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The Schrödinger equation on non-stationary domains *

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Abstract

We investigate the dynamical effects of non-stationary boundaries on the stability of a quantum Hamiltonian system described by a periodic family $\{H(\gamma,t), t\in[0,\Gamma], \Gamma>0\}$ of Sturm-Liouville operators, a Schrödinger equation $i\partial_t\psi = H(\gamma,t)\psi$ defined on

$$\Omega(a) = \left\{(t,x)\in\mathbb{R}^2 : x\in(a(t),\infty), a\in\mathcal{C}^3(\mathbb{R}), a(t)=a(t+k\Gamma), k\in\mathbb{Z}\right\},$$

as well as boundary conditions at $x=a(t)$ modeled by the Γ-periodic function γ. Employing extended Hilbert space methods, stability conditions for the spectra of the evolution operators $\mathcal{U}(a,\gamma,\Gamma,0)$ to the families $\{H(\gamma,t)\}$ under perturbations induced by variations of boundary oscillations, respectively conditions, are derived.

In particular, it is shown that the existence of a pure point finitely degenerate realization $\mathcal{U}(a,\hat\gamma,\Gamma,0))$ implies pure point $\mathcal{U}(a,\gamma,\Gamma,0)$ for all $\gamma\in\mathcal{C}^1(\mathbb{R})$, $a\in\mathcal{C}^3(\mathbb{R})$, whereas in case of infinitely degenerate $\sigma_{pp}\big(\mathcal{U}(a,\hat\gamma,\Gamma,0)\big)$ the existence of $\sigma_{\mathrm{ac}}\big(\mathcal{U}(a,\gamma,\Gamma,0)\big)\neq\emptyset$, respectively $\sigma_{sc}\big(\mathcal{U}(a,\gamma,\Gamma,0)\big)\neq\emptyset$, is possible.

1 Dynamical Preliminaries

This article is concerned with the stability (as introduced in Definition 1.2 below) of certain quantum systems under specific, time-dependent perturbations. In particular, the following one degree-of-freedom models will be studied (for the exact definition, see Hypothesis 1.1): Let a family of Sturm-Liouville differential operators $\{\tilde H(t), t\in\mathbb{R}\}$ be defined on Hilbert spaces $\tilde{\mathcal{H}}(t)=\mathcal{L}^2([a(t),\infty),dx)$ and the boundary motion described by the non-negative function $a\in\mathcal{C}^3(\mathbb{R})$ with $a(t)=a(t+k\Gamma)$, $k\in\mathbb{Z}$ and some period $\Gamma>0$. Then there exists a (heuristic) Schrödinger equation of the type

$$i\partial_t\phi(t,y) = \tilde H(t)\phi(t,y) \tag{1.1}$$

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valid on $\tilde{\Omega}(a) = \{(t,y) \in \mathbb{R}^2 : y \in (a(t),\infty), t \in \mathbb{R}\}$. Together with the given initial condition $\phi(t=t_0)$ and boundary condition function $\gamma \in \mathcal{C}^1(\mathbb{R})$. Equation (1.1) represents the initial-boundary value problem $\tilde{\mathcal{S}}(a,\gamma)$. In the sequel, assume that the Hamiltonians $\tilde{H}$ are chosen such that $\tilde{H}(t) = \tilde{H}(t+k\Gamma)$ and the allowed boundary functions obey $\gamma(t) = \gamma(t+k\Gamma)$ for all $k \in \mathbb{Z}$. Furthermore, assume that for a given boundary motion $\hat{a}$ there exists a family of boundary conditions $\hat{\gamma}$ such that the system $\tilde{\mathcal{S}}(\hat{a},\hat{\gamma})$ is stable in the sense of Definition 1.2 below. Now the main question discussed in this article can be phrased as follows:

> Are there perturbations of $\tilde{\mathcal{S}}(\hat{a},\hat{\gamma})$ induced by a change in the boundary function $\hat{\gamma} \to \gamma$, while the boundary motion $\hat{a}$ is kept fixed, which break the confinement in $\tilde{\mathcal{S}}(\hat{a},\hat{\gamma})$, i.e. allow for instabilities in the perturbed system $\tilde{\mathcal{S}}(\hat{a},\gamma)$?

In physical terms, one might think of different forms of energy exchange (between moving boundary and quantum system) being represented by different functions γ, see [1] for instance.

To obtain a first impression of the influence of the boundary oscillations and to simplify matters, map (1.1) onto $\mathbb{R} \times \mathbb{R}^+ \setminus \{0\}$ with the aid of (point-wise) unitary transformations generated by $(t,y) \mapsto (t, x = y - a(t))$ and assume that the families $\{\tilde{H}(t), t \in \mathbb{R}\}$ behave under these shifts such that

$$i\partial_t \psi(t,x) = \{H_c + i\dot{a}(t)\partial_x + V(t)\}\psi(t,x),$$

respectively the equivalent form

$$i\partial_t \phi(t,x) = \{H_c + V(t) + x\ddot{a}(t)/2 - \dot{a}^2(t)/4\}\phi(t,x). \tag{1.2}$$

result on $\mathbb{R} \times \mathbb{R}^+ \setminus \{0\}$, where H_c is now time-constant. To ensure unitary time evolutions of initial states $\psi(t=t_0) \in \mathcal{L}^2(\mathbb{R}^+, dx)$, the bracketed expression in (1.2) has to be a self-adjoint operator on $\mathcal{L}^2(\mathbb{R}^+, dx)$ for each $t \in \mathbb{R}$. In order to address problems in quantum dynamics, [1] for instance, and to focus on the influence of the non-stationary boundary rather than the properties of the potentials, the following assumptions will be valid throughout this article.

Hypothesis 1.1 (A1) For each $t \in \mathbb{R}$, $h(a,t) := h_c + v(t, \cdot + a(t)) + x\ddot{a}(t)/2$ is a minimal Sturm-Liouville differential operator, i.e. $h_c\psi = -(p\ \psi')' + w\ \psi$, where $\psi' := d\psi/dx$ and $p \in \mathcal{C}^1(\mathbb{R}^+), p > 0$ everywhere, $w \in \mathcal{C}^1(\mathbb{R}^+), v(\circ, \cdot + a(\circ)) \in \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R}^+)$ and all three functions are real-valued. The potentials w and v are chosen such at $x = 0$ the limit circle case, respectively at $x = \infty$ the limit point case are present as well as $\mathcal{D}(h(a,t)) = \mathcal{D}(h(a,t=0)) = \{f \in \mathcal{L}^2(\mathbb{R}^+, dy) : \operatorname{supp} f \subset \mathbb{R}^+ \text{and } h(a,t=0)f \in \mathcal{L}^2(\mathbb{R}^+, dy)\}$. (See [2, 3], for instance.) The boundary oscillation function a obeys $a \in \mathcal{C}^3_\Gamma(\mathbb{R})$, where $\mathcal{C}^k_\Gamma(\mathbb{R}) := \{f \in \mathcal{C}^k(\mathbb{R}) : f(t) = f(t+\Gamma), k \in \mathbb{Z}, \dot{f} := df/dt \neq 0\ a.e.\}$.

(A2) For each $t \in \mathbb{R}$, any self-adjoint realization $H(a,\alpha,t)$ of $h(a,t)$ is purely discrete.

(A3) The theory of Sturm-Liouville operators implies that self-adjointness of $H(a,\alpha,t)$ is achieved by requiring

$$\big(\psi_x/\psi\big)(t)\big|_{x=0} = \alpha(t) \in \mathbb{R} \tag{1.3}$$

for each $t \in \mathbb{R}$ and $\psi \in \mathcal{D}\big(H(a,\alpha,t)\big)$. In the sequel, the boundary condition function α will be assumed to be time-periodic, i.e. $\alpha \in \mathcal{C}^1_\Gamma(\mathbb{R})$ piece-wise, so the Dirichlet condition $\alpha = \infty$ is allowed as well.

Choosing boundary motion, boundary function and external perturbation of the same period Γ naturally restricts the number of physical realizations. Yet, as time-periodic systems are intensively explored in the search for *quantum realizations of chaos*, the present investigation covers an important class of quantum mechanical systems, see for instance [4 - 9].

Now the notion of stability can be defined as follows.

Definition 1.2 The system represented by $H(a,\alpha,t=0)$ is said to be stable if

$$\sup_{t\geq 0} |\big\langle \mathcal{U}(a,\alpha,t,0)\Psi(0), H(a,\alpha,t=0)\,\mathcal{U}(a,\alpha,t,0)\Psi(0)\big\rangle_{\mathcal{H}}| < \infty$$

for a total set of initial conditions $\Psi(0)$.

Here $\mathcal{U}(a,\alpha,t,s)$ is the propagator (see Definition 2.4) corresponding to the family $\big\{H(a,\alpha,t)\big\}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is the canonical scalar product on $\mathcal{L}^2\left(\mathbb{R}^+,dx\right)$. Following [10], the unitary family $\big\{\mathcal{U}(a,\alpha,t,0), t \in \mathbb{R}\big\}$ is also said to have time-bounded energy $H(a,\alpha,t=0)$, see [4, 5] as well. A remarkable feature of time-periodic families such as $\big\{H(a,\alpha,t)\big\}$, which is proven in [10], is the fact that stability leads to a pure point *Floquet operator* $\mathcal{U}(a,\alpha,\Gamma,0)$. The converse, however, is not true since a pure point spectrum of the Floquet operator does not necessarily imply that any solution of (1.2), which remains in the domain of $H(a,\alpha,t=0)$ for all $t>0$, has time-bounded energy, see [4, 5, 10].

The main findings of this article (as expressed in Theorem 3.2, Proposition 3.4 and Theorem 3.6) demonstrate the *generic* stability of the systems {(1.2), (A1)-(A3)} in the following sense:

> Suppose there exists a realization of (1.2) such that the corresponding Floquet operator $\mathcal{U}_0(a,\Gamma,0)$ is pure point with finitely degenerate eigenvalues, then the generic Floquet operator $\mathcal{U}(a,\alpha,\Gamma,0)$, where α is of the type (A3), is pure point as well.

To be more precise, finite degeneracy of $\sigma_{pp}\big(\mathcal{U}_0(a,\Gamma,0)\big)$ excludes absolutely continuous spectrum for all $\mathcal{U}(a,\alpha,\Gamma,0)$ under assumption (A3). Singular continuous spectrum, however, might be present as a limiting case $\mathcal{U}(a,\alpha,\Gamma,0)$ of a (pure) absolutely continuous sequence $\big\{\mathcal{U}(a_j,\alpha_j,\Gamma,0), j \in \mathbb{N}\big\}$ if $(a_j,\alpha_j,) \to (a,\alpha)$ in $\mathcal{C}^3_\Gamma(\mathbb{R}) \times \mathcal{C}^1_\Gamma(\mathbb{R})$. For that reason merely pure point Floquet operators are called generic. (Remark that detailed conditions are part of Theorem 3.6). Section 3 contains as well exact requirements for the existence of

$\sigma_{\rm ac}\big(\mathcal{U}(a_j,\alpha_j,\Gamma,0)\big) \neq \emptyset$. As mentioned, a necessary condition for the latter is the infinite degeneracy of (a subset of) $\sigma_{pp}\big(\mathcal{U}_0(a_j,\Gamma,0)\big)$, i.e. the presence of a *resonance situation* in the reference model.

An implication of these results concerns the relationship between quantum system and classical analogon. As discussed in [1], see also [11], there is at least one rigorously studied classical model, corresponding to $H_c = -\frac{d^2}{dx^2}+x, V(t) \equiv 0$ in (1.2), which exhibits unbounded phase space trajectories ($\equiv$ unlimited energy gain) for a set of initial conditions with non-zero Lebesgue measure. Since this unlimited acceleration basically stems from the *nearly random* time-distribution of the collisions between moving boundary and classical system, one might think that regular boundary conditions of the type (A3) in the quantum model are not entirely adequate to capture the variety of the classical dynamics. On the other hand, there are no experiments which determine the behaviour of a quantum wave packet *colliding* with a moving, *impenetrable wall.* (Whatever some of these words might mean in quantum theory?) Therefore, the answer to the choice of *physically correct* boundary conditions has to be left open to discussions.

A natural extension of the above scheme is the choice of different periods such as $\Gamma_\alpha \neq \Gamma_a$. That may lead to quasi-periodic models and based on earlier investigations [5, 7], instabilities can be expected in such systems.

An even more general case is the assumption of boundary conditions randomly distributed in time. As mentioned, that situation resembles the mechanism which leads to the chaotic effects encountered in certain near-integrable classical systems [1, 6, 11] and probably quantum manifestations of classical chaos might be observed in a set-up with random boundary conditions. (See [12, 13], where potentials with random time-dependence have been employed.)

Remark that the techniques applied in the sequel cover as well problems with non-local *kick conditions* imposed periodically in time. A prominent example is the kicked rotor model (or quantum standard map), for instance [5, 14], where the condition $\Psi(k\Gamma_+,x) = \exp\big(-iV(x)\big)\ \Psi(k\Gamma_-,x)$ for all $x \in \mathcal{S}^1,\ k \in \mathbb{Z}$ is imposed on the evolution of the quantum system under consideration. A detailed discussion of the latter will appear somewhere else.

2 Extended Hilbert space dynamics

An appropriate frame for the study of (1.2) in case of a common period Γ of boundary-oscillations, -conditions and external potentials is the so-called *extended Hilbert space formalism*, which provides a Hilbert space structure in the time-variable as well [15]. (This idea is borrowed from the theory of dynamical systems, where it runs under the notion *extended phase space*, [5] for instance.)

Introduce the extended Hilbert space $\mathbf{H}$ as the direct integral

$$\mathbf{H} = \int_{\pi\Gamma}^{\oplus} ds\, \mathcal{H}(s) \tag{2.1}$$

with canonical norm $\|\Psi\|^2_{\mathbf{H}} = \int_0^\Gamma ds\ \|\Psi(s)\|^2_{\mathcal{H}(s)}$, where $\mathcal{H}(s) \equiv \mathcal{L}^2(\mathbb{R}^+, dx)$, $\pi_\Gamma := \big\{(0,\Gamma], s=0 \text{ identified with } s=\Gamma\big\}$. On $\mathbf{H}$ define the symmetric family $\mathsf{H}(a) := \big\{\mathbb{H}(a,s), s \in \pi_\Gamma\big\}$ by

$$\mathbb{H}(a,s) = h_c + v\big(s, \cdot + a(s)\big) + \ddot{a}(s)\ x/2 \tag{2.2}$$

where $\mathbb{H}(a,s)$ fulfills assumption (A1), i.e. the boundary perturbation is represented by a certain function a obeying (A1). In addition, introduce on $\mathbf{H}$ the differential operators $\dot{\mathsf{D}} = -id/ds \otimes \mathbb{I}$ and $\dot{\mathsf{K}}(a) = \dot{\mathsf{D}} + \mathsf{H}(a)$. The latter, respectively its various self-adjoint realizations $\mathsf{K}(a,\alpha)$, which are called *Floquet Hamiltonians*, are the cornerstones in the investigation of the dynamical structure of the models sketched by $\big\{(1.2), (A.1)-(A.3)\big\}$.

Self-adjoint realizations of $\mathbb{H}(a,s)$ are introduced as follows.

Lemma 2.1 *(i) Assume (A1)-(A3) and define* $\mathbb{H}(a,\alpha,s)$ *by*

$$\begin{aligned}\mathcal{D}\left(\mathbb{H}(a,\alpha,s)\right) = \big\{\Psi \in \mathcal{H}(s), \mathbb{H}(a,\alpha,s)\Psi \in \mathcal{H}(s) :\\ \Psi_x/\Psi(s,x=0) = \alpha(s)\ a.e.\big\}\\ \mathbb{H}(a,\alpha,s)\ \Psi = \mathbb{H}(a,s)^*\ \Psi \qquad \forall\Psi \in \mathcal{D}\left(\mathbb{H}(\alpha,s)\right).\end{aligned}$$

Then $\mathsf{H}(a,\alpha) = \int^\oplus_{\pi_\Gamma} ds\, \mathbb{H}(a,\alpha,s)$ *and the fibers* $\mathbb{H}(a,\alpha,s)$ *are self-adjoint on* $\mathbf{H}$*, respectively* $\mathcal{H}(s)$*.*

(ii) The self-adjoint realization D *of* $\dot{\mathsf{D}}$ *is the closure of* $\dot{\mathsf{D}}$ *when defined on functions obeying the periodic boundary condition* $\Psi(\Gamma) = \Psi(0)$*.*

Proof The claims for $\mathsf{H}(a,\alpha)$ and D follow from the direct integral structure. The fibers $\mathbb{H}(a,\alpha,s)$ are covered by the Sturm-Liouville theory. □

As stated in the introduction, the aim of this article is the classification of boundary functions α which provide pure point evolution operators $\mathbb{U}(a,\alpha;\Gamma,0)$ corresponding to the families $\big\{\mathbb{H}(a,\alpha,s), s \in \pi_\Gamma\big\}$ in presence of a given boundary motion a. This task is simplified under the assumption that there exists at least one self-adjoint Floquet Hamiltonian $\mathsf{K}_0(a)$ with pure point spectrum such that the corresponding propagator $\mathbb{U}_0(a;t,0)$ has time-bounded energy, see Definition 1.2. Remark that the common opinion justifies that assumption in the sense that most Floquet Hamiltonians in two dimensions are expected to be pure point, see [5, 8, 9] for instance. In [1] a concrete example, for which the following hypothesis is fulfilled, is presented.

Hypothesis 2.2 (A4) Assume the existence of a self-adjoint pure point Floquet Hamiltonian $\mathsf{K}_0(a) = \overline{\mathsf{D} + \mathsf{H}_0(a)} \supset \mathsf{K}(a) := \mathsf{D} + \mathsf{H}(a)$, where $\mathsf{H}_0(a)$ is a self-adjoint realization of $\mathsf{H}(a)$ defined in (2.2) and the corresponding propagator $\big\{\mathbb{U}_0(a;t,s); (s,t) \in \mathbb{R}^2\big\}$ is strongly differentiable with eigenvalues $\big\{\epsilon_k/\Gamma, k \in \mathcal{K}\big\}$ obeying $\epsilon^{-1}_{k_n} \underset{n\to\infty}{=} o\big(n^{-1/2}\big)$ for all sequences $\big\{k_n, n \in \mathbb{N}\big\}$ of pair-wise different elements of $\mathcal{K}$. (See Definition 2.4 and Theorem 2.5 below.)

In that case there is an entire family of self-adjoint Floquet Hamiltonians parametrized by the boundary functions α. (As the boundary motion a is fixed in the sequel, the dependence on a is omitted for the rest of this section.)

Lemma 2.3 *Assume (A1)-(A4) and define* $\mathsf{H}(\alpha)$ *and* D *as in Lemma 2.1. Then there exist self-adjoint Floquet Hamiltonians* $\mathsf{K}(\alpha)$ *given by the operator closures of* $\dot{\mathsf{K}}(\alpha)$*, which are defined as* $\dot{\mathsf{K}}(\alpha) = \mathsf{D} + \mathsf{H}(\alpha)$ *on* $\mathcal{D}(\mathsf{D}) \cap \mathcal{D}(\mathsf{H}(\alpha))$.

Proof General principles [16] yield $\overline{\mathsf{K}} \subset \mathsf{K}_0 \subset \mathsf{K}^*$, which implies that the defect indices of $\overline{\mathsf{K}}$ are equal and bigger than one [2]. Thus, among others, there exists a family of self-adjoint extensions of K parametrized by α when the latter is chosen according to (A3). Its elements are denoted by $\mathsf{K}(\alpha)$ with

$$\mathsf{K}(\alpha)\Psi = \big(-i\frac{\partial}{\partial s} \otimes \mathbb{I} + \int_{\pi\Gamma}^{\oplus} ds\, \mathbb{H}(s)^*\big)\Psi$$

for all $\Psi \in \mathcal{D}\big(\mathsf{K}(\alpha)\big)$. Obviously, $\mathsf{K} \subset \dot{\mathsf{K}}(\alpha) \subset \mathsf{K}(\alpha)$ and the form of $\mathsf{K}(\alpha)$ implies that $\mathcal{D}\big(\mathsf{K}(\alpha)\big)$ and $\mathcal{D}\big(\dot{\mathsf{K}}(\alpha)\big)$ are characterized by the same boundary conditions. Hence, to every $\Phi \in \mathcal{D}(\mathsf{K}(\alpha))$ there exists a sequence $\big\{\Psi_n \in \mathcal{D}\left(\dot{\mathsf{K}}(\alpha)\right)\big\}_{n\in\mathbb{N}}$ such that $\Psi_n \stackrel{s}{\to} \Phi$ and $\left\{\|\dot{\mathsf{K}}(\alpha)\Psi_n\|_{\mathbf{H}}\right\}_{n\in\mathbb{N}} < M(\Phi) < \infty$, since no diverging contributions from different boundary values of Φ and Ψ_n arise. Thus, $\dot{\mathsf{K}}(\alpha)\Psi_k \stackrel{w}{\to} \mathsf{K}(\alpha)\Phi$ for $k \to \infty$ follows.

Let $\xi \in \ker(\dot{\mathsf{K}}(\alpha)^* - z)$. Then $\big\langle \dot{\mathsf{K}}(\alpha)^*\xi, \Psi_k \big\rangle_{\mathbf{H}} = \big\langle \xi, \overline{z}\, \Psi_k \big\rangle_{\mathbf{H}} = \big\langle \xi, \dot{\mathsf{K}}(\alpha)\Psi_k \big\rangle_{\mathbf{H}}$ for all $\Psi_k \in \mathcal{D}(\dot{\mathsf{K}}(\alpha))$ implies $\big\langle \xi, [\overline{z} - \mathsf{K}(\alpha)]\, \Phi \big\rangle_{\mathbf{H}} = 0$ for all $\Phi \in \mathcal{D}(\mathsf{K}(\alpha))$. Therefore, $\xi \equiv 0$ and $\dot{\mathsf{K}}(\alpha)$ has deficiency indices $(0, 0)$. $\square$

On account of Lemma 2.3 the *Trotter product formula* [2] applies:

$$\exp(-i\tau\, \mathsf{K}(\alpha)) = \mathrm{s} - \lim_{n\to\infty} \big[\exp(-i\, \mathsf{D}\tau/n) \cdot \exp(-i\, \mathsf{H}(\alpha)\tau/n)\big]^n \tag{2.3}$$

and a straightforward calculation yields the existence and uniqueness of solutions of the initial-boundary value problem {(1.2), (A.1)-(A.3)} as expressed in the next theorem. Before that, however,

Definition 2.4 [15] A two-parameter unitary family $\big\{\mathfrak{V}(t, s); s, t \in \mathbb{R}\big\}$ is called a propagator if, for all $s, t \in \mathbb{R}$

(i) $\mathfrak{V}(r, t) = \mathfrak{V}(r, s)\, \mathfrak{V}(s, t)$ (groupoid)

(ii) $\mathfrak{V}(t, t) = \mathbb{I}$

(iii) $\mathfrak{V}(t, s)$ is jointly strongly continuous in s and t.

The next statement connects Floquet Hamiltonians $\mathsf{K}(\alpha)$ and propagators to the corresponding Schrödinger equations.

Theorem 2.5 *Define* $\mathsf{K}(\alpha), \mathsf{D}$*, and* $\mathbb{H}(\alpha, s)$ *as in Lemmata 2.1 and 2.3. Then*

$$\exp\left(-i\tau\ \mathsf{K}(\alpha)\right) = \mathsf{U}(\alpha,\tau)\exp\left(-i\tau\ \mathsf{D}\right)$$

with unitary multiplications $\mathsf{U}(\alpha,\tau) = \big\{\mathbb{U}(\alpha; s+\tau, s) : \tau \in \mathbb{R})\big\}$ *where the fibers* $\mathbb{U}(\alpha; t, r)$ *are part of the unique propagator to*

$$i\frac{\mathbb{U}(\alpha(t); t, r)\Psi(r)}{dt} = \mathbb{H}(\alpha(t), t)\ \mathbb{U}(\alpha(t); t, r)\Psi(r)$$

In particular, $\exp\left(-i\,\Gamma\ \mathsf{K}(\alpha)\right) = \mathsf{U}(\alpha, \Gamma)$ *for entire periods* $\tau = \Gamma$*.*

Proof Based on the shifts $\exp\left(-i\tau\ \mathsf{D}\right)\Psi(s) = \Psi(s-\tau)$ and the direct integral properties $\big[\exp\left(-i\tau\ \mathsf{H}(\alpha)\right)\Psi\big](s) = \exp\left[-i\tau\ \mathbb{H}(\alpha, s)\right]\Psi(s)$, a short evaluation provides for almost every $s \in \pi_\Gamma$

$$\begin{aligned}&\big[\exp\left(-i\ \mathsf{D}\tau/n\right)\cdot\exp\left(-i\ \mathsf{H}(\alpha)\tau/n\right)\big]^n\Psi(s)\\ &\quad= \prod_{k=0}^{n-1}\exp\left[-i\ \mathbb{H}\big(\alpha, s-\tau/(n-k)\big)\tau/(n-k)\right]\ \exp\left(-i\tau\ \mathsf{D}\right)\Psi\ (s).\end{aligned} \tag{2.4}$$

Hence, with (2.3) and the definition (2.1) of the norm on **H**

$$\mathrm{s}-\lim_{n\to\infty}\prod_{k=0}^{n-1}\exp\left[-i\ \mathbb{H}\big(\alpha, \cdot-\tau/(n-k)\big)\tau/(n-k)\right] = \mathbb{U}(\alpha; \cdot+\tau, \cdot)$$

almost everywhere. Differentiation on $\mathcal{D}(\dot{\mathsf{K}}(\alpha))$ shows that $\mathbb{U}(\alpha; \cdot+\tau, \cdot)$ solves the given Schrödinger equation. The propagator properties of $\big\{\mathbb{U}(\alpha; t, r); t, r \in \mathbb{R}\big\}$ as stated in Definition 2.4 follow from the group structure of $\big\{\exp\left(-i\tau\ \mathsf{K}(\alpha)\right), \tau \in \mathbb{R}\big\}$. (See [15] as well.) $\square$

Remark Theorem 2.5 implies the spectral equivalence of $\mathsf{K}(\alpha)$ and $\mathbb{U}(\alpha; \Gamma, 0)$. For instance, if $\lambda \in \sigma_{pp}\big(\mathsf{K}(\alpha)\big)$ with corresponding Γ-periodic eigenfunctions Ψ_λ,

$$\big(\exp\left(-i\ \Gamma\ \mathsf{K}(\alpha)\right)\Psi_\lambda\big)(\Gamma) = \exp\left(-i\ \Gamma\lambda\right)\Psi_\lambda(0) = \mathbb{U}(\alpha; \Gamma, 0)\Psi_\lambda(0). \tag{2.5}$$

Hence, it suffices to study the spectrum $\sigma\big(\mathsf{K}(\alpha)\big)$ in order to understand the dynamics of the models (1.2), (A.1)-(A.3), i.e. the stability of the system.

Having settled existence and uniqueness of solutions to (1.2), the next question addresses the long-term behaviour of these solutions. On account of Theorem 2.5 and (2.5), the latter is characterized by $\sigma\big(\mathsf{K}(\alpha)\big)$.

The determination of $\sigma\big(\mathsf{K}(\alpha)\big)$ presented in the next section is based upon the assumed properties of K_0 as stated in (A4) and Krein's formula [17] for the resolvent difference of two distinct self-adjoint extensions of a given symmetric operator. Thus, the spectral nature of $\mathsf{K}(\alpha)$ is revealed through a comparison between $\mathsf{K}(\alpha)$ and the given operator K_0 as a starting point. The confirmation of the existence of such a reference system K_0 in individual examples, such as the *kicked rotor* or *Pustylnikov's model*, is an independent problem left open in this article. (See for instance [1, 5, 14].)

3 Spectral properties of $\mathsf{K}(a,\alpha)$, Krein's formula

This section deals with the determination of $\sigma\big(\mathsf{K}(a,\alpha)\big)$ for boundary motions and functions $(a,\alpha) \in \mathcal{C}^3_\Gamma(\mathbb{R}) \times \mathcal{C}^1_\Gamma(\mathbb{R})$ in the sense that a pure-point reference operator $\mathsf{K}_0(a)$ is assumed to exist (Hypothesis A.4) and the operators $\mathsf{K}(a,\alpha)$ are determined through alterations of the boundary functions α in presence of the common boundary motion a.

The technical instrument employed is Krein's formula. According to [17], the latter is introduced as follows: Denote by $\dot{k}(a)$ the maximal symmetric common part of the operators $\mathsf{K}(a,\alpha)$ and $\mathsf{K}_0(a)$ as defined in Lemma 2.3 and Hypothesis 2.2 respectively. (I.e. $\dot{k}(a)$ is an extension of every $k \subset \mathsf{K}(a,\alpha)$ and $k \subset \mathsf{K}_0(a)$, in particular of $\mathsf{K}(a)$ defined in (A4).) Introduce an orthonormal basis $\{\Phi_k(a,z), k \in \mathcal{K}\}$ of the (infinite-dimensional) Hilbert space $\ker(\dot{k}^*(a)-z)$. (Relevant properties of such a basis are discussed in the Appendix, see Lemma A.1.) Then Krein's formula for the resolvent difference between $\mathsf{K}(a,\alpha)$ and $\mathsf{K}_0(a)$ is given by

$$(\mathsf{K}(a,\alpha)-z)^{-1} f = (\mathsf{K}_0(a)-z)^{-1} f + \sum_k \sum_l \lambda_{kl}(a,\alpha,z) \big\langle \Phi_l(a,\overline{z}), f \big\rangle_{\mathcal{H}} \Phi_k(a,z) \tag{3.1}$$

for all $f \in \mathcal{H}$ and $\Im z \neq 0$. The coefficients $\lambda_{kl}(a,\alpha,z)$ are uniquely determined by the domain properties of $\mathsf{K}(a,\alpha)$ and $\mathsf{K}_0(a)$ as well as the choice of the basis $\big\{\Phi_k(a,z), k \in \mathcal{K}\big\}$. In particular,

$$\lambda_{kl}(a,\alpha,z) = \left\langle \Phi_k(a,z), \left[(\mathsf{K}(a,\alpha)-z)^{-1} - (\mathsf{K}_0(a)-z)^{-1}\right] \Phi_l(a,\overline{z}) \right\rangle_{\mathcal{H}}. \tag{3.2}$$

(More information concerning $\lambda_{kl}(a,\alpha,z)$ is contained in the Appendix. Merely remark that $\left[(\mathsf{K}(a,\alpha)-z)^{-1} - (\mathsf{K}_0(a)-z)^{-1}\right] g \equiv 0$ for all $g \in \operatorname{ran}(\dot{k}(a)-\overline{z})$, see [17] for a proof.) In the determination of the spectral properties of $\mathsf{K}(a,\alpha)$, Krein's formula is useful in conjunction with the following result going back to *de la Vallèe Poussin*. (Whenever obvious, the dependence on a is suppressed in the sequel.)

Proposition 3.1 *Define $\mathsf{K}(\alpha)$ as in Lemma 2.3 and denote by $\varrho_{\mathrm{ac}}(\mathsf{K}(\alpha),\phi)$ the absolutely continuous spectral measure associated with ϕ and by $\varrho_{\mathrm{sing}}(\mathsf{K}(\alpha),\phi)$ the singular measure. Introduce*

$$\mathbb{S}_{\mathrm{sing}}(\alpha,\phi) := \left\{ y \in \mathbb{R} : \lim_{\epsilon\downarrow 0} \Im\big(\big\langle \phi, (\mathsf{K}(\alpha)-y-i\varepsilon)^{-1} \phi \big\rangle_{\mathcal{H}}\big) = \infty \right\}$$

and

$$\mathbb{S}_{\mathrm{ac}}(\alpha,\phi) := \left\{ y \in \mathbb{R} : \lim_{\epsilon\downarrow 0} \Im\big(\big\langle \phi, (\mathsf{K}(\alpha)-y-i\varepsilon)^{-1} \phi \big\rangle_{\mathcal{H}}\big) = \xi(y) \neq 0 \right\}.$$

Then

(i) $\operatorname{supp}\bigl(\varrho_{\text{sing}}(\mathsf{K}(\alpha),\phi)\bigr) = \mathbb{S}_{\text{sing}}(\alpha,\phi)$

(ii) $\operatorname{supp}\bigl(\varrho_{\text{ac}}(\mathsf{K}(\alpha),\phi)\bigr) \supset \mathbb{S}_{\text{ac}}(\alpha,\phi), |\operatorname{supp}\bigl(\varrho_{\text{ac}}(\mathsf{K}(\alpha),\phi)\bigr)| = |\mathbb{S}_{\text{ac}}(\alpha,\phi)|$

(iii) $\mathbb{S}_{\text{ac}}(\alpha,\phi) \cap \mathbb{S}_{\text{sing}}(\alpha,\phi) = \emptyset$.

(For a proof see [18, 19], for instance.) As demonstrated in the sequel, (3.1) and Proposition 3.1 are cornerstones in the determination of $\sigma\bigl(\mathsf{K}(\alpha)\bigr)$.

The first in a series of statements on the spectral properties of $\mathsf{K}(\alpha)$ concerns the absence of absolutely continuous spectrum. The following Theorem 3.2 provides a sufficient condition for $\sigma_{\text{ac}}\bigl(\mathsf{K}(\alpha)\bigr) = \emptyset$ in terms of the behaviour of the functions $\bigl(\alpha\Phi_k - \frac{\partial\Phi_k}{\partial x}\bigr)(z;\cdot,x=0)$. In most cases, however, the latter cannot be explicitly determined. Therefore, Proposition 3.4 below relates the above features to the reference spectrum $\sigma_{pp}\bigl(\mathsf{K}_0\bigr)$ and it turns out that a finitely degenerate $\sigma_{pp}\bigl(\mathsf{K}_0\bigr)$ is sufficient for $\sigma_{\text{ac}}\bigl(\mathsf{K}(\alpha)\bigr) = \emptyset$ for all $\alpha \in \mathcal{C}^1_\Gamma(\mathbb{R})$. (The point spectrum $\sigma_{pp}\bigl(\mathsf{K}_0\bigr)$ is defined as the (in general $\mathbb{R}$-dense) set of eigenvalues of K_0.) Note that the resolvent difference in (3.1) in general is not trace class. Therefore, the Birman-Krein argument cannot be used to deduce $\sigma_{\text{ac}}\bigl(\mathsf{K}(\alpha)\bigr) = \emptyset$ from the emptiness of $\sigma_{\text{ac}}\bigl(\mathsf{K}_0\bigr)$. (The method of trace class differences between the resolvents or Floquet operators of pure point reference operators and the (spectrally) unknown operators has been used in [8, 18], for instance.)

Theorem 3.2 *Define $\mathsf{K}(\alpha)$ by Lemma 2.3, assume (A1)-(A4) and*

$$\Bigl|\Bigl(\alpha\,\Phi_k - \frac{\partial\Phi_k}{\partial x}\Bigr)(z;s,x=0)\Bigr| > 0$$

for all $k \in \mathcal{K}\backslash\mathcal{K}_f$, where the set $\mathcal{K}_f$ is finite, all $s \in \pi_\Gamma$, $\Re z \notin \bigl\{\bigl(\sigma_{sc} \cup \sigma_{pp}\bigr)\bigl(\mathsf{K}(\alpha)\bigr) \cup \sigma_{pp}\bigl(\mathsf{K}_0\bigr)\bigr\}$ and $0 \le \Im z \le c$ for some $c > 0$. (Here $\{\Phi_k(z), k \in \mathcal{K}\}$ is an orthonormal basis of $\ker(\dot{k}^* - z)$ in the sense of Lemma A.1.) Then $\mathsf{K}(\alpha)$ is purely singular, i.e.

$$\sigma_{\text{ac}}\bigl(\mathsf{K}(\alpha)\bigr) = \emptyset.$$

Proof Let $z = y + i\kappa, \kappa \neq 0$ and $y \notin \sigma_{pp}\bigl(\mathsf{K}_0\bigr)$. At first, note that $\chi_l(\alpha,z) := \bigl(\mathsf{K}(\alpha)-z\bigr)^{-1}\Phi_l(\overline{z})$ and $\chi_{l,0}(z) := \bigl(\mathsf{K}_0 - z\bigr)^{-1}\Phi_l(\overline{z})$ are elements of $\mathcal{D}\left(\mathsf{H}(\alpha)\right)$ and $\mathcal{D}\left(\mathsf{H}_0\right)$, respectively, as seen from the following: (Formal) application of $-i\partial_t$ on $\left(\mathsf{K}(\alpha)-z\right)\chi_l(\alpha,z)$, (A1) and Lemma A.1 (i) yield

$$-i\partial_t\chi_l(\alpha,z) = \bigl(\mathsf{K}(\alpha)-z\bigr)^{-1}\Bigl\{-i\partial_t\Phi_l(\overline{z}) + \bigl(ix\ddot{a}(t)/2 + iv_t(t)\bigr)\chi_l(\alpha,z)\Bigr\} \in \mathcal{H}.$$

Now the claim follows with Lemma 2.3 and Hypothesis2.2. Therefore, with (3.1), both $\sum_k \lambda_{kl}(\alpha,z)\Phi_k(z)\big|_{x=0}$ and $\frac{\partial}{\partial x}\bigl(\sum_k \lambda_{kl}(\alpha,z)\Phi_k(z)\bigr)\big|_{x=0}$ are well-defined and for all $f \in linspan\,\{\Phi_l(\overline{z}), l \in \mathcal{L}\}$, Krein's formula (3.1) implies

$$\begin{aligned}
&\sum_l \left\{ \frac{\partial}{\partial x}\left((\mathsf{K}_0 - z)^{-1}\,\Phi_l(\overline{z})\right)\Big|_{x=0} - \alpha\,(\mathsf{K}_0 - z)^{-1}\,\Phi_l(\overline{z})\big|_{x=0} \right\} \langle \Phi_l(\overline{z}), f\rangle_{\mathcal{H}} \\
&= \quad \sum_l \{\Psi_l(z,\alpha\}\big|_{x=0} \langle \Phi_l(\overline{z}), f\rangle_{\mathcal{H}} \qquad (3.3)\\
&= \quad \sum_l \left\{ \alpha \left(\sum_k \lambda_{kl}(\alpha,z)\Phi_k(z)\right)\Big|_{x=0} - \frac{\partial}{\partial x}\left(\sum_k \lambda_{kl}(\alpha,z)\Phi_k(z)\right)\Big|_{x=0} \right\} \times \\
&\quad\ \langle \Phi_l(\overline{z}), f\rangle_{\mathcal{H}}\,. \qquad (3.4)
\end{aligned}$$

As f is arbitrary from a dense subset of $ker\big(\dot{k}^* - \overline{z}\big)$, the equality of the expressions in the parenthesis follows for all $\kappa \neq 0$.

The existence of $\lim_{\kappa\to 0} \Psi_l(y \pm i\kappa, \alpha)\big|_{x=0} =: \Psi_l(y,\alpha)\big|_{x=0}$ for all $l \in \mathcal{K}$ is deduced as follows: Lemma A.2 provides the existence of $\mathrm{s} - \lim_{\Im\, z\to 0} (\mathsf{K}_0 - z)^{-1}\,\Phi_l(\overline{z})$. (Lemma A.2 applies in particular in case of dense $\sigma_{pp}\big(\mathsf{K}\big)$. For discrete, however infinitely degenerate, eigenvalues of K_0 the statement is obvious.) Together with the absolute continuity of $(\mathsf{K}_0 - z)^{-1}\,\Phi_l(\overline{z})$ for all $\kappa \neq 0$ the latter implies for all $s \in \pi_\Gamma$ the existence of $\lim_{\Im\, z\to 0} (\mathsf{K}_0 - z)^{-1}\,\Phi_l(\overline{z})\big|_{x=0}$ as well as of $\lim_{\Im\, z\to 0}\left(\frac{\partial}{\partial x}\,(\mathsf{K}_0 - z)^{-1}\,\Phi_l(\overline{z})\big|_{x=0}\right)$. Thus, with (3.3)

$$\begin{aligned}
\Psi_l(y,\alpha;s)\big|_{x=0} \quad &= \quad \lim_{\Im\, z\to 0} \left\{ \alpha(s) \left(\sum_k \lambda_{kl}(\alpha,z)\Phi_k(z)\right)(s, x=0) - \right.\\
&\qquad \left. - \frac{\partial}{\partial x}\left(\sum_k \lambda_{kl}(\alpha,z)\Phi_k(z)\right)(s,x=0) \right\}
\end{aligned}$$

and with Lemma A.3 (i), that becomes

$$\begin{aligned}
\Psi_l(y,\alpha;s)\big|_{x=0} \quad &= \quad \lim_{\Im\, z\to 0} \sum_{k\in\mathcal{K}} \lambda_{kl}(\alpha,z)\left(\alpha(\cdot)\Phi_k(z) - \frac{\partial \Phi_k(z)}{\partial x}\right)(s,x=0)\\
&= \quad \lim_{\Im\, z\to 0} \mathcal{S}_{\mathcal{K},l}(\alpha,z,s)
\end{aligned}$$

for all $s \in \pi_\Gamma$ and $l \in \mathcal{K}$.

Now restrict to $y \notin \big\{ \big(\sigma_{sc} \cup \sigma_{pp}\big)\big(\mathsf{K}(\alpha)\big) \cup \sigma_{pp}\big(\mathsf{K}_0\big)\big\}$. Then, with Lemma A.1 (i), (ii) and Lemma A.3 (ii),

$$\lim_{\Im\, z\to 0} \mathcal{S}_{\mathcal{K}_f,l}(\alpha,z,s) = \sum_{k\in\mathcal{K}_f} \lambda_{kl}(\alpha,y)\left(\alpha\Phi_k(y) - \frac{\partial\Phi_k(y)}{\partial x}\right)(s,x=0) \qquad (3.5)$$

follows for all finite subsets $\mathcal{K}_f \subset \mathcal{K}$. Thus,

$$\begin{aligned}
&\lim_{\Im\, z\to 0} \big(\mathcal{S}_{\mathcal{K},l}(\alpha,z,s) - \mathcal{S}_{\mathcal{K}_f,l}(\alpha,z,s)\big)\\
&\quad = \quad \lim_{\Im\, z\to 0} \sum_{k\in\mathcal{K}\setminus\mathcal{K}_f} \lambda_{kl}(\alpha,z)\left(\alpha\Phi_k(z) - \frac{\partial\Phi_k(z)}{\partial x}\right)(s,x=0) \qquad (3.6)
\end{aligned}$$

exists for all $s \in \pi_\Gamma$, $l \in \mathcal{K}$ and $\mathcal{K}_f \subset \mathcal{K}$. Together with the properties of $\{\Phi_k(z), k \in \mathcal{K}\}$ described in Lemma A.1 (iii), (3.6) yields $\lim_{\Im\, z\to 0} \sum_{k\in\mathcal{K}\setminus\mathcal{K}_f} |\lambda_{kl}(\alpha,z)| \in \mathbb{R}$ for all $\mathcal{K}_f \subset \mathcal{K}$ and with Lemma A.3 (i) the uniform boundedness in $\mathcal{K}_f$ of the latter follows. Hence, $\lim_{\Im\, z\to 0} \sum_{k\in\mathcal{K}} |\lambda_{kl}(\alpha,z)| \geq \sum_{k\in\mathcal{K}} |\lambda_{kl}(\alpha,y)|$ results, i.e. $\big\{\lambda_{kl}(y), k \in \mathcal{K}\big\} \in \ell^2$ for all $l \in \mathcal{K}$. Now, with Lemma A.3 (iii), $\sigma_{\rm ac}\big(\mathsf{K}(\alpha)\big) = \emptyset$ is implied. □

Theorem 3.2 demonstrates that the non-resonant behaviour of infinitely many $\Phi_k(z)$ in some neighborhood of the real axis essentially causes the absence of $\sigma_{\rm ac}\big(\mathsf{K}(\alpha)\big)$. That statement is underlined by the following example of a Floquet Hamiltonian $\big(\mathsf{K}(\alpha)\big)$ with non-empty absolutely continuous spectrum. The latter is essentially caused by resonances between some $\mathbb{H}(\alpha,s)$ and $\mathbb{H}_0(s)$, due to the specific form of the boundary function α, and occurring away from a set $\epsilon_0(\alpha) \subset \pi_\Gamma$ where $\mathbb{H}(\alpha,s) = \mathbb{H}_0(s)$ is possible.

Theorem 3.3 *Define $\mathsf{K}(\alpha)$ by Lemma 2.3, assume (A1) - (A4) and, in addition, $\mathbb{H}(\alpha,s) \neq \mathbb{H}_0(s)$ on $\pi_\Gamma\backslash\epsilon_0(\alpha)$ as well as the presence of (at most) countable sets $\mathcal{D}(\alpha,y) \subset \pi_\Gamma$, with $\mathcal{D}(\alpha,y)\cap\epsilon_0(\alpha) = \emptyset$, and intervals $\mathcal{A}(\alpha) = [a,b]$. If*

$$\alpha(s)\ \Phi_k(y;s,x=0) = \frac{\partial\Phi_k}{\partial x}(y;s,x=0).$$

for each $s \in \mathcal{D}(\alpha,y)$, $y \in \mathcal{A}(\alpha)$ and $k \in \mathcal{K}$, then $\sigma_{\rm ac}\big(\mathsf{K}(\alpha)\big) \supset \mathcal{A}(\alpha)$.

Proof At first, remark that $\mathcal{A}(\alpha)\cap\sigma_{pp}\big(\mathsf{K}(\alpha)\big) = \emptyset$ on account of the resonance condition since eigenfunctions ξ_j of $\mathsf{K}(\alpha)$ obey the boundary conditions $\alpha\xi_j\big|_{x=0} = \frac{\partial\xi_j}{\partial x}\big|_{x=0}$ everywhere, which contradicts the countability of $\mathcal{D}(\alpha,y)$.

Choose $y \in \mathcal{A}(\alpha)$ such that $y \notin \big\{\sigma_{sc}\big(\mathsf{K}(\alpha)\big)\cup\sigma_{pp}\big(\mathsf{K}_0\big)\big\}$ and assume $\big\{\lambda_{kl}(\alpha,y) = \lim_{\Im z\to 0}\lambda_{kl}(\alpha,z)\big\}_{k\in\mathcal{K}} \in l^2$ for all $l \in \mathcal{K}$. As the latter implies with Lemma A.3 (iii a) that $y \notin \sigma_{\rm ac}\big(\mathsf{K}(\alpha)\big)$, $dist\big(\sigma\big(\mathsf{K}(\alpha)\big),y\big) > \epsilon$ for some $\epsilon > 0$ follows. Therefore, with (3.1) and Lemma A.2, $\sum_{k\in\mathcal{K}}\lambda_{kl}(\alpha,z)\Phi_k(z) \underset{\Im z\to 0}{\overset{s}{\to}} \sum_{k\in\mathcal{K}_f}\lambda_{kl}(\alpha,y)\Phi_k(y)$ for all $l \in \mathcal{K}$ and with the smoothness properties described in the proof of Lemma A.3 (i), respectively the resonance condition

$$\begin{aligned}
&\alpha(\hat{s})\lim_{\Im z\to 0}\sum_{k\in\mathcal{K}}\lambda_{kl}(\alpha,z)\Phi_k(z)(\hat{s},x=0)\\
&= \alpha(\hat{s})\lim_{\mathcal{K}_f\to\mathcal{K}}\sum_{k\in\mathcal{K}_f}\lambda_{kl}(\alpha,y)\Phi_k(y,\hat{s},x=0)\\
&= \lim_{\mathcal{K}_f\to\mathcal{K}}\frac{\partial}{\partial x}\sum_{k\in\mathcal{K}_f}\lambda_{kl}(\alpha,y)\Phi_k(y,\hat{s},x=0)\\
&= \frac{\partial}{\partial x}\sum_{k\in\mathcal{K}}\lambda_{kl}(\alpha,y)\Phi_k(y)(\hat{s},x=0)\\
&= \lim_{\Im z\to 0}\frac{\partial}{\partial x}\sum_{k\in\mathcal{K}}\lambda_{kl}(\alpha,z)\Phi_k(z)(\hat{s},x=0)
\end{aligned}$$

results for all $\hat{s} \in \mathcal{D}(\alpha, y)$. Hence, with the definition of $\mathcal{S}_{\mathcal{K},l}(z,\alpha,\hat{s})$ from Theorem 3.2

$$\lim_{\Im z \to 0} \mathcal{S}_{\mathcal{K},l}(z,\alpha,\hat{s}) = 0$$

for all $\hat{s} \in \mathcal{D}(\alpha, y)$. However, in contradiction, the assumption $\mathbb{H}(\alpha,s) \neq \mathbb{H}_0(s)$ on $\mathcal{D}(\alpha, y)$, (3.3) and (3.4) yield that $\lim_{\Im z \to 0} \mathcal{S}_{\mathcal{K},l}(z,\alpha,s) \neq 0$ on $\mathcal{D}(\alpha, y)$. Therefore, Lemma A.3 (iii) implies that $y \in \mathcal{A}(\alpha)$ is an accumulation point of $\sigma\big(\mathsf{K}(\alpha)\big)$. Now $\sigma_{\mathrm{ac}}\big(\mathsf{K}(\alpha)\big) \neq \emptyset$ follows from the closedness of $\sigma_{sc}\big(\mathsf{K}(\alpha)\big)$. $\square$

Hence, the resonance condition $\alpha\Phi_j(y)\big|_{x=0} = \frac{\partial\Phi_j(y)}{\partial x}\big|_{x=0}$ on some (countable) subset of π_Γ indeed has a decisive influence on the nature of $\sigma\big(\mathsf{K}(\alpha)\big)$. (The countability is necessary since certain eigenfunctions of $\mathsf{K}(\alpha)$ may fulfill $\alpha\xi_j\big|_{x=0} = \frac{\partial\xi_j}{\partial x}\big|_{x=0}$ on some interval $\mathcal{B}(\alpha,y)$ and $\alpha\xi_j\big|_{x=0} = \frac{\partial\xi_j}{\partial x}\big|_{x=0} = 0$ on $\pi_\Gamma\backslash\mathcal{B}(\alpha,y)$.) Unfortunately, this condition is hard to verify in explicit examples. Therefore, a connection between the defect functions $\Phi_k(z)$ and the given structure of $\sigma_{pp}\big(\mathsf{K}_0\big)$ is desirable. The next statement provides such a relationship.

Proposition 3.4 *Define $\mathsf{K}(\alpha)$ by Lemma 2.3 and assume (A1)–(A4). If K_0 is pure point with finitely degenerate eigenvalues, then there is a non-resonant set of defect functions $\big\{\Phi_k(y), k \in \mathcal{K}\big\}$, with the properties described in Lemma A.1, for all $y \in \mathbb{R}$, i.e. the assumptions of Theorem 3.3 are nowhere fulfilled.*

The proof of Proposition 3.4 is part of the Appendix. As infinite degeneracy of $\sigma_{pp}\big(\mathsf{K}_0\big)$ is *not* the rule, the appearance of absolutely continuous $\mathsf{K}(\alpha)$ can be seen as exceptional. (See below.) However, the set of *exceptional realizations* might be considerable. For example, any $\Gamma = 2\pi(p/q)$ with $(p,q) \in \mathbb{N}^2$ in the kicked rotor model (mentioned at the end of Section 1) yields pure absolutely continuous Floquet operators, see [14] for instance.

It remains to determine a distinction between the singular operators $\mathsf{K}(\alpha)$. To this end, introduce

Hypothesis 3.5 (A5) Assume that there is a sequence of pairs $(a_j, \alpha_j) \underset{j\to\infty}{\to} (a, \alpha)$ in $\mathcal{C}^3_\Gamma(\mathbb{R}) \times \mathcal{C}^1_\Gamma(\mathbb{R})$ obeying (A1) and (A3) for each $j \in \mathbb{N}$ such that

(i)
$$\big[\alpha_j(\hat{s})\ \Phi_k(y, a_j; \hat{s}) - \frac{\partial\Phi_k}{\partial x}\big(y, a_j; \hat{s}\big)\big]\big|_{x=0} = 0$$
for infinitely many $k \in \mathcal{K}(j,\hat{s})$, $\hat{s} \in \mathcal{D}(\alpha_j)$, $y \in \mathcal{A}(\alpha_j)$ and all $j \in \mathbb{N}$.

(ii)
$$\big[\alpha_j(s)\ \Phi_k(y, a; s) - \frac{\partial\Phi_k}{\partial x}\big(y, a; s\big)\big]\big|_{x=0} = 0$$
for finitely many $k \in \mathcal{K}(s)$, all $s \in \pi_\Gamma$ and $y \in \mathbb{R}$. (Here $\{\Phi_k(z,a), k \in \mathcal{K}\}$ and $\{\Phi_k(z,a_j), k \in \mathcal{K}\}$ are orthonormal basis of $\ker(\dot{k}^*(a) - z)$ and $\ker(\dot{k}^*(a_j) - z)$ respectively, see Lemma A.1.)

(iii)

$$\big(\mathsf{K}_0(a_j)-z\big)^{-1}\Phi_l(a_j,\overline{z})\big|_{x=0} \underset{j\to\infty}{\overset{s}{\to}} \big(\mathsf{K}_0(a)-z\big)^{-1}\Phi_l(a,\overline{z})\big|_{x=0}$$

exists for all $\Im z \neq 0$.

The idea behind (A5) is to characterize certain singular continuous Floquet Hamiltonians $\mathsf{K}(a,\alpha)$ as limiting cases of sequences of pure absolutely continuous operators $\mathsf{K}(a_j,\alpha_j)$.

Theorem 3.6 *Define $\mathsf{K}(a,\alpha)$ by Lemma 2.3, assume (A1)-(A5) and, in particular, that $\sigma\big(\mathsf{K}(a_j,\alpha_j)\big)=\sigma_{\rm ac}\big(\mathsf{K}(a_j,\alpha_j)\big)$ for all $j\in\mathbb{N}$. Then $\sigma\big(\mathsf{K}(a,\alpha)\big)=\sigma_{sc}\big(\mathsf{K}(a,\alpha)\big)$ iff the following condition is fulfilled:*

$$\lim_{j\to\infty}\Psi_l(y\pm i\epsilon,a_j,\alpha_j;s)\big|_{x=0}=\Psi_l(y\pm i\epsilon,a,\alpha;s)\big|_{x=0} \qquad (C)$$

is uniform in $\epsilon\geq 0$, $y\notin \underset{j\in\mathbb{N}}{\cup}\sigma_{pp}\big(\mathsf{K}_0(a_j)\big)$, $s\in\pi_\Gamma$ and $l\in\mathcal{K}$. (See (3.3) for the definition of $\Psi_l(z,a,\alpha)$.)

Proof (i) Theorem 3.2 and (A5) imply that $\mathsf{K}(a,\alpha)$ is pure singular. Condition (C), respectively (3.3) provides that the sequence

$$\Big\{\sum_{k\in\mathcal{K}}\lambda_{kl}(a_j,\alpha_j;z)\big[\alpha_j\Phi_k(a_j,z)-\frac{\partial\Phi_k}{\partial x}(a_j,z)\big] \\ -\lambda_{kl}(a,\alpha;z)\big[\alpha\Phi_k(a,z)-\frac{\partial\Phi_k}{\partial x}(a,z)\big]\Big\}_{j\in\mathbb{N}}$$

is uniformly in $\Im z\geq 0$, respectively $\Im z\leq 0$, convergent for all $s\in\pi_\Gamma$ and $y\notin\cup_{j\in\mathbb{N}}\sigma_{pp}\big(\mathsf{K}_0(a_j)\big)$. Together with the assumption of absolutely continuous $\mathsf{K}(a_j,\alpha_j)$ the above yields a s-dependent divergence rate of the sequence $\big\{\sum_{k\in\mathcal{K}}\lambda_{kl}(a,\alpha,z)\Phi_k(a,z)\big\}$, whereas by assumption $\lim_{\Im z\to 0}\sum_{k\in\mathcal{K}}\lambda_{kl}(a_j,\alpha_j,z)\Phi_k(a_j,z)$ exists for all $s\notin\mathcal{D}(\alpha_j)$ (See also Lemma A.3 (iv).) These features exclude $\sigma_{pp}\big(\mathsf{K}(a,\alpha)\big)\neq\emptyset$, since in that case the existence of $\Psi_l(z,a,\alpha)$ in the limit $\Im z\to 0$, with $\Re z:=\mathcal{E}_{\hat{j}}$, would origin from s-constant cancellations between the diverging functions $\alpha\big(\mathcal{E}_{\hat{j}}-z\big)^{-1}\big\langle\xi_{\hat{j}},\Phi_l(\overline{z})\big\rangle_{\mathcal{H}}\xi_{\hat{j}}$ and $\big(\mathcal{E}_{\hat{j}}-z\big)^{-1}\big\langle\xi_{\hat{j}},\Phi_l(\overline{z})\big\rangle_{\mathcal{H}}\frac{\partial\xi_{\hat{j}}}{\partial x}$, with $\big(\mathcal{E}_{\hat{j}},\xi_{\hat{j}}\big)$ a certain eigenvalue/function of $\mathsf{K}(a,\alpha)$.

If condition (C) is violated, remark that $\big\{\Psi_l(y,a_j,\alpha_j),j\in\mathbb{N}\big\}$ is unbounded everywhere on account of (A5), (iii). In addition, if $\mathsf{K}(a,\alpha)$ is pure singular continuous, $\ker(\mathsf{K}(a,\alpha)-y)=\emptyset$. Thus, none of the cancellation mechanisms of part (i) applies, whereas (3.4) is supposed to hold for all $y\notin\sigma_{pp}\big(\mathsf{K}_0(a)\big)$. However, as some of the sequences $\big\{\sum_{k\in\mathcal{K}}\lambda_{kl}(a,\alpha,z)\Phi_k(a,z)\big\}$ are diverging for $\Im z\to 0$, see Lemma A.3 (iv), this requirement cannot be fulfilled for all $y\notin\sigma_{pp}\big(\mathsf{K}_0(a)\big)$ and $l\in\mathcal{K}$. □

The mechanism generating singular continuous Floquet Hamiltonians as described above seems to be present in the kicked rotor models, whenever irrational periods Γ of Liouville type are approximated by sequences of rationals, see [14]. However, the more general investigation in [21], which uses Baire category arguments, indicates that the occurrence of singular continuous quasi-energies is more frequent and caused as well by conditions different from the one given in Theorem 3.6. Therefore, as a natural question, the size of the set of boundary perturbations which provide the stability of *generic* systems $\{(1.2), (A.1) - (A.3)\}$, i.e. in situations where the assumptions of Theorem 3.6 do not hold, has to be addressed.

The presence of absolutely continuous $\mathsf{K}(a, \alpha)$ seems basic to the existence of singular continuous Floquet Hamiltonians - which is obvious in the above discussion, however, Baire category arguments such as in [21] indicate that this is a more general statement. Hence, the size of the set of (un-)stable pairs (a, α) is connected to the (in-)finite degeneracy of $\sigma_{pp}(\mathsf{K}_0(a))$ and the resonance condition of Theorem 3.3. The proof of Proposition 3.4 gives a vague idea of that relationship, in particular (4.1) and (4.2), however, more detailed information (as for quantum twist maps, see [14, 21]) is hard to obtain as long as no precise knowledge of the defect functions $\{\Phi_k(z)\}$ is available. Yet, it should be noted that the influence of the boundary at $x = 0$ can be (partially) circumvented (in an unphysical way) by introducing a Hilbert space $\mathcal{H} =: \mathcal{L}^2(\mathbb{R}^+) \oplus \mathcal{L}^2(\mathbb{R}^+)$ and defining a self-adjoint *momentum operator* on $\mathcal{H}$. Then the generator $i\mathsf{P}$ of unitary shifts on $\mathcal{H}$ can be introduced by

$$\mathsf{P} = \begin{pmatrix} -i\frac{\partial}{\partial x} & 0 \\ 0 & i\frac{\partial}{\partial y} \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathsf{P}) = \left\{ \Psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} : \psi^+, \psi^- \text{absolutely continuous}, \psi^+(x=0) = \psi^-(y=0) \right\}.$$

In the same way a Laplacian P^2 is introduced. In [1] this idea has been applied to the case of $H_c = -\frac{d^2}{dx^2} + x$, where it turns out that no infinitely degenerate $\mathsf{K}_0(a)$ exists for all $a \in \mathcal{C}^3_\Gamma(\mathbb{R})$ as long as $|\ddot{a}(t)| < 2$ everywhere. It is conceivable that this method can be extended to the corresponding class of potential introduced in [9], thus providing a pool of pure point reference systems as well as estimates on the size of any exceptional set.

Finally, remark that the presence of singular continuous $\mathsf{K}(a, \alpha)$ as indicated by Hypothesis 3.5 and Theorem 3.6 seems to be very unstable to *independent* alterations of a, respectively α, since the defect functions $\{\Phi_k(z)\}$ do not depend on the boundary function α, whereas the resonance condition in Theorem 3.3 combines influences from a and α.

As a concluding observation, note that *generic* (i.e. pure point) systems are indeed stable:

Proposition 3.7 *Define* $\mathsf{K}(a, \alpha)$ *and* $\mathbb{U}(a, \alpha; \Gamma, 0)$ *as in Lemma 2.3 and Theorem 2.5, respectively, and choose* (a, α) *generic. Then* $\mathbb{H}(a, \alpha, 0)$ *has time-bounded energy in the sense that*

$$\sup k \in \mathbb{N} \left| \left\langle \mathbb{U}(a, \alpha; k\Gamma, 0)\phi_0, \mathbb{H}(a, \alpha, 0)\, \mathbb{U}(a, \alpha; k\Gamma, 0)\phi_0 \right\rangle_{\mathcal{H}(t=0)} \right| < \infty$$

for all $\phi_0 \in \operatorname{linspan}\big\{\psi_j = \textit{eigenfunctions to } \mathbb{U}(a,\alpha;\Gamma,0)\big\}$.

Proof The eigenfunctions to $\mathsf{K}(a,\alpha)$ are given by

$$\big\{\Psi_j(a,\alpha;s,x) = \exp\big(i\lambda_j(s-s')\big)\big(\mathbb{U}(a,\alpha;s,s')\psi_j\big)(x), j \in \mathcal{J}\big\},$$

see [15]. Thus, $\mathbb{U}(a,\alpha;k\Gamma,0)\psi_j \in \mathcal{D}\big(\mathbb{H}(a,\alpha,0)\big)$ for all $(j,k) \in \mathcal{J}\times\mathbb{N}$ and $\mathbb{H}(a,\alpha,0)+c$ is non-negative for sufficiently large c. That implies

$$\begin{aligned}
&\|\big(\mathbb{H}(a,\alpha,0)+c\big)^{1/2}\mathbb{U}\big(a,\alpha;k\Gamma,0\big)\phi_0\| \\
&= \ \|\sum_{j\in\mathcal{J}}\exp\big(i\lambda_j k\Gamma\big)\langle\phi_0,\psi_j\rangle\big(\mathbb{H}(a,\alpha,0)+c\big)^{1/2}\psi_j\| \\
&\leq \ \sum_{j\in\mathcal{J}}|\langle\phi_0,\psi_j\rangle|\|\big(\mathbb{H}(a,\alpha,0)+c\big)^{1/2}\psi_j\| < \infty
\end{aligned}$$

for all $\phi_0 \in \operatorname{linspan}\big\{\psi_j, j\in\mathcal{J}\big\}$. □

4 Appendix

The Appendix contains a collection of auxiliary statements employed in the proofs of the main results in Section 3. (The dependence on the boundary motion a is suppressed.)

Lemma A. 1 *Assume (A1)-(A4), define* $\dot{k}$ *as the maximal symmetric common part of the operators* $\mathsf{K}(\alpha)$ *and* K_0 *as well as* $\dot{\mathsf{H}}$ *to be the the maximal common part of* $\mathsf{H}(\alpha)$ *and* H_0. *Then there exists an orthonormal basis* $\{\Phi_k(z), k\in\mathcal{K}\}$ *of* $\ker(\dot{k}^*-z)$ *such that*

(i) $\Phi_k(y\pm i\epsilon) \in \mathcal{D}(\mathsf{D})\cap\mathcal{D}(\dot{\mathsf{H}}^*)\ \forall\epsilon\geq 0.$

(ii) $\Phi_k(y\pm i\epsilon) \xrightarrow{s} \Phi_k(y)$

(iii) Does not exist a sequence of pairwise different elements $\{k_j, j\in\mathbb{N}\}$ *such that*

$$\big[\alpha(s)\ \Phi_{k_j}(y\pm i\epsilon;s) - \frac{\partial\Phi_{k_j}}{\partial x}\big(y\pm i\epsilon;s\big)\big]\big|_{x=0} \underset{j\to\infty}{\to} 0 \textit{ for all } s\in\pi_\Gamma, \epsilon>0\,.$$

Proof (i) The operator $\dot{k}^*$ is bounded on the Hilbert space $\ker(\dot{k}^*-z)$ and $(\mathcal{D}(\mathsf{D})\cap\mathcal{D}(\dot{\mathsf{H}}^*))\cap\ker(\dot{k}^*-z)$ forms a core. Thus, as a pre-Hilbert space, the latter contains an orthonormal basis of $\ker(\dot{k}^*-z)$, which is denoted by $\{\Phi_k(z), k\in\mathcal{K}\}$ in the sequel.

(ii) Let $f_{\ker}(z)$ be the orthogonal projection of $f\in\mathcal{H}$ onto $\ker(\dot{k}^*-z)$. As $\varphi(\overline{z}) \xrightarrow{s} \varphi(\Re z)$ for all $\varphi(\overline{z}) \in \operatorname{ran}\big(k-\overline{z}\big)$, $f_{\ker}(z) \xrightarrow{s} f_{\ker}(\Re z)$ for $\Im z\to 0$ follows. Hence, the projections obey $\langle f,\Phi_k(z)\rangle_{\mathcal{H}} \to \alpha_k(f)$ for $\Im z\to 0$ and from $\|\Phi_k(z)\|_{\mathcal{H}} = 1$ for all $\Im z > 0$,$\alpha_k(f) = \langle f, \chi_k(\Re z)\rangle_{\mathcal{H}}$ results, i.e. $\Phi_k(z) \overset{w}{\to} \chi_k(\Re z)$. Now, from

$$0 = \langle \ker\left(\dot{k} - \overline{z}\right)\xi, \Phi_k(z)\rangle_{\mathcal{H}} \underset{Imz\to 0}{\to} 0 = \langle \xi, \ker\left(\dot{k}^* - z\right)\chi_k(\Re z)\rangle_{\mathcal{H}}$$

for all $\xi \in \mathcal{D}(\dot{k})$, the identification $\chi_k(\Re z) \equiv \phi_k(\Re z)$ follows.

To prove strong continuity, introduce k as $k := \overline{\dot{k} \restriction \mathcal{C}_0^\infty\big((0,\Gamma)\times(0,\infty)\big)}$, which implies $\dot{k}^* \subset k^*$ see [14]. An orthonormal basis of $\ker(k^* - z)$ is formed by $\big\{\xi_n(z; s, x) = c(z)\exp(izs)\ \mathfrak{U}_0(s,0)\ \phi_n(x), n \in \mathcal{N}\big\}$, where $\big\{\phi_n, n \in \mathcal{N}\big\}$ is the complete set of eigenfunctions to the Floquet operator $\mathfrak{U}_0(\Gamma, 0)$ introduced in (A4), see also [13], and $c(z)$ is the normalization.

Obviously, $\operatorname*{s-lim}_{\Im z\to 0}\ \xi_n(z) = \xi_n(\Re z)$ and $\{\Phi_k(z), k \in \mathcal{K}\} \subset ker\,(k^* - z)$. Therefore, $\sum_{n\in\mathcal{N}_f} |\langle \xi_n(z), \Phi_k(z)\rangle_{\mathcal{H}}|^2 \to \sum_{n\in\mathcal{N}_f} |\langle \xi_n(\Re z), \chi_k(\Re z)\rangle_{\mathcal{H}}|^2$ for all (finite) subsets $\mathcal{N}_f \subset \mathcal{N}$ and the convergence rate is N-uniform on account of the form of all $\xi_n(z)$. Thus, $\|\Phi_k(z)\|_{\mathcal{H}} \to \|\chi_k(\Re z)\|_{\mathcal{H}}$.

(iii) Remark that merely eigenfunctions $\xi(\alpha, y)$ to some $\mathsf{K}(\alpha)$ obey $\big[\alpha\xi(y) - \frac{\partial \xi}{\partial x}(y)\big]\big|_{x=0} = 0$ for all $s \in \pi_\Gamma$ and $y \in \sigma_{pp}\big(\mathsf{K}(\alpha)\big)$. In addition, $\Phi_k(z)\big|_{x=0}$ and $\frac{\partial \Phi_k}{\partial x}(z)\big|_{x=0}$ cannot converge simultaneously to zero since only $\chi \in \mathcal{D}\big(\dot{\mathsf{D}}\big) \cap \mathcal{D}\big(\mathsf{H}\big)$ obeys $\chi\big|_{x=0} = \frac{\partial \chi_k}{\partial x}\big|_{x=0} = 0$, see (2.2). □

Lemma A. 2 *Define* K_0 *as in Hypothesis 2.2 and* $\{\Phi_k(z), k \in \mathcal{K}\}$ *as in Lemma A.1. Then* $\operatorname*{s-lim}_{\Im z\to 0}\big(\mathsf{K}_0 - z\big)^{-1}\Phi_l(\overline{z}) := \big(\mathsf{K}_0 - \Re z\big)^{-1}\Phi_l(\Re z)$ *exists for all* $\Re z \notin \sigma_{pp}\big(\mathsf{K}_0\big)$ *and* $(k, l) \in \mathcal{K}^2$.

Proof Using the Floquet representation of the propagator $\mathfrak{U}_0(s, 0)$ introduced in (A4), i.e. $\mathfrak{U}_0(s,0) = \mathfrak{P}_0(s)\ \exp\big(-is\mathfrak{G}_0\big)\big)$ with self-adjoint $\mathfrak{G}_0$ and strongly differentiable family $\big\{\mathfrak{P}_0(s), s \in [0,\Gamma]\big\}$ obeying $\mathfrak{P}_0(0) = \mathfrak{P}_0(\Gamma) = \mathbb{I}$, the eigenfunctions of K_0 are represented by

$$\big\{\xi_\gamma\big(E_\gamma; s, x\big) = \Gamma^{-1/2}\exp(2\pi ijs/\Gamma)\ \mathfrak{P}_0(s)\ \phi_k(x), \gamma := (j,k) \in \mathbb{Z}\times\mathcal{K}\big\}\,.$$

(Here $\big\{\phi_k, k \in \mathcal{K}\big\}$ is the set of normalized eigenfunctions to $\mathfrak{G}_0$.)

Let $y := \Re z \notin \sigma_{pp}\big(\mathsf{K}_0\big)$. As $\lambda_k := E_\gamma - 2\pi ij/\Gamma \in \sigma\big(\mathfrak{G}_0\big)$ is of order $o\big(k^{-1/2}\big)$ by assumption (A4), the convergence of $\big\{\big\|\big(\mathsf{K}_0 - z\big)^{-1}\Phi_l(\overline{z})\big\|, \Im z \geq 0\big\}$ follows with Theorem 3.1 from [19] and Lemma A.1, since both together imply

$$\sum_{j\in\mathbb{Z}} \sup_{k\in\mathcal{K}}\left(\big(2\pi j/\Gamma - y\big){\lambda_k}^{-1} + 1\right)^{-2} \sup_{k\in\mathcal{K}}\big|\big\langle \xi_{(j,k)}, \Phi_l(y)\big\rangle_{\mathcal{H}}\big|^2 < \infty.$$

In addition, $\big\langle \chi, (\mathsf{K}_0 - z)^{-1}\,\Phi_l(\overline{z})\big\rangle_{\mathcal{H}} \to \big\langle \chi, (\mathsf{K}_0 - y)^{-1}\,\Phi_l(y)\big\rangle_{\mathcal{H}}$ for $\Im z \to 0$ and all $\chi \in linspan\,\big\{\xi_\gamma\big\}$. □

Lemma A. 3 *Define* $\lambda_{kl}(\alpha, z)$ *by (3.2),* $\mathsf{K}(\alpha)$ *as in Lemma 2.3 and assume (A4).*

(i) Let $\Im z \neq 0$*. Then* $\{\lambda_{kl}(\alpha, z), k \in \mathcal{K}\} \in \ell^1$*.*

(ii) If $y := \Re z \notin \mathbb{S}(\alpha) := \left\{ \left(\sigma_{pp} \cup \sigma_{sc}\right)(\mathsf{K}(\alpha)) \cup \sigma_{pp}(\mathsf{K}_0) \right\}$ *then* $\lim_{\Im z \to 0} \lambda_{kl}(\alpha, z) := \lambda(\alpha, y) \in \mathbb{C}$*.*

(iii) Let $y \notin \mathbb{S}(\alpha)$*.*

(a) If $\{\lambda_{kl}(\alpha, y), k \in \mathcal{K}\} \in l^2$ *for all* $l \in \mathcal{K}$*, then* $y \notin \sigma_{\mathrm{ac}}(\mathsf{K}(\alpha))$*.*

(b) If $\{\lambda_{kl}(\alpha, y), k \in \mathcal{K}\} \notin l^2$ *for some* $l \in \mathcal{K}$*, then* y *is an accumulation point of* $\sigma(\mathsf{K}(\alpha))$*.*

(iv) Let $y \in \sigma_{\mathrm{sing}}(\mathsf{K}(\alpha))$ *and* $y \notin \{\sigma_{\mathrm{ac}}(\mathsf{K}(\alpha)) \cup \sigma_{pp}(\mathsf{K}_0)\}$*. Then* $\{\lambda_{kl}(\alpha, z)\}$ *diverges to infinity for some* $(k, l) \in \mathcal{K}^2$*.*

Proof (i) From (3.1) and the first part of the proof of Theorem 3.2 it follows that $\{\lambda_{kl}(\alpha, z), k \in \mathcal{K}\} \in \ell^2$ and $\sum_{k\in\mathcal{K}} \lambda_{kl}(\alpha, z)\Phi_k(z)$ is absolutely continuous w.r.t. $s \in \pi_\Gamma$ and $\mathcal{H}^2_2$ w.r.t. $x \in \mathbb{R}^+$. Therefore,

$$\lim_{\mathcal{K}_f \to \mathcal{K}} \Big[\sum_{k\in\mathcal{K}\setminus\mathcal{K}_f} \lambda_{kl}(\alpha, z)\Phi_k(z)\Big](s, x) = 0$$

for all $(s, x) \in (\pi_\Gamma, \mathbb{R}^+)$. This fact and the absolute continuity of $\partial/\partial x\big[\sum_{k\in\mathcal{K}\setminus\mathcal{K}_f} \lambda_{kl}(\alpha, z)\Phi_k(z)\big]$ for all $\mathcal{K}_f$ provide that $\sum_{k\in\mathcal{K}} \lambda_{kl}(\alpha, z)\Phi_k(z, s, x = 0)$ and $\sum_{k\in\mathcal{K}} \lambda_{kl}(\alpha, z)\frac{\partial \Phi_k}{\partial x}(z, s, x = 0)$ exist. Now the claim follows by applying the arguments used in the Proof of Lemma A.1 (iii) to $\sum_{k\in\mathcal{K}} \lambda_{kl}(\alpha, z)\big(\alpha\Phi_k(z) - \frac{\partial \Phi_k}{\partial x}(z)\big)(s, x = 0)$.

(ii) Define $\varphi_k(z) := \mathsf{P}_{\mathrm{ac}}(\alpha)\Phi_k(z)$, with $\mathsf{P}_{\mathrm{ac}}(\alpha)\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(\mathsf{K}(\alpha))$ and let $\mu_{kl}(\alpha, z) := \langle \varphi_k(z), (\mathsf{K}(\alpha) - z)^{-1} \varphi_l(\overline{z})\rangle_{\mathcal{H}}$. Spectral and Radon-Nikodym theorems imply

$$\begin{aligned} \mu_{kl}(\alpha, z) &= \int_{\mathbb{R}} \big(\lambda - z\big)^{-1} d\langle \varphi_k(z), E_{\mathrm{ac}}(\alpha, \lambda)\varphi_l(\overline{z})\rangle_{\mathcal{H}} \\ &= \int_{\mathbb{R}} \big(\lambda - z\big)^{-1} f_{kl}(\alpha, \lambda, z)\, d\lambda \end{aligned}$$

with integrable, non-negative $f_{kl}(\alpha, z) \to f_{kl}(\alpha, y)$ a.e. on account of Lemma A.1 (ii). That information and the distributional limit

$$\lim_{\varepsilon\downarrow 0} \big(\lambda - y - i\varepsilon\big)^{-1} = \mathcal{P}\big((\lambda - y)^{-1}\big) - i\pi\delta(\lambda - y)$$

where $\mathcal{P}(\cdot)$ is Cauchy's principal value and δ the Dirac distribution. [20], for example, demonstrates the existence of $\lim_{\varepsilon\downarrow 0} \mu_{kl}(\alpha, y + i\varepsilon)$. The existence of the remaining limits

$$\lim_{\varepsilon\downarrow 0} \big\{ \langle \varphi_k(z)^\perp, (\mathsf{K}(\alpha) - z)^{-1} \varphi_l(\overline{z})^\perp\rangle_{\mathcal{H}} - \langle \Phi_k(z), (\mathsf{K}_0 - z)^{-1} \Phi_l(\overline{z})\rangle_{\mathcal{H}} \big\}$$

respectively $\lim_{\varepsilon \downarrow 0} \lambda_{kl}(y+i\varepsilon)$ in case of $y \notin \sigma_{\text{ac}}(\mathsf{K}(\alpha)) \cup \mathbb{S}(\alpha,0)$ are implied by Proposition 3.1. An analogue procedure applies to $\Im z \uparrow 0$.

(iii) (a) Equations (3.1), (3.2), Lemma A.2 and the assumption imply that $\lim_{\Im z \to 0} \left\| \left(\mathsf{K}(\alpha)-z\right)^{-1} \Phi_l(\overline{z}) \right\|_{\mathcal{H}}$ exists. Now the claim follows with the spectral theorem, see part (ii).

(iii) (b) In this case the corresponding assumption implies $\left\| \left(\mathsf{K}(\alpha)-z\right)^{-1} \Phi_l(\overline{z}) \right\|_{\mathcal{H}} \to \infty$, i.e. $\operatorname{dist}\left(z, \sigma\left(\mathsf{K}(\alpha)\right)\right) \to 0$ for $\Im z \to 0$.

(iv) Let $y \in \sigma_{\text{sing}}(\mathsf{K}(\alpha))$ and $\xi_k(y) := \mathsf{P}_{sc}(\alpha)\Phi_k(y)$ with $\xi_k(y) \neq 0$. Then the claimed $\left\langle \xi_k(y), \left(\mathsf{K}(\alpha)-y-i\varepsilon\right)^{-1} \xi_l(y) \right\rangle_{\mathcal{H}} \underset{\varepsilon \downarrow 0}{\to} \infty$ is implied by spectral representation, polarization identity, Proposition 3.1 and Lemma A.1 (ii) $\square$

Finally, as the last statement of the Appendix, the Proof of Proposition 3.4, which has been omitted in the main text.

Proof of Proposition 3.4 Let $\mathfrak{U}_0(s,0)$ and $\mathfrak{B}_0 := \left\{\xi_\gamma, \gamma \in \mathbb{Z} \times \mathcal{N}\right\}$ be as in the Proof of Lemma A.2 and expand $\Phi_k(y) - \frac{\partial \Phi_k}{\partial x}(y)$ into the orthonormal basis $\mathfrak{B}_0$. As $\mathfrak{P}_0(s)$ is strongly differentiable by assumption and $\Phi_k(y)$, $\frac{\partial \Phi_k}{\partial x}(y) \in \mathcal{D}(\mathsf{D})$ according to Lemma A.1,

$$\begin{aligned}\frac{\partial}{\partial s}\left(\alpha \Phi_k(y) - \frac{\partial \Phi_k}{\partial x}(y)\right) &= \Bigg[\dot{\alpha}\,\Phi_k + \frac{2\pi i}{\Gamma} \sum_{\gamma=(j,k)} j\left(\alpha \langle \xi_\gamma, \Phi_k\rangle_{\mathcal{H}} - \langle \xi_\gamma, \frac{\partial \Phi_k}{\partial x}\rangle_{\mathcal{H}}\right)\xi_\gamma \\ &\qquad + \frac{\partial \mathfrak{P}_0(s)}{\partial s}[\mathfrak{P}_0(s)]^{-1}\left(\alpha \Phi_k(y) - \frac{\partial \Phi_k}{\partial x}\right)\Bigg](y).\end{aligned}$$

Now assume $\left(\alpha \Phi_k(\hat{y}) - \frac{\partial \Phi_k}{\partial x}(\hat{y})\right)\big|_{x=0}(\hat{s}) = 0$ for some $\hat{s} \in \pi_\Gamma$, $\hat{y} \in \mathbb{R}$ and infinitely many $k \in \hat{\mathcal{K}}$. Then the above yields

$$\frac{\partial}{\partial s}\left(\alpha \Phi_k - \frac{\partial \Phi_k}{\partial x}\right)\Big|_{x=0}(\hat{y}, \hat{s}) = \Delta_k \neq 0 \tag{4.1}$$

with $\Delta_{k_i} \neq \Delta_{k_j}$ for infinitely many pairwise different $(i,j) \in \mathbb{N}^2$, since otherwise

$$\dot{\alpha} + \alpha\, 2\pi j/\Gamma = \langle \xi_\gamma, \frac{\partial \Phi_k}{\partial x}\rangle_{\mathcal{H}} \left[\langle \xi_\gamma, \Phi_k \rangle_{\mathcal{H}}\right]^{-1} \tag{4.2}$$

respectively Φ_k replaced by $\Phi_{k_i} - \Phi_{k_j}$, for all γ and $k \in \hat{\mathcal{K}}$. Now the claim follows from contradiction with the implicit function theorem, which implies that the locations of the zeroes of $\left(\alpha \Phi_k(y) - \frac{\partial \Phi_k}{\partial x}(y)\right)\big|_{x=0}(s)$ are k-dependent.

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