(z,\alpha) = 2z^{1+\alpha}\left(2\alpha + z - (1+\alpha)z^{1-\alpha} - \alpha z^\alpha\right).$$

It is sufficient to show that

$$g(z) := 2\alpha + z - (1+\alpha)z^{1-\alpha} - \alpha z^{\alpha} \geq 0.$$

Observe that $g(1) = 0$,

$$g'(z) = 1 - (1-\alpha^2)z^{-\alpha} - \alpha^2 z^{\alpha-1}, \quad g'(1) = 0,$$
$$g''(z) = \alpha(1-\alpha^2)z^{-\alpha-1} + (1-\alpha)\alpha^2 z^{\alpha-2} \geq 0$$

for $z \geq 0$ and $0 < \alpha < 1$. Thus, g is convex and hence using the convexity property

$$g(z) \geq g(1) + g'(1)(z-1),$$

we arrive at the conclusion that $g(z) \geq 0$. Finally, we set $z = t/s$ in (2.13) to complete the proof. □

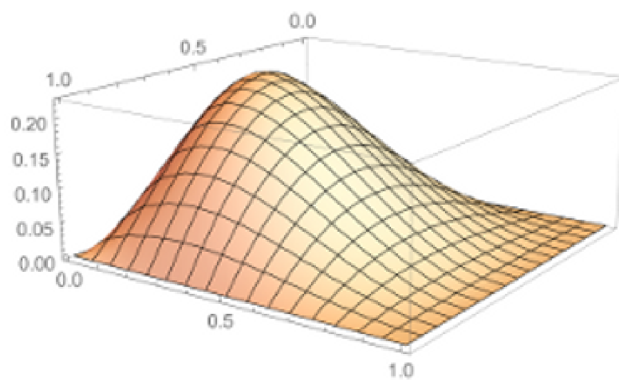

Figure 1. Graph of $h(z,\alpha)$ for $0 \leq z \leq 1$ and $0 < \alpha < 1$.

Remark 2.2. Note that if $1 < \alpha < 2$ then g becomes concave and the inequality in (2.12) is reversed.

3. Proof of the main result

Now, we are ready to prove Theorem 1.2 by using Lemma 2.1.

Proof of Theorem 1.2. Assume that (2.2)-(2.7) hold. Consider the decreasing, absolutely continuous function $\varrho_k : [0,\infty) \to [0,\infty)$ defined by (2.10). We know that $0 \leq -\varrho_k'(t) \leq \Lambda$ for $t \geq 0$ where $\Lambda > 0$ is given by (2.9). Since $\varrho_k(0) > 0$ due to (2.1) let us first define

$$\Theta_k(t) := \frac{1}{\varrho_k(0)}\varrho_k\Big(\frac{\varrho_k(0)}{\Lambda}t\Big). \tag{3.1}$$

Note that Θ_k is positive, $\Theta_k(0) = 1$ and $0 \leq -\Theta_k'(t) \leq 1$. To simplify the notation, we also set $\theta_k(t) := -\Theta_k'(t)$ for $t \geq 0$. Hence, $0 \leq \theta_k(t) \leq 1$ for $t \geq 0$ and

$$\int_0^\infty \theta_k(t)\,dt = \Theta_k(0) = 1.$$

Now, set

$$\xi_k = \int_0^\infty t\Theta_k(t)\,dt \quad \text{and} \quad \delta_k = \int_0^\infty t^{\alpha+1}\Theta_k(t)\,dt. \tag{3.2}$$

Using (2.2) we obtain

$$k = \int_{\mathbb{R}^2} \mathcal{U}_k(\mu)\, d\mu = \int_{\mathbb{R}^2} \mathcal{U}_k^*(\mu)\, d\mu = 2\pi \int_0^\infty t\varrho_k(t)\, dt. \tag{3.3}$$

Moreover, since the map $\mu \mapsto |\mu|^\alpha$ is radial and increasing, by (2.4), we obtain

$$\begin{aligned} \sum_{j=1}^{k} \lambda_j^{(\alpha)} &= \int_{\mathbb{R}^2} |\mu|^\alpha \mathcal{U}_k(\mu)\, d\mu \\ &\geq \int_{\mathbb{R}^2} |\mu|^\alpha \mathcal{U}_k^*(\mu)\, d\mu \\ &= 2\pi \int_0^\infty t^{\alpha+1} \varrho_k(t)\, dt. \end{aligned} \tag{3.4}$$

Substitution of (3.1) into (3.2) yields

$$\begin{gathered} \xi_k = \frac{\Lambda^2}{\varrho_k(0)^3} \int_0^\infty t\varrho_k(t)\, dt = \frac{\Lambda^2 k}{2\pi \varrho_k(0)^3}, \\ \delta_k = \frac{\Lambda^{\alpha+2}}{\varrho_k(0)^{\alpha+3}} \int_0^\infty t^{\alpha+1} \varrho_k(t)\, dt \leq \frac{\Lambda^{\alpha+2} \sum_{j=1}^k \lambda_j^{(\alpha)}}{2\pi \varrho_k(0)^{\alpha+3}}. \end{gathered} \tag{3.5}$$

Observe that Fubini's theorem with

$$\Theta_k(s) = \int_s^\infty \theta_k(t)\, dt$$

leads to

$$\begin{aligned} \frac{1}{x+2} \int_0^\infty t^{x+2} \theta_k(t)\, dt &= \int_0^\infty \Big(\int_0^t s^{x+1}\, ds \Big) \theta_k(t)\, dt \\ &= \int_0^\infty s^{x+1} \Big(\int_s^\infty \theta_k(t)\, dt \Big) ds \\ &= \int_0^\infty s^{x+1} \Theta_k(s)\, ds, \end{aligned}$$

which together with $x = 0$ and $x = \alpha$ respectively yield

$$\int_0^\infty t^2 \theta_k(t)\, dt = 2\xi_k \quad \text{and} \quad \int_0^\infty t^{\alpha+2} \theta_k(t)\, dt = (\alpha+2)\delta_k. \tag{3.6}$$

Notice that

$$\big(t^2 - 1\big)\big(\theta_k(t) - \chi_{[0,1]}(t)\big) \geq 0, \quad t \in [0, \infty). \tag{3.7}$$

Integrating (3.7) from 0 to ∞ gives

$$\int_0^\infty t^2 \theta_k(t)\, dt \geq \frac{1}{3} = \psi(0),$$

where $\psi : [0, \infty) \to (0, \infty)$ is defined by

$$\psi(x) = \int_x^{x+1} t^2\, dt.$$

Since ψ is continuous and non-decreasing and $\psi(x) \to \infty$ as $x \to \infty$, the Intermediate Value Theorem provides us with the existence of $\epsilon \geq 0$ such that

$$\psi(\epsilon) = \int_\epsilon^{\epsilon+1} t^2\, dt = \int_0^\infty t^2 \theta_k(t)\, dt.$$

which, by (3.6), implies that

$$\int_{\epsilon}^{\epsilon+1} t^2\,dt = 2\xi_k. \tag{3.8}$$

Now consider the polynomial

$$V(t) = t^{\alpha+2} - \zeta_1 t^2 + \zeta_2 = t^2(t^\alpha - \zeta_1) + \zeta_2$$

where

$$\zeta_1 = \frac{(\epsilon+1)^{\alpha+2} - \epsilon^{\alpha+2}}{2\epsilon+1} > 0, \quad \zeta_2 = \frac{(\epsilon+1)^{\alpha+2} - \epsilon^{\alpha+2}}{2\epsilon+1}\epsilon^2 - \epsilon^{\alpha+2} \geq 0$$

are chosen so that $V(\epsilon) = 0$ and $V(\epsilon+1) = 0$ and V becomes negative on $(\epsilon, \epsilon+1)$ and positive on $[0,\infty)\backslash[\epsilon, \epsilon+1]$. Considering the intervals $(\epsilon, \epsilon+1)$ and $[0,\infty)\backslash[\epsilon, \epsilon+1]$ separately, it is not difficult to infer that

$$V(t)\left(\chi_{[\epsilon,\epsilon+1]}(t) - \theta_k(t)\right) \leq 0 \quad \text{on } [0,\infty). \tag{3.9}$$

Integration of (3.9) on $[0,\infty)$ leads to

$$\int_{\epsilon}^{\epsilon+1} t^{\alpha+2}\,dt \leq \int_0^\infty t^{\alpha+2}\theta_k(t)\,dt - \zeta_1\Big(\int_0^\infty t^2\theta_k(t)\,dt - \int_{\epsilon}^{\epsilon+1} t^2\,dt\Big),$$

simplifying to

$$\int_{\epsilon}^{\epsilon+1} t^{\alpha+2}\,dt \leq \int_0^\infty t^{\alpha+2}\theta_k(t)\,dt. \tag{3.10}$$

Using (3.6), we derive that

$$\int_{\epsilon}^{\epsilon+1} t^{\alpha+2}\,dt \leq (\alpha+2)\delta_k. \tag{3.11}$$

Also, Jensen's inequality leads to

$$2\xi_k = \int_{\epsilon}^{\epsilon+1} t^2\,dt \geq \Big(\int_{\epsilon}^{\epsilon+1} t\,dt\Big)^2 \geq \Big(\int_0^1 t\,dt\Big)^2 = \frac{1}{4}. \tag{3.12}$$

Notice that (2.12) gives the key inequality in the proof of this lemma. Indeed, integrating (2.12) in t from ϵ to $\epsilon+1$ we obtain

$$\begin{aligned}\int_{\epsilon}^{\epsilon+1} t^{\alpha+2}\,dt \geq \frac{\alpha+2}{2}s^\alpha \int_{\epsilon}^{\epsilon+1} t^2\,dt - \frac{\alpha}{2}s^{\alpha+2} + \frac{\gamma\alpha}{2}s^\alpha \int_{\epsilon}^{\epsilon+1} (t-s)^2\,dt \\ + \gamma\alpha s^{1-\alpha}\int_{\epsilon}^{\epsilon+1} t(t^\alpha - s^\alpha)^2\,dt,\end{aligned} \tag{3.13}$$

which holds for $0 < \gamma \leq 1$. Observe that

$$\int_{\epsilon}^{\epsilon+1} (t-s)^2\,dt \geq \min_{\epsilon\geq 0}\int_{\epsilon}^{\epsilon+1} (t-s)^2\,dt = \int_{s-\frac{1}{2}}^{s+\frac{1}{2}} (t-s)^2\,dt = \frac{1}{12}. \tag{3.14}$$

Moreover, in view of $s \geq 1/2$, we also observe that

$$\begin{aligned}\int_{\epsilon}^{\epsilon+1} t\left(t^\alpha - s^\alpha\right)^2\,dt &\geq s^{2\alpha}\min_{\epsilon\geq 0}\int_{\epsilon}^{\epsilon+1} t\,dt + \min_{\epsilon\geq 0}\int_{\epsilon}^{\epsilon+1} t^{\alpha+1}\left(t^\alpha - 2s^\alpha\right)\,dt \\ &\geq s^{2\alpha}\int_0^1 t\,dt + \int_0^1 \left(t^{2\alpha+1} - 2t^{\alpha+1}s^\alpha\right)\,dt,\end{aligned}$$

yielding

$$\int_{\epsilon}^{\epsilon+1} t(t^{\alpha}-s^{\alpha})^2\,dt \ge \frac{\alpha^2}{2(\alpha+1)(\alpha+2)^2}. \tag{3.15}$$

Since $2\xi_k \ge 1/4$ by (3.12), setting $s=(2\xi_k)^{1/2}\ge 1/2$ and using (3.8), (3.11), (3.14) and (3.15), we deduce that (3.13) simplifies to

$$\delta_k \ge \frac{1}{\alpha+2}(2\xi_k)^{1+\frac{\alpha}{2}} + \frac{\gamma\alpha}{24(\alpha+2)}(2\xi_k)^{\alpha/2} + \frac{\gamma\alpha^3}{2(\alpha+1)(\alpha+2)^3}(2\xi_k)^{\frac{1-\alpha}{2}} \tag{3.16}$$

for any $0<\gamma\le 1$. Equations in (3.5) turn (3.16) into

$$\begin{aligned}\sum_{j=1}^{k}\lambda_j^{(\alpha)} \ge{}& \frac{2}{\alpha+2}\pi^{-\frac{\alpha}{2}}\varrho_k(0)^{-\frac{\alpha}{2}}k^{1+\frac{\alpha}{2}} + \frac{\gamma\alpha}{12(\alpha+2)}\Lambda^{-2}\pi^{1-\frac{\alpha}{2}}\varrho_k(0)^{3-\frac{\alpha}{2}}k^{\alpha/2}\\ &+ \frac{\gamma\alpha^3}{(\alpha+1)(\alpha+2)^3}\Lambda^{-1-2\alpha}\pi^{\frac{\alpha+1}{2}}\varrho_k(0)^{\frac{5\alpha+3}{2}}k^{\frac{1-\alpha}{2}}.\end{aligned} \tag{3.17}$$

Inserting $\Lambda=\frac{1}{2\pi^2}|\Omega|^{1/2}I(\Omega)^{1/2}$ leads to

$$\begin{aligned}\sum_{j=1}^{k}\lambda_j^{(\alpha)} \ge{}& \frac{2}{\alpha+2}\pi^{-\frac{\alpha}{2}}\varrho_k(0)^{-\frac{\alpha}{2}}k^{1+\frac{\alpha}{2}} + \frac{\gamma\alpha}{3(\alpha+2)}\frac{\pi^{5-\frac{\alpha}{2}}\varrho_k(0)^{3-\frac{\alpha}{2}}}{|\Omega|I(\Omega)}k^{\alpha/2}\\ &+ \frac{\gamma\alpha^3 2^{2\alpha+1}}{(\alpha+1)(\alpha+2)^3}\frac{\pi^{\frac{9\alpha+5}{2}}\varrho_k(0)^{\frac{5\alpha+3}{2}}}{|\Omega|^{\frac{2\alpha+1}{2}}I(\Omega)^{\frac{2\alpha+1}{2}}}k^{\frac{1-\alpha}{2}}\end{aligned} \tag{3.18}$$

for any auxiliary parameter $\gamma\in(0,1]$. Next we shall minimize the right-hand side of (3.18) over $\varrho_k(0)$. To do this, let us first set $x=\varrho_k(0)>0$. By (2.3) we know that $0<x\le|\Omega|/(4\pi^2)$, then we define

$$\begin{gathered}\varphi_1(x)=\frac{\pi^{-\frac{\alpha}{2}}}{\alpha+2}k^{1+\frac{\alpha}{2}}x^{-\frac{\alpha}{2}} + \frac{\gamma\alpha}{3(\alpha+2)}\frac{\pi^{5-\frac{\alpha}{2}}k^{\alpha/2}}{|\Omega|I(\Omega)}x^{3-\frac{\alpha}{2}},\\ \varphi_2(x)=\frac{\pi^{-\frac{\alpha}{2}}}{\alpha+2}k^{1+\frac{\alpha}{2}}x^{-\frac{\alpha}{2}} + \frac{\gamma\alpha^3 2^{2\alpha+1}}{(\alpha+1)(\alpha+2)^3}\frac{\pi^{\frac{9\alpha+5}{2}}k^{\frac{1-\alpha}{2}}}{|\Omega|^{\frac{2\alpha+1}{2}}I(\Omega)^{\frac{2\alpha+1}{2}}}x^{\frac{5\alpha+3}{2}}.\end{gathered}$$

Next, we shall prove that $\varphi(x)=\varphi_1(x)+\varphi_2(x)$ is decreasing on $(0,|\Omega|/(4\pi^2)]$ even if $\gamma=1$. To this end, it is enough to show both $\varphi_1,\varphi_2:(0,|\Omega|/(4\pi^2)]\to(0,\infty)$ are decreasing. Differentiating φ_1 and φ_2, we observe that $\varphi_1(x)$ is decreasing when

$$0<x\le\Big(\frac{3k|\Omega|I(\Omega)}{\gamma(6-\alpha)\pi^5}\Big)^{1/3},$$

while $\varphi_2(x)$ is decreasing when

$$0<x\le\Big(\frac{(\alpha+1)(\alpha+2)^2\big(k|\Omega|I(\Omega)\big)^{\frac{2\alpha+1}{2}}}{\gamma\alpha^2(5\alpha+3)\pi^{\frac{10\alpha+5}{2}}2^{2\alpha+1}}\Big)^{\frac{2}{6\alpha+3}}.$$

Therefore, we obtain that φ is decreasing on $(0,|\Omega|/(4\pi^2)]$ when we have

$$\frac{|\Omega|}{4\pi^2}\le\min\Big\{\Big(\frac{3k|\Omega|I(\Omega)}{\gamma(6-\alpha)\pi^5}\Big)^{1/3},\Big(\frac{(\alpha+1)(\alpha+2)^2\big(k|\Omega|I(\Omega)\big)^{\frac{2\alpha+1}{2}}}{\gamma\alpha^2(5\alpha+3)\pi^{\frac{10\alpha+5}{2}}2^{2\alpha+1}}\Big)^{\frac{2}{6\alpha+3}}\Big\} \tag{3.19}$$

for any $k \geq 1$. In other words, in view of $2\pi I(\Omega) \geq |\Omega|^2$ we may take $\varrho_k(0) = |\Omega|/(4\pi^2)$ for

$$\gamma \leq \min_{0\leq\alpha\leq 1} \Big\{ \frac{96}{6-\alpha}, \frac{(\alpha+1)(\alpha+2)^2 2^{\frac{6\alpha+3}{2}}}{\alpha^2(5\alpha+3)} \Big\} = 16.$$

Note that this minimum is due to the first term as the the second term has the minimum value 46.8907 at $\alpha = 0.7584$. Since $\gamma \in (0,1]$, φ is decreasing on $(0, |\Omega|/(4\pi^2)]$ and so we conclude that $\varphi(x) \geq \varphi(|\Omega|/(4\pi^2))$ even if $\gamma = 1$. Setting $\varrho_k(0) = |\Omega|/(4\pi^2)$ in (3.18) with $\gamma = 1$ leads to (1.14). This completes the proof. □

Let us briefly explain how Corollary 1.4 falls out as a by-product of the above discussion.

Proof of Corollary 1.4. Defining $\mathcal{U}_k$ as in (2.1), we obtain (2.2) immediately. Let us re-write (2.4) as

$$\begin{aligned}\sum_{j=1}^k \lambda_j &= \sum_{j=1}^k \langle u_j, \lambda_j u_j\rangle \\ &= \sum_{j=1}^k \langle u_j, -\mathcal{E}_f u_j\rangle \\ &= \sum_{j=1}^k \int_{\mathbb{R}^2} S_\alpha(\mu)|\hat{u}_j(\mu)|^2\, d\mu \\ &\geq \sigma \int_{\mathbb{R}^2} |\mu|^\alpha \mathcal{U}_k(\mu)\, d\mu.\end{aligned} \tag{3.20}$$

where we used from [12, Proposition 3.3]) that

$$S_\alpha(\mu) = \int_{\mathbb{R}^2} (1-\cos(y\cdot\mu)) f(y)\, dy \geq \sigma \mathcal{A}_\alpha \int_{\mathbb{R}^2} \frac{1-\cos(y\cdot\mu)}{|y|^{\alpha+2}}\, dy = \sigma|\mu|^\alpha.$$

Having (3.20) in hand, we notice that (3.4) changes as follows

$$\sum_{j=1}^k \lambda_j \geq \sigma \int_{\mathbb{R}^2} |\mu|^\alpha \mathcal{U}_k(\mu) d\mu \geq \sigma \int_{\mathbb{R}^2} |\mu|^\alpha \mathcal{U}_k^*(\mu) d\mu = 2\pi\sigma \int_0^\infty t^{\alpha+1} \varrho_k(t)\, dt. \tag{3.21}$$

and proceeding exactly as before using (3.21) in place of (3.4), and taking into account that $\lambda_j \leq \lambda_k$ for each $j \leq k$, we readily obtain the estimates in Corollary 1.4. □

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