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REMARKS ON THE GRADIENT OF AN INFINITY-HARMONIC FUNCTION

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Abstract. In this work we (i) prove a maximum principle for the modulus of the gradient of infinity-harmonic functions, (ii) prove some local properties of the modulus, and (iii) prove that if the modulus is constant on the boundary of a planar disc then it is constant inside.

1. Introduction

In this work we discuss some local properties of the modulus of the gradient of the gradient of an infinity-harmonic function. Differentiability remains an open problem, except in the planar case [11]; however, a quantity, which would be the modulus should differentiability hold, does exist. Our effort in this note is to prove a maximum principle for the modulus and record some local properties of an infinity-harmonic function at points where the modulus is a maximum. In particular, we prove that if the modulus is constant on the boundary of a planar disc then it is constant inside.

We start with some notations. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, will denote a bounded domain, the origin o will be assumed to lie in Ω. Let $B_r(x)$, $x \in \mathbb{R}^n$, be the ball of radius r with center x. Let $\bar{A}$ denote the closure of a set A and $A^c = \mathbb{R}^n \setminus A$. An upper semicontinuous function u, defined in Ω, is infinity-subharmonic in Ω if it solves

$$\Delta_\infty u(x) = \sum_{i,j=1}^{n} D_i u(x) D_j u(x) D_{ij} u(x) \geq 0, \quad x \in \Omega, \tag{1.1}$$

in the viscosity sense. A lower semicontinuous function u is infinity-superharmonic in Ω if $\Delta_\infty u(x) \leq 0$, $x \in \Omega$, in the viscosity sense. Moreover, u is infinity-harmonic in Ω if it is both infinity-subharmonic and infinity-superharmonic in Ω. Our work exploits the cone comparison property satisfied by u, see [6]. Also see [1, 3, 4, 5, 8] in this connection. For $x \in \Omega$ and $B_r(x) \Subset \Omega$, for $0 \leq t \leq r$, we define $M_x(t) = \sup_{B_t(x)} u$, $m_x(t) = \inf_{B_t(x)} u$. For infinity-subharmonic functions, $M_x(t) = \sup_{\partial B_t(x)} u$, and for infinity-superharmonic functions, $m_x(t) = \inf_{\partial B_t(x)} u$.

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The existence of the following limits is well known [6, Lemma 2.7],

$$\begin{aligned} &\lim_{t\downarrow 0}\frac{M_x(t)-u(x)}{t}=\Lambda^+(x), \quad \text{when } u \text{ is infinity-subharmonic,}\\ &\lim_{t\downarrow 0}\frac{u(x)-m_x(t)}{t}=\Lambda^-(x), \quad \text{when } u \text{ is infinity-superharmonic.} \end{aligned} \tag{1.2}$$

Moreover, if u is infinity-harmonic then $\Lambda^+(x)=\Lambda^-(x)=\Lambda(x)$, and if also differentiable at x, then $\Lambda(x)=|Du(x)|$. See [1, 4, 6, 7]. We now state the two main results of this work.

Theorem 1.1 (Maximum Principle)**.** *Let $\Omega\subset\mathbb{R}^n$ and $\Omega_1\Subset\Omega$. Recall the statements in* (1.2). *(i) If u is infinity-subharmonic in Ω, then $\sup_{x\in\bar\Omega_1}\Lambda^+(x)=\sup_{x\in\partial\Omega_1}\Lambda^+(x)$, and (ii) if u is infinity-superharmonic in Ω, then $\sup_{x\in\bar\Omega_1}\Lambda^-(x)=\sup_{x\in\partial\Omega_1}\Lambda^-(x)$. In particular, if u is infinity-harmonic then $\sup_{x\in\bar\Omega_1}\Lambda(x)=\sup_{x\in\partial\Omega_1}\Lambda(x)$.*

The anonymous referee pointed out this general version of Theorem 1.1. An older version of this theorem was stated only for infinity-harmonic functions. A proof will be presented in Section 2. The main idea of the proof is to exploit the result about increasing slope estimate in [6, Lemma 3.3]. In [4], these have been referred to as Hopf derivatives in the case of infinity-harmonic functions. The properties of the latter will be used to prove the second main result of this work.

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^2$ and $B_r(x)\Subset\Omega$. Let u be infinity-harmonic in Ω. Suppose that for some $L>0$ and for every $y\in\partial B_r(x)$, $\Lambda(y)=|Du(y)|=L$, then*

- (i) *for any $w\in B_r(x)$, $|Du(w)|=L$, and*
- (ii) *given any point $z\in B_r(x)$ there is a straight segment T, with its end points on $\partial B_r(x)$ and containing z, such that u is linear on T. Also, if e_T is a unit vector parallel to T then for any ξ on T either $Du(\xi)=Le_T$, or $Du(\xi)=-Le_T$. In addition, if T_1 and T_2 are any two such segments then either T_1 coincides with T_2 or they are distinct.*

At this time it is unclear whether or not this holds in $\mathbb{R}^n$ with $n\geq 3$. Theorem 1.2 does not hold in general and the convexity of the domain seems to play a role in the proof this result. Consider the example $u(x,y)=x^{4/3}-y^{4/3}$ on $\mathbb{R}^2$, where a point is described as (x,y). Then $\Lambda(x,y)=|Du(x,y)|=4/3\sqrt{x^{2/3}+y^{2/3}}$, and consider the regions D_c of the type bounded by $x^{2/3}+y^{2/3}=c>0$. While $|Du(x,y)|$ is constant on ∂D_c, $|Du(o)|=0$ and $|Du(x,y)|<4c^2/3$, $(x,y)\in D_c$.

We have divided our work as follows. Section 2 presents a proof of Theorem 1.1. In Section 3, we study the behaviour of an infinity-harmonic function u near points of maximum of $\Lambda(x)$. In Section 4, we prove the rigidity result in Theorem 1.2.

2. Proof of main results

We first state results we will use in the proof of Theorem 1.1. Recall the statements in (1.2). Let $\Omega\subset\mathbb{R}^n$ and $B_r(x)\Subset\Omega$. Let (a) $p_t\in\partial B_t(x)$, $t\leq r$, denote a point of maximum of $u(p)$ on $B_t(x)$, when u is infinity-subharmonic in Ω, and (b) $q_t\in\partial B_t(x)$ denote a point of minimum of u on $B_t(x)$, when u is infinity-superharmonic in Ω. For part (i) of the theorem we will use the following.

$$\Lambda^+(x)\leq\frac{M_x(t)-u(x)}{t}\leq\inf_{p_t}\Lambda^+(p_t), \quad \text{and } \Lambda^+ \text{ is upper-semicontinuous.} \tag{2.1}$$

While for part (ii) we use

$$\Lambda^-(x) \le \frac{u(x) - m_x(t)}{t} \le \inf_{q_t} \Lambda^-(q_t), \quad \text{and } \Lambda^-(x) \text{ is upper-semicontinuous.} \tag{2.2}$$

See [6, Lemma 3.3]. We also point out that minor modifications of the arguments in [3, 4] will also yield (2.1) and (2.2). We now prove Theorem 1.1 by employing the above repeatedly.

Proof of Theorem 1.1. We first prove part (i). Let $L = \sup_{x\in\bar\Omega_1} \Lambda^+(x)$, we assume that $L > 0$. Since Λ^+ is upper semi-continuous, for some $y \in \bar\Omega_1$ we have $\Lambda(y) = L$. If $y \in \partial\Omega_1$ then we are done. Assume then that $y \in \Omega_1$. We will show that in this case there is a point $\bar y \in \partial\Omega_1$ with $\Lambda(\bar y) = L$. Set $y_1 = y$ and let $d_1 = \operatorname{dist}(y_1, \Omega_1^c)$. Clearly, $B_{d_1}(y_1) \subset \Omega_1$; by (2.1), $\Lambda^+(p) \ge (u(p) - u(y_1))/d_1 \ge \Lambda^+(y_1) = L$, for any $p \in \partial B_{d_1}(y_1)$ with $u(p) = M_{y_1}(d_1)$. Thus $\Lambda^+(p) = L$ and $u(p) = u(y_1) + Ld_1$. If $p \in \partial\Omega_1$ then we are done, otherwise set $y_2 = p$. As already noted $u(y_2) = u(y_1) + Ld_1$. Let $d_2 = \operatorname{dist}(y_2, \Omega_1^c)$, and any $p \in \partial B_{d_2}(y_2)$ be such that $u(p) = M_{d_2}(y_2)$. Again by (2.1), $\Lambda^+(p) \ge (u(p) - u(y_2))/d_2 \ge \Lambda^+(y_2) = L$. Thus $\Lambda^+(p) = L$ and $u(p) = u(y_1) + L(d_1 + d_2)$. If $p \in \partial\Omega_1$ then we are done. Suppose now that we have obtained sequences $\{y_i\}_{i=1}^k$, $\{d_i\}_{i=1}^k$ such that

(a) $d_i = \operatorname{dist}(y_i, \Omega_1^c)$, $y_i \in \partial B_{d_{i-1}}(y_{i-1})$ and $y_i \notin \partial\Omega_1$, $2 \le i < k$,
(b) $u(y_i) = M_{y_{i-1}}(d_{i-1}) = u(y_1) + L\sum_{j=1}^{i-1} d_j$, $2 \le i \le k$, and
(c) $\Lambda^+(y_i) = \Lambda^+(y_j) = L$, for all $i, j = 1, 2, \dots, k$.

Suppose that $y_k \notin \partial\Omega_1$. For any $p \in \partial B_{d_k}(y_k)$, with $u(p) = M_{d_k}(y_k)$, (2.1) implies $\Lambda^+(p) \ge (u(p) - u(y_k))/d_k \ge \Lambda^+(y_k) = L$. Thus $\Lambda^+(p) = L$, $u(p) = u(y_k) + Ld_k$. If $p \in \partial\Omega_1$ then we are done otherwise set $y_{k+1} = p$ and note that $u(y_{k+1}) = u(y_1) + L\sum_{i=1}^k d_i$ and $\Lambda^+(y_{k+1}) = L$. By the maximum principle, for every k, $u(y_1) + L\sum_{i=1}^k d_i = u(y_{k+1}) \le \sup_{\Omega_1} u < \infty$. Thus $\sum_{i=1}^\infty d_i < \infty$ and $d_i \to 0$. Moreover, for $i < j$, $|y_i - y_j| \le \sum_{l=i}^{j-1} |y_l - y_{l+1}| = \sum_{l=i}^{j-1} d_l$ is small if i is large. Thus for some $\bar y \in \partial\Omega_1$, $y_i \to \bar y$ and $\Lambda^+(\bar y) \ge \limsup_{k\to\infty} \Lambda^+(y_k) = L$. Part (ii) may now be proved analogously by using points of minima and (2.2). The conclusion follows. ☐

Remark 2.1. In the case of infinity-harmonic functions, we can show using Lemma 3.1 (see Section 2) that the points $y_1, y_2, \dots.$ all lie on a straight segment terminating at $\bar y$. We also mention in passing that if $\Lambda^s(r) = \sup_{x\in\partial B_r(o)} \Lambda(x)$ then the upper-semicontinuity of Λ and Theorem 1.1 implies that $\Lambda^s(r)$ is right continuous.

3. Comments on the function Λ

For the remainder of this work u will denote an infinity-harmonic function. Our effort in this section will be to describe the behaviour of u near points of maximum of Λ. We recall some previously defined notations for ease of presentation. Let $B_r(x) \Subset \Omega$; a point $p_t \in \partial B_t(x)$, $t \le r$, will denote a point of maximum of u on $B_t(x)$. The direction $(p_t - x)/t$ will be denoted by ω_t. The quantities $M_x(t)$ and $m_x(t)$ continue to denote the maximum and the minimum of u on $B_t(x)$. Note that $M_x(t)$ and $-m_x(t)$ are convex in t. We will drop the subscript x when $x = o$. Next we summarize the properties of the Hopf-derivaties which will be used repeatedly in the rest of this work, see [4, Theorems 1 and 2]. We work in $B_r(o)$.

(i) $\frac{M(t)-u(o)}{t}$ decreases to $\Lambda(o)$ as $t \downarrow 0$,

(ii) for $0 \le s \le \tau \le t \le r$, we have $\Lambda(o) \le \sup_{p_s \in \partial B_s(o)} \Lambda(p_s) \le \Lambda(p_\tau) \le \inf_{p_t \in \partial B_t(o)} \Lambda(p_t) \le \Lambda(p_r)$, and

$$\lim_{t \downarrow 0} \sup_{p_t \in \partial B_t(o)} \Lambda(p_t) = \Lambda(o), \quad t \le r, \tag{3.1}$$

(iii) $\Lambda(o) \le \frac{M(r)-u(t\omega_r)}{r-t} \le \Lambda(p_r)$, and $\frac{M(r)-u(t\omega_r)}{r-t}$ increases to $\Lambda(p_r)$ as $t \uparrow r$.

Moreover,

(i) u is differentiable at any $p_t \in \partial B_t(o)$ and $Du(p_t) = \Lambda(p_t)\omega_t$, $t \le r$,

(ii)
$$M'(t-) \le \inf_{p_t \in \partial B_t(o)} \Lambda(p_t) \le \sup_{p_t \in \partial B_t(o)} \Lambda(p_t) \le M'(t+), \tag{3.2}$$

(iii) There exists $p_t \in \partial B_t(o)$ such that $\Lambda(p_t) = M'(t+)$.

Analogous statements also hold for q_t and $m(t)$. Moreover, for any pair of sequences $r_k \downarrow 0, \omega_{r_k} \in S^{n-1}$, with $\omega_{r_k} \to \omega$ (by compactness such pairs do exist, also see [7]), we have

$$\begin{gathered} \lim_{r_k \downarrow 0} \frac{u(r_k \omega_{r_k}) - u(o)}{r_k} = \lim_{r_k \downarrow 0} \frac{u(r_k \omega) - u(o)}{r_k} = \Lambda(o), \\ \lim_{r_k \downarrow 0} \frac{u(\theta r_k \eta) - u(o)}{r_k} = \theta \Lambda(o) \langle \omega, \eta \rangle, \quad \forall\, \eta \in S^{n-1}, \end{gathered} \tag{3.3}$$

for any fixed $\theta > 0$. Note that $M(r_k) = u(r_k \omega_{r_k})$. The above statements also apply to points of minima. In particular, if $\nu_t = q_t/t$ where $u(q_t) = m(t)$, then $\nu_{r_k} \to -\omega$. If ω is the only limit point of ω_t as $t \downarrow 0$, then u is differentiable at o, see [7]. Also, if $\omega \in S^{n-1}$ is such that (3.3)(ii) holds for any sequence then ω is a gradient direction and u is differentiable at o. We now prove the following result.

Lemma 3.1. *Let $u \ne 0$ be infinity-harmonic in Ω and $B_r(o) \Subset \Omega$.*

(a) *If $\Lambda(o) = (M(r) - u(o))/r$, then u is differentiable at o and $Du(o) = \Lambda(o)\omega$, for some $\omega \in S^{n-1}$. Moreover, for $0 \le t \le r$, $M(t) = u(t\omega) = u(o) + t\Lambda(o)$, and for very $t > 0$ there is exactly one point $p_t \in \partial B_t(o)$ such that $u(p_t) = M(t)$.*

(b) *If $p_r \in \partial B_r(o)$ is such that $\Lambda(p_r) = \Lambda(o)$ then the same conclusion holds for u with $\omega = p_r/r$, and $M(t) = u(t\omega) = u(o) + t\Lambda(o)$, $0 \le t \le r$.*

Furthermore, if x is any point on the segment op_r then $Du(x) = \Lambda(o)\omega$.

Proof. We prove part (a). Recall that $M(t)$ is convex in t, thus by 5(i) and the first part of (3.1)(iii),

$$\Lambda(o) \le \frac{M(t) - u(o)}{t} \le \frac{M(r) - u(o)}{r} = \Lambda(o), \quad 0 \le t \le r. \tag{3.4}$$

Thus $M(t) = u(o) + t\Lambda(o)$, and $u(t\omega) \le u(o) + t\Lambda(o)$, for all $0 \le t \le r$. For $0 < t < r$, let $p_t \in \partial B_o(t)$ be any point of maximum of u, set $\omega_t = p_t/t$. Since $M'(t) = \Lambda(o)$, using (3.2)(ii) and (3.4), we have that $\Lambda(p_t) = \Lambda(o)$. For a fixed $t \le r$, an application of (3.1)(iii) to the ball $B_t(o)$ results in

$$\Lambda(o) \le \frac{M(t) - u(s\omega_t)}{t - s} \le \Lambda(p_t) = \Lambda(o), \quad \forall\, 0 \le s < t.$$

Thus $u(s\omega_t) = u(o) + s\Lambda(o)$, $0 \le s \le t$, and this holds for every $0 \le t < r$. Clearly, for a fixed $0 < t < r$, $(u(s\omega_t) - u(o))/s \to \Lambda(o)$ as $s \downarrow 0$. By the comments following (3.3), u is differentiable at o and the gradient direction is ω_t. This is true of any ω_t and any $0 < t < r$. Clearly, $\omega = \omega_t$, $0 < t \le r$, is unique. Moreover,

$M(t) = u(t\omega) = u(o) + t\Lambda(o)$, $0 \le t \le r$. By the second part of (3.1)(iii) and (3.2)(iii), $\Lambda(r\omega) = \Lambda(o) = M'(r-)$, for any $p_r \in B_r(o)$. It is also clear that $r\omega$ is a point of maximum of u on $\partial B_r(o)$.

Now suppose that $\omega_1 \in S^{n-1}$ is such that $u(r\omega_1) = M(r)$. Using the special nature of $M(t)$, we see from the second part of (3.1)(iii) that $\Lambda(r\omega_1) = \Lambda(o)$. Using (3.1)(i) and arguing as above we see that ω_1 is another gradient direction at o. Thus by (3.3), $\omega_1 = \omega$. Also note that by (3.2)(iii), $M'(r) = \Lambda(o)$. This also proves part (b). To show the last statement let $0 \le s \le r$ be such that $x = s\omega$. Then $B_\rho(x) \subset B_{\rho+s}(o)$, $u(x) = u(o) + s\Lambda(o)$ and $M_x(\rho) = \sup_{B_\rho(x)} u = M(\rho + s) = u(x) + \rho\Lambda(o)$, $\rho \le r - s$. The rest now follows from the comments following (3.3), see [7]. □

Remark 3.2. An analogous version of Lemma 3.1 holds for the case of minima.

In the rest of this work we will have occasion to use a version of Rolle's property. We refer the reader to the Appendix for a proof in the case $n \ge 3$.

Remark 3.3. Suppose that $B_r(o) \subset \Omega$; let $p_r \in B_r(o)$ be any point such that $u(p_r) = M(r)$. Set $\omega_r = p_r/r$; we claim that for $0 \le \rho < r$, $\Lambda(\rho\omega) \le \Lambda(p_r)$. To see this note that $B_{(r-\rho)}(\rho\omega) \subset B_r(o)$ and $M_{\rho\omega}(r-\rho) = M(r)$. Thus using the first part of (3.1)(iii) in $B_{(r-\rho)}(\rho\omega)$, we see that $\Lambda(\rho\omega) \le (M(r) - u(\rho\omega))/(r - \rho) \le \Lambda(p_r)$. Moreover, we claim that there is a sequence of points x_k on the line segment op_r such that $x_k \to p_r$ and $\Lambda(x_k) \uparrow \Lambda(p_r)$. To see this note that for $0 \le s \le t < r$, (3.1)(iii) implies

$$\sup(\Lambda(o), \Lambda(sw)) \le \frac{M(r) - u(s\omega)}{r - s} \le \frac{M(r) - u(t\omega)}{r - t} \le \Lambda(p_r). \tag{3.5}$$

An application of Rolle's property to $u(p_r) - u(s\omega)$ and $u(p_r) - u(t\omega)$ in (3.5) implies there are $\theta_s \in (s, r)$, $\theta_t \in (t, r)$ and ω_{θ_s}, $\omega_{\theta_t} \in S^{n-1}$ such that $\Lambda(o) \le \Lambda(\theta_s\omega)\langle\omega_{\theta_s}, \omega\rangle \le \Lambda(\theta_t\omega)\langle\omega_{\theta_t}, \omega\rangle \le \Lambda(p_r)$. Also from (3.5), we see that $\Lambda(o) \le \Lambda(s\omega) \le \Lambda(\theta_s\omega) \le \Lambda(p_r)$. We iterate the latter using (3.5) as follows. Starting with $s \ge 0$ and setting $s_1 = s$, $s_2 = \theta_{s_1}$ and $s_{k+1} = \theta_{s_k}$, $k = 2, \dots$, we employ

$$\Lambda(s_k\omega) \le \frac{M(r) - u(s_k\omega)}{r - s_k} = \Lambda(s_{k+1}\omega)\langle\omega, \omega_{s_{k+1}}\rangle \le \Lambda(s_{k+1}\omega) \le \Lambda(p_r), \tag{3.6}$$

to see that (i) $s_k \uparrow r$, (ii) $\Lambda(s_1\omega) \le \Lambda(s_1\omega) \le \Lambda(s_2\omega) \le \dots. \le \Lambda(s_k\omega) \le \dots \le \Lambda(p)$. To see (i) suppose that $s_k \uparrow s < r$, then (3.1)(i) and the second inequality in (3.6) would then imply that $\Lambda(s) \le [M(r) - u(s\omega)]/(r - s) \le \Lambda(s)$. Lemma 3.1 would then hold and for $s < t < r$, $\Lambda(t) = \Lambda(s) = \Lambda(p_r)$. We may now select $s_k \uparrow r$. Finally, employing the definition of s_k and the second part of (3.1)(iii) in (3.6), we obtain that $\Lambda(s_k\omega) \uparrow \Lambda(p)$ as $s_k \uparrow r$ and $\omega_{s_k} \to \omega$.

Next we discuss the nature of u near points of maximum of Λ. We recall (3.3) and the discussion just following it. Let $B_r(o) \Subset \Omega$; for $0 < t \le r$, again p_t will denote any point of maximum of u on $\partial B_t(o)$ and q_t any point of minimum. Once again set $\omega_t = p_t/t$ and $\nu_t = q_t/t$. We restate (3.3) for ease of presentation. If

$t_k \downarrow 0$ with $\omega_{t_k} \to \omega$ then $\nu_{t_k} \to \nu = -\omega$ and

$$\begin{aligned} \lim_{t_k\downarrow 0} \frac{u(t_k\omega)-u(o)}{t_k} &= \lim_{t_k\downarrow 0} \frac{u(t_k\omega_{t_k})-u(o)}{t_k} \\ &= -\lim_{t_k\downarrow 0} \frac{u(t_k\nu_{t_k})-u(o)}{t_k} \\ &= -\lim_{t_k\downarrow 0} \frac{u(t_k\nu)-u(o)}{t_k} = \Lambda(o). \end{aligned} \tag{3.7}$$

Also $M(t_k)$ and $m(t_k)$ occur near $t_k\omega$ and $-t_k\omega$ when t_k is small. We refer to ω, ν as limit directions.

Lemma 3.4. *Let u be infinity-harmonic in Ω and $B_r(o) \Subset \Omega$. Also set $\Lambda^s = \sup_{x\in \bar{B}_r(o)} \Lambda(x) > 0$, let $y \in \partial B_r(o)$ be such that $\Lambda(y) = \Lambda^s$. Let H_y denote the $n-1$ dimensional plane tangential to $\partial B_r(o)$ at y. Then only one of the following happens.*
Case(a): There is a straight segment xy with $x \in \partial B_r(o)$ such that u is a linear function on xy. More precisely, for every $0 \le t \le |x-y|$, either (i) $u(y+te) = u(y)+t\Lambda^s$, or (ii) $u(y+te) = u(y)-t\Lambda^s$, where $e = (x-y)/|x-y|$. Moreover, u is differentiable on the segment xy, and if $z \in xy$ then in (i) $Du(z) = \Lambda^s e$, and in (ii) $Du(z) = -\Lambda^s e$.
Case (b): For every $s > 0$, all the points of extrema of u on $\partial B_s(y)$ lie outside $\bar{B}_r(o)$. In particular all limit directions ω, ν (see comment following (3.7)) lie in H_y. Moreover, if ω is a limit direction, $s_k \downarrow 0$ the corresponding sequence, $\eta \in S^{n-1}$ and $y_k \in \partial B_r(o)$ is the point nearest to $y+s_k\eta$ then $\lim_{s_k\downarrow 0}(u(y_k)-u(y))/s_k = \Lambda^s\langle\omega,\eta\rangle$.

Proof. Assume that Case (b) is not true. There is a ball $B_\delta(y)$ and a point $p \in \partial B_\delta(y) \cap \bar{B}_r(o)$ such that $u(p) = M_y(\delta)$. Our assumption of a point of maximum of u on $\partial B_\delta(y)$, lying in $\bar{B}_r(o)$, is not restrictive and the arguments we use will apply equally to a minimum. By (3.1)(iii) or even (2.1), $\Lambda(p) \ge \Lambda(y)$ implying that $\Lambda(p) = \Lambda^s$. Set $\omega = (p-y)/\delta$; by Lemma 3.1, we see that (i) $u(y+t\omega) = u(y)+t\Lambda^s$, $0 \le t \le \delta$, (ii) u is differentiable everywhere on the segment yp with $Du(z) = \Lambda^s\omega$, for any z on yp, and (iii) p is the only point of maximum on $\partial B_\delta(y)$. If $p \in \partial B_r(o)$, then $x = p$ and the lemma holds. Assume that $p \in B_r(o)$; set $y_1 = y$, $y_2 = p$, $\omega_1 = \omega$ and $d_1 = \delta$. Note that ω_1 points into $B_r(o)$. We repeat the argument at y_2 as follows. Set $d_2 = r - |y_2|$ and $y_3 \in \partial B_{d_2}(y_2)$ be a point of maximum. By 5(iii), $\Lambda(y_3) \ge \Lambda(y_2) = \Lambda^s$ implying $\Lambda(y_3) = \Lambda^s$; set $\omega_2 = (y_3 - y_2)/d_2$. Again by Lemma 3.1, u is differentiable on y_2y_3 with $u(y_2+t\omega_2) = u(y_2)+t\Lambda^s = u(y_1)+(t+d_1)\Lambda^s$, $0 \le t \le d_2$. By the uniqueness of gradient direction at y_2, $\omega_2 = \omega_1 = \omega$ and y_1y_3 is a straight segment. If $y_3 \in \partial B_r(o)$ the process stops. Otherwise assume that we have a sequence of points $\{y_i\}_{i=1}^k \in B_r(o)$, with $\omega_i = (y_{i+1}-y_i)/d_i = \omega$, $i = 1, 2, \dots, k-1$; i.e., y_1y_k a straight segment parallel to ω, and $u(y_1+t\omega) = u(y_1)+t\Lambda^s$, $0 \le t \le \sum_{i=1}^k d_i$. Moreover, u is differentiable at every point z on y_1y_k and $Du(z) = \Lambda^s\omega$. Now let $d_{k+1} = r - |y_k|$ and $y_{k+1} \in \partial B_{d_{k+1}}(y_k)$ such that $u(y_{k+1}) = M_{y_k}(d_{k+1})$. Set $\omega_{k+1} = (y_{k+1}-y_k)/d_k$; then by Lemma 3.1, $\Lambda(y_{k+1}) = \Lambda(y_k) = \Lambda^s$, y_ky_{k+1} is a straight segment and u is differentiable on y_ky_{k+1}. Thus $\omega_{k+1} = \omega$, i.e., y_1y_{k+1} is a straight segment parallel to ω. Moreover, on y_1y_{k+1}, $u(y_1+t\omega) = u(y_1)+t\Lambda^s$, $0 \le t \le \sum_{i=1}^{k+1} d_i$, and $Du(z) = \Lambda^s\omega$, for any z on y_1y_{k+1}. Either $y_{k+1} \in \partial B_r(o)$ in which case the process stops or we continue. By the maximum principle, $u(y_1)+\Lambda^s\sum_{i=1}^k d_i \le M_o(r) < \infty$, for all $k \ge 1$. Thus

$d_i \to 0$, $y_k \to x$ where $x \in \partial B_r(o)$. Thus we obtain a straight segment xy where $x \in \partial B_r(o)$ and the conclusions of Case (a) hold with $e = \omega$.

Now assume that Case (b) holds. We suppose that for every $s > 0$, the points of extrema of u on $\partial B_s(o)$ lie outside $\bar{B}_r(o)$. Given any sequence $s_k \downarrow 0$ and $\omega_{s_k} \to \omega$ we also have $\nu_{s_k} \to -\omega$, see remarks preceding (3.7). Thus any limit direction ω lies in H_y. Let ω be a limit direction and $s_k \downarrow 0$ be such that $(u(y + s_k\omega) - u(y))/s_k \to \Lambda(y)$. For $k = 1, 2, \dots$, let $y_k \in \partial B_R(o) \cap \partial B_{s_k}(y)$ be the point nearest to $y + s_k\omega$. Thus $y_k = y + s_k\zeta_k$, where $\zeta_k \in S^{n-1}$. Since the sphere is C^2 at y, $|y_k - (y + s_k\omega)|/s_k \to 0$ and $\langle \omega, \zeta_k \rangle \to 1$. Thus we have that, near y,

$$\begin{aligned} \Big|\frac{u(y_k) - u(y)}{s_k} - \Lambda^s\Big| &\le \Big|\frac{u(y_k) - u(y + s_k\omega)}{s_k}\Big| + \Big|\frac{u(y + s_k\omega) - u(y)}{s_k} - \Lambda^s\Big| \\ &\le C|\zeta_k - \omega| + \Big|\frac{u(y + s_k\omega) - u(y)}{s_k} - \Lambda^s\Big|, \end{aligned}$$

where $C > 0$ is the local Lipschitz constant. Clearly, the conclusion holds when $\eta = \omega$ by letting $s_k \to 0$. The statement for general η may now be derived by using (3.3). $\square$

Remark 3.5. In Case (a) of Lemma 3.4, if z is any point in the interior of the segment xy and $B_s(z) \subset B_r(o)$, then u has exactly one point of maximum and one point of minimum on $\partial B_s(z)$. Both these lie on xy. One may show this by applying (3.1)(iii) or (2.2). Lemma 3.4 also holds if a limit direction ω or $-\omega$, at y, points into $\bar{B}_r(o)$. One can find a small $\delta > 0$ such that $M_y(\delta)$ occurs near $\delta\omega$ (analogous for a minimum) and hence lies in $\bar{B}_r(o)$. See discussion at the beginning of this section. Using Lemma 3.1, one may show that xy is parallel to ω.

Remark 3.6. By (3.2)(iii) there is at least one point $p_r \in \partial B_r(o)$, where $\Lambda(p_r) = M'(r+)$. Thus $\Lambda^s \ge M'(r+)$. The existence of a straight line segment on which u is linear need not imply that u is affine. Take $u(x) = |x|$, $x \ne 0$. Also see capacitary rings [3].

Remark 3.7. If $y \in \partial B_r(o)$ is a point of extrema of u and $\Lambda(y) = \Lambda^s$, then by (3.2)(i) $Du(y) = \pm\Lambda^s\omega$, where $\omega = y/r$. Clearly, case (a) of Lemma 3.4 applies and u is linear and differentiable on xy, where $x = -y$. For $0 \le t \le r$, either $u(x + t\omega) = u(x) + t\Lambda^s$ or $u(x + t\omega) = u(x) - t\Lambda^s$. Since $\Lambda(y) = \Lambda(o) = \Lambda^s$, by Lemma 3.1 and Remark 3.2, for $0 \le t \le r$, we have $M'(t) = -m'(t) = \Lambda^s$; we also have $|M(t) - m(t)| \le 2t\Lambda^s$. Assume that $u(y) = M(r)$; linearity implies that for any $0 \le t \le r$, $u(o) = M(t) - t\Lambda^s$, $m(t) = u(-t\omega) = M(t) - 2t\Lambda^s$, and in particular, $u(x) = m(r) = M(r) - 2r\Lambda^s$. Employing Lemma 3.1, we see that $t\omega$, $-t\omega$ are the only points of extrema on $\partial B_t(o)$, $t\omega$ being the maximum and $-t\omega$ being the minimum. Thus for every $0 < t \le r$, $m(t) < u(x) < M(t)$, $x \in \partial B_t(o) \setminus \{\pm t\omega\}$.

Next we show a property of u in the situation when Case (a) of Lemma 3.4 holds. For $z \in \mathbb{R}^n$ and $e \in S^{n-1}$, let $\gamma(z, e)$ be the interior of the cone that has vertex z, aperture $\pi/3$ and opens along e.

Lemma 3.8. *Let $y \in \partial B_r(o)$ be such that $\Lambda(y) = \Lambda^s$. Assume Case (a) of Lemma 3.4 holds, that is, there is a segment xy in $\bar{B}_r(o)$, with $x \in \partial B_r(o)$, such that u is linear and differentiable on xy. Assume that $u(y + te) = u(y) + t\Lambda^s$, $0 \le t \le d$, where $d = |x-y|$ and $e = (x-y)/d$. Let $y_t = y+te$, $0 \le t < d$, then (i) $u(z) \ge u(y_t)$, $z \in \gamma(y_t, e) \cap B_r(o) \cap B_{d-t}(y_t)$, and (ii) $u(z) \le u(y_t)$, $z \in \gamma(y_t, -e) \cap B_r(o) \cap B_t(y_t)$. The case when $u(y + te) = u(y) - t\Lambda^s$, $0 \le t \le d$, is analogous.*

Proof. Let $0 \le \varepsilon \le d-t$, set $y_{t+\varepsilon} = y_t + \varepsilon e$. Now select $z \in B_r(o)$ such that $|z-y_t| = \varepsilon$ and set $e_\varepsilon = (z-y_{t+\varepsilon})/|z-y_{t+\varepsilon}|$. Let θ be the angle between segments zy_t and xy_t. By the Rolle's property, for some point a on the straight segment zy_t and limit direction ω, we have $u(z) - u(y_{t+\varepsilon}) = u(z) - u(y_t) - \varepsilon\Lambda^s = 2\varepsilon\Lambda(a)\langle\omega, e_\varepsilon\rangle \sin(\theta/2)$. Thus $u(z)-u(y_t) = 2\varepsilon\left(\Lambda^s + \Lambda(a)\langle\omega, e_\varepsilon\rangle\sin(\theta/2)\right) \ge \varepsilon\Lambda^s\left(1-2\sin(\theta/2)\right)$. It follows that $u(z) \ge u(y_t)$, if $\theta \le \pi/3$. We now take $y_{t-\varepsilon} = y_t - \varepsilon e$, $z \in B_r(o)$ with $|z-y_t| = \varepsilon$ and $\bar{e}_\varepsilon = (z-y_{t-\varepsilon})/|z-y_{t-\varepsilon}|$. With θ as defined before, argue similarly to see that for some $\bar{a}$ on $zy_{t-\varepsilon}$ and a limit direction $\bar{\omega}$, $u(z)-u(y_{t-\varepsilon}) = u(z)-u(y_t)+\varepsilon\Lambda^s = 2\varepsilon\Lambda(\bar{a})\langle\bar{\omega}, \bar{e}_\varepsilon\rangle \sin[(\pi-\theta)/2]$. Thus $u(z)-u(y_t) \le \varepsilon\Lambda^s(-1+2\sin[(\pi-\theta)/2])$. If $\theta \ge 2\pi/3$ then $u(z) \le u(y_t)$. $\square$

Remark 3.9. Let $B_r(o)$, x, y and e and be as in Lemma 3.8. Set $2l = |x-y|$ and consider the triangle $\triangle oyx$. The angles $\angle oyx = \angle oxy \le \pi/3$ if and only if $l \ge r/2$. Let $l \ge r/2$ and $y_t = y+te$ be such that $\angle oy_tx = \pi/3$ then $t = l - \sqrt{(r^2-l^2)/3}$. Since o lies in the cone $\gamma(y_t, e)$, Lemma 3.8 implies

$$u(y) + \Lambda^s[l - \sqrt{(r^2-l^2)/3}] \le u(o) \le u(x) - \Lambda^s[l - \sqrt{(r^2-l^2)/3}].$$

Also $u(o) - r\Lambda^s \le u(y) \le u(x) \le u(o) + r\Lambda^s$. If $l \uparrow r$, we have $u(y) \to u(o) - r\Lambda^s (= m(r))$ and $u(x) \to u(o) + r\Lambda^s (= M(r))$. See Remark 3.7.

4. Proof of Theorem 1.2

Let $D \subset \mathbb{R}^2$ be the unit disc centered at o. We will often describe a point $z \in \mathbb{R}^2$ as $z = (x, y)$. Also set e_1 and e_2 to be the unit vectors along the positive x-axis and the positive y-axis. Let u be infinity-harmonic in a domain $\Omega \subset \mathbb{R}^2$ and $D \Subset \Omega$. Recall that u is C^1 [11], and the use of this fact simplifies our presentation. However, a proof can be worked out without using this fact. Without any loss of generality, assume that $u(o) = 0$. Let $M = \sup_D u$ and $m = \inf_D u$. Also let $p,\ q \in \partial D$ be such that $u(p) = M$ and $u(q) = m$. By Theorem 1.1, $L = \sup_{x \in \bar{D}} |Du(x)|$. By Remark 3.7, p and q are antipodal points and we may take both of them on the y-axis with $p = (0,1)$ and $q = (0,-1)$. Also $u(0,t) = m + (t+1)L = M - (1-t)L$, $-1 \le t \le 1$. Moreover, for $-1 \le t \le 1$, and $Du(0,t) = Le_2$. Let $H+ = \{z \in \mathbb{R}^2 :\ x(z) \ge 0\}$ denote the right half disc and $H- = \{z \in \mathbb{R}^2 :\ x(z) \le 0\}$ the left half-disc. Let the right semi-circle be denoted by $I+ = \partial D \cap H+$ and the left semi-circle by $I- = \partial D \cap H-$. We will work in $H-$ and the analysis is analogous in $H+$. Let $a,\ b \in I-$ with $a \ne b$. We will denote the circular arc on ∂D, with end points a and b, by $\widehat{ab}$, and use $\bar{ab}$ for the straight segment with end points a and b. Also $l(a,b)$ will denote the arc length of $\widehat{ab}$.

Step 1. Let $a,\ b \in I-$ with $a \ne b$. Then

(i) there is a point point $c \in D$, on the straight segment ab such that $u(a) - u(b) = \langle Du(c), a-b\rangle$, and (ii) there is a point $d \in \partial D$, on $\widehat{ab}$, and a vector $e_d \in S^1$, with e_d tangential to ∂D (perpendicular to the segment od) at d, such that $u(a) - u(b) = \langle Du(d), e_d\rangle l(a,b)$. (4.1)

In (4.1)(ii), if $u(a) = u(b)$ then $\langle Du(d), e_d\rangle = 0$, implying $Du(d) \perp e_d$ and parallel to od. Noting that $Du(d) = L$, by Case (a) of Lemma 3.4, we have a straight segment originating at d, along od and lying in D, on which u is linear. Since this segment terminates on ∂D, it passes through o, and differentiability of u at o implies that $\omega_d = Du(d)/L = e_2$. Thus either $a = b = p$ or $a = b = q$.

Also see Remark 3.7 and the remarks preceding Step 1. Clearly, $u(a) \neq u(b)$ if $a,\ b \in I-$ and $a \neq b$. Since $u(p) > u(q)$, we see that $u(z) = u(x,y)$, $z \in I-$, is increasing in y. Recalling (4.1)(i) and (ii), we see that for $a,\ b \in I-$, $a \neq b$, $u(a) - u(b) = \langle Du(d), e_d\rangle l(a,b) = \langle Du(c), a-b\rangle \neq 0$. Let ω_d denote the gradient direction of u at d. Noting that $|Du(d)| = L \geq |Du(c)|$ and $l(a,b) > |a-b|$, it follows that $\langle \omega_d, e_d\rangle \neq 0, \pm 1$. This implies that ω_d does not lie in the tangent space of ∂D at d nor is it parallel to segment od. Case(a) of Lemma 3.4 now applies and we have a straight segment originating from d and terminating at $\bar{d} \in \partial D$ such that u is linear on the segment $d\bar{d}$, and $|Du(z)| = L$, $z \in d\bar{d}$, and if $\zeta = (d-\bar{d})/|d-\bar{d}|$ then $Du(z) = \pm L\zeta$.

From here on T will denote a segment of the type $d\bar{d}$, as described in Step 1. Let $z_T = (x_T, y_T)$ and $\bar{z}_T = (\bar{x}_T, \bar{y}_T)$ denote the two end points that lie on the unit circle ∂D. We set z_T to be the higher end point and $\bar{z}_T$ will denote the lower end point, i.e., $y_T \geq \bar{y}_T$. Also set e_T to be the unit vector parallel to T and pointing towards z_T. By the comments in Step 1, $u(z_T) \geq u(\bar{z}_T)$, u is linear on T and $Du(x) = Le_T$ for any x on T. Also let $\lambda(T)$ denote the length of T. ¿From now on we will call such segments T, as described in Step 1, as segments of type S.

Step 2. By taking arbitrary points $a,\ b \in I-$, $a \neq b$ in (4.1)(ii), we see that the points d, on the arc $\widehat{ab}$ form a dense set in the unit circle ∂D. By Step 1, we obtain infinitely many such segments T of type S. By the uniqueness of gradient directions any two such segments intersect if and only if they are identical. By the discussion preceding Step 1, pq is one such segment. It also follows then that segments T of type S either lie completely in $H+$ or in $H-$. Suppose that T_1 and T_2 are two such non-overlapping segments in $H-$ then one lies to the "left" of the other. More precisely, if $y_{T_1} > y_{T_2}$, then

$$\bar{y}_{T_1} < \bar{y}_{T_2}, \quad \lambda(T_1) > \lambda(T_2), \quad \operatorname{dist}(o, T_1) < \operatorname{dist}(o, T_2). \tag{4.2}$$

An analogous property holds in $H+$.

Step 3. For $k = 1, 2, 3, \dots$ let T_k be a segment of type S in $H-$ such that $y_{T_k} \uparrow 1$. Since the end points z_{T_k} and $\bar{z}_{T_k}$ lie on the unit circle, $z_{T_k} \to p$ and $x_{T_k} \uparrow 0$. Moreover by Step 2 and (4.2), $\bar{y}_{T_k} \downarrow y_\infty \geq -1$ and $\bar{x}_{T_k} \to x_\infty$. Set $e_\infty = (-x_\infty, 1 - y_\infty)/\sqrt{x_\infty^2 + (1-y_\infty)^2}$, clearly, $e_{T_k} \to e_\infty$. Thus the segments T_k tend to the segment T_∞ with end points $z_{T_\infty} = (0,1)$ and $\bar{z}_{T_\infty} = (x_\infty, y_\infty)$. Also by Step 1, for every k and any $0 \leq t \leq \lambda(T_k)$, $u(z_{T_k} - te_{T_k}) = u(z_{T_k}) - tL$, and $Du(z_{T_k} - te_{T_k}) = Le_{T_k}$. Since u is C^1 we see that for any $0 \leq t \leq \lambda(T_\infty)$, $u(p - te_\infty) = M - tL$, $Du(p - te_\infty) = Le_\infty$, and T_∞ is of type S. By the comments preceding Step 1, $Du(p) = Le_2 = Le_\infty$, and $(x_0, y_0) = q$. Thus the segments T_k move right to the segment pq. As noted in Step 1, since the set of end points z_T and $\bar{z}_T$, of segments T of type S, are dense in ∂D, it is clear now that we can always find segments T arbitrarily close to the segment pq and lying in $H-$.

Step 4. Suppose now that there is an $a \in D$ such that $|Du(a)| < L$, then there is a disc $D_\varepsilon(a) \subset D$ such that $|Du(w)| < L$, $w \in D_\varepsilon(a)$. Since $D_\varepsilon(a)$ cannot intersect the segment pq, it lies either in $H+$ or in $H-$. Assume that $D_\varepsilon(a) \subset H-$. Let $\eta_a = a/|a|$ and w_ε be the point on $\partial D_\varepsilon(a)$ nearest to o, i.e., $w_\varepsilon = (|a| - \varepsilon)\eta_a$. By the comment made at the end of Step 3, there are segments T of type S that intersect the segment ow_ε. These lie completely in $H-$. Consider now the set of such segments T and set y_0 to be the infimum of y_T's(y-coordinates of the higher end points) of these segments. Let $z_0 = (x_0, y_0) \in I-$. Also by (4.2), the supremum

$\bar{y}_0$ of the $\bar{y}_T$'s(y-coordinates of the lower end points) of these particular segments exists. Clearly, $\bar{y}_0 \le y_0$; set $\bar{z}_0 = (\bar{x}_0, \bar{y}_0) \in I-$. By employing (4.2), one can easily find a sequence segments T_k of type S, that intersect ow_ε, such that T_k's tend to the segment $z_0\bar{z}_0$, i.e., $e_{T_k} \to e$, where $e = (z_0 - \bar{z}_0)/|z_0 - \bar{z}_0|$. Morever, since u is C^1, the straight segment $z_0\bar{z}_0$ is of type S, it intersects ow_ε and

$$u(z_0 - te) = u(z_0) - tL, \quad 0 \le t \le |z_0 - \bar{z}_0|, \quad Du(z_0 - te) = Le. \tag{4.3}$$

Now let T_k be segments of type S with $z_{T_k} \to z_0$ (this is possible by the density of z_T's). We choose these to lie to the left of $z\bar{z}$, i.e., $y_{T_k} \uparrow y_0$ (see above). By the definition of z_0 and our assumption about $D_\varepsilon(a)$, the segments T_k neither intersect ow_ε nor $D_\varepsilon(a)$. We now consider the lower end points $\bar{z}_{T_k}$ of these T_k's. Since $y_{T_k} \le y_0$, (4.2) implies that $\inf_k \bar{y}_{T_k} > \bar{y}_0$ and $\inf_k \operatorname{dist}(o, T_k) > \operatorname{dist}(o, z_0\bar{z}_0)$. Let $\bar{y}_1 = \inf_k \bar{y}_{T_k}$ and $\bar{z}_1 = (\bar{x}_1, \bar{y}_1) \in I-$. It follows easily that the segment $z_0\bar{z}_1$ is type S. Let $\bar{e} = (z_0 - \bar{z}_1)/|z_0 - \bar{z}_1|$, then $e \ne \bar{e}$ since $\bar{z}_0 \ne \bar{z}_1$. It now follows that on the segment $z_0\bar{z}_1$,

$$u(z_0 - t\bar{e}) = u(z_0) - tL, \quad 0 \le t \le |z_0 - \bar{z}_1|, \quad Du(z_0 - t\bar{e}) = L\bar{e}.$$

By (4.3), $Du(z_0) = Le = L\bar{e}$ and we have a contradiction. Thus the theorem holds and $|Du(w)| = L$, for all $w \in D$.

5. Appendix

We now prove a version of the Rolle's property in $\mathbb{R}^n$, $n \ge 3$.

Lemma 5.1. *Let u be infinity-harmonic in $\Omega \subset \mathbb{R}^n$, $n \ge 3$. Let x, $y \in \Omega$ and $\sigma(s)$, $0 \le s \le 1$ be a C^1 curve that lies completely in Ω with $\sigma(0) = x$ and $\sigma(1) = y$. Let $l(s)$ denote the arclength of the curve from $\sigma(0)$ to $\sigma(s)$. Then for some $0 < \tau < 1$, and vector $\omega_\tau \in S^{n-1}$, we have*

$$u(y) - u(x) = \Lambda(\sigma(\tau))l(1)\langle \omega_\tau, \sigma'(\tau)\rangle/|\sigma'(\tau)|.$$

Proof. The proof utilizes simple calculus ideas and (3.3)(i). Without any loss of generality, take $x = o$, $u(o) = 0$, and set $v(s) = u(\sigma(s)) - u(y)l(s)/l(1)$, $0 \le s \le 1$. Then $v(s)$ is continuous and $v(0) = v(1) = 0$. Suppose that v has a positive maximum at some $0 < \tau < 1$. Thus $u(\sigma(\tau)) - u(y)l(\tau)/l(1) \ge u(\sigma(s)) - u(y)l(s)/l(1)$, $0 \le s \le 1$, and

$$u(\sigma(s)) - u(\sigma(\tau)) \le u(y)(l(s) - l(\tau))/l(1), \quad 0 \le s \le 1. \tag{5.1}$$

Set $z = \sigma(\tau)$ and $e = \sigma'(\tau)/|\sigma'(\tau)|$. By (3.3)(i), there exists a limit direction $\omega_\tau \in S^{n-1}$ and $r_k \downarrow 0$ such that $\lim_{r_k \downarrow 0}(u(z + r_k\omega_\tau) - u(z))/r_k = \Lambda(z)$. Let $z_k = z - r_k e$, $\xi_k = z + r_k e$; denote by s_k, the largest value of $s \le \tau$ such that $\sigma(s) \in \partial B_{r_k}(z)$, and by $\bar{s}_k$, the smallest value of $s \ge \tau$ such that $\sigma(\bar{s}_k) \in \partial B_{r_k}(z)$. Since σ is C^1 and u is locally Lipschitz, the following hold for small r_k:

$$\begin{gathered}
|\sigma'(\tau)|(\tau - s_k), \quad |\sigma'(\tau)|(\bar{s}_k - \tau) \approx r_k, \\
|\sigma(s) - z| - |\sigma'(\tau)(s - \tau)| = o(|s - \tau|), \\
|\sigma(s_k) - z_k|, \quad |\sigma(\bar{s}_k) - \xi_k| = o(r_k), \\
|u(z_k) - u(\sigma(s_k))|, \quad |u(\xi_k) - u(\sigma(\bar{s}_k))| = o(r_k).
\end{gathered} \tag{5.2}$$

From (5.1),

$$\frac{u(\sigma(s_k)) - u(z)}{r_k} \le -\frac{u(y)[l(\tau) - l(s_k)]}{l(1)r_k}, \quad \frac{u(\sigma(\bar{s}_k)) - u(z)}{r_k} \le \frac{u(y)[l(\bar{s_k}) - l(\tau)]}{l(1)r_k}.$$

Using (5.2) and taking limits in the above stated inequalities, we obtain that

$$\begin{aligned} \lim_{r_k \downarrow 0} \frac{u(\sigma(s_k)) - u(z)}{r_k} &= \lim_{r_k \downarrow 0} \frac{u(z_k) - u(z)}{r_k} = -\Lambda(z)\langle \omega_\tau, e\rangle \\ &\le \lim_{r_k \downarrow 0} -\frac{u(y)(l(\tau) - l(s_k))}{l(1) r_k} = -\frac{u(y)}{l(1)}. \end{aligned} \tag{5.3}$$

Using $\bar{s}_k$ and ξ_k, and taking limits as in (5.3), we see that $\Lambda(z)\langle \omega_z, e\rangle \le u(y)/l(1)$. The conclusion of the lemma holds. The analyses when $v(s) = 0$, for all $s > 0$, or when $v(s)$ has a negative minimum are analogous. □

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