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LOCALIZED NODAL SOLUTIONS FOR SEMICLASSICAL QUASILINEAR CHOQUARD EQUATIONS WITH SUBCRITICAL GROWTH

BO ZHANG, XIANGQING LIU

Abstract. In this article, we study the existence of localized nodal solutions for semiclassical quasilinear Choquard equations with subcritical growth

$$-\varepsilon^p \Delta_p v + V(x)|v|^{p-2}v = \varepsilon^{\alpha-N}|v|^{q-2}v \int_{\mathbb{R}^N} \frac{|v(y)|^q}{|x-y|^\alpha}\,dy, \quad x \in \mathbb{R}^N\,,$$

where $N \geq 3$, $1 < p < N$, $0 < \alpha < \min\{2p, N-1\}$, $p < q < p_\alpha^*$, $p_\alpha^* = \frac{p(2N-\alpha)}{2(N-p)}$, V is a bounded function. By the perturbation method and the method of invariant sets of descending flow, for small ε we establish the existence of a sequence of localized nodal solutions concentrating near a given local minimum point of the potential function V.

1. Introduction

In this article, we are interested in localized nodal solutions of the quasilinear Choquard equation

$$\begin{gathered} -\varepsilon^p \Delta_p v + V(x)|v|^{p-2}v = \varepsilon^{\alpha-N}|v|^{q-2}v \int_{\mathbb{R}^N} \frac{|v(y)|^q}{|x-y|^\alpha}\,dy, \quad x \in \mathbb{R}^N\,, \\ v(x) \to 0 \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.1}$$

where $\Delta_p = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ is the p-Laplacian operator, $N \geq 3$, $1 < p < N$, $0 < \alpha < \min\{2p, N-1\}$, $p < q < p_\alpha^*$, $p_\alpha^* = \frac{p(2N-\alpha)}{2(N-p)}$ is the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, $\varepsilon > 0$ is a small parameter. The potential function V satisfies the following assumptions:

(A1) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and there exist $b > a > 0$ such that
$$a \leq V(x) \leq b, \quad \forall x \in \mathbb{R}^N\,.$$

(A2) There exists a bounded domain $\mathcal{M} \subset \mathbb{R}^N$ with the smooth boundary $\partial\mathcal{M}$ such that
$$\langle \vec{n}(x), \nabla V(x)\rangle > 0\,, \quad \forall x \in \partial\mathcal{M}\,,$$
where $\vec{n}(x)$ is the outer normal of $\partial\mathcal{M}$ at x.

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In the previous decades, the Choquard type equation has been widely studied, we refer to [28, 11, 12, 16, 27, 20] and references therein. The Choquard equation

$$- \Delta u + u = u \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}\, dy\,, \quad x \in \mathbb{R}^3, \tag{1.2}$$

which goes back to the description of the quantum theory of a polaron at rest by Pekar [28], and which emerged in the work of Choquard on the modeling of an electron trapped in its own hole, as a certain approximation to Hartree-Fock theory of one-component plasma [11]. For the Choquard equation

$$\begin{gathered} -\Delta u + V(x)u = |u|^{p-2}u \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\alpha}\, dy, \quad x \in \mathbb{R}^N\,, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.3}$$

where $\frac{2N-\alpha}{N} \le p \le \frac{2N-\alpha}{N-2}$. When the potential V is a positive constant, Lieb [12] obtained the existence and uniqueness of positive radial ground states for (1.3), Lions [16] extended Lieb's results to a more general case and established the existence of infinitely many radial solutions, Ma and Zhao [20] studied the radial symmetry and uniqueness of positive ground states for (1.3) in higher dimension space via the method of moving planes. For more related results, we refer to [23, 24, 22, 21, 25] and references therein. For the semiclassical Choquard equation

$$\begin{gathered} -\varepsilon^2\Delta u + V(x)u = \varepsilon^{\alpha-N} g(x,u) \int_{\mathbb{R}^N} \frac{G(y,u(y))}{|x-y|^\alpha}\, dy, \quad x \in \mathbb{R}^N\,, \\ u(x) \to 0 \quad \text{as } |x| \to \infty\,. \end{gathered} \tag{1.4}$$

Alves and Yang [3] proved the existence, multiplicity and concentration of solutions for (1.4) by using the Lyapunov-Schmidt reduction method [10]. The existence and qualitative properties of solutions for (1.4) have been also studied extensively for recent decades by variational methods [30, 1, 2]. When $p = 2$, Moroz and Van Schaftingen [26] constructed the single spike solution concentrating around the local minimum of the potential V by using the nonlocal penalization method. Cassani and Zhang [5] considered the existence of ground states for (1.4) involving a critical nonlinearity in the sense of Hardy-Littlewood-Sobolev. Yang and Zhang [32] investigated the existence and concentration of solutions under the local potential well condition. Under conditions (A1) and (/a2), Li and Ma [18] proved the existence and concentration of infinite many solutions for the subcritical Choquard equation by using the penalization method [4] and symmetric mountain pass lemma.

In this article, we consider the existence of localized nodal solutions for the semiclassical Choquard equation (1.1) by using the method of invariant sets of descending flow and the perturbation method. Byeon-Wang type penalization method [4] can be used to deal with multiple localized nodal solutions for semiclassical Schrödinger equations. Additional coercive term [17] can be used to make the perturbed functional has necessary compactness properties in changed space.

Under the condition (A2), the critical set

$$\mathcal{A} = \{x \in \mathcal{M} : \nabla V(x) = 0\} \neq \emptyset,$$

and without loss of generality we assume $0 \in \mathcal{A}$. For any set $B \subset \mathbb{R}^N$ and any $\delta > 0$ we denote

$$B_\delta = \{x \in \mathbb{R}^N : \delta x \in B\},$$

$$B^\delta = \{x \in \mathbb{R}^N \,|\, \operatorname{dist}(x, B) := \inf_{y \in B} |x - y| < \delta\}.$$

Theorem 1.1. *Assume that* (A1) *and* (A2) *hold. Then for any positive integer k there exists $\varepsilon_k > 0$ such that if $0 < \varepsilon < \varepsilon_k$, equation* (1.1) *has at least k pairs of sign-changing solutions $\pm v_{j,\varepsilon}$, $j = 1, \dots, k$. Moreover, for any $\delta > 0$ there exist $\mu > 0$, $C = C_k > 0$ and $\varepsilon_k(\delta) > 0$ such that if $0 < \varepsilon < \varepsilon_k(\delta)$, then it holds that*

$$|v_{j,\varepsilon}(x)| \le C \exp\{-\frac{\mu}{\varepsilon} \operatorname{dist}(x, \mathcal{A}^\delta)\} \quad \text{for } x \in \mathbb{R}^N,\ j = 1, \dots, k.$$

Denoting $u(x) = v(\varepsilon x)$, equation (1.1) is equivalent to

$$-\Delta_p u + V(\varepsilon x)|u|^{p-2}u = |u|^{q-2}u \int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x - y|^\alpha}\, dy \quad x \in \mathbb{R}^N. \tag{1.5}$$

The energy functional associated with (1.5) is

$$I_\varepsilon(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p)\, dx - \frac{1}{2q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x - y|^\alpha}\, dx\, dy, \tag{1.6}$$

for $u \in W^{1,p}(\mathbb{R}^N)$.

To obtain multiple localized nodal solutions of (1.5), we use the penalization method due to Byeon and Wang [4]. Let $\zeta \in C^\infty(\mathbb{R}^N)$ be a cut-off function, $\zeta(t) = 0$ for $t \le 0$, $\zeta(t) = 1$ for $t \ge 1$, $0 \le \zeta'(t) \le 2$ and $0 \le \zeta(t) \le 1$. Define

$$\chi_\varepsilon(x) = \varepsilon^{-p} \zeta(\operatorname{dist}(x, \mathcal{M}_\varepsilon)).$$

Since the imbedding from $W^{1,p}(\mathbb{R}^N)$ to $L^s(\mathbb{R}^N)$ $(p < s < p^*)$ is continuous but not compact, we choose a suitable space as working space such that the functional I_ε recovers compactness on the changed space. We denote $X_\varepsilon = W^{1,p}(\mathbb{R}^N) \bigcap L^m_\varepsilon(\mathbb{R}^N)$, where $L^m_\varepsilon(\mathbb{R}^N)$ is a weighted space defined as

$$L^m_\varepsilon(\mathbb{R}^N) = \{u \in L^m(\mathbb{R}^N) : \int_{\mathbb{R}^N} \exp\{(m - p) \operatorname{dist}(\varepsilon x, M)\} |u|^m\, dx < +\infty\}$$

endowed with the norm

$$\|u\|_{L^m_\varepsilon(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} \exp\{(m - p) \operatorname{dist}(\varepsilon x, M)\} |u|^m\, dx \Big)^{1/m}.$$

We define

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} (|\nabla u|^p + E(\varepsilon x)|u|^p)\, dx \Big)^{1/p},$$
$$\|u\|_{X_\varepsilon} = \|u\|_{W^{1,p}(\mathbb{R}^N)} + \|u\|_{L^m_\varepsilon(\mathbb{R}^N)}.$$

Meanwhile, we introduce one additional coercive term such that I_ε has the necessary compactness property on X_ε. We need some auxiliary functions. Let $\xi(t) \in C^\infty(\mathbb{R}, [0, 1])$ be a smooth, even function, such that $\xi(t) = 1$ if $|t| \le 1$, $\xi(t) = 0$ if $|t| \ge 2$, $0 \le \xi(t) \le 1$, and ξ is decreasing in $[1, 2]$. For $\varepsilon \in (0, 1]$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, we define

$$b_\varepsilon(x, t) = \xi(\varepsilon \exp\{\operatorname{dist}(\varepsilon x, M)\} t), \quad m_\varepsilon(x, t) = \int_0^t b_\varepsilon(x, \tau)\, d\tau,$$

$$k_\varepsilon(x, t) = \Big(\frac{t}{m_\varepsilon(x, t)} \Big)^{m-p} |t|^{p-2} t, \quad K_\varepsilon(x, t) = \int_0^t k_\varepsilon(x, \tau)\, d\tau,$$

where $p < m < \min\{2, q\}$ if $1 < p < 2$; $p < m < q$ if $p \geq 2$. We define

$$\begin{aligned} \Gamma_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + E(\varepsilon x)|u|^p)\,dx + \sigma\int_{\mathbb{R}^N} K_\varepsilon(x,u)\,dx \\ + \frac{1}{p\beta}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^\beta - \frac{1}{2q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}\,dx\,dy, \end{aligned} \tag{1.7}$$

for $u \in X_\varepsilon$, where $p < p\beta < q$, $E(x) = V(x) - \sigma$, for small σ, E satisfies the assumptions (A1) and (A2) (with a different constant $a' = a - \sigma > 0$).

We will use the method of invariant sets of descending flow [19] to prove the existence of sign-changing critical points of $\Gamma_{\varepsilon,\lambda}$, but the method of invariant sets of descending flow can not fit well for the functional (1.7). So we use the perturbation method [9] to overcome this difficulty. For $t \in \mathbb{R}^+$, we define

$$\begin{gathered} b_\lambda(t) = \xi(\lambda t), \quad m_\lambda(t) = \int_0^t b_\lambda(\tau)\,d\tau, \\ g_\lambda(t) = \frac{m_\lambda(t)}{t}, \quad h_\lambda(t) = g_\lambda(t) + b_\lambda(t). \end{gathered}$$

Now we define

$$\begin{aligned} \Gamma_{\varepsilon,\lambda}(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + E(\varepsilon x)|u|^p)\,dx + \sigma\int_{\mathbb{R}^N} K_\varepsilon(x,u)\,dx \\ + \frac{1}{p\beta}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^\beta - \frac{1}{2q}g_\lambda(\varphi^{1/2}(u))\varphi(u), \quad u \in X_\varepsilon, \end{aligned} \tag{1.8}$$

where

$$\varphi(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^q|u(x)|^q}{|x-y|^\alpha}\,dx\,dy.$$

By Hardy-Littlewood-Sobolev inequality and the Sobolev inequality, we have

$$\varphi^{1/2}(u) \leq C_0\|u\|^q_{W^{1,p}(\mathbb{R}^N)}.$$

Note that if

$$\begin{gathered} \|u\|_{W^{1,p}(\mathbb{R}^N)} \leq \big(\frac{1}{C_0\lambda}\big)^{1/q}, \\ |u(x)| \leq \frac{1}{\varepsilon}\exp\{-\operatorname{dist}(\varepsilon x, M)\} \quad \text{for } x \in \mathbb{R}^N, \\ \Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+ = 0, \end{gathered}$$

for sufficiently small ε, λ, then $\Gamma_{\varepsilon,\lambda}(u) = I_\varepsilon(u)$ and $D\Gamma_{\varepsilon,\lambda}(u) = DI_\varepsilon(u)$. Hence we obtain solutions of (1.5).

In the following we use c to denote various constants, c_ε denotes constants depending on ε and c, c_ε may be used from line to line for different constants but independent of the arguments. This article is organized as follows. In Section 2 we prove preliminary results and verify the Palais-Smale condition for the functional $\Gamma_{\varepsilon,\lambda}$. In Section 3 we construct a sequence of sign-changing critical points of $\Gamma_{\varepsilon,\lambda}$ by using the method of invariant sets of descending flow. In Section 4 prove Theorem 1.1. In Section 5 we prove the uniform bounds on the critical points obtained in Section 3.

2. Palais-Smale condition

First collect some elementary results about the auxiliary functions and we prove that $\Gamma_{\varepsilon,\lambda}$ satisfies the (PS) condition.

Lemma 2.1 (Hardy-Littlewood-Sobolev inequality [13]). *Suppose $\alpha \in (0,N)$, and $s,r>1$ with $\frac{1}{s}+\frac{1}{r}=\frac{2N-\alpha}{N}$. Let $g\in L^s(\mathbb{R}^N), h\in L^r(\mathbb{R}^N)$, there exists a sharp constant $C(s,r,\alpha,N)$, independently of g,h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(x)h(y)}{|x-y|^\alpha}\,dx\,dy \le C(s,r,\alpha,N)\|g\|_{L^r(\mathbb{R}^N)}\|h\|_{L^s(\mathbb{R}^N)}.$$

Lemma 2.2. *For $x\in\mathbb{R}^N$ and $t\in\mathbb{R}$, it holds:*

1. $0\le b_\varepsilon(x,t)\le \frac{m_\varepsilon(x,t)}{t}\le 1$;
2. $m_\varepsilon(x,t)=t$, *if* $|t|<\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,M)\}$;
3. $\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,M)\}\le |m_\varepsilon(x,t)|\le c\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,M)\}$, *if* $\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,M)\}\le|t|\le 2\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,M)\}$;
4. $|m_\varepsilon(x,t)|=c\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,M)\}$, *if* $|t|\ge 2\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,M)\}$, *where* $c=\int_0^\infty\xi(\tau)\,d\tau$;
5. $c_1(1+\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x,M)\}|t|^{m-p})|t|^{p-2}t\le k_\varepsilon(x,t)$ $\le c_2(1+\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x,M)\}|t|^{m-p})|t|^{p-2}t$;
6. $\frac{1}{m}tk_\varepsilon(x,t)\le K_\varepsilon(x,t)\le\frac{1}{p}tk_\varepsilon(x,t)$;
7. $(k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2))(t_1-t_2)\ge c\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x,M)\}|t_1-t_2|^m(p\ge 2)$; $(k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2))(t_1-t_2)\ge c\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x,M)\}|t_1-t_2|^2(|t_1|^{2-m}-|t_2|^{2-m})^{-1}(1<p<2)$;
8. $|k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2)|\le c(|t_1|^{p-2}+|t_2|^{p-2}+\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x,M)\}$ $(|t_1|^{m-2}+|t_2|^{m-2}))|t_1-t_2|(p\ge 2)$, $|k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2)|\le c(|t_1-t_2|^{p-1}+\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x,M)\}|t_1-t_2|^{m-1})(1<p<2)$.

Proof. The proof is straightforward. We only prove (6), (7) and (8).

(6) Let $f(x,t)=K_\varepsilon(x,t)-\frac{1}{p}tk_\varepsilon(x,t)$, $g(x,t)=K_\varepsilon(x,t)-\frac{1}{m}tk_\varepsilon(x,t)$, since $f(x,0)=0$, $\frac{\partial f(x,t)}{\partial t}\le 0$ for $t\ge 0$; $\frac{\partial f(x,t)}{\partial t}\ge 0$ for $t\le 0$ and $g(x,0)=0$, $\frac{\partial g(x,t)}{\partial t}\ge 0$ for $t\ge 0$; $\frac{\partial g(x,t)}{\partial t}\le 0$ for $t\le 0$. So (6) holds.

We use the following elementary inequalities (see [8]). For $p\ge 2$ and $\xi,\eta\in\mathbb{R}^N$,

$$|\xi-\eta|^p\le d_1(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta,\xi-\eta), \tag{2.1}$$

$$|\,|\xi|^{p-2}\xi-|\eta|^{p-2}\eta|\le d_2(|\xi|+|\eta|)^{p-2}|\xi-\eta|^2. \tag{2.2}$$

For $1<p<2$ and $\xi,\eta\in\mathbb{R}^N$,

$$|\xi-\eta|^p\le d_3\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta,\xi-\eta\big)^{p/2}(|\xi|^p+|\eta|^p)^{\frac{2-p}{2}}, \tag{2.3}$$

$$|\,|\xi|^{p-2}\xi-|\eta|^{p-2}\eta|\le d_4|\xi-\eta|^{p-1}. \tag{2.4}$$

(7) For $p\ge 2$, by (2.1) we have

$$\begin{aligned}
&(k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2))(t_1-t_2)\\
&=\int_0^1\frac{\partial k_\varepsilon(x,\theta t_1+(1-\theta)t_2)}{\partial t}d\theta(t_1-t_2)^2\\
&\ge c\int_0^1\Big(\frac{\theta t_1+(1-\theta)t_2}{m_\varepsilon(x,\theta t_1+(1-\theta)t_2)}\Big)^{m-p}|\theta t_1+(1-\theta)t_2|^{p-2}d\theta(t_1-t_2)^2
\end{aligned}$$

$$
\begin{aligned}
&\geq c\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}\int_0^1 |\theta t_1+(1-\theta)t_2|^{m-2}d\theta(t_1-t_2)^2\\
&\geq c\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}|t_1-t_2|^m.
\end{aligned}
$$

For $1<p<2$, by (2.3) we have

$$
\begin{aligned}
&(k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2))(t_1-t_2)\\
&\geq c\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}\int_0^1 |\theta t_1+(1-\theta)t_2|^{m-2}d\theta(t_1-t_2)^2\\
&\geq c\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}|t_1-t_2|^2(|t_1|^{2-m}+|t_2|^{2-m})^{-1}.
\end{aligned}
$$

(8) For $p\geq 2$, by (2.2) we have

$$
\begin{aligned}
&|k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2)|\\
&=\Big|\int_0^1 \frac{\partial k_\varepsilon(x,\theta t_1+(1-\theta)t_2)}{\partial t}d\theta(t_1-t_2)\Big|\\
&\leq c\int_0^1\Big(\frac{\theta t_1+(1-\theta)t_2}{m_\varepsilon(x,\theta t_1+(1-\theta)t_2)}\Big)^{m-p}|\theta t_1+(1-\theta)t_2)|^{p-2}d\theta|t_1-t_2|\\
&\leq c\int_0^1(1+\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}|\theta t_1+(1-\theta)t_2|^{m-p})\\
&\quad\times|\theta t_1+(1-\theta)t_2|^{p-2}d\theta|t_1-t_2|\\
&\leq c\int_0^1|\theta t_1+(1-\theta)t_2|^{p-2}d\theta|t_1-t_2|+c\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}\\
&\quad\times\int_0^1|\theta t_1+(1-\theta)t_2|^{m-2}d\theta|t_1-t_2|\\
&\leq c\big(|t_1|^{p-2}+|t_1|^{p-2}+\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}\\
&\quad\times(|t_1|^{m-2}+|t_2|^{m-2})\big)|t_1-t_2|.
\end{aligned}
$$

For $1<p<2$, by (2.4) we have

$$
\begin{aligned}
&|k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2)|\\
&\leq c\int_0^1|\theta t_1+(1-\theta)t_2|^{p-2}d\theta|t_1-t_2|+c\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}\\
&\quad\times\int_0^1|\theta t_1+(1-\theta)t_2|^{m-2}d\theta|t_1-t_2|\\
&\leq c(|t_1-t_2|^{p-1}+\varepsilon^{m-p}\exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}|t_1-t_2|^{m-1}).
\end{aligned}
$$

□

Lemma 2.3. *For $t\in\mathbb{R}^+$, it holds*

(1) $g_\lambda(t)=1, g'_\lambda(t)=0$ *if* $0<t<\frac{1}{\lambda}$;
(2) $b_\lambda(t)t\leq g_\lambda(t)t\leq c_\lambda$, *where* $c_\lambda=\frac{\int_0^\infty\xi(\tau)\,d\tau}{\lambda}$;
(3) $g'_\lambda(t)t+g_\lambda(t)=b_\lambda(t)$.

The proof of the above lemma is easy, we omit it.

Lemma 2.4. *The imbedding $X_\varepsilon\hookrightarrow L^r(\mathbb{R}^N))$ $(1\leq r<p^*)$ is compact.*

Proof. Let $\{u_n\} \subset X_\varepsilon$ be bounded, then $u_n \rightharpoonup u$ in X_ε, up to a subsequence if necessary, $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)(1 \le r < p^*)$. We first prove $u_n \to u$ in $L^1(\mathbb{R}^N)$. For $R > 0$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_R(0)} |u|\,dx \\
&\le \Big(\int_{\mathbb{R}^N\setminus B_R(0)} e^{(m-p)dist(\varepsilon x,M)}|u|^m\,dx\Big)^{1/m}\Big(\int_{\mathbb{R}^N\setminus B_R(0)} e^{-\frac{m-p}{m-1}dist(\varepsilon x,M)}\,dx\Big)^{\frac{m-1}{m}} \\
&\le \|u\|_{L^m_\varepsilon(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N\setminus B_R(0)} e^{-\frac{m-p}{m-1}dist(\varepsilon x,M)}\,dx\Big)^{\frac{m-1}{m}} \\
&= o_R(1).
\end{aligned}
$$

Hence

$$
\int_{\mathbb{R}^N} |u_n-u|\,dx = \int_{B_R(0)} |u_n-u|\,dx + \int_{\mathbb{R}^N\setminus B_R(0)} |u_n-u|\,dx = o_n(1)+o_R(1)
$$

as $n\to\infty$. For $1<r<p^*$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |u_n-u|^r\,dx \\
&= \int_{\mathbb{R}^N} |u_n-u|^{r\theta+(1-\theta)r}\,dx \\
&\le \Big(\int_{\mathbb{R}^N} |u_n-u|^{r\theta\frac{1}{r\theta}}\,dx\Big)^{r\theta}\Big(\int_{\mathbb{R}^N} |u_n-u|^{(1-\theta)r\frac{p^*}{(1-\theta)r}}\,dx\Big)^{\frac{(1-\theta)r}{p^*}} \\
&\le c\Big(\int_{\mathbb{R}^N} |u_n-u|\,dx\Big)^{r\theta},
\end{aligned}
$$

where $0<\theta<1$, $\frac{1}{r}=\theta+\frac{1-\theta}{p^*}$. □

Lemma 2.5. *Let $\{u_n\} \subset X_\varepsilon$ be a Palais-Smale sequence of the functional $\Gamma_{\varepsilon,\lambda}$, then $\{u_n\}$ is bounded in X_ε.*

Proof. Since

$$
\begin{aligned}
&\langle D\Gamma_{\varepsilon,\lambda}(u),v\rangle \\
&= \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u\nabla v + E(\varepsilon x)|u|^{p-2}uv)\,dx + \sigma\int_{\mathbb{R}^N} k_\varepsilon(x,u)v\,dx \\
&\quad + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^{p-2}uv\,dx \\
&\quad - \frac{1}{2}h_\lambda(\varphi^{1/2}(u))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^q|u(x)|^{q-2}u(x)v(x)}{|x-y|^\alpha}\,dx\,dy,
\end{aligned}
\tag{2.5}
$$

for any $v\in X_\varepsilon$. By Lemma 2.2, we have

$$
\begin{aligned}
&\Gamma_{\varepsilon,\lambda}(u) - \frac{1}{q}\langle D\Gamma_{\varepsilon,\lambda}(u),u\rangle \\
&= (\frac{1}{p}-\frac{1}{q})\int_{\mathbb{R}^N} (|\nabla u|^p + E(\varepsilon x)|u|^p)\,dx + \sigma\int_{\mathbb{R}^N} K_\varepsilon(x,u)\,dx \\
&\quad - \frac{\sigma}{q}\int_{\mathbb{R}^N} k_\varepsilon(x,u)u\,dx + \frac{1}{p\beta}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^2\,dx-1\Big)_+^{\beta}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{q}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\,dx \\
&+ \frac{1}{2q} h_\lambda\big(\varphi^{1/2}(u)\big)\varphi(u) - \frac{1}{2q} g_\lambda\big(\varphi^{1/2}(u)\big)\varphi(u) \\
\geq{}& \big(\frac{1}{p} - \frac{1}{q}\big) \int_{\mathbb{R}^N} (|\nabla u|^p + E(\varepsilon x)|u|^p)\,dx + \sigma\big(\frac{1}{m} - \frac{1}{q}\big) \int_{\mathbb{R}^N} k_\varepsilon(x,u)u\,dx \\
&+ c\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^{\beta} - c \\
\geq{}& c\big(\|u\|^p_{W^{1,p}(\mathbb{R}^N)} + \|u\|^m_{L^m_\varepsilon(\mathbb{R}^N)}\big) + c\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^{\beta} - c.
\end{aligned}
$$

Hence the Palais-Smale sequence $\{u_n\}$ is bounded in X_ε. □

Lemma 2.6. *For every $\varepsilon > 0$, $\Gamma_{\varepsilon,\lambda}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset X_\varepsilon$ be a Palais-Smale sequence of the functional $\Gamma_{\varepsilon,\lambda}$. By Lemma 2.5, $\{u_n\}$ is bounded in X_ε. By Lemma 2.4, we can assume $u_n \to u$ in $L^r(R^N)$, $1 \leq r < p^*$. By Lemma 2.1 and Lemma 2.4, we obtain

$$
\begin{aligned}
o(1) ={}& \langle D\Gamma_{\varepsilon,\lambda}(u_k) - D\Gamma_{\varepsilon,\lambda}(u_l), u_k - u_l\rangle \\
={}& \int_{\mathbb{R}^N} (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_l|^{p-2}\nabla u_l, \nabla u_k - \nabla u_l)\,dx \\
&+ \int_{\mathbb{R}^N} E(\varepsilon x)(|u_k|^{p-2}u_k - |u_l|^{p-2}u_l)(u_k - u_l)\,dx \\
&+ \sigma \int_{\mathbb{R}^N} (k_\varepsilon(x,u_k) - k_\varepsilon(x,u_l))(u_k - u_l)\,dx \\
&+ \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u_k|^p\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)|u_k|^{p-2}u_k(u_k - u_l)dx \\
&+ \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u_l|^p\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)|u_l|^{p-2}u_l(u_k - u_l)dx \\
&- \frac{1}{2} h_\lambda(\varphi(u_k)) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(y)|^q|u_k(x)|^{q-2}u_k(x)(u_k(x) - u_l(x))}{|x-y|^\alpha}\,dx\,dy \\
&+ \frac{1}{2} h_\lambda(\varphi(u_l)) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_l(y)|^q|u_l(x)|^{q-2}u_l(x)(u_k(x) - u_l(x))}{|x-y|^\alpha}\,dx\,dy \\
\geq{}& \int_{\mathbb{R}^N} (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_l|^{p-2}\nabla u_l)\nabla(u_k - u_l)\,dx \\
&+ \int_{\mathbb{R}^N} E(\varepsilon x)(|u_k|^{p-2}u_k - |u_l|^{p-2}u_l)(u_k - u_l)\,dx \\
&+ \sigma \int_{\mathbb{R}^N} (k_\varepsilon(x,u_k) - k_\varepsilon(x,u_l))(u_k - u_l)\,dx + o(1).
\end{aligned}
$$

So

$$
\begin{gathered}
\int_{\mathbb{R}^N} (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_l|^{p-2}\nabla u_l, \nabla u_k - \nabla u_l)\,dx \to 0, \quad \text{as } k,l \to \infty, \\
\int_{\mathbb{R}^N} (|u_k|^{p-2}u_k - |u_l|^{p-2}u_l)(u_k - u_l)\,dx \to 0, \quad \text{as } k,l \to \infty, \\
\int_{\mathbb{R}^N} (k_\varepsilon(x,u_k) - k_\varepsilon(x,u_l))(u_k - u_l)\,dx \to 0, \quad \text{as } k,l \to \infty.
\end{gathered}
$$

For $p \geq 2$, by (2.1) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(u_k - u_l)|^p \, dx \\
&\leq c \int_{\mathbb{R}^N} (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_l|^{p-2}\nabla u_l, \nabla u_k - \nabla u_l) \, dx \to 0, \quad \text{as } k, l \to \infty,
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_k - u_l|^p \, dx \\
&\leq c \int_{\mathbb{R}^N} (|u_k|^{p-2}u_k - |u_l|^{p-2}u_l)(u_k - u_l) \, dx \to 0, \quad \text{as } k, l \to \infty,
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}^N} \exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}|u_k - u_l|^m \, dx \\
&\leq c \int_{\mathbb{R}^N} (k_\varepsilon(x, u_k) - k_\varepsilon(x, u_l))(u_k - u_l) \, dx \to 0, \quad \text{as } k, l \to \infty.
\end{aligned}$$

For $1 < p < 2$, by (2.3) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(u_k - u_l)|^p \, dx \\
&\leq c\Big(\int_{\mathbb{R}^N} (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_l|^{p-2}\nabla u_l, \nabla u_k - \nabla u_l) \, dx\Big)^{p/2} \\
&\quad \times \Big(\int_{\mathbb{R}^N} (|\nabla u_k|^p + |\nabla u_l|^p) \, dx\Big)^{\frac{2-p}{2}} \\
&\leq c\Big(\int_{\mathbb{R}^N} (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_l|^{p-2}\nabla u_l, \nabla u_k - \nabla u_l) \, dx\Big)^{p/2} \to 0, \quad \text{as } k, l \to \infty,
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}^N} E(\varepsilon x)|u_k - u_l|^p \, dx \\
&\leq c\Big(\int_{\mathbb{R}^N} (|u_k|^{p-2}u_k - |u_l|^{p-2}u_l)(u_k - u_l) \, dx\Big)^{p/2} \\
&\quad \times \Big(\int_{\mathbb{R}^N} (|u_k|^p + |u_l|^p) \, dx\Big)^{\frac{2-p}{2}} \\
&\leq c\Big(\int_{\mathbb{R}^N} (|u_k|^{p-2}u_k - |u_l|^{p-2}u_l)(u_k - u_l) \, dx\Big)^{p/2} \to 0, \quad \text{as } k, l \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}|(u_k - u_l)|^m \, dx \\
&\leq c\Big(\int_{\mathbb{R}^N} (k_\varepsilon(x, u_k) - k_\varepsilon(x, u_l))(u_k - u_l) \, dx\Big)^{m/2} \\
&\quad \times \Big(\int_{\mathbb{R}^N} \varepsilon^{m-p} \exp\{(m-p)\operatorname{dist}(\varepsilon x, M)\}(|u_k|^m + |u_l|^m) \, dx\Big)^{\frac{2-m}{2}} \\
&\leq c\Big(\int_{\mathbb{R}^N} (k_\varepsilon(x, u_k) - k_\varepsilon(x, u_l))(u_k - u_l) \, dx\Big)^{m/2} \to 0, \quad \text{as } k, l \to \infty.
\end{aligned}$$

So $\{u_n\}$ is a Cauchy sequence in X_ε. $\square$

3. Existence of solutions

In this section we construct a sequence of critical points of the functional $\Gamma_{\varepsilon,\lambda}$ by using the method of invariant sets of a descending flow. Firstly we define an operator $A : X \to X$. The vector field $u - Au$ will be used as pseudo-gradient vector field of the functional $\Gamma_{\varepsilon,\lambda}$. In order to obtain multiple sign-changing critical points of $\Gamma_{\varepsilon,\lambda}$, we introduce the abstract critical point theorem [14, Theorem 2.5], see also [6, Theorem 3.2].

Let X be a Banach space, f be an even C^1-functional on X. Let P, Q be two open convex sets of X, $Q = -P$. Set

$$W = P \cup Q, \quad \Sigma = \partial P \cap \partial Q.$$

Assume

- (A3) f satisfies the (PS) condition.
- (A4) $c^* = \inf_{x\in\Sigma} f(x) > 0$.

Assume there exists an odd continuous map $A : X \to X$ satisfying

- (A5) Given c_0, $b_0 > 0$, there exists $b = b(c_0, b_0) > 0$ such that if $\|Df(x)\| \ge b_0$, $|f(x)| \le c_0$, then
$$\langle Df(x), x - Ax\rangle \ge b\|x - Ax\|_X > 0\,.$$
- (A6) $A(\partial P) \subset P$, $A(\partial Q) \subset Q$.

We define

$$\Gamma_j = \{E \subset X : E \text{ is compact}, -E = E,\ \gamma(E \cap \eta^{-1}(\Sigma)) \ge j \text{ for } \eta \in \Lambda\},$$
$$\Lambda = \{\eta \in C(X, X) : \eta \text{ is odd}, \eta(P) \subset P,\ \eta(Q) \subset Q,\ \eta(x) = x \text{ if } f(x) < 0\}$$

where γ is the genus of symmetric sets,

$$\gamma(E) = \inf\big\{n : \text{there exists an odd map } \eta : E \to \mathbb{R}^n\backslash\{0\}\big\}\,.$$

Assume

- (A7) Γ_j is nonempty.

We define

$$c_j = \inf_{E\in\Gamma_j} \sup_{x\in E\backslash W} f(x), \quad j = 1, 2, \dots\,;$$
$$K_c = \{x : Df(x) = 0,\ f(x) = c\}, \quad K_c^* = K_c \setminus W\,.$$

Theorem 3.1. *Assume* (A3)–(A7) *hold. Then*

1. *$c_j \ge c^*$, $K^*_{c_j} \ne \emptyset$.*
2. *$c_j \to \infty$ as $j \to \infty$.*
3. *If $c_j = c_{j+1} = \dots = c_{j+k-1} = c$, then $\gamma(K^*_c) \ge k$.*

Lemma 3.2. *For any $v \in L^s(\mathbb{R}^N)$, $s \in (1, \frac{N}{N-\alpha})$, $\int_{\mathbb{R}^N} \frac{v(y)}{|x-y|^\alpha}\,dy \in L^{\frac{Ns}{N-Ns+\alpha s}}(\mathbb{R}^N)$. Moreover*

$$\Big(\int_{\mathbb{R}^N} \Big|\int_{\mathbb{R}^N} \frac{v(y)}{|x-y|^\alpha}\,dy\Big|^{\frac{Ns}{N-Ns+\alpha s}}\,dx\Big)^{\frac{N-Ns+\alpha s}{Ns}}\,dx \le c(s, N, \alpha)\|v\|_{L^s(\mathbb{R}^N)}.$$

By Hardy-Littlewood-Sobolev inequality, the proof of Lemma 3.2 is straightforward, we omit it.

Lemma 3.3 ([31, Theorem 4.2.7]). *Let $\Omega \subseteq \mathbb{R}^N$ be a domain and let $\{u_n\}$ be bounded in $L^q(\Omega)$ for some $q > 1$, If $u_n(x) \to u(x)$ a.e. in Ω as $n \to \infty$, then $u_n \rightharpoonup u$ in $L^q(\Omega)$ as $n \to \infty$.*

Lemma 3.4. *If $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$, $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$, then for $p < q < p^*_\alpha$,*

(1) $\int_{\mathbb{R}^N} \big||u_n|^q - |u_n - u|^q - |u|^q\big|^{\frac{2N}{2N-\alpha}}\, dy \to 0$ *as* $n \to \infty$.
(2) $\int_{\mathbb{R}^N} \big||u_n|^q + |u_n - u|^q - |u|^q\big|^{\frac{2N}{2N-\alpha}}\, dy \to 0$ *as* $n \to \infty$.
(3) $\int_{\mathbb{R}^N} \big||u_n|^{q-2}u_n - |u_n - u|^{q-2}(u_n - u) - |u|^{q-2}u\big|^{\frac{2Np}{2Np-\alpha p-2N+2p}}\, dx \to 0$ *as* $n \to \infty$.

The proof of the above lemma is similar to that of [33, Theorem 2.5], we omit it.

Lemma 3.5 ([7, Theorem 2.6]). *Let $\alpha \in (0, N)$, $s \in (1, \frac{N}{N-\alpha})$ and let $\{u_n\} \subset L^1(\mathbb{R}^N) \bigcap L^s(\mathbb{R}^N)$ be bounded and such that, up to a subsequence, for any bounded domain $\Omega \subset \mathbb{R}^N$, $u_n \to 0$ in $L^s(\mathbb{R}^N)$ as $n \to \infty$. Then, up to a subsequence if necessary, $\int_{\mathbb{R}^N} \frac{u_n(y)}{|x-y|^\alpha}\, dy \to 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$.*

Lemma 3.6. *Let $\{u_n\} \subset W^{1,p}(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Then, up to a subsequence if necessary,*
$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^q|u_n(x)|^q}{|x-y|^\alpha}\, dx\, dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)-u(y)|^q|u_n(x)-u(x)|^q}{|x-y|^\alpha}\, dx\, dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^q|u(x)|^q}{|x-y|^\alpha}\, dx\, dy + o_n(1).
\end{aligned} \tag{3.1}
$$
$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^q|u_n(x)|^{q-2}u_n(x)\phi(x)}{|x-y|^\alpha}\, dx\, dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)-u(y)|^q|u_n(x)-u(x)|^{q-2}(u_n(x)-u(x))\phi(x)}{|x-y|^\alpha}\, dx\, dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^q|u(x)|^{q-2}u(x)\phi(x)}{|x-y|^\alpha}\, dx\, dy + o_n(1)\|\phi\|_{W^{1,p}(\mathbb{R}^N)},
\end{aligned} \tag{3.2}
$$
where $0 < \alpha < \min\{2p, N\}$, $p < q < p^*_\alpha$, $\phi(x) \in C_0^\infty(\mathbb{R}^N)$ and $o_n(1) \to 0$ as $n \to \infty$.

From Lemmas 3.2-3.5, we can prove this lemma according to [7, Lemma 2.2, Lemma 2.4], we omit it. We define
$$
J_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + E(\varepsilon x)|u|^p)\, dx + \sigma\int_{\mathbb{R}^N} K_\varepsilon(x,u)\, dx.
$$

Definition 3.7. Given $u \in X_\varepsilon$, define $v = Au$ by the equation
$$
\begin{aligned}
&\frac{1}{2}\langle DJ_\varepsilon(u) + DJ_\varepsilon(v) - DJ_\varepsilon(u-v), \eta\rangle \\
&+ \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\, dx - 1\Big)_+^{\beta-1}\int_{\mathbb{R}^N} \chi_\varepsilon(x)|v|^{p-2}v\eta\, dx \\
&= \frac{1}{2}h_\lambda\big(\varphi^{1/2}(u)\big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^q|u(x)|^{q-2}u(x)\eta(x)}{|x-y|^\alpha}\, dx\, dy, \quad \eta \in X_\varepsilon.
\end{aligned} \tag{3.3}
$$

Lemma 3.8. *If $\|u\|_{X_\varepsilon}$ is bounded, then $\|Au\|_{X_\varepsilon}$ is bounded.*

Proof. By (3.3)

$$\begin{aligned}
&\|Au\|^p_{W^{1,p}(\mathbb{R}^N)}+\|Au\|^m_{L^m_\varepsilon(\mathbb{R}^N)}\\
&\le c\langle DJ_\varepsilon(v),v\rangle\\
&\le ch_\lambda(\varphi^{1/2}(u))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^q|u(x)|^{q-2}u(x)v(x)}{|x-y|^\alpha}\,dx\,dy\\
&\le c\|u\|^{2q-1}_{W^{1,p}(\mathbb{R}^N)}\|Au\|_{W^{1,p}(\mathbb{R}^N)}\\
&\le c\|Au\|_{W^{1,p}(\mathbb{R}^N)},
\end{aligned}$$

so $\|Au\|_{X_\varepsilon}$ is bounded. □

Lemma 3.9. *For u, $v\in X_\varepsilon$, the following holds: (1) For $p\ge 2$,*

$$\begin{aligned}
&\langle DJ_\varepsilon(u)-DJ_\varepsilon(v),\phi\rangle\\
&\le c\big(\|u\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}+\|v\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\big)\|u-v\|_{W^{1,p}(\mathbb{R}^N)}\|\phi\|_{W^{1,p}(\mathbb{R}^N)}\\
&\quad+c\big(\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}+\|v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\big)\|u-v\|_{L^m_\varepsilon(\mathbb{R}^N)}\|\phi\|_{L^m_\varepsilon(\mathbb{R}^N)}.
\end{aligned}$$

(2) For $1<p<2$,

$$\begin{aligned}
&\langle DJ_\varepsilon(u)-DJ_\varepsilon(v),\phi\rangle\\
&\le c\big(\|u-v\|^{p-1}_{W^{1,p}(\mathbb{R}^N)}\|\phi\|_{W^{1,p}(\mathbb{R}^N)}+\|u-v\|^{m-1}_{L^m_\varepsilon(\mathbb{R}^N)}\|\phi\|_{L^m_\varepsilon(\mathbb{R}^N)}\big).
\end{aligned}$$

(3) For $p>1$,

$$\langle DJ_\varepsilon(u-v),\phi\rangle\le c\big(\|u-v\|^{p-1}_{W^{1,p}(\mathbb{R}^N)}\|\phi\|_{W^{1,p}(\mathbb{R}^N)}+\|u-v\|^{m-1}_{L^m_\varepsilon(\mathbb{R}^N)}\|\phi\|_{L^m_\varepsilon(\mathbb{R}^N)}\big).$$

Proof. We only verify (1). By (2.2) and the Hölder inequality, for $p\ge 2$, we have

$$\begin{aligned}
&\langle DJ_\varepsilon(u)-DJ_\varepsilon(v),\phi\rangle\\
&\le c\int_{\mathbb{R}^N}(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v,\nabla\phi)\,dx+c\int_{\mathbb{R}^N}(|u|^{p-2}u-|v|^{p-2}v)\phi\,dx\\
&\quad+c\int_{\mathbb{R}^N}(k_\varepsilon(x,u)-k_\varepsilon(x,v))\phi\,dx\\
&\le c\int_{\mathbb{R}^N}(|\nabla u|^{p-2}+|\nabla v|^{p-2})|\nabla(u-v)||\nabla\phi|\,dx\\
&\quad+c\int_{\mathbb{R}^N}(|u|^{p-2}+|v|^{p-2})|u-v||\phi|\,dx\\
&\quad+c\int_{\mathbb{R}^N}\exp\{(m-p)\operatorname{dist}(\varepsilon x,M)\}(|u|^{m-2}+|v|^{m-2})|u-v||\phi|\,dx\\
&\le c\big(\|u\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}+\|v\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\big)\|u-v\|_{W^{1,p}(\mathbb{R}^N)}\|\phi\|_{W^{1,p}(\mathbb{R}^N)}\\
&\quad+c\big(\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}+\|v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\big)\|u-v\|_{L^m_\varepsilon(\mathbb{R}^N)}\|\phi\|_{L^m_\varepsilon(\mathbb{R}^N)}.
\end{aligned}$$

□

Lemma 3.10. *A is odd, well defined, and continuous on X_ε.*

Proof. It is easy to see that A is odd. We define

$$G(v)=\frac12\langle DJ_\varepsilon(u),v\rangle+\frac12 J_\varepsilon(v)+\frac12 J_\varepsilon(u-v)$$

$$
\begin{aligned}
&+\frac{1}{p}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^p\,dx\\
&-\frac{1}{2}h_\lambda\big(\varphi^{1/2}(u)\big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^q|u(x)|^{q-2}u(x)v(x)}{|x-y|^\alpha}\,dx\,dy,\ v\in X_\varepsilon.
\end{aligned}
$$

Equation (3.3) has a unique solution $v=Au$, which can be obtained by solving the minimization problem $\inf\{G(v):v\in X_\varepsilon\}$. Since

$$
G(v)\ge \frac{1}{2}\langle DJ_\varepsilon(u),v\rangle+\frac{1}{2}J_\varepsilon(v)-c_\lambda\ge c_1\|v\|^p_{W^{1,p}(\mathbb{R}^N)}-c_2\|v\|_{W^{1,p}(\mathbb{R}^N)}-c_\lambda,
$$

G is coercive.

Let $\{v_n\}\subset X_\varepsilon$ be a minimizing sequence for the functional G, $v_n\rightharpoonup v$ in X_ε. By the lower semicontinuity

$$
G(v)=\liminf_{n\to\infty}G(v_n)=\inf\{G(v)\big|v\in X_\varepsilon\},
$$

so v is a solution of (3.3). Assume v_1, v_2 are solutions of (3.3), then taking (v_1-v_2) as the test function, we have

$$
\begin{aligned}
&\frac{1}{2}\langle DJ_\varepsilon(v_1)-DJ_\varepsilon(v_2),v_1-v_2\rangle-\frac{1}{2}\langle DJ_\varepsilon(u-v_1)-DJ_\varepsilon(u-v_2),v_1-v_2\rangle\\
&+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)(|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_1-v_2)\,dx=0.
\end{aligned}
$$

Hence

$$
\langle DJ_\varepsilon(v_1)-DJ_\varepsilon(v_2),v_1-v_2\rangle=0.
$$

For $p\ge 2$, by (2.1) we have

$$
\langle DJ_\varepsilon(v_1)-DJ_\varepsilon(v_2),v_1-v_2\rangle\ge c(\|v_1-v_2\|^p_{W^{1,p}(\mathbb{R}^N)}+\|v_1-v_2\|^m_{L^m_\varepsilon(\mathbb{R}^N)}).
$$

For $1<p<2$, by (2.3) we have

$$
\begin{aligned}
&\langle DJ_\varepsilon(v_1)-DJ_\varepsilon(v_2),v_1-v_2\rangle\\
&\ge c\Big(\int_{\mathbb{R}^N}|\nabla(v_1-v_2)|^p\,dx\Big)^{2/p}\Big(\int_{\mathbb{R}^N}(|\nabla v_1|^p+|\nabla v_2|^p)\,dx\Big)^{\frac{p-2}{p}}\\
&\quad+c\Big(\int_{\mathbb{R}^N}|v_1-v_2|^p\,dx\Big)^{2/p}\Big(\int_{\mathbb{R}^N}(|v_1|^p+|v_2|^p)\,dx\Big)^{\frac{p-2}{p}}\\
&\quad+c\Big(\int_{\mathbb{R}^N}\exp\{(m-p)\operatorname{dist}(\varepsilon x,M)\}|v_1-v_2|^m\,dx\Big)^{\frac{2}{m}}\\
&\quad\times\Big(\int_{\mathbb{R}^N}\exp\{(m-p)\operatorname{dist}(\varepsilon x,M)\}(|v_1|^m+|v_2|^m)\,dx\Big)^{\frac{m-2}{m}}\\
&\ge c\|v_1-v_2\|^2_{W^{1,p}(\mathbb{R}^N)}\big(\|v_1\|^{2-p}_{W^{1,p}(\mathbb{R}^N)}+\|v_2\|^{2-p}_{W^{1,p}(\mathbb{R}^N)}\big)^{-1}\\
&\quad+\|v_1-v_2\|^2_{L^m_\varepsilon(\mathbb{R}^N)}\big(\|v_1\|^{2-m}_{L^m_\varepsilon(\mathbb{R}^N)}+\|v_2\|^{2-m}_{L^m_\varepsilon(\mathbb{R}^N)}\big)^{-1}.
\end{aligned}
$$

Then we have $v_1=v_2$ in X_ε. So (3.3) has a unique solution $v=Au$. Denoting

$$
\psi_\varepsilon(u)=\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx-1\Big)_+^{\beta-1},
$$

and taking $\eta = v_n - v$ in (3.3), we have

$$
\begin{aligned}
&\frac{1}{2}\langle DJ_\varepsilon(u-v) - DJ_\varepsilon(u_n - v_n), (u-v) - (u_n - v_n)\rangle \\
&+ \frac{1}{2}\langle DJ_\varepsilon(v_n) - DJ_\varepsilon(v), v_n - v\rangle \\
&+ \psi_\varepsilon(u_n)\int_{\mathbb{R}^N} \chi_\varepsilon(x)(|v_n|^{p-2}v_n - |v|^{p-2}v)(v_n - v)\,dx \\
&= \frac{1}{2}\langle DJ_\varepsilon(u-v) - DJ_\varepsilon(u_n - v_n), u - u_n\rangle + \frac{1}{2}\langle DJ_\varepsilon(u_n) - DJ_\varepsilon(u), v - v_n\rangle \\
&\quad + \big(\psi_\varepsilon(u_n) - \psi_\varepsilon(u)\big)\int_{\mathbb{R}^N} \chi_\varepsilon(x)|v|^{p-2}v(v - v_n)\,dx \\
&\quad + \frac{1}{2}h_\lambda(\varphi^{1/2}(u_n))t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(\big(|u_n(y)|^q|u_n(x)|^{q-2}u_n(x) - |u(y)|^q|u(x)|^{q-2}u(x)\big) \\
&\quad \times \big(v_n(x) - v(x)\big)\Big)/|x-y|^\alpha\,dx\,dy \\
&\quad + \frac{1}{2}(h_\lambda(\varphi^{1/2}(u_n)) - h_\lambda(\varphi^{1/2}(u))) \\
&\quad \times \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^q|u(x)|^{q-2}u(x)(v_n(x) - v(x))}{|x-y|^\alpha}\,dx\,dy.
\end{aligned}
\tag{3.4}
$$

Now we estimate the right-hand side of (3.4). By Lemma 3.8 and Lemma 3.9, suppose $u_n \to u$ in X_ε, for $p \geq 2$, we have

$$
\begin{aligned}
&\langle DJ_\varepsilon(u-v) - DJ_\varepsilon(u_n - v_n), u - u_n\rangle + \langle DJ_\varepsilon(u_n) - DJ_\varepsilon(u), v - v_n\rangle \\
&\leq c\big(\|u-v\|^{p-2}_{W^{1,p}(\mathbb{R}^N)} + \|u_n - v_n\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\big)\|u - v - u_n + v_n\|_{W^{1,p}(\mathbb{R}^N)} \\
&\quad \times \|u - u_n\|_{W^{1,p}(\mathbb{R}^N)} \\
&\quad + c\big(\|u-v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)} + \|u_n - v_n\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\big)\|u - v - u_n + v_n\|_{L^m_\varepsilon(\mathbb{R}^N)} \\
&\quad \times \|u - u_n\|_{L^m_\varepsilon(\mathbb{R}^N)} \\
&\quad + c\big(\|u_n\|^{p-2}_{W^{1,p}(\mathbb{R}^N)} + \|u\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\big)\|u_n - u\|_{W^{1,p}(\mathbb{R}^N)}\|v - v_n\|_{W^{1,p}(\mathbb{R}^N)} \\
&\quad + c\big(\|u_n\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)} + \|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\big)\|u_n - u\|_{L^m_\varepsilon(\mathbb{R}^N)}\|v - v_n\|_{L^m_\varepsilon(\mathbb{R}^N)} \\
&\leq c\|u - u_n\|_{X_\varepsilon} = o_n(1).
\end{aligned}
\tag{3.5}
$$

For $1 < p < 2$, we have

$$
\begin{aligned}
&\langle DJ_\varepsilon(u-v) - DJ_\varepsilon(u_n - v_n), u - u_n\rangle + \langle DJ_\varepsilon(u_n) - DJ_\varepsilon(u), v - v_n\rangle \\
&\leq c\big(\|u - v - u_n + v_n\|^{p-1}_{W^{1,p}(\mathbb{R}^N)} + \|u - v - u_n + v_n\|^{m-1}_{L^m_\varepsilon(\mathbb{R}^N)}\big)\|u - u_n\|_{X_\varepsilon} \\
&\quad + c\big(\|u_n - u\|^{p-1}_{W^{1,p}(\mathbb{R}^N)} + \|u_n - u\|^{m-1}_{L^m_\varepsilon(\mathbb{R}^N)}\big)\|v - v_n\|_{X_\varepsilon} = o_n(1).
\end{aligned}
\tag{3.6}
$$

By Lemmas 2.1and 3.8 and the continuity of $\psi_\varepsilon(u)$ and $h_\lambda\big(\varphi^{1/2}(u)\big)$ for u, we have

$$
\begin{aligned}
&\big(\psi_\varepsilon(u_n) - \psi_\varepsilon(u)\big)\int_{\mathbb{R}^N} \chi_\varepsilon(x)|v|^{p-2}v(v - v_n)\,dx \\
&+ \frac{1}{2}\big(h_\lambda(\varphi^{1/2}(u_n)) - h_\lambda(\varphi^{1/2}(u))\big) \\
&\times \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^q|u(x)|^{q-2}u(x)(v_n(x) - v(x))}{|x-y|^\alpha}\,dx\,dy = o_n(1).
\end{aligned}
\tag{3.7}
$$

By Lemmas 2.1,3.6 and 3.8, we have

$$\begin{aligned} &\frac{1}{2}h_\lambda\big(\varphi^{1/2}(u_n)\big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(\big(|u_n(y)|^q|u_n(x)|^{q-2}u_n(x)\\ &\quad -|u(y)|^q|u(x)|^{q-2}u(x)\big)\big(v_n(x)-v(x)\big)\Big)/|x-y|^\alpha\,dx\,dy\\ &=\frac{1}{2}h_\lambda(\varphi^{1/2}(u_n))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(|u_n(y)-u(y)|^q|u_n(x)-u(x)|^{q-2}\\ &\quad\times(u_n(x)-u(x))(v_n(x)-v(x))\Big)/|x-y|^\alpha\,dx\,dy+o_n(1)=o_n(1). \end{aligned} \tag{3.8}$$

So the right-hand side of (3.4) satisfies

$$RHS=o_n(1). \tag{3.9}$$

Next, we estimate the left-hand side of (3.4), for $p\ge 2$, by (2.1),

$$\begin{aligned} LHS&\ge\langle DJ_\varepsilon(v_n)-DJ_\varepsilon(v),v_n-v\rangle\\ &\ge c\big(\|v_n-v\|^p_{W^{1,p}(\mathbb{R}^N)}+\|v_n-v\|^m_{L^m_\varepsilon(\mathbb{R}^N)}\big). \end{aligned} \tag{3.10}$$

For $1<p<2$, by (2.3),

$$\begin{aligned} LHS\ge&\langle DJ_\varepsilon(v_n)-DJ_\varepsilon(v),v_n-v\rangle\\ \ge& c\|v_n-v\|^2_{W^{1,p}(\mathbb{R}^N)}\big(\|v_n\|^{2-p}_{W^{1,p}(\mathbb{R}^N)}+\|v\|^{2-p}_{W^{1,p}(\mathbb{R}^N)}\big)^{-1}\\ &+\|v_n-v\|^2_{L^m_\varepsilon(\mathbb{R}^N)}\big(\|v_n\|^{2-m}_{L^m_\varepsilon(\mathbb{R}^N)}+\|v\|^{2-m}_{L^m_\varepsilon(\mathbb{R}^N)}\big)^{-1}. \end{aligned} \tag{3.11}$$

By (3.9)-(3.11), for $p>1$, we obtain $\|v_n-v\|_{X_\varepsilon}\to 0$ as $n\to\infty$. □

Lemma 3.11. *Let $u\in X_\varepsilon$, $v=A(u)$, then the following holds:*

1. $\langle D\Gamma_{\varepsilon,\lambda}(u),u-v\rangle\ge c(\|u-v\|^p_{W^{1,p}(\mathbb{R}^N)}+\|u-v\|^m_{L^m_\varepsilon(\mathbb{R}^N)})$;
2. $\|D\Gamma_{\varepsilon,\lambda}(u)\|\le c(1+|\Gamma_{\varepsilon,\lambda}(u)|+\|u-v\|_{X_\varepsilon})^\gamma\|u-v\|_{X_\varepsilon}$ $(\gamma>1)$.

Proof. (1) We denote $v=Au$. For $\eta\in X_\varepsilon$, we have

$$\begin{aligned} &\langle D\Gamma_{\varepsilon,\lambda}(u),\eta\rangle\\ &=\frac{1}{2}\langle DJ_\varepsilon(u)-DJ_\varepsilon(v),\eta\rangle+\frac{1}{2}\langle DJ_\varepsilon(u-v),\eta\rangle\\ &\quad+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)(|u|^{p-2}u-|v|^{p-2}v)\eta\,dx\,. \end{aligned} \tag{3.12}$$

Hence

$$\begin{aligned} &\langle D\Gamma_{\varepsilon,\lambda}(u),u-v\rangle\\ &=\frac{1}{2}\langle DJ_\varepsilon(u)-DJ_\varepsilon(v),u-v\rangle+\frac{1}{2}\langle DJ_\varepsilon(u-v),u-v\rangle\\ &\quad+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)(|u|^{p-2}u-|v|^{p-2}v)(u-v)\,dx\\ &\ge c\big(\|u-v\|^p_{W^{1,p}(\mathbb{R}^N)}+\|u-v\|^m_{L^m_\varepsilon(\mathbb{R}^N)}\big). \end{aligned}$$

(2) By (3.3) and Lemma 3.9, we have

$$
\begin{aligned}
&\Gamma_{\varepsilon,\lambda}(u) - \frac{1}{2q}\langle DJ_\varepsilon(u) - DJ_\varepsilon(v), u\rangle \\
&= \big(\frac{1}{p} - \frac{1}{q}\big)\int_{\mathbb{R}^N} (|\nabla u|^p + E(\varepsilon x)|u|^p)\,dx + \sigma \int_{\mathbb{R}^N} K_\varepsilon(x,u)\,dx \\
&\quad - \frac{\sigma}{q}\int_{\mathbb{R}^N} k_\varepsilon(x,u)u\,dx + \frac{1}{p\beta}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^\beta \\
&\quad - \frac{1}{q}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^{\beta-1}\int_{\mathbb{R}^N} \chi_\varepsilon(x)|v|^{p-2}vu\,dx \\
&\quad + \frac{1}{2q} b_\lambda\big(\varphi^{1/2}(u)\big)\varphi(u) + \frac{1}{2q}\langle DJ_\varepsilon(u-v), u\rangle \\
&\geq c\big(\|u\|^p_{W^{1,p}(\mathbb{R}^N)} + \|u\|^m_{L^m_\varepsilon(\mathbb{R}^N)}\big) + c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^\beta \\
&\quad - c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u-v|^p\,dx\Big)^\beta \\
&\quad - c\big(\|u\|^{p-2}_{W^{1,p}(\mathbb{R}^N)} + \|v\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\big)\|u-v\|_{W^{1,p}(\mathbb{R}^N)}\|u\|_{W^{1,p}(\mathbb{R}^N)} \\
&\quad - c\big(\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)} + \|v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\big)\|u-v\|_{L^m_\varepsilon(\mathbb{R}^N)}\|u\|_{L^m_\varepsilon(\mathbb{R}^N)} - c,
\end{aligned}
\tag{3.13}
$$

where we have used the estimate

$$
\begin{aligned}
&\frac{1}{p\beta}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^\beta - \frac{1}{q}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^{p-2}vu\,dx \\
&= \frac{1}{p\beta}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^\beta - \frac{1}{q}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx \\
&\quad + \frac{1}{q}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)(|u|^{p-2}u - |v|^{p-2}v)u\,dx \\
&\geq c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^\beta - c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)(|u|^{p-2} + |v|^{p-2})|u-v||u|\,dx\Big)^\beta - c \\
&\geq c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^\beta - c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u-v|^p\,dx\Big)^\beta - c.
\end{aligned}
$$

On the other hand, by Lemma 3.9,

$$
\begin{aligned}
&\Gamma_{\varepsilon,\lambda}(u) - \frac{1}{2q}\langle DJ_\varepsilon(u) - DJ_\varepsilon(v), u\rangle \\
&\leq |\Gamma_{\varepsilon,\lambda}(u)| + c\big(\|u\|^{p-2}_{W^{1,p}(\mathbb{R}^N)} + \|v\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\big)\|u-v\|_{W^{1,p}(\mathbb{R}^N)}\|u\|_{W^{1,p}(\mathbb{R}^N)} \\
&\quad + c\big(\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)} + \|v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\big)\|u-v\|_{L^m_\varepsilon(\mathbb{R}^N)}\|u\|_{L^m_\varepsilon(\mathbb{R}^N)}.
\end{aligned}
\tag{3.14}
$$

By (3.13), (3.14) and the Young's inequality, we have

$$
\begin{aligned}
&\|u\|^p_{W^{1,p}(\mathbb{R}^N)} + \|u\|^m_{L^m_\varepsilon(\mathbb{R}^N)} + \Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^\beta \\
&\leq c(1 + |\Gamma_{\varepsilon,\lambda}(u)| + \|u-v\|^p_{W^{1,p}(\mathbb{R}^N)} + \|u-v\|^m_{L^m_\varepsilon(\mathbb{R}^N)} + \|u-v\|^{p\beta}_{W^{1,p}(\mathbb{R}^N)}).
\end{aligned}
\tag{3.15}
$$

By Lemma 3.9, (3.12), (3.15) and Young's inequality, we have

$$
\begin{aligned}
&\|D\Gamma_{\varepsilon,\lambda}(u)\| \\
&\leq c\Big(\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\Big)_+^{\beta-1} + 1\Big)(\|u\|^{p-2}_{W^{1,p}(\mathbb{R}^N)} + \|v\|^{p-2}_{W^{1,p}(\mathbb{R}^N)})\|u-v\|_{X_\varepsilon}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + c\big(\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)} + \|v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\big)\|u-v\|_{X_\varepsilon} \\
&\le c\big(1+|\Gamma_{\varepsilon,\lambda}(u)| + \|u-v\|^p_{W^{1,p}(\mathbb{R}^N)} + \|u-v\|^m_{L^m_\varepsilon(\mathbb{R}^N)} + \|u-v\|^{p\beta}_{W^{1,p}(\mathbb{R}^N)}\big)^2 \\
&\quad \times \|u-v\|_{X_\varepsilon} \\
&\le c(1+|\Gamma_{\varepsilon,\lambda}(u)| + \|u-v\|_{X_\varepsilon})^\gamma \|u-v\|_{X_\varepsilon}.
\end{aligned}
$$

□

Corollary 3.12. *For all $b_0, c_0 > 0$, there exists $b = b(b_0, c_0) > 0$ such that if $|\Gamma_{\varepsilon,\lambda}(u)| \le c_0$ and $\|D\Gamma_{\varepsilon,\lambda}(u)\| \ge b_0$, then $u - Au \ne 0$ and*

$$\langle D\Gamma_{\varepsilon,\lambda}(u), u - Au\rangle \ge b\|u - Au\|_{X_\varepsilon} > 0.$$

Now we define the convex open sets

$$
\begin{aligned}
P &= \{u \big| u \in X_\varepsilon,\ \|u_-\|_{W^{1,p}(\mathbb{R}^N)} < \delta\}, \\
Q &= \{u \big| u \in X_\varepsilon,\ \|u_+\|_{W^{1,p}(\mathbb{R}^N)} < \delta\},
\end{aligned}
$$

where δ is a positive constant, $u_- = \min\{u, 0\}$ and $u_+ = \max\{u, 0\}$. We denote

$$D(f,g) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\alpha}\,dx\,dy.$$

Lemma 3.13 ([13, Theorem 9.8])**.** *Let $N \ge 3$, $0 < \alpha < N$ and $D(f,f), D(g,g) < \infty$, then*

$$|D(f,g)|^2 \le D(f,f)D(g,g)$$

with equality for $g \ne 0$ only when $f = cg$ for some constant c.

Lemma 3.14. *There exists $\delta_\lambda > 0$ such that for $0 < \delta < \delta_\lambda$,*

$$A(\partial P) \subset P, \quad A(\partial Q) \subset Q.$$

Proof. We only prove $A(\partial Q) \subset Q$. For $u \in \partial Q$, let $v = Au$. By Lemma s2.3 and 3.13, we have

$$
\begin{aligned}
&\|v_+\|^p_{W^{1,p}(\mathbb{R}^N)} \\
&\le c\langle J_\varepsilon(v), v_+\rangle \\
&\le ch_\lambda\big(\varphi^{1/2}(u)\big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u(y)|^q|u(x)|^{q-2}u(x)v_+(x)}{|x-y|^\alpha}\,dx\,dy \\
&\le ch_\lambda\big(\varphi^{1/2}(u)\big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u(y)|^q|u_+(x)|^{q-2}u_+(x)v_+(x)}{|x-y|^\alpha}\,dx\,dy \\
&\le ch_\lambda\big(\varphi^{1/2}(u)\big)\varphi^{1/2}(u)\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_+(x)|^{q-1}|v_+(x)||u_+(y)|^{q-1}|v_+(y)|}{|x-y|^\alpha}\,dx\,dy\Big)^{1/2} \\
&\le c_\lambda \|u_+\|^{q-1}_{W^{1,p}(\mathbb{R}^N)}\|v_+\|_{W^{1,p}(\mathbb{R}^N)},
\end{aligned}
$$

taking $\delta_\lambda \le c_\lambda^{-\frac{1}{q-p}}$ the conclusion follows. □

Lemma 3.15. *There exist $\delta_0 > 0$ and $c^* = c^*(\delta)$, such that for any $0 < \delta < \delta_0$,*

$$\Gamma_{\varepsilon,\lambda}(u) \ge c^* > 0 \quad \text{for all } u \in \partial P \cap \partial Q.$$

Proof. For $u \in \partial P \cap \partial Q$, we have

$$\begin{aligned}
\Gamma_{\varepsilon,\lambda}(u) &\geq \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + E(\varepsilon x)|u|^p)\,dx - \frac{1}{2q}g_\lambda(\varphi(u))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}\,dx\,dy \\
&\geq c_1\|u\|^p_{W^{1,p}(\mathbb{R}^N)} - c_2\|u\|^{2q}_{L^{\frac{2Nq}{2N-\alpha}}(\mathbb{R}^N)} \\
&\geq c_1 p\|u\|^p_{W^{1,p}(\mathbb{R}^N)} - c_2\delta^{2q-p}\|u\|^p_{W^{1,p}(\mathbb{R}^N)} \\
&\geq \frac{c_1}{2}\|u\|^p_{W^{1,p}(\mathbb{R}^N)} \geq \frac{c_1}{2}\delta^p := c^*,
\end{aligned}$$

where $\delta_0 = (\frac{c_1}{2c_2})^{\frac{1}{2q-p}}$. □

Assume $B = \{x \in \mathbb{R}^N \big| |x| \leq R\} \subset \mathcal{M}$. Let $\{e_n\}_{n=1}^\infty$ be a family of linearly independent functions in $C_0^\infty(B)$. There exists an increasing sequence R_n such that

$$J_0(u) < 0, \quad \forall u \in H_n,\ \|u\|_{X_\varepsilon} \geq R_n$$

where $H_n := span\{e_1, \dots, e_n\}$ and

$$J_0(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + b|u|^p)\,dx + \sigma\int_{\mathbb{R}^N} e^{(m-p)|x|}|u|^m\,dx - \frac{1}{2q}g_\lambda(\varphi^{1/2}(u))\varphi(u).$$

We define $\varphi_n \in C(B_n, C_0^\infty(B))$ as

$$\varphi_n(t) = R_n\sum_{i=1}^n t_i e_i, \quad t = (t_1, \dots, t_n) \in B_n = \{t | t \in \mathbb{R}^N, |t| \leq 1\},$$

where R_n is also chosen such that $\|\varphi_n(t)\|_{X_\varepsilon} \geq R_n$ for $t \in \partial B_n$. Let

$$\begin{gathered}
\Gamma_j = \{E \subset X_\varepsilon : E \text{ is compact},\ -E = E,\ \gamma(E \cap \eta^{-1}(\Sigma)) \geq j \text{ for } \eta \in \Lambda\}, \\
\Lambda = \{\eta \in C(X_\varepsilon, X_\varepsilon) : \eta \text{ is odd},\ \eta(P) \subset P,\ \eta(Q) \subset Q,\ \eta(u) = u \text{ if } \Gamma_{\varepsilon,\lambda}(u) \leq 0\}.
\end{gathered}$$

Lemma 3.16. *The set Γ_j is nonempty.*

For the proof of the above lemma, we refer to [15, Lemma 5.6].

Theorem 3.17. *Assume that conditions* (A1) *and* (A2) *hold. Then there exist $0 < \overline{\varepsilon} < 1$ and $0 < \overline{\lambda} < 1$, such that if $0 < \varepsilon < \overline{\varepsilon}$, $0 < \lambda < \overline{\lambda}$, then the functional $\Gamma_{\varepsilon,\lambda}$ has infinitely many sign-changing critical points, the corresponding critical values are*

$$c_j(\varepsilon,\lambda) = \inf_{E\in\Gamma_j}\sup_{u\in E\setminus W}\Gamma_{\varepsilon,\lambda}(u), \quad j = 1, 2, \dots. \tag{3.16}$$

Moreover

1. *there exist m_j, $j = 1, \dots,$ independent of ε, λ such that*
$$c_j(\varepsilon,\lambda) \leq m_j, \quad j = 1, 2, \dots. \tag{3.17}$$
2. *If $c_j(\varepsilon,\lambda) = \dots = c_{j+k}(\varepsilon,\lambda) = c$, then $\gamma(K_c^*) \geq k+1$.*

Proof. All the assumptions of Theorem 3.1 are satisfied, so we only need to prove estimate (3.17). Since $E_j = \varphi_{j+1}(B_{j+1}) \in \Gamma_j$, for $t \in B_{j+1}$, $u = \varphi_{j+1}(t)$, there exist $0 < \overline{\varepsilon} < 1$ and $0 < \overline{\lambda} < 1$, such that $\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p dx - 1\right)_+^\beta = 0$, and $\Gamma_{\varepsilon,\lambda}(u) \leq J_0(u)$, for $u \in \varphi_{j+1}(B_{j+1})$, if $0 < \varepsilon < \overline{\varepsilon}$, $0 < \lambda < \overline{\lambda}$. Hence

$$c_j(\varepsilon,\lambda) \leq m_j := \sup_{u\in E_j} J_0(u).$$

□

4. Proof of Theorem 1.1

Theorem 4.1. *(1) Assume $\Gamma_{\varepsilon,\lambda}(u) \le L$, $D\Gamma_{\varepsilon,\lambda}(u) = 0$. Then there exists a constant $H = H(L)$ such that*

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} \le H.$$

(2) Assume $\Gamma_\varepsilon(u) \le L$, $D\Gamma_\varepsilon(u) = 0$. Then there exist constants $\mu > 0$, $c = c(L)$ such that, for any $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$, for $0 < \varepsilon < \varepsilon(\delta)$

$$|u(x)| \le c \exp\{-\mu \operatorname{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\} \quad \text{for } x \in \mathbb{R}^N.$$

The proof of the above theorem is given in Section 5.

Corollary 4.2. *(1) Assume $\Gamma_{\varepsilon,\lambda}(u) \le L$, $D\Gamma_{\varepsilon,\lambda}(u) = 0$. Then there exists $\overline{\lambda} = \overline{\lambda}(L)$ such that $\Gamma_{\varepsilon,\lambda}(u) = \Gamma_\varepsilon(u)$ and $D\Gamma_\varepsilon(u) = 0$ if $0 < \lambda \le \overline{\lambda}$.*

(2) Assume $\Gamma_\varepsilon(u) \le L$, $D\Gamma_\varepsilon(u) = 0$. Then there exists $\overline{\varepsilon} = \overline{\varepsilon}(L)$ such that $\Gamma_\varepsilon(u) = I_\varepsilon(u)$ and $DI_\varepsilon(u) = 0$ if $0 < \varepsilon \le \overline{\varepsilon}$.

Proof. (1) By Theorem 4.1 (1), if $0 < \lambda < \overline{\lambda}(L) = \frac{1}{C_0 H^q}$, then

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} \le \Big(\frac{1}{C_0 \lambda}\Big)^{1/q}.$$

It follows that $\Gamma_{\varepsilon,\lambda}(u) = \Gamma_\varepsilon(u)$ and $D\Gamma_\varepsilon(u) = 0$.

(2) By Theorem 4.1 (2), there exist constants $\mu, c = c(L)$ such that, for any $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$, for $0 < \varepsilon < \varepsilon(\delta)$

$$|u(x)| \le c \exp\{-\mu \operatorname{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\}$$

for $x \in \mathbb{R}^N$. Let $\overline{\varepsilon} = \overline{\varepsilon}(L) \le \min\{\mu, \frac{1}{c}\}$, then for $0 < \varepsilon \le \overline{\varepsilon}$,

$$\begin{aligned} |u(x)| &\le c \exp\{-\mu \operatorname{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\} \\ &\le \frac{1}{\varepsilon} \exp\{-\varepsilon \operatorname{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\} \\ &\le \frac{1}{\varepsilon} \exp\{- \operatorname{dist}(\varepsilon x, \mathcal{M})\}. \end{aligned}$$

Hence $m_\varepsilon(x, u) = u$ for $x \in \mathbb{R}^N$. Moreover we denote $D = \max\{|y| \,\big|\, y \in \overline{\mathcal{M}}\}$, $d = \operatorname{dist}(\mathcal{A}^\delta, \partial\mathcal{M})$. Choose an integer $l > 1$ such that $ld \ge D$, then for $x \notin \overline{\mathcal{M}}_\varepsilon$

$$\begin{aligned} l \operatorname{dist}(x, (\mathcal{A}^\delta)_\varepsilon) &\ge l \operatorname{dist}((\mathcal{A}^\delta)_\varepsilon, \partial\mathcal{M}_\varepsilon) + \operatorname{dist}(x, \partial\mathcal{M}_\varepsilon) \\ &\ge \frac{l}{\varepsilon} d + |x| - \frac{D}{\varepsilon} \ge |x|, \end{aligned}$$

hence

$$\begin{aligned} |u(x)| &\le c \exp\{-c \operatorname{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\} \\ &\le c \exp\{-\frac{c}{l} |x|\}, \quad for \;\; x \notin \mathcal{M}_\varepsilon. \end{aligned}$$

We have

$$\begin{aligned} \int_{\mathbb{R}^N} \chi_\varepsilon(x) |u|^p \, dx &\le c\varepsilon^{-p} \int_{|x| \ge c\varepsilon^{-1}} \exp\{-\frac{c}{l}|x|\} \, dx \\ &\le c\varepsilon^{-N-p+1} \exp\{-\frac{c}{\varepsilon}\} < 1 \end{aligned}$$

and

$$\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) |u|^p \, dx - 1 \Big)_+ = 0$$

for $0 < \varepsilon < \varepsilon(\delta)$ sufficiently small. It follows that $\Gamma_\varepsilon(u) = I_\varepsilon(u)$ and $DI_\varepsilon(u) = 0$. $\square$

The proof of Theorem 1.1. For each positive integer k, by Theorem 3.17, there exist $0 < \tilde{\varepsilon} < 1$ and $0 < \tilde{\lambda} < 1$, such that if $0 < \varepsilon < \tilde{\varepsilon}$ and $0 < \lambda < \tilde{\lambda}$, the functional $\Gamma_{\varepsilon,\lambda}$ has k pairs of sign-changing critical points $\pm u_j, j = 1, \dots, k$. The corresponding critical values satisfy

$$0 < c_1(\varepsilon, \lambda) \le \dots \le c_k(\varepsilon, \lambda) \le m_k.$$

By Corollary 4.2 (2), there exists $\varepsilon_k = \varepsilon_k(m_k)$, such that if

$$0 < \varepsilon < \tilde{\varepsilon}_k = \min\{\varepsilon_k, \tilde{\varepsilon}\}, \, \Gamma_\varepsilon(u) \le m_k, \quad D\Gamma_\varepsilon(u) = 0,$$

then

$$\Gamma_\varepsilon(u) = I_\varepsilon(u), \quad DI_\varepsilon(u) = 0.$$

Fixed $\overline{\varepsilon} \in (0, \tilde{\varepsilon}_k)$. By Corollary 4.2 (1), there exists $\lambda_k = \lambda_k(m_k)$, such that if

$$0 < \lambda < \tilde{\lambda}_k = \min\{\lambda_k, \tilde{\lambda}\}, \, \Gamma_{\overline{\varepsilon},\lambda}(u) \le m_k, \, DI_{\overline{\varepsilon},\lambda}(u) = 0,$$

then

$$\Gamma_{\overline{\varepsilon},\lambda}(u) = \Gamma_{\overline{\varepsilon}}(u), \, D\Gamma_{\overline{\varepsilon}}(u) = 0.$$

Now for $0 < \varepsilon < \tilde{\varepsilon}_k$, $0 < \lambda < \tilde{\lambda}_k$, $u_{j,\varepsilon} = u_j(\varepsilon, \lambda)$, $j = 1, \dots, k$ are critical points of the functional I_ε. Moreover, by Theorem 4.1, there exist constants $\mu > 0$, $c = c(m_k)$, such that for any $\delta > 0$, there exists $\overline{\varepsilon}_k(\delta)$ such that for $0 < \varepsilon < \overline{\varepsilon}_k(\delta)$ it holds

$$|u_{j,\varepsilon}| \le c \exp\{-\mu \operatorname{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\}, \quad x \in \mathbb{R}^N,$$

hence

$$|v_{j,\varepsilon}| \le c \exp\{-\frac{\mu}{\varepsilon} \operatorname{dist}(x, \mathcal{A}^\delta)\}, x \in \mathbb{R}^N. \qquad \square$$

5. Uniform bound

In this section we prove Theorem 4.1. It is easy to obtain part (1). So we only prove part (2).

Lemma 5.1. *Assume $\Gamma_\varepsilon(u) < L, D\Gamma_\varepsilon(u) = 0$. Then*

(1) *there exists c_L, such that $|u(x)| \le c_L$ for $x \in \mathbb{R}^N$;*
(2) *there exists d, such that $\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\alpha} \, dy \le d$ for $x \in \mathbb{R}^N$;*
(3) *for any $\delta > 0$ there exists $c = c(\delta, L)$ such that $|u(x)| \le c\varepsilon$ for $x \in \mathbb{R}^N \backslash (\mathcal{M}_\varepsilon)^\delta$.*

Proof. (1) It is easy to show that u is bounded in $W^{1,p}(\mathbb{R}^N)$ and $(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \, dx - 1)_+^\beta$ is bounded. Choose $\phi = |u_T|^{p(k-1)}u$ as the test function in $\langle D\Gamma_\varepsilon(u), \phi\rangle = 0$ where $k \ge 1$, $T > 0$ and $u_T(x) = \pm T$ if $\pm u(x) \ge T$, $u_T(x) = u(x)$ if $|u(x)| \le T$. By $\langle D\Gamma_\varepsilon(u), \phi\rangle = 0$, it is easy to obtain the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx \le c \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y)|^q |u(x)|^{q-2} u(x) \phi(x)}{|x-y|^\alpha} \, dx \, dy. \tag{5.1}$$

First, let us estimate the right-hand side of the inequality (5.1). According to Lemma 2.1 and Hölder's inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^q)|u(x)|^{q-2}u(x)\phi(x)}{|x-y|^\alpha}\,dx\,dy \\ &\le c\Big(\int_{\mathbb{R}^N}|u|^{\frac{2Nq}{2N-\alpha}}\,dx\Big)^{\frac{(2N-\alpha)(2q-p)}{2Nq}}\Big(\int_{\mathbb{R}^N}(|u||u_T|^{k-1})^{\frac{2Nq}{2N-\alpha}}\,dx\Big)^{\frac{p(2N-\alpha)}{2Nq}} \\ &\le c\Big(\int_{\mathbb{R}^N}(|u||u_T|^{k-1})^{\frac{2Nq}{2N-\alpha}}\,dx\Big)^{\frac{p(2N-\alpha)}{2Nq}}. \end{aligned} \tag{5.2}$$

The left-hand side of (5.1) satisfies

$$\begin{aligned} \int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\phi\,dx &\ge \int_{\mathbb{R}^N}|\nabla u|^p|u_T|^{p(k-1)}\,dx \\ &\ge \frac{c}{k^p}\int_{\mathbb{R}^N}|\nabla(|u||u_T|^{k-1})|^p\,dx \\ &\ge \frac{c}{k^p}\big(\int_{\mathbb{R}^N}(|u||u_T|^{k-1})^{p^*}\,dx\big)^{\frac{p}{p^*}}. \end{aligned} \tag{5.3}$$

By (5.2) and (5.3), we have

$$\Big(\int_{\mathbb{R}^N}(|u||u_T|^{k-1})^{p^*}\,dx\Big)^{\frac{p}{p^*}} \le ck^p\Big(\int_{\mathbb{R}^N}(|u||u_T|^{k-1})^{\frac{2Nq}{2N-\alpha}}\,dx\Big)^{\frac{p(2N-\alpha)}{2Nq}}. \tag{5.4}$$

Letting $T\to\infty$ in (5.4), we obtain

$$\Big(\int_{\mathbb{R}^N}|u|^{p^*k}\,dx\Big)^{\frac{p}{p^*}} \le ck^p\Big(\int_{\mathbb{R}^N}|u|^{\frac{2Nqk}{2N-\alpha}}\,dx\Big)^{\frac{p(2N-\alpha)}{2Nq}}.$$

We write $\chi=\frac{p(2N-\alpha)}{2q(N-p)}>1$. By using iteration, starting from $k_1=\frac{p(2N-\alpha)}{2q(N-p)}>1$, we have

$$\Big(\int_{\mathbb{R}^N}|u|^{p^*\chi^n}\,dx\Big)^{\frac{1}{p^*\chi^n}} \le (c\chi^{pn})^{\frac{1}{p\chi^n}}\Big(\int_{\mathbb{R}^N}|u|^{p^*\chi^{n-1}}\,dx\Big)^{\frac{1}{p^*\chi^{n-1}}}, \tag{5.5}$$

for $n=1,2,\dots$. Hence by (5.5) we have

$$\|u\|_{L^\infty(\mathbb{R}^N)}\le c\|u\|_{L^{p^*}(\mathbb{R}^N)}\le c_L. \tag{5.6}$$

(2) In view of $0<\alpha<N-1$, for $x\in\mathbb{R}^N$ we have

$$\begin{aligned} \int_{R^N}\frac{|u(y)|^q}{|x-y|^\alpha}\,dy &\le c\Big(\int_{|x-y|\ge1}\frac{|u(y)|^q}{|x-y|^\alpha}\,dy+\int_{|x-y|<1}\frac{|u(y)|^q}{|x-y|^\alpha}\,dy\Big) \\ &\le c\Big(\|u\|^q_{L^q(\mathbb{R}^N)}+\int_{|x-y|<1}\frac{1}{|x-y|^\alpha}\,dx\|u\|^q_{L^\infty(\mathbb{R}^N)}\Big) \\ &\le c\big(\|u\|^q_{L^q(\mathbb{R}^N)}+\|u\|^q_{L^\infty(\mathbb{R}^N)}\big)\le C. \end{aligned}$$

(3) For $x_0\in\mathbb{R}^N$, $0<\rho<R\le1$. Choose $\eta\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\eta(x)=0$ for $x\notin B_R(x_0)$; $\eta(x)=1$ for $x\in B_\rho=B_\rho(x_0)$ and $|\nabla\eta|\le\frac{c}{R-\rho}$. Take $\varphi=u|u|^{p(k-1)}\eta^p$, $p\ge1$ as the test function in $\langle D\Gamma_{\varepsilon,\lambda}(u),\varphi\rangle=0$, we have

$$\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx\le c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^q|u(x)|^{q-2}u(x)\varphi(x)}{|x-y|^\alpha}\,dx\,dy. \tag{5.7}$$

The left-hand side of (5.7) satisfies

$$
\begin{aligned}
&LHS\\
&\geq \int_{\mathbb{R}^N} |\nabla u|^p |u|^{p(k-1)}\eta^p\,dx + p\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u |u|^{p(k-1)} u\eta^{p-1}\nabla\eta\,dx\\
&\geq \int_{\mathbb{R}^N} |\nabla u|^p |u|^{p(k-1)}\eta^p\,dx - p\int_{\mathbb{R}^N} |\nabla u|^{p-1}|u|^{(p-1)(k-1)}|u|^k\eta^{p-1}|\nabla\eta|\,dx\\
&\geq c\int_{\mathbb{R}^N} |\nabla u|^p |u|^{p(k-1)}\eta^p\,dx - c\int_{\mathbb{R}^N} |u|^{pk}|\nabla\eta|^p\,dx\\
&\geq \frac{c}{k^p}\int_{\mathbb{R}^N} |\nabla(|u|^k\eta)|^p\,dx - c\int_{\mathbb{R}^N} |u|^{pk}|\nabla\eta|^p\,dx\\
&\geq \frac{c}{k^p}(\int_{B_\rho} |u|^{p^*k}\,dx)^{\frac{p}{p^*}} - \frac{c}{(R-\rho)^p}\int_{B_R} |u|^{pk}\,dx.
\end{aligned}
\tag{5.8}
$$

By (5.6), the right-hand side of (5.7) satisfies

$$
RHS \leq c\int_{\mathbb{R}^N} |u|^{q-p}|u|^{pk}\eta^p\,dx \leq c_L\int_{B_R} |u|^{pk}\,dx. \tag{5.9}
$$

By (5.8) and (5.9), we have

$$
\Big(\int_{B_\rho} |u|^{p^*k}\,dx\Big)^{p/p^*} \leq \frac{c_L k^p}{(R-\rho)^p}\int_{B_R} |u|^{pk}\,dx \quad \text{for } k\geq 1.
$$

By iteration we obtain

$$
\|u\|_{L^\infty(B(x,R/2))} \leq c_L\|u\|_{L^p(B(x,R))}. \tag{5.10}
$$

Since

$$
\int_{\mathbb{R}^N\backslash(\mathcal{M}_\varepsilon)^\delta} |u|^p\,dx \leq c_\delta\varepsilon^p,
$$

by (5.10), we have $|u(x)|\leq c(\delta,L)\varepsilon$ for $x\in\mathbb{R}^N\backslash(\mathcal{M}_\varepsilon)^\delta$. $\square$

Let $\varepsilon_n\to 0$, assume $u_n\in W^{1,p}(\mathbb{R}^N)$, $D\Gamma_{\varepsilon_n}(u_n)=0$, $\Gamma_{\varepsilon_n}(u_n)\leq L$. Since $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$, we have the following profile decomposition [29],

$$
u_n = \sum_{k\in\Lambda} U_k(\cdot - y_{n,k}) + r_n, \tag{5.11}
$$

where Λ is an index set, $y_{n,k}\in\mathbb{R}^N$,

(1) $u_n(\cdot + y_{n,k}) \rightharpoonup U_k$ in $W^{1,p}(\mathbb{R}^N)$ as $n\to\infty$.
(2) $|y_{n,k}-y_{n,l}|\to\infty$ as $n\to\infty$ for $k\neq l$.
(3) $\|u_n\|^p_{W^{1,p}(\mathbb{R}^N)} = \sum_{k\in\Lambda}\|U_k\|^p_{W^{1,p}(\mathbb{R}^N)} + \|r_n\|^p_{W^{1,p}(\mathbb{R}^N)} + o(1)$ as $n\to\infty$.
(4) $\|r_n\|_{L^s(\mathbb{R}^N)}\to 0$ as $n\to\infty$, $p<s<p^*$, $\|u_n\|^s_{L^s(\mathbb{R}^N)} = \sum_{k\in\Lambda}\|U_k\|^s_{L^s(\mathbb{R}^N)} + o(1)$ as $n\to\infty$.

By Lemma 5.1 (3) we have

$$
\lim_{n\to\infty} \operatorname{dist}(y_{n,k},\mathcal{M}_{\varepsilon_n}) < +\infty.
$$

We denote

$$
y_k^* = \lim_{n\to\infty}\varepsilon_n y_{n,k}.
$$

Since $\operatorname{dist}(y_{n,k},\mathcal{M}_{\varepsilon_n}) = \varepsilon_n^{-1}\operatorname{dist}(\varepsilon_n y_{n,k},\mathcal{M})$, we have

$$
\operatorname{dist}(y_k^*,\overline{\mathcal{M}}) = 0, \quad \text{i.e. } y_k^*\in\overline{\mathcal{M}}. \tag{5.12}
$$

Lemma 5.2. *Assume $\varepsilon_n \to 0$, $D\Gamma_{\varepsilon_n}(u_n) = 0$, $\Gamma_{\varepsilon_n}(u_n) \le L$. Let $\tilde u_n = u_n(\cdot + y_n) \rightharpoonup U$ in $W^{1,p}(\mathbb{R}^N)$, $y_n \in \mathbb{R}^N$, $\lim_{n\to\infty} \varepsilon_n y_n = y^*$. Then $Z = |U|$ satisfies*

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla Z|^{p-2}\nabla Z \nabla\varphi\,dx + \int_{\mathbb{R}^N} |Z|^{p-1}\varphi\,dx \\ &\le c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Z(y)|^q |Z(x)|^{q-1}\varphi(x)}{|x-y|^\alpha}\,dx\,dy, \end{aligned} \tag{5.13}$$

for $\varphi \in W^{1,p}(\mathbb{R}^N)$, $\varphi \ge 0$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Selecting $\varphi_n(x) = \varphi(x - y_n)$ as a test function in $\langle D\Gamma_{\varepsilon_n}(u_n), \varphi_n\rangle = 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(|\nabla \tilde u_n|^{p-2}\nabla\tilde u_n \nabla\varphi + E(\varepsilon_n(x+y_n))|\tilde u_n|^{p-2}\tilde u_n\varphi\big)\,dx \\ &+ \sigma\int_{\mathbb{R}^N} k_{\varepsilon_n}(x+y_n,\tilde u_n)\varphi\,dx + \psi_{\varepsilon_n}(u_n)\int_{\mathbb{R}^N}\chi_{\varepsilon_n}(x+y_n)|\tilde u_n|^{p-2}\tilde u_n\varphi\,dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde u_n(y)|^q|\tilde u_n(x)|^{q-2}\tilde u_n(x)\varphi(x)}{|x-y|^\alpha}\,dx\,dy. \end{aligned} \tag{5.14}$$

Let $R > 0$, such that $\varphi(x) = 1$ for $|x| \le R$ and $\varphi(x) = 0$ for $|x| \ge 2R$. The sequence $\{\tilde u_n\}$ converges in $L^q_{\rm loc}(\mathbb{R}^N)$, $p < q < p^*$. By Lemma 3.6, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla\tilde u_k|^{p-2}\nabla\tilde u_k - |\nabla\tilde u_l|^{p-2}\nabla\tilde u_l, \nabla\tilde u_k - \nabla\tilde u_l)\varphi\,dx \\ &\le c\Big(\int_{\mathbb{R}^N}(|\nabla\tilde u_k|^p + |\nabla\tilde u_l|^p)\,dx\Big)^{\frac{p-1}{p}}\cdot\Big(\int_{B_{2R}(0)}|\tilde u_k - \tilde u_l|^p\,dx\Big)^{1/p} \\ &\quad + c\Big(\int_{\mathbb{R}^N}(|\tilde u_k|^p + |\tilde u_l|^p)\,dx\Big)^{\frac{p-1}{p}}\Big(\int_{B_{2R}(0)}|\tilde u_k - \tilde u_l|^p\,dx\Big)^{1/p} \\ &\quad + c\Big(\int_{\mathbb{R}^N}(|\tilde u_k|^m + |\tilde u_l|^m)\,dx\Big)^{\frac{m-1}{m}}\Big(\int_{B_{2R}(0)}|\tilde u_k - \tilde u_l|^m\,dx\Big)^{1/m} \\ &\quad + c\Big(\int_{\mathbb{R}^N}(|\tilde u_k|^{\frac{2Nq}{2N-\alpha}} + |\tilde u_l|^{\frac{2Nq}{2N-\alpha}})\,dx\Big)^{\frac{2N-\alpha}{2N}} \\ &\quad \times\Big(\int_{B_{2R}(0)}|\tilde u_k - \tilde u_l|^{\frac{2Nq}{2N-\alpha}}\,dx\Big)^{\frac{2N-\alpha}{2N}} + o(1) \\ &\le c\|\tilde u_k - \tilde u_l\|_{L^p(B_{2R}(0))} + c\|\tilde u_k - \tilde u_l\|_{L^m(B_{2R}(0))} \\ &\quad + c\|\tilde u_k - \tilde u_l\|_{L^{\frac{2Nq}{2N-\alpha}}(B_{2R}(0))} + o(1) \\ &\to 0 \quad \text{as } k,l\to\infty\,. \end{aligned} \tag{5.15}$$

For $p \ge 2$, by (2.1) and (5.15) we have

$$\begin{aligned} &\int_{B_R(0)} |\nabla(\tilde u_k - \tilde u_l)|^p\,dx \\ &\le c\int_{\mathbb{R}^N}(|\nabla\tilde u_k|^{p-2}\nabla\tilde u_k - |\nabla\tilde u_l|^{p-2}\nabla\tilde u_l, \nabla(u_k - u_l))\varphi\,dx \to 0, \end{aligned}$$

as $k,l \to \infty$.

For $1 < p < 2$, by (2.3) and (5.15) we have

$$\begin{aligned}
&\int_{B_R(0)} |\nabla(\tilde{u}_k - \tilde{u}_l)|^p \,dx \\
&\leq c\Big(\int_{\mathbb{R}^N} (|\nabla\tilde{u}_k|^{p-2}\nabla\tilde{u}_k - |\nabla\tilde{u}_l|^{p-2}\nabla\tilde{u}_l, \nabla(\tilde{u}_k - \tilde{u}_l))\varphi \,dx\Big)^{p/2} \\
&\quad\times \Big(\int_{\mathbb{R}^N} (|\nabla\tilde{u}_k|^p + |\nabla\tilde{u}_l|^p)\varphi \,dx\Big)^{\frac{2-p}{2}} \\
&\leq c\Big(\int_{\mathbb{R}^N} (|\nabla\tilde{u}_k|^{p-2}\nabla\tilde{u}_k - |\nabla\tilde{u}_l|^{p-2}\nabla\tilde{u}_l)\nabla(\tilde{u}_k - \tilde{u}_l)\varphi \,dx\Big)^{p/2} \to 0 \quad \text{as } k, l \to \infty.
\end{aligned}$$

Since $\varphi(x) = 1$ in $B_R(0)$ and $\varphi \geq 0$. Hence $\tilde{u}_n \to u$ in $W^{1,p}_{\rm loc}(\mathbb{R}^N)$. Let $z_n = |\tilde{u}_n|$, $w_{n,\delta} = (\tilde{u}_n^2 + \delta^2)^{1/2} - \delta$, then it follows from Lebesgue dominated convergence theorem that $w_{n,\delta} \in W^{1,p}(\mathbb{R}^N)$, and $w_{n,\delta} \to z_n$ in $W^{1,p}(\mathbb{R}^N)$ as $\delta \to 0$. Now for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$, we have $\varphi_\delta = \varphi\tilde{u}_n(\tilde{u}_n^2 + \delta^2)^{-\frac{1}{2}} \in W^{1,p}_{loc}(\mathbb{R}^N)$, and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(|\nabla\tilde{u}_n|^{p-2}\nabla w_{n,\delta}\nabla\varphi + E(\varepsilon_n(x+y_n))|\tilde{u}_n|^{p-2}w_{n,\delta}\varphi\big)\,dx \\
&= \int_{\mathbb{R}^N} |\nabla\tilde{u}_n|^{p-2}\tilde{u}_n\nabla\tilde{u}_n\nabla\varphi(\tilde{u}_n^2 + \delta^2)^{-\frac{1}{2}}\,dx \\
&\quad + \int_{\mathbb{R}^N} E(\varepsilon_n(x+y_n))|\tilde{u}_n|^{p-2}\big((\tilde{u}_n^2 + \delta^2)^{1/2} - \delta\big)\varphi\,dx \\
&= \int_{\mathbb{R}^N} (|\nabla\tilde{u}_n|^{p-2}\nabla\tilde{u}_n\nabla\varphi_\delta - |\nabla\tilde{u}_n|^p\varphi(\tilde{u}_n^2 + \delta^2)^{-\frac{3}{2}}\delta^2 \\
&\quad + E(\varepsilon_n(x+y_n))|\tilde{u}_n|^{p-2}((\tilde{u}_n^2 + \delta^2)^{1/2} - \delta)\varphi)\,dx \\
&\leq \int_{\mathbb{R}^N} (|\nabla\tilde{u}_n|^{p-2}\nabla\tilde{u}_n\nabla\varphi_\delta + E(\varepsilon_n(x+y_n))|\tilde{u}_n|^{p-2}\tilde{u}_n\varphi_\delta)\,dx \\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z_n(y)|^q|z_n(x)|^{q-1}|\varphi_\delta(x)|}{|x-y|^\alpha}\,dx\,dy.
\end{aligned} \tag{5.16}$$

Let $\delta \to 0$ in (5.16), for $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\nabla z_n|^{p-2}\nabla z_n\nabla\varphi + |z_n|^{p-1}\varphi)\,dx \\
&\leq c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z_n(y)|^q|z_n(x)|^{q-1}\varphi(x)}{|x-y|^\alpha}\,dx\,dy.
\end{aligned} \tag{5.17}$$

By $\tilde{u}_n \to u$ in $W^{1,p}_{loc}(\mathbb{R}^N)$ as $n \to \infty$, we have $z_n \to Z$ in $W^{1,p}_{loc}(\mathbb{R}^N)$ as $n \to \infty$. Hence we complete the proof by a denseness argument. □

Lemma 5.3. *Λ is a finite set.*

Proof. $Z_k = |U_k|$ satisfies (5.13) and take $\varphi = Z_k$ in (5.13), we have

$$\|Z_k\|^p_{W^{1,p}(\mathbb{R}^N)} \leq c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Z_k(y)|^q|Z_k(x)|^q}{|x-y|^\alpha}\,dx\,dy \leq c\|Z_k\|^{2q}_{W^{1,p}(\mathbb{R}^N)}. \tag{5.18}$$

So there exists $m > 0$ such that $\|U_k\|_{W^{1,p}(\mathbb{R}^N)} \geq m$. By the property (3) of the profile decomposition (5.11), we know Λ is a finite set. □

Assume that the sequence $\{u_n\}$ has the profile decomposition (5.11). Define

$$\Lambda = \{1, \dots, k\}, \quad \Omega_R^{(n)} = \mathbb{R}^N \backslash \{\cup_{k\in\Lambda} B(y_{n,k}, R) \cup B(0, R)\}.$$

Lemma 5.4. *Assume $D\Gamma_{\varepsilon_n}(u_n) = 0, \Gamma_{\varepsilon_n}(u_n) < L$. Then there exist $c = c(L), \mu$, independent of n, such that*

$$\int_{\Omega_R^{(n)}} G_{\varepsilon_n}(x, u_n, \nabla u_n)\, dx \le ce^{-\mu R} \text{ for } x \in \Omega_R^{(n)},$$

where

$$\begin{aligned} &G_{\varepsilon_n}(x, u_n, \nabla u_n) \\ &= |\nabla u_n|^p + |u_n|^p + k_{\varepsilon_n}(x, u_n)u_n + \Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x)|u_n|^p\, dx - 1\Big)_+^{\beta-1} \chi_{\varepsilon_n}(x)|u_n|^p. \end{aligned}$$

Moreover, we have

$$|u_n(x)| \le ce^{-\mu R} \quad \text{for } x \in \Omega_R^{(n)}.$$

Proof. By the decomposition (5.11) we have

$$\|u_n\|_{L^q(\Omega_R^{(n)})} = o_R(1), \quad p < q < p^*,$$

where $o_R(1) \to 0$ as $R \to +\infty$, by Moser's iteration we have

$$\|u_n\|_{L^\infty(\Omega_R^{(n)})} = o_R(1).$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ such that $\eta(x) = 0$ for $x \notin \Omega_R^{(n)}$; $\eta(x) = 1$ for $x \in \Omega_{R+1}^{(n)}$ and $|\nabla \eta| \le 2$. Take $\varphi_n = u_n\eta^p$ as test function in $\langle D\Gamma_{\varepsilon_n}(u_n), \varphi\rangle = 0$, we have

$$\begin{aligned} &\int_{\Omega_R^{(n)}} \big(|\nabla u_n|^p + E(\varepsilon_n x)|u_n|^p\big)\eta^p\, dx + \sigma \int_{\Omega_R^{(n)}} k_{\varepsilon_n}(x, u_n)u_n\eta^p\, dx \\ &+ \Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x)|u_n|^p\, dx - 1\Big)_+^{\beta-1} \int_{\Omega_R^{(n)}} \chi_{\varepsilon_n}(x)|u_n|^p\eta^p\, dx \\ &= \int_{\Omega_R^{(n)}} \int_{\mathbb{R}^N} \frac{|u_n(y)|^q|u_n(x)|^q\eta^p(x)}{|x-y|^\alpha}\, dx\, dy \\ &\quad - p\int_{\Omega_R^{(n)}\backslash\Omega_{R+1}^{(n)}} |\nabla u_n|^{p-2}\nabla u_n u_n\eta^{p-1}\nabla\eta\, dx. \end{aligned}$$

By Lemma 5.1 (2) and $\|u_n\|_{L^\infty(\Omega_R^{(n)})} = o_R(1)$ we obtain

$$\int_{\Omega_R^{(n)}} dx \int_{\mathbb{R}^N} \frac{|u_n(y)|^q|u_n(x)|^q\eta^p(x)}{|x-y|^\alpha}\, dy \le \frac{1}{2}\int_{\Omega_R^{(n)}} E(\varepsilon_n x)|u_n|^p\eta^p\, dx.$$

Also

$$\begin{aligned} &\Big|\int_{\Omega_R^{(n)}\backslash\Omega_{R+1}^{(n)}} |\nabla u_n|^{p-2}\nabla u_n u_n\eta^{p-1}\nabla\eta\, dx\Big| \\ &\le \tau\int_{\Omega_R^{(n)}} |\nabla u_n|^p\eta^p + c_\tau \int_{\Omega_R^{(n)}\backslash\Omega_{R+1}^{(n)}} |u_n|^p\eta^p\, dx. \end{aligned}$$

So, we have

$$\int_{\Omega_{R+1}^{(n)}} G_{\varepsilon_n}(x, u_n, \nabla u_n)\, dx \le c\int_{\Omega_R^{(n)}\backslash\Omega_{R+1}^{(n)}} G_{\varepsilon_n}(x, u_n, \nabla u_n)\, dx.$$

Consequently,

$$\int_{\Omega_{R+1}^{(n)}} G_{\varepsilon_n}(x,u_n,\nabla u_n)\,dx \le \theta \int_{\Omega_R^{(n)}} G_{\varepsilon_n}(x,u_n,\nabla u_n)\,dx,$$

where $\theta = \frac{c}{c+1} < 1$. Finally

$$\int_{\Omega_R^{(n)}} G_{\varepsilon_n}(x,u_n,\nabla u_n)\,dx \le ce^{-\mu R},$$

where $\mu = -\ln\theta > 0$. And by Moser's iteration, we have $|u_n(x)| \le ce^{-\mu R}$ for $x \in \Omega_R^{(n)}$. □

Lemma 5.5. *For every $k \in \Lambda$, it holds $y_k^* = \lim_{n\to\infty} \varepsilon_n y_{n,k} \in \bar{\mathcal{A}}$.*

Proof. If not, we assume that there exist $k \in \Lambda$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$ and $\operatorname{dist}(y_k^*, \bar{\mathcal{A}}) > 0$. Let $t_k = \nabla V(y_k^*) \ne 0$. Then by condition (A2) there exists $\delta_1 > 0$ such that

$$\big(t_k, \nabla V(x)\big) \ge \frac{1}{2}|t_k|^2 > 0, \quad \big(t_k, \nabla \operatorname{dist}(x,\mathcal{M})\big) \ge 0 \quad \text{for } x \in B_{\delta_1}(y_k^*). \tag{5.19}$$

Set

$$\delta_2 = \min\{|y_k^* - y_l^*| y_k^* \ne y_l^*, \ \ k, \ \ l = 0,1,\dots,k_0, \ \ y_0^* = 0\}.$$

Let

$$0 < \delta < \min\{\frac{1}{2}\delta_1, \frac{1}{100}\delta_2\}.$$

Denote

$$B_n = \{x||x - y_{n,k}| \le 2\delta\varepsilon_n^{-1}\},$$
$$T_n = x|\delta\varepsilon_n^{-1} \le |x - y_{n,k}| \le 2\delta\varepsilon_n^{-1}.$$

Choose $\eta \in C_0^\infty(\mathbb{R}^N)$ such that $\eta(x) = 0$ if $|x - y_{n,k}| \ge 2\delta\varepsilon_n^{-1}$; $\eta(x) = 1$ if $|x - y_{n,k}| \le \delta\varepsilon_n^{-1}$ and $|\nabla\eta| \le \frac{2}{\delta}\varepsilon_n (\le 1)$. By $\langle D\Gamma_{\varepsilon_n}(u_n), \varphi\rangle = 0$ for $\varphi \in W^{1,p}(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi + E(\varepsilon_n x)|u_n|^{p-2}u_n\varphi\big)\,dx + \sigma\int_{\mathbb{R}^N} k_{\varepsilon_n}(x,u_n)\varphi\,dx \\
&+ \Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x)|u_n|^p\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x)|u_n|^{p-2}u_n\varphi\,dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^q|u_n(x)|^{q-2}u_n(x)\varphi(x)}{|x-y|^\alpha}\,dx\,dy.
\end{aligned} \tag{5.20}$$

Choosing $\varphi = (t_k, \nabla u_n)\eta$ as test function in (5.20), we obtain the local Pohožaev identity

$$\begin{aligned}
&\frac{\varepsilon_n}{p}\int_{\mathbb{R}^N} \big(t_k, \nabla E(\varepsilon_n x)\big)|u_n|^p\eta\,dx + \sigma\int_{\mathbb{R}^N} \big(t_k, \nabla_x k_{\varepsilon_n}(x,u_n)\big)\eta\,dx \\
&+ \frac{1}{p}\Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x)|u_n|^p\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \big(t_k, \nabla\chi_{\varepsilon_n}(x)\big)|u_n|^p\eta\,dx \\
&= \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}(\nabla u_n, \nabla\eta)(t_k, \nabla u_n)\,dx \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u_n|^p + E(\varepsilon_n x))|u_n|^p)(t_k, \nabla\eta)\,dx - \sigma\int_{\mathbb{R}^N} k_{\varepsilon_n}(x,u_n)u_n(t_k,\nabla\eta)\,dx \\
&\quad - \frac{1}{p}\Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x)|u_n|^p\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x)|u_n|^p(t_k, \nabla\eta)\,dx
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\alpha}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(t_k, x-y)\frac{|u_n(y)|^q|u_n(x)|^q\eta(x)}{|x-y|^{\alpha+2}}\,dx\,dy \\
&+ \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(t_k,\nabla\eta)\frac{|u_n(y)|^q|u_n(x)|^q}{|x-y|^{\alpha}}\,dx\,dy. \qquad (5.21)
\end{aligned}
$$

Next, we estimate all terms of the above inequality. By (5.19), we have

$$
\varepsilon_n\int_{\mathbb{R}^N}\big(t_k,\nabla E(\varepsilon_n x)\big)|u_n|^p\eta\,dx \ge c\varepsilon_n,
$$

and

$$
\Big(\int_{\mathbb{R}^N}\chi_{\varepsilon_n}(x)|u_n|^p\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\big(\nabla\chi_{\varepsilon_n}(x),t_k\big)|u_n|^p\eta\,dx\ge 0.
$$

Hence the left-hand side of (5.21), satisfies

$$
LHS \ge c\varepsilon_n. \qquad (5.22)
$$

We estimate the right-hand side of (5.21), by

$$
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(t_k,x-y)\frac{|u_n(y)|^q|u_n(x)|^q\eta(x)\eta(y)}{|x-y|^{\alpha+2}}\,dx\,dy=0,
$$

then

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(t_k,x-y)\frac{|u_n(y)|^q|u_n(x)|^q\eta(x)}{|x-y|^{\alpha+2}}\,dx\,dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(t_k,x-y)\frac{|u_n(y)|^q|u_n(x)|^q\eta(x)(1-\eta(y))}{|x-y|^{\alpha+2}}\,dx\,dy \\
&\le c\iint_{\substack{|y-y_{n,k}|\ge\delta\varepsilon_n^{-1}\\ |x-y_{n,k}|\le 2\delta\varepsilon_n^{-1}}}\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^{\alpha+1}}\,dx\,dy \\
&\le c\iint_{\substack{\delta\varepsilon_n^{-1}\le|y-y_{n,k}|\le 3\delta\varepsilon_n^{-1}\\ |x-y_{n,k}|\le 2\delta\varepsilon_n^{-1}}}\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^{\alpha+1}}\,dx\,dy \\
&\quad + c\iint_{\substack{|y-y_{n,k}|\ge 3\delta\varepsilon_n^{-1}\\ |x-y_{n,k}|\le 2\delta\varepsilon_n^{-1}}}\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^{\alpha+1}}\,dx\,dy \\
&=: I+II,
\end{aligned}
$$

where

$$
II\le c\iint_{\substack{|y-y_{n,k}|\ge 3\delta\varepsilon_n^{-1}\\ |x-y_{n,k}|\le 2\delta\varepsilon_n^{-1}}}|u_n(y)|^q|u_n(x)|^q\frac{1}{\delta^{\alpha+1}}\varepsilon_n^{\alpha+1}\,dx\,dy\le c\varepsilon_n^{\alpha+1}.
$$

The region $\tilde{T_n}=\{y|\delta\varepsilon_n^{-1}\le|y-y_{n,k}|\le 3\delta\varepsilon_n^{-1}\}$ is contained in $\Omega^{(n)}_{\delta\varepsilon_n^{-1}}$, by Lemma 5.4, we have

$$
|u_n(y)|\le ce^{-\mu\delta\varepsilon_n^{-1}}, y\in\tilde{T_n}.
$$

Then

$$
\begin{aligned}
I &\le ce^{-q\mu\delta\varepsilon_n^{-1}}\iint_{\substack{\delta\varepsilon_n^{-1}\le|y-y_{n,k}|\le 3\delta\varepsilon_n^{-1}\\ |x-y_{n,k}|\le 2\delta\varepsilon_n^{-1}}}\frac{|u_n(x)|^q}{|x-y|^{\alpha+1}}\,dx\,dy \\
&\le ce^{-q\mu\delta\varepsilon_n^{-1}}\iint_{\substack{|x-y|\le 5\delta\varepsilon_n^{-1}\\ |x-y_{n,k}|\le 2\delta\varepsilon_n^{-1}}}\frac{1}{|x-y|^{\alpha+1}}|u_n(x)|^q\,dy\,dx \\
&\le ce^{-q\mu\delta\varepsilon_n^{-1}}\varepsilon_n^{-N+\alpha+1}\le c\varepsilon_n^{\alpha+1}.
\end{aligned}
$$

By Lemmas 5.1 and 5.4, we obtain that the right-hand side of (5.21), satisfies

$$\begin{aligned} RHS &\le c\int_{T_n} G_{\varepsilon_n}(x,u_n,\nabla u_n)\,dx + c\varepsilon_n^{\alpha+1} \\ &\le ce^{-\mu\delta\varepsilon_n^{-1}} + c\varepsilon_n^{\alpha+1} \le c\varepsilon_n^{\alpha+1}. \end{aligned}$$

Therefore $c\varepsilon_n \le c\varepsilon_n^{\alpha+1}$. Since $0<\alpha<\min\{N-1,2p\}$, we arrive at a contradiction as $n\to\infty$. The proof is complete. $\square$

The proof of Theorem 4.1 (2). By Lemma 5.4,

$$|u_n(x)| \le ce^{-\mu R} \quad \text{for } x\in\Omega_R^{(n)}.$$

Let $R_n(x)=\min\{|x-y_{n,k}|k\in\Lambda\}$. Then

$$|u_n(x)| \le ce^{-\mu R_n(x)} \quad \text{for } x\in\Omega_{R_n}^{(n)}.$$

Since $\varepsilon_n y_{n,k} \to y_k^* \in \mathcal{A}$, for any δ, there exists $\varepsilon(\delta)$ such that for $\varepsilon_n \le \varepsilon(\delta)$, $\varepsilon_n y_{n,k} \in \mathcal{A}^\delta$, hence

$$|u_n(x)| \le ce^{-\mu R_n} \le ce^{-\mu\,\mathrm{dist}(x,(\mathcal{A}^\delta)_{\varepsilon_n})}, \quad x\in\mathbb{R}^N.$$

$\square$

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Bo Zhang
Department of Mathematics, Yunnan Normal University, Kunming, Yunnan 650500, China
Email address: zhangbo371013@163.com

Xiangqing Liu
Department of Mathematics, Yunnan Normal University, Kunming, Yunnan 650500, China
Email address: lxq8u8@163.com