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EXISTENCE OF SOLUTIONS TO SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS HAVING FINITE LIMITS AT $\pm\infty$

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Abstract. In this article, we study the boundary-value problem

$$\ddot{x} = f(t, x, \dot{x}), \quad x(-\infty) = x(+\infty), \quad \dot{x}(-\infty) = \dot{x}(+\infty).$$

Under adequate hypotheses and using the Bohnenblust-Karlin fixed point theorem for multivalued mappings, we establish the existence of solutions.

1. Introduction

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous mapping. Consider the infinite boundary-value problem

$$\ddot{x} = f(t, x, \dot{x}), \tag{1.1}$$

$$x(-\infty) = x(+\infty), \quad \dot{x}(-\infty) = \dot{x}(+\infty), \tag{1.2}$$

where $x(\pm\infty)$ and $\dot{x}(\pm\infty)$ denote the limits

$$x(\pm\infty) = \lim_{t\to\pm\infty} x(t) \quad \text{and} \quad \dot{x}(\pm\infty) = \lim_{t\to\pm\infty} \dot{x}(t), \tag{1.3}$$

which are assumed to be finite. Problem (1.1)-(1.2) may be considered as a generalization of problem (1.1) with boundary condtions

$$x(a) = x(b), \quad \dot{x}(a) = \dot{x}(b), \tag{1.4}$$

as $a \to -\infty$ and $b \to +\infty$. The bilocal boundary-value problem (1.1)-(1.4) is closely related to the problem of finding periodic solutions to (1.1). The reader is referred to [17, 19, 20] where extensive use of topological degree theory is made to study this problem.

Problem (1.1)-(1.2) is related to the so-called *convergent solutions*, i.e. the solutions defined on $\mathbb{R}_+ = [0, +\infty)$ (or $\mathbb{R}$) and having finite limit to $+\infty$ (respectively $-\infty$), see [4, 5, 14, 15, 16]. For studies on (1.1)-(1.2) using variational methods, we refer the reader to [1, 2, 3, 13, 20, 21]. In [12] the existence of the solutions to the equation (1.1) with the boundary conditions $x(\infty) = \dot{x}(\infty) = 0$ is studied for $f(t, u, v) = g(t)v - u + h(t, u)$. Through the Schauder-Tychonoff and Banach fixed point Theorems estimates for the solutions are found.

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When f is a differentiable function, (1.1) can be written as

$$\ddot{x} = a(t, x, \dot{x})\dot{x} + b(t, x, \dot{x})x + c(t), \tag{1.5}$$

where $a,\ b : \mathbb{R}^3 \to \mathbb{R}$, $c : \mathbb{R} \to \mathbb{R}$, $a(t, u, v) := \int_0^1 \frac{\partial f}{\partial u}(t, su, sv)\, ds$, $b(t, u, v) := \int_0^1 \frac{\partial f}{\partial v}(t, su, sv)\, ds$ and $c(t) := f(t, 0, 0)$, for all $t,\ u,\ v \in \mathbb{R}$.

Sufficient conditions for the existence of solutions to the linear problem

$$\ddot{x} = a(t)\dot{x} + b(t)x + c(t), \tag{1.6}$$

with boundary condition (1.2), were given in [11]. By using this result, in the real Banach space

$$X := \{x \in C^2(\mathbb{R}) : (\exists)\, x(\pm\infty),\ (\exists)\, \dot{x}(\pm\infty)\}$$

endowed with the uniform convergence topology on $\mathbb{R}$ one defines an operator $T : X \to 2^X$ which maps $u \in X$ into the set of the solutions to the problem (1.7)-(1.2), where

$$\ddot{x} = a(t, u(t), \dot{u}(t))\dot{x} + b(t, u(t), \dot{u}(t))x + c(t). \tag{1.7}$$

Next one considers the restriction of T to a bounded, convex and closed set M, conveniently chosen so that the Bohnenblust-Karlin Theorem can be applied. The compactness of $T(M)$ is established by using a characterization developed by the the first author in [4, 6].

The use of a multivalued operator T is motivated by the fact that one cannot determine a solution to the problem (1.7)-(1.2) through an "initial" condition independent of u.

2. Main result

Let $a,\ b : \mathbb{R}^3 \to \mathbb{R}$, $c : \mathbb{R} \to \mathbb{R}$ be continuous functions, and let

$$\alpha_1(t) := \inf_{u,v\in\mathbb{R}} \{a(t, u, v)\}, \quad \alpha_2(t) := \sup_{u,v\in\mathbb{R}} \{a(t, u, v)\},$$

$$\beta(t) := \sup_{u,v\in\mathbb{R}} \{b(t, u, v)\}, \quad A_i(t) := \exp\Big(\int_0^t \alpha_i(s)\, ds\Big),$$

for $i \in \{1, 2\}$ and $t \in \mathbb{R}$. We shall assume that $\alpha_1,\ \alpha_2,\ \beta$ are defined on $\mathbb{R}$.

Consider the following hypotheses, where the integrals are considered in the Riemann sense:

- (A1) The mappings α_1 and α_2 are bounded on $\mathbb{R}$, and $\lim_{t\to\pm\infty} \alpha_i(t) = 0$, for $i \in \{1, 2\}$
- (A2) $\lim_{t\to\pm\infty} A_i(t) = 0$ for $i \in \{1, 2\}$
- (B1) $0 \le b(t, u, v)$ for every $t,\ u,\ v \in \mathbb{R}$ and $\lim_{t\to\pm\infty} \beta(t) = 0$
- (B2) $\int_{-\infty}^{+\infty} \big(A_i(t) \cdot \int_0^t \frac{\beta(s)}{A_i(s)} ds\big)dt \in \mathbb{R}$ for $i \in \{1, 2\}$
- (B3) $\int_{-\infty}^{+\infty} \frac{\beta(t)}{A_i(t)} dt < +\infty$, for $i \in \{1, 2\}$
- (C1) $\int_{-\infty}^{+\infty} |c(t)| dt < +\infty$
- (C2) $\int_{-\infty}^{+\infty} \Big(\int_{-t}^{t} \frac{|c(s)|}{A_i(s)} ds\Big)dt \in \mathbb{R}$ for $i \in \{1, 2\}$.

Our main result is as follows:

Theorem 2.1. *If the hypotheses (A1)–(A2), (B1)–(B3), (C1)–(C2) are satisfied, then* (1.5)-(1.2) *has a solution.*

Since

$$\lim_{t\to\pm\infty}\frac{A_i(t)}{A_i(t)\cdot\int_0^t\frac{\beta(s)}{A_i(s)}ds}=\lim_{t\to\pm\infty}\frac{1}{\int_0^t\frac{\beta(s)}{A_i(s)}ds}$$

is a real number by hypothesis (B3), it follows by hypothesis (B2), via a well known convergence criterion for Riemann integrals, that for each $i\in\{1,2\}$,

$$\int_{-\infty}^{+\infty}A_i(t)dt<+\infty\,. \tag{2.1}$$

Similarly, by hypothesis (A2),

$$\lim_{t\to\pm\infty}\frac{\beta(t)}{\frac{\beta(t)}{A_i(t)}}=0,\quad \lim_{t\to\pm\infty}\frac{|c(t)|}{\frac{|c(t)|}{A_i(t)}}=0,$$

it follows, by hypothesis (B3), that

$$\int_{-\infty}^{+\infty}\beta(t)dt<+\infty, \tag{2.2}$$

and, by hypothesis (C1),

$$\int_{-\infty}^{+\infty}\frac{|c(t)|}{A_i(t)}dt<+\infty, \tag{2.3}$$

for each $i\in\{1,2\}$.

Remark 2.2. (i) One can replace the hypothesis (B2) by

(B2') $\int_{-\infty}^{+\infty}A_i(t)dt<+\infty\,.$

(ii) Assumption (B2') does not imply (C2).

(i) Indeed, since (B3) implies the boundedness of the mapping $\int_0^{(\cdot)}\frac{\beta(s)}{A_i(s)}ds$ and therefore, (B2') implies (B2).
(ii) It is sufficient to choose $c(t)=A_i(t)$, for all $t\in\mathbb{R}$, where $i=1$ or $i=2$.

For proving our main result we use the following theorem.

Theorem 2.3 (Bohnenblust-Karlin [22, p. 452])**.** *Let X be a Banach space and $M\subset X$ be a convex closed subset of it. Suppose that $T:X\to 2^X$ is a multivalued operator on X satisfying the following hypotheses:*

(a) *$T(M)\subset 2^M$ and T is upper semicontinuous*
(b) *the set $T(M)$ is relatively compact*
(c) *for every $x\in M$, $T(x)$ is a non-empty convex closed set.*

Then T admits fixed points.

Recall that $T:M\to 2^M$ is *upper semicontinuous* if for every closed subset A of M, the set

$$T^{-1}(A):=\{x\in M:T(x)\cap A\neq\emptyset\}$$

is also a closed subset of M. Another useful result is the following Lemma.

Lemma 2.4 (Barbălat)**.** *If $f:[0,+\infty)\to\mathbb{R}$ satisfies: (a) f is uniformly continuous and (b) the integral $\int_0^{+\infty}f(t)\,dt$ exists and is finite, then* $\lim_{t\to+\infty}f(t)=0$.

The main idea of this paper is to build a multivalued operator T defined on an adequate space which satisfies the hypotheses of the Bohnenblust-Karlin Theorem. We define

$$X := \{x \in C^2(\mathbb{R}) : (\exists)\ x(\pm\infty) \text{ and } \dot{x}(\pm\infty)\},$$

which, endowed with the usual norm,

$$\|x\| := \sup_{t\in\mathbb{R}} \max\{|x(t)|, |\dot{x}(t)|\},$$

becomes a real Banach space. The relative compactness of the set $T(M)$ be will be proved by using the following Proposition.

Proposition 2.5 (Avramescu [4, 6]). *A set $\mathcal{A} \subset X$ is relatively compact if and only if the following conditions are fulfilled:*

(a) *There exist h_1, $h_2 \geq 0$ such that for every $x \in \mathcal{A}$ and $t \in \mathbb{R}$, we have $|x(t)| \leq h_1$ and $|\dot{x}(t)| \leq h_2$*
(b) *For every $K = [a, b] \subset \mathbb{R}$ and $\varepsilon > 0$ there exists $\delta = \delta(K, \varepsilon) > 0$ such that for every $x \in \mathcal{A}$ and t_1, $t_2 \in K$ with $|t_1 - t_2| < \delta$, we have $|x(t_1) - x(t_2)| < \varepsilon$ and $|\dot{x}(t_1) - \dot{x}(t_2)| < \varepsilon$*
(c) *For every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that for every t_1, t_2 with $|t_1|$, $|t_2| > T$ and $t_1 \cdot t_2 > 0$, and for every $x \in \mathcal{A}$, we have $|x(t_1) - x(t_2)| < \varepsilon$ and $|\dot{x}(t_1) - \dot{x}(t_2)| < \varepsilon$.*

3. Construction of the multivalued operator T

Let $u \in C^2(\mathbb{R})$ be arbitrary. Consider the problem

$$\begin{gathered} \ddot{x} = a_u(t)\dot{x} + b_u(t)x + c(t) \\ x(+\infty) = x(-\infty), \quad \dot{x}(+\infty) = \dot{x}(-\infty), \end{gathered} \tag{3.1}$$

where $a_u(t) := a(t, u(t), \dot{u}(t))$ and $b_u(t) = b(t, u(t), \dot{u}(t))$. Consider the homogeneous problem

$$\begin{gathered} \ddot{x} = a_u(t)\dot{x} + b_u(t)x \\ x(+\infty) = x(-\infty), \quad \dot{x}(+\infty) = \dot{x}(-\infty). \end{gathered} \tag{3.2}$$

Since

$$x(t) = \exp\Big(\int_0^t y(s)ds\Big), \quad t \in \mathbb{R}$$

is a solution to $\ddot{x} = a_u(t)\dot{x} + b_u(t)x$ if and only if y is a solution to

$$\dot{y} = a_u y + b_u - y^2, \tag{3.3}$$

we have $a_u(t)y - y^2 \leq \dot{y} \leq a_u(t)y + b_u(t)$, for every $t \in \mathbb{R}$.

Let v, w satisfy

$$\begin{gathered} \dot{v} = a_u(t)v - v^2 \\ v(0) = \xi \end{gathered} \tag{3.4}$$

and

$$\begin{gathered} \dot{w} = a_u(t)w + b_u(t) \\ w(0) = \xi\,. \end{gathered} \tag{3.5}$$

Hence

$$\begin{gathered} \dot{y} = a_u y + b_u - y^2 \\ y(0) = \xi, \end{gathered}$$

which implies

$$\begin{aligned} v(t) \le y(t) \le w(t), &\quad \text{if } t \ge 0,\\ w(t) \le y(t) \le v(t), &\quad \text{if } t \le 0. \end{aligned}$$

Let $\alpha_u(t) := \exp\big(\int_0^t a_u(s)ds\big)$, for every $t \in \mathbb{R}$. Thus

$$\begin{gathered} v(t) = \frac{\xi\alpha_u(t)}{1+\xi\int_0^t \alpha_u(s)ds}\\ w(t) = \alpha_u(t)\big[\xi + \int_0^t \frac{b_u(s)}{\alpha_u(s)}ds\big]. \end{gathered} \tag{3.6}$$

Therefore,

$$\begin{gathered} \frac{\xi\alpha_u(t)}{1+\xi\int_0^t \alpha_u(s)ds} \le y(t) \le \alpha_u(t)\Big[\xi + \int_0^t \frac{b_u(s)}{\alpha_u(s)}ds\Big], \quad \text{if } t \ge 0,\\ \alpha_u(t)\Big[\xi + \int_0^t \frac{b_u(s)}{\alpha_u(s)}ds\Big] \le y(t) \le \frac{\xi\alpha_u(t)}{1+\xi\int_0^t \alpha_u(s)ds}, \quad \text{if } t \le 0. \end{gathered}$$

We write

$$g_u(t) \le y(t) \le G_u(t), \quad \text{for } t \in \mathbb{R}, \tag{3.7}$$

where

$$g_u(t) := \begin{cases} \frac{\xi\alpha_u(t)}{1+\xi\int_0^t \alpha_u(s)ds}, & \text{if } t \ge 0\\ \alpha_u(t)\Big[\xi + \int_0^t \frac{b_u(s)}{\alpha_u(s)}ds\Big], & \text{if } t \le 0 \end{cases} \tag{3.8}$$

and

$$G_u(t) := \begin{cases} \alpha_u(t)\Big[\xi + \int_0^t \frac{b_u(s)}{\alpha_u(s)}ds\Big], & \text{if } t \ge 0\\ \frac{\xi\alpha_u(t)}{1+\xi\int_0^t \alpha_u(s)ds}, & \text{if } t \le 0\,. \end{cases} \tag{3.9}$$

Let y_u denote the solution to the equation (3.3) with the initial condition

$$y_u(0) = \xi\,.$$

Hence, $g_u(t) \le y_u(t) \le G_u(t)$, for every $t \in \mathbb{R}$. From (3.6) we see that y_u is defined for all $t \in \mathbb{R}$ if and only if

$$\xi \in \Big(-\frac{1}{\int_0^{+\infty}\alpha_u(s)ds}, \frac{1}{\int_{-\infty}^0 \alpha_u(s)ds}\Big) := (\lambda_u, \mu_u).$$

We let $\lambda := \sup_{u\in C^2(\mathbb{R})}\{\lambda_u\}$ and $\mu := \inf_{u\in C^2(\mathbb{R})}\{\mu_u\}$. Since

$$\begin{aligned} A_1(t) \le \alpha_u(t) \le A_2(t), &\quad \text{for every } t \ge 0 \text{ and } u \in C^2(\mathbb{R})\\ A_2(t) \le \alpha_u(t) \le A_1(t), &\quad \text{for every } t \le 0 \text{ and } u \in C^2(\mathbb{R}) \end{aligned} \tag{3.10}$$

it follows that

$$-\frac{1}{\int_0^{+\infty} A_1(t)dt} \le -\frac{1}{\int_0^{+\infty}\alpha_u(s)ds} \le -\frac{1}{\int_0^{+\infty} A_2(t)dt} := \lambda$$

and

$$\mu := \frac{1}{\int_{-\infty}^0 A_1(t)dt} \le \frac{1}{\int_{-\infty}^0 \alpha_u(s)ds} \le \frac{1}{\int_{-\infty}^0 A_2(t)dt}.$$

Therefore,

$$-\frac{1}{\int_0^{+\infty} A_2(t)dt} := \lambda < 0 < \mu := \frac{1}{\int_{-\infty}^{0} A_1(t)dt}. \tag{3.11}$$

Let

$$g(t) := \inf_{u\in C^2(\mathbb{R})} g_u(t) \quad \text{and} \quad G(t) := \sup_{u\in C^2(\mathbb{R})} G_u(t), \text{ for } t \in \mathbb{R}.$$

For $t \le 0$, we have

$$g_u(t) \ge \alpha_u(t)\Big[\lambda + \int_0^t \frac{b_u(s)}{\alpha_u(s)}ds\Big] \ge A_1(t)\Big[\lambda + \int_0^t \frac{\beta(s)}{A_2(s)}ds\Big]$$

and for $t \ge 0$,

$$g_u(t) \ge \frac{\lambda\alpha_u(t)}{1+\lambda\int_0^t \alpha_u(s)ds} \ge \frac{\lambda A_2(t)}{1+\lambda\int_0^t A_2(s)ds}.$$

Thus

$$g(t) := \begin{cases} \frac{\lambda A_2(t)}{1+\lambda\int_0^t A_2(s)ds}, & \text{if } t \ge 0\\ A_1(t)\Big[\lambda + \int_0^t \frac{\beta(s)}{A_2(s)}ds\Big], & \text{if } t \le 0\,. \end{cases} \tag{3.12}$$

Similarly

$$G(t) := \begin{cases} A_2(t)\Big[\mu + \int_0^t \frac{\beta(s)}{A_1(s)}ds\Big], & \text{if } t \ge 0\\ \frac{\mu A_1(t)}{1+\mu\int_0^t A_1(s)ds}, & \text{if } t \le 0\,. \end{cases} \tag{3.13}$$

By hypothesis (A2), one has $g(\pm\infty) = G(\pm\infty) = 0$. Thus for every $\xi \in (\lambda,\mu)$ and for every y solution to the equation (3.3) with the initial condition $y(0) = \xi$, we have

$$g(t) \le y(t) \le G(t), \quad \text{for every } t \in \mathbb{R}. \tag{3.14}$$

Let $\xi_1,\ \xi_2 \in (\lambda,\mu)$, $\xi_1 \ne \xi_2$ be arbitrary, and y_i^u be the solution to the problem

$$\begin{gathered} \dot y = a_u(t)y + b_u(t) - y^2\\ y(0) = \xi_i \end{gathered}$$

where $i \in \{1,2\}$ and $u \in C^2(\mathbb{R})$. Let $x_i^u(t) := \exp(\int_0^t y_i^u(s)ds)$, for $t \in \mathbb{R}$, $i \in \{1,2\}$ and $u \in C^2(\mathbb{R})$. Then $x_i^u(0) = 1$, $\dot x_i^u(0) = \xi_i$, $\dot x_i^u(t) = y_i^u(t)\cdot x_i^u(t)$, for $t \in \mathbb{R}$, $i \in \{1,2\}$ and $u \in C^2(\mathbb{R})$.

Let us prove that, for every $i \in \{1,2\}$ and $u \in C^2(\mathbb{R})$, $x_i^u(\pm\infty)$, $\dot x_i^u(\pm\infty)$, exist and are finite. Indeed, by relation (2.1),

$$\begin{aligned} x_i^u(+\infty) &= \exp\Big(\int_0^{+\infty} y_i^u(t)dt\Big)\\ &\le \exp\Big(\int_0^{+\infty} A_2(t)\big[\mu + \int_0^t \frac{\beta(s)}{A_1(s)}ds\big]dt\Big)\\ &\le \exp\Big\{\big(\int_0^{+\infty} A_2(t)dt\Big)\cdot\Big[\mu + \int_0^{+\infty}\frac{\beta(s)}{A_1(s)}ds\Big]\Big\} < +\infty, \end{aligned}$$

and

$$\begin{aligned} x_i^u(-\infty) &= \exp\Big(\int_0^{-\infty} y_i^u(t)dt\Big)\\ &\le \exp\Big\{\Big(\int_0^{-\infty} A_1(t)dt\Big)\cdot\Big[\lambda + \int_0^{-\infty}\frac{\beta(s)}{A_1(s)}ds\Big]\Big\} < +\infty, \end{aligned}$$

for every $i \in \{1,2\}$ and $u \in C^2(\mathbb{R})$. For $i \in \{1,2\}$ and $u \in C^2(\mathbb{R})$,

$$|x_i^u(t)| = \exp\Big(\int_0^t y_i^u(s)ds\Big).$$

Hence, for $t \geq 0$,

$$\exp\Big(\int_0^t y_i^u(s)ds\Big) \leq \exp\Big\{(\int_0^{+\infty} A_2(t)dt)\cdot\Big[\mu + \int_0^{+\infty}\frac{\beta(s)}{A_1(s)}ds\Big]\Big\} =: \delta_1$$

and for $t \leq 0$,

$$\exp\Big(\int_0^t y_i^u(s)ds\Big) \leq \exp\Big\{(\int_0^{-\infty} A_1(t)dt)\cdot\Big[\lambda + \int_0^{-\infty}\frac{\beta(s)}{A_1(s)}ds\Big]\Big\} =: \delta_2.$$

Therefore, taking $M_1 := \max\{\delta_1, \delta_2\} > 0$, we have $|x_i^u(t)| \leq M_1$, for every $t \in \mathbb{R}$, $i \in \{1,2\}$, and $u \in C^2(\mathbb{R})$.

Since g and G are continuous with $g(\pm\infty) = G(\pm\infty) = 0$ it follows that they are bounded on $\mathbb{R}$. But

$$g(t) \leq y_i^u(t) \leq G(t), \quad \text{for every } t \in \mathbb{R},\ i \in \{1,2\} \text{ and } u \in C^2(\mathbb{R}).$$

Hence, there exists a constant $\delta_3 > 0$ such that

$$|y_i^u(t)| \leq \delta_3, \quad \text{for } t \in \mathbb{R},\ i \in \{1,2\} \text{ and } u \in C^2(\mathbb{R})$$

and so

$$|\dot{x}_i^u(t)| \leq M_1 \cdot \delta_3 =: M_2, \quad \text{for } t \in \mathbb{R},\ i \in \{1,2\} \text{ and } u \in C^2(\mathbb{R}).$$

For $u \in C^2(\mathbb{R})$ the general solution to the nonhomogeneous equation

$$\ddot{x} = a_u(t)\dot{x} + b_u(t)x + c(t) \tag{3.15}$$

is

$$\begin{aligned} x(t) = \gamma_1^u x_1^u(t) + \gamma_2^u x_2^u(t) + x_2^u(t)\cdot\int_0^t x_1^u(s)\cdot\frac{c(s)}{(\xi_2-\xi_1)\alpha_u(s)}ds \\ - x_1^u(t)\cdot\int_0^t x_2^u(s)\cdot\frac{c(s)}{(\xi_2-\xi_1)\alpha_u(s)}ds, \end{aligned} \tag{3.16}$$

with γ_1^u, $\gamma_2^u \in \mathbb{R}$. From the condition $x(+\infty) = x(-\infty)$, we have

$$\begin{aligned} &\gamma_1^u\cdot[x_1^u(+\infty) - x_1^u(-\infty)] + \gamma_2^u\cdot[x_2^u(+\infty) - x_2^u(-\infty)] \\ &= x_1^u(+\infty)\cdot\int_0^{+\infty} x_2^u(s)\cdot\frac{c(s)}{(\xi_2-\xi_1)\alpha_u(s)}\,ds \\ &\quad - x_1^u(-\infty)\cdot\int_0^{-\infty} x_2^u(s)\cdot\frac{c(s)}{(\xi_2-\xi_1)\alpha_u(s)}\,ds \\ &\quad + x_2^u(-\infty)\cdot\int_0^{-\infty} x_1^u(s)\cdot\frac{c(s)}{(\xi_2-\xi_1)\alpha_u(s)}\,ds \\ &\quad - x_2^u(+\infty)\cdot\int_0^{+\infty} x_1^u(s)\cdot\frac{c(s)}{(\xi_2-\xi_1)\alpha_u(s)}\,ds. \end{aligned} \tag{3.17}$$

Now we prove that the relation (3.17) is satisfied by infinitely many pairs (γ_1^u, γ_2^u), $u \in C^2(\mathbb{R})$. Indeed, if we denote by

$$d_1 := x_1^u(+\infty) - x_1^u(-\infty), \quad d_2 := x_2^u(+\infty) - x_2^u(-\infty),$$

and d_3 the right hand side of (3.17), then we have to consider only three cases.

Case 1. If $d_1 \neq 0$ and $d_2 = 0$, it follows that $\gamma_1^u = \frac{d_3}{d_1}$ and $\gamma_2^u \in \mathbb{R}$; similarly, if $d_1 = 0$ and $d_2 \neq 0$, it follows that $\gamma_1^u \in \mathbb{R}$ and $\gamma_2^u = \frac{d_3}{d_2}$.
Case 2. If $d_1 \neq 0$ and $d_2 \neq 0$, it follows that

$$\gamma_1^u = \frac{d_3 - d_2\gamma_2^u}{d_1} \quad \text{and} \quad \gamma_2^u \in \mathbb{R}.$$

Case 3. If $d_1 = 0$ and $d_2 = 0$, we show that $d_3 = 0$ (and so the solutions are γ_1^u, $\gamma_2^u \in \mathbb{R}$).

Indeed, in this case, $x_1^u(+\infty) = x_1^u(-\infty)$ and $x_2^u(+\infty) = x_2^u(-\infty)$, and we have to prove that

$$x_1^u(+\infty) \cdot \int_{-\infty}^{+\infty} x_2^u(s) \cdot \frac{c(s)}{\alpha_u(s)} ds = x_2^u(+\infty) \cdot \int_{-\infty}^{+\infty} x_1^u(s) \cdot \frac{c(s)}{\alpha_u(s)} ds. \tag{3.18}$$

To prove (3.18) we shall apply Lemma 2.4 to the mapping $f : [0, +\infty) \to \mathbb{R}$, defined by

$$f(t) := x_1^u(t) \cdot \int_{-t}^{+t} x_2^u(s) \cdot \frac{c(s)}{\alpha_u(s)} ds - x_2^u(t) \cdot \int_{-t}^{+t} x_1^u(s) \cdot \frac{c(s)}{\alpha_u(s)} ds.$$

Thus

$$\begin{aligned} \frac{df}{dt}(t) &= \dot{x}_1^u(t) \cdot \int_{-t}^{+t} x_2^u(s) \cdot \frac{c(s)}{\alpha_u(s)}\, ds - \dot{x}_2^u(t) \cdot \int_{-t}^{+t} x_1^u(s) \cdot \frac{c(s)}{\alpha_u(s)}\, ds \\ &\quad + \frac{c(-t)}{\alpha_u(-t)} \left[x_1^u(t) \cdot x_2^u(-t) - x_2^u(t) \cdot x_1^u(-t) \right]. \end{aligned}$$

Since $\dot{x}_i^u(\pm\infty) = x_i^u(\pm\infty) \cdot y_i^u(\pm\infty) = 0$, $i \in \{1, 2\}$, the mapping $\frac{c}{\alpha_u}$ is bounded on $\mathbb{R}$ (see hypothesis (C2)), and

$$\lim_{t \to +\infty} \left[x_1^u(t) \cdot x_2^u(-t) - x_2^u(t) \cdot x_1^u(-t) \right] = 0,$$

it follows that $\lim_{t\to+\infty} \frac{df}{dt}(t) = 0$. Therefore f is uniformly continuous on $[0, +\infty)$, being Lipschitz on $[0, +\infty)$. Since x_i^u, $i \in \{1, 2\}$ are bounded, from (C2) it follows that $\int_0^{+\infty} f(t)\, dt$ exists and is finite. Hence, by Lemma 2.4 we obtain

$$\lim_{t\to+\infty} f(t) = 0.$$

Now we define the multivalued operator $T : X \to 2^X$, by

$$\begin{aligned} Tu := \Big\{ &\gamma_1^u x_1^u(\cdot) + \gamma_2^u x_2^u(\cdot) + x_2^u(\cdot) \cdot \int_0^{(\cdot)} x_1^u(s) \cdot \frac{c(s)}{(\xi_2 - \xi_1)\alpha_u(s)}\, ds \\ &- x_1^u(\cdot) \cdot \int_0^{(\cdot)} x_2^u(s) \cdot \frac{c(s)}{(\xi_2 - \xi_1)\alpha_u(s)}\, ds, \\ &\text{with } |\gamma_1^u| + |\gamma_2^u| \leq 1,\ \gamma_1^u,\ \gamma_2^u \text{ satisfying (3.17)} \Big\}, \end{aligned}$$

for every $u \in X$. By (3.15)-(3.16) we have

$$|x(t)| \leq 2M_1 + \frac{M_1}{|\xi_2 - \xi_1|} \Big(\Big| \int_0^t x_1^u(s) \frac{c(s)}{\alpha_u(s)} ds \Big| + \Big| \int_0^t x_2^u(s) \frac{c(s)}{\alpha_u(s)} ds \Big| \Big).$$

Hence $|x(t)| \leq k_1$, for every $t \in \mathbb{R}$, where

$$k_1 := \max \Big\{ 2M_1 + \frac{2M_1^2}{|\xi_2 - \xi_1|} \int_0^{+\infty} \frac{|c(s)|}{A_1(s)} ds, 2M_1 + \frac{2M_1^2}{|\xi_2 - \xi_1|} \int_{-\infty}^0 \frac{|c(s)|}{A_2(s)} ds \Big\}.$$

Similarly

$$|\dot{x}(t)| = \Big|\gamma_1^u \dot{x}_1^u(t) + \gamma_2^u \dot{x}_2^u(t) + \dot{x}_2^u(t)\int_0^t x_1^u(s)\frac{c(s)}{\alpha_u(s)}ds - \dot{x}_1^u(t)\int_0^t x_2^u(s)\frac{c(s)}{\alpha_u(s)}ds\Big|,$$

and there exists another constant $k_2 \geq 0$,

$$k_2 := \max\Big\{2M_2 + \frac{2M_1M_2}{|\xi_2-\xi_1|}\int_0^{+\infty}\frac{|c(s)|}{A_1(s)}ds, 2M_2 + \frac{2M_1M_2}{|\xi_2-\xi_1|}\int_{-\infty}^0\frac{|c(s)|}{A_2(s)}ds\Big\},$$

such that $|\dot{x}(t)| \leq k_2$, for every $t \in \mathbb{R}$. Remark that, by relation (2.3), k_1, k_2 are finite. We let $k := \max\{k_1,\ k_2\}$, and

$$M := \big\{x \in C^2(\mathbb{R}),\ |x(t)| \leq k,\ |\dot{x}(t)| \leq k,\ \text{for every } t \in \mathbb{R}\big\}.$$

4. Proof of main result

To prove Theorem 2.1 it is sufficient to prove that the operator T has a fixed point. We do this in three steps.

Step 1: For every $u \in M$, $T(u)$ is a non-empty convex closed set. Let $u \in M$ be arbitrary.

From the definition of T we see that $T(u)$ is non-empty and convex.

Let $(x^n)_{n\in\mathbb{N}} \subset T(u)$ be such that $x^n \to x$ and $\dot{x}^n \to \dot{x}$ uniformly on $\mathbb{R}$ as $n \to \infty$. We have

$$x^n(t) := \gamma_{1,n}^u x_1^u(t) + \gamma_{2,n}^u x_2^u(t) + H^u(t),$$

for every $n \in \mathbb{N}$, with $|\gamma_{1,n}^u| + |\gamma_{2,n}^u| \leq 1$, $\gamma_{1,n}^u, \gamma_{2,n}^u$ satisfying (3.17), and

$$H^u(t) := x_2^u(t)\cdot\int_0^t x_1^u(s)\cdot\frac{c(s)}{(\xi_2-\xi_1)\alpha_u(s)}ds - x_1^u(t)\cdot\int_0^t x_2^u(s)\cdot\frac{c(s)}{(\xi_2-\xi_1)\alpha_u(s)}\,ds.$$

Then there exist subsequences such that $\gamma_{1,k_n}^u \to \gamma_1^u$ and $\gamma_{2,k_n}^u \to \gamma_2^u$, as $n \to \infty$.

Since $(x^{k_n})_{n\in\mathbb{N}}$ converges uniformly to $y := \gamma_1^u x_1^u + \gamma_2^u x_2^u + H^u$, it follows that $x = y$. Also

$$\dot{x}^{k_n} \to \dot{y} = \dot{x}, \quad \text{as } n \to \infty.$$

So $x \in T(u)$, that is $T(u)$ is a closed set.

Step 2: $T(M)$ is relatively compact. The relative compactness of $T(M)$ will be proved by using Proposition 2.5.

From the definitions of T and M we see that $|x(t)| \leq k$, $|\dot{x}(t)| \leq k$, for all $t \in \mathbb{R}$. Thus the first condition of Proposition 2.5 is fulfilled with $h_1 = h_2 = k$.

Conditions (b) and (c) of Proposition 2.5 are implied by the following assumption:

(d) *There exist f_1, $f_2 : \mathbb{R} \to \mathbb{R}_+$ integrable on $\mathbb{R}$ such that for every $x \in \mathcal{A}$*

$$|\dot{x}(t)| \leq f_1(t) \quad \textit{and} \quad |\ddot{x}(t)| \leq f_2(t), \quad \textit{for } t \in \mathbb{R}.$$

This last assertion follows from the fact that, for every $t_1, t_2 \in \mathbb{R}$,

$$x(t_1) - x(t_2) = \int_{t_1}^{t_2}\dot{x}(t)dt \quad\text{and}\quad \dot{x}(t_1) - \dot{x}(t_2) = \int_{t_1}^{t_2}\ddot{x}(t)\,dt.$$

For $i \in \{1,2\}$ let

$$g_{1i}(t) := \begin{cases} \max\Big\{A_2(t)\big[\mu + \int_0^t \frac{\beta(s)}{A_1(s)}ds\big], \frac{|\xi_i|A_2(t)}{|1+\xi_i\int_0^t A_2(s)ds|}\Big\}, & t \geq 0\\ \max\Big\{\frac{|\xi_i|A_1(t)}{|1+\xi_i\int_0^t A_1(s)ds|}, A_1(t)\big[-\lambda + \int_t^0 \frac{\beta(s)}{A_2(s)}ds\big]\Big\}, & t \leq 0. \end{cases}$$

Hence $|\dot{x}_i^u|$ is bounded by the integrable function $M_1 \cdot g_{1i}$, $i \in \{1,2\}$. Furthermore, since

$$|\int_0^t x_2^u(s) \cdot \frac{c(s)}{(\xi_2 - \xi_1) \cdot \alpha_u(s)} ds|$$

is bounded (on the positive semiaxis by $\frac{M_1}{|\xi_2 - \xi_1|} \cdot \int_0^{+\infty} \frac{|c(s)|}{A_1(s)} ds$ and on the negative semiaxis by $\frac{M_1}{|\xi_2 - \xi_1|} \cdot \int_{-\infty}^0 \frac{|c(s)|}{A_1(s)} ds$), and $|\dot{x}_1^u|$ is bounded by an integrable function, we see that

$$|\dot{x}_1^u \cdot \int_0^{(\cdot)} x_2^u(s) \cdot \frac{c(s)}{(\xi_2 - \xi_1) \cdot \alpha_u(s)} ds|$$

is bounded by an integrable function. Similarly,

$$|\dot{x}_2^u \cdot \int_0^{(\cdot)} x_1^u(s) \cdot \frac{c(s)}{(\xi_2 - \xi_1) \cdot \alpha_u(s)} ds|$$

is bounded by an integrable function. Therefore, the existence of f_1 in assertion (d) follows. Now, since

$$\ddot{x}(t) = a_u(t)\dot{x}(t) + b_u(t)x + c(t),$$

a_u is bounded (hypothesis (A1)), $|\dot{x}|$ is bounded by an integrable function, $|x|$ is bounded (by k), b_u is integrable on $\mathbb{R}$ (by relation (2.2), hypothesis (B1), and $|c|$ is integrable on $\mathbb{R}$ (by hypothesis (C1)), we see that $|\ddot{x}|$ is bounded by an integrable function. This proves the existence of f_2, and hence assertion (d) is verified.

Step 3: T is upper semicontinuous. Let A be a closed subset of M. Hence if $(u_n)_n \subset A$ such that $u_n \to u$ and $\dot{u}_n \to \dot{u}$ uniformly on $\mathbb{R}$, as $n \to \infty$, it follows that $u \in A$.

Let $z_n \in T^{-1}(A)$ be such that $z_n \to z$ and $\dot{z}_n \to \dot{z}$ uniformly on $\mathbb{R}$, as $n \to \infty$. We have to prove that $z \in T^{-1}(A)$. Since $z_n \in T^{-1}(A)$ there exists $x_n \in A$, $x_n \in Tz_n$. Thus

$$\ddot{x}_n = a(t, z_n, \dot{z}_n)\dot{x}_n + x(t, z_n, \dot{z}_n)x_n + c(t), \quad n \in \mathbb{N} \tag{4.1}$$

and

$$x_n(+\infty) = x_n(-\infty), \quad \dot{x}_n(+\infty) = \dot{x}_n(-\infty), \quad n \in \mathbb{N}. \tag{4.2}$$

Since $x_n \in T(M)$ and $T(M)$ is relatively compact, the sequence x_n contains subsequence converging in C^2 to some x. One can assume that $x_n \to x$, $\dot{x}_n \to \dot{x}$ uniformly on $\mathbb{R}$, as $n \to \infty$.

Since $a(t, z_n(t), \dot{z}_n(t)) \to a(t, z(t), \dot{z}(t))$ and $b(t, z_n(t), \dot{z}_n(t)) \to b(t, z(t), \dot{z}(t))$, uniformly on compact subsets of $\mathbb{R}$, it follows that x is solution to the equation

$$\ddot{x} = a(t, z(t), \dot{z}(t))\dot{z} + b(t, z(t), \dot{z}(t))z + c(t),$$

with

$$x(0) = \lim_{n\to\infty} x_n(0) \quad \text{and} \quad \dot{x}(0) = \lim_{n\to\infty} \dot{x}_n(0).$$

Furthermore, by (4.2) we find, by passing to the limit as $n \to \infty$,

$$x(+\infty) = x(-\infty) \quad \text{and} \quad \dot{x}(+\infty) = \dot{x}(-\infty).$$

Since the set A is closed, $x \in A$. Therefore, $z \in T^{-1}(A)$, which completes the proof of Theorem 2.1.

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