Electronic Journal of Differential Equations, Vol. 2005(2005), No. 47, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ASYMPTOTIC PROPERTIES OF SOLUTIONS TO THREE-DIMENSIONAL FUNCTIONAL DIFFERENTIAL SYSTEMS OF NEUTRAL TYPE

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Abstract. In this paper, we study the behavior of solutions to three-dimensional functional differential systems of neutral type. We find sufficient conditions for solutions to be oscillatory, and to decay to zero. The main results are presented in three theorems and illustrated with one example.

1. Introduction

We consider neutral functional differential systems

$$\begin{aligned} \big[y_1(t) - a(t)y_1(g(t))\big]' &= p_1(t)y_2(t), \\ y_2'(t) &= p_2(t)y_3(t), \\ y_3'(t) &= -p_3(t)f(y_1(h(t))), \quad t \ge t_0. \end{aligned} \tag{1.1}$$

The following conditions are assumed:

(a) $a : [t_0, \infty) \to (0, \infty]$ is a continuous function;
(b) $g : [t_0, \infty) \to \mathbb{R}$ is a continuous and increasing function and $\lim_{t\to\infty} g(t) = \infty$;
(c) $p_i : [t_0, \infty) \to [0, \infty)$, $i = 1, 2, 3$ are continuous functions; p_3 not identically equal to zero in any neighbourhood of infinity, $\int^\infty p_j(t)\,dt = \infty$, $j = 1, 2$;
(d) $h : [t_0, \infty) \to \mathbb{R}$ is a continuous and increasing function and $\lim_{t\to\infty} h(t) = \infty$;
(e) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $uf(u) > 0$ for $u \ne 0$ and $|f(u)| \ge K|u|$, where K is a positive constant.

For $t_1 \ge t_0$, we define

$$\widetilde{t}_1 = \min\{t_1, g(t_1), h(t_1)\}\,.$$

A function $y = (y_1, y_2, y_3)$ is a solution of the system (1.1) if there exists a $t_1 \ge t_0$ such that y is continuous on $[\widetilde{t}_1, \infty)$, $y_1(t) - a(t)y_1(g(t)), y_i(t)$, $i = 2, 3$ are continuously differentiable on $[t_1, \infty)$ and y satisfies (1.1) on $[t_1, \infty)$. Denote by W

2000 *Mathematics Subject Classification.* 34K25, 34K40.
Key words and phrases. Differential system of neutral type; asymptotic properties of solutions.

Submitted February 12, 2005. Published April 27, 2005.
Supported by grant VEGA No. 2/3205/23 of Scientific Grant Agency of Ministry of Education of Slovak Republic and Slovak Academy of Sciences.

the set of all solutions $y = (y_1, y_2, y_3)$ of the system (1.1) which exist on some ray $[T_y, \infty) \subset [t_0, \infty)$ and satisfy

$$\sup \Big\{ \sum_{i=1}^{3} |y_i(t)| : t \ge T \Big\} > 0 \quad \text{for any } T \ge T_y.$$

A solution $y \in W$ is considered to be non-oscillatory if there exists a $T_y \ge t_0$ such that every component is different from zero for $t \ge T_y$. Otherwise a solution $y \in W$ is said to be oscillatory.

The purpose of this article is to study asymptotic properties of solutions to the three-dimensional functional differential systems of neutral type (1.1) and also the special asymptotic properties of solutions whose first component is bounded. The asymptotic and oscillatory properties of solutions to differential systems with deviating arguments has been studied for example in the papers [1, 2, 4, 8, 10, 11].

For a $y_1(t)$, we define

$$z_1(t) = y_1(t) - a(t) y_1(g(t)). \tag{1.2}$$

Denote

$$P_1(s,t) = \int_t^s p_1(x)\,dx, \quad P_{1,2}(s,t) = \int_t^s p_1(v) \int_t^v p_2(x)\,dx\,dv,$$
$$P_2(s,t) = \int_t^s p_2(x)\,dx, \quad P_{2,1}(s,t) = \int_t^s p_2(v) \int_t^v p_1(x) dx\,dv, \quad s \ge t \ge t_0.$$

2. Classification of non-oscillatory solutions

Lemma 2.1 ([6, Lemma 1]). *Let $y \in W$ be a solution of (1.1) with $y_1(t) \ne 0$ on $[t_1, \infty)$, $t_1 \ge t_0$. Then y is non-oscillatory and $z_1(t), y_2(t), y_3(t)$ are monotone on some ray $[T, \infty)$, $T \ge t_1$.*

Let $y \in W$ be a non-oscillatory solution of (1.1). From (1.1) and (c) it follows that the function $z_1(t)$ from (1.2) has to be eventually of constant sign, so that either

$$y_1(t) z_1(t) > 0 \tag{2.1}$$

or

$$y_1(t) z_1(t) < 0 \tag{2.2}$$

for sufficiently large t. Assume first that (2.1) holds. From [6, Lemma 4] it follows the statement in Lemma 2.2.

Lemma 2.2. *Let $y = (y_1, y_2, y_3) \in W$ be a non-oscillatory solution of (1.1) on $[t_1, \infty)$ and that (2.1) holds. Then there exists a $t_2 \ge t_1$ such that for $t \ge t_2$ either*

$$\begin{gathered} y_1(t) z_1(t) > 0 \\ y_2(t) z_1(t) < 0 \\ y_3(t) z_1(t) > 0 \end{gathered} \tag{2.3}$$

or

$$y_i(t) z_1(t) > 0, \quad i = 1, 2, 3. \tag{2.4}$$

Denote by N_1^+ the set of non-oscillatory solutions of (1.1) satisfying (2.3), and by N_3^+ the non-oscillatory solutions of (1.1) satisfying (2.4). Now assume that (2.2) holds. With the aid of the Kiguradze's Lemma is easy to prove Lemma 2.3.

Lemma 2.3. *Let $y = (y_1, y_2, y_3) \in W$ be a non-oscillatory solution of (1.1) on $[t_1, \infty)$ and (2.2) holds. Then there exists a $t_2 \geq t_1$ such that for $t \geq t_2$ either*

$$\begin{gathered} y_1(t)z_1(t) < 0 \\ y_2(t)z_1(t) > 0 \\ y_3(t)z_1(t) < 0 \end{gathered} \tag{2.5}$$

or

$$\begin{gathered} y_1(t)z_1(t) < 0 \\ y_i(t)z_1(t) > 0, \quad i = 2, 3. \end{gathered} \tag{2.6}$$

Denote by N_2^- the sets of non-oscillatory solutions of (1.1) satisfying (2.5), and by N_3^- the non-oscillatory solutions of (1.1) satisfying (2.6). Denote by N the set of all non-oscillatory solutions of (1.1). Obviously by Lemmas 2.2 and 2.3, we have

$$N = N_1^+ \cup N_3^+ \cup N_2^- \cup N_3^-. \tag{2.7}$$

Lemma 2.4. *Suppose that $a(t)$ is bounded on $[t_2, \infty)$ and $y \in W$ be a non-oscillatory solution of the system (1.1) with $y_1(t)$ bounded on $[t_2, \infty)$, $t_2 \geq t_0$. Then*

$$y \in N_1^+ \cup N_2^-.$$

Proof. We must show that the set $N_3^+ \cup N_3^-$ is empty. Let $y \in W$ be a non-oscillatory solution of (1.1) with $y_1(t)$ bounded on $[t_2, \infty)$ and $y \in N_3^+ \cup N_3^-$. Without loss of generality we suppose that $y_1(t) > 0$ on $[t_2, \infty)$. Because $a(t)$ and $y_1(t)$ are bounded, $z_1(t)$ is bounded on $[t_3, \infty)$, where $t_3 \geq t_2$ is sufficiently large. If $y \in N_3^+ \cup N_3^-$ then a function $|y_2(t)|$ is nondecreasing and

$$|y_2(t)| \geq M, \quad 0 < M = \text{const.} \quad \text{for } t \geq t_3.$$

Integrating the first equation of (1.1) from s to t and using the last inequality we get

$$|z_1(t)| - |z_1(s)| \geq M \int_s^t p_1(u) du, \quad t > s \geq t_3. \tag{2.8}$$

From (2.8) and (c) we have $\lim_{t\to\infty} |z_1(t)| = \infty$. This contradicts the fact that $z_1(t)$ is bounded and $N_3^+ \cup N_3^- = \emptyset$. The proof is complete. □

Lemma 2.5 ([3, Lemma 2.2]). *In addition to the conditions (a) and (b) suppose that*

$$1 \leq a(t) \quad \text{for } t \geq t_0.$$

Let $y_1(t)$ be a continuous non-oscillatory solution of the functional inequality

$$y_1(t)[y_1(t) - a(t)y_1(g(t))] > 0$$

defined in a neighbourhood of infinity. Suppose that $g(t) > t$ for $t \geq t_0$. Then $y_1(t)$ is bounded. If, moreover,

$$1 < \lambda_\star \leq a(t), \quad t \geq t_0$$

for some positive constant $\lambda_\star$, then $\lim_{t\to\infty} y_1(t) = 0$.

Lemma 2.6 ([9, Lemma 4]). *Assume that $q : [t_0, \infty) \to [0, \infty)$ and $\delta : [t_0, \infty) \to \mathbb{R}$ are continuous functions, $\lim_{t\to\infty} \delta(t) = \infty$, $\delta(t) > t$ for $t \geq t_0$, and*

$$\liminf_{t\to\infty} \int_t^{\delta(t)} q(s)\, ds > \frac{1}{e}.$$

Then the functional inequality

$$x'(t) - q(t)x(\delta(t)) \geq 0, \quad t \geq t_0,$$

has no eventually positive solution, and

$$x'(t) - q(t)x(\delta(t)) \leq 0, \quad t \leq t_0$$

has no eventually negative solution.

3. Oscillation theorems

Theorem 3.1. *Suppose that*

$$a(t) \text{ is bounded for } t \geq t_0, \tag{3.1}$$

$$g(t) < h(t) < t < \alpha(t) \quad \text{for } t \geq t_0, \tag{3.2}$$

where $\alpha : [t_0, \infty) \to \mathbb{R}$ is a continuous function,

$$\limsup_{t\to\infty} \int_t^{h^{-1}(t)} KP_{2,1}(u,t)p_3(u)\, du > 1, \tag{3.3}$$

$$\liminf_{t\to\infty} \int_t^{g^{-1}(h(t))} p_1(v) \int_v^{\alpha(v)} \frac{KP_2(u,v)p_3(u)\, du\, dv}{a(g^{-1}(h(u)))} > \frac{1}{e}, \tag{3.4}$$

where $g^{-1}(t)$ is the inverse function of $g(t)$. Then every solution $y = (y_1, y_2, y_3) \in W$ of (1.1) with $y_1(t)$ bounded is oscillatory.

Proof. Let $y \in W$ be a non-oscillatory solution of (1.1) with $y_1(t)$ bounded. From Lemma 2.4 we have $y \in N_1^+ \cup N_2^-$ on $[t_2, \infty)$. Without loss of generality we may suppose that $y_1(t)$ is positive for $t \geq t_2$.
I) Let $y \in N_1^+$ on $[t_2, \infty)$. In this case

$$y_1(t) > 0, \quad z_1(t) > 0, \quad y_2(t) < 0, \quad y_3(t) > 0 \quad \text{for } t \geq t_2. \tag{3.5}$$

Integrating $\int_t^s P_{2,1}(u,t)y_3'(u)\, du$ by parts with $f(u) = P_{2,1}(u,t)$, $g(u) = y_3(u)$, and one gets

$$\int_t^s P_{2,1}(u,t)\ y_3'(u)\, du = P_{2,1}(s,t)\ y_3(s) - \int_t^s P_1(u,t)\ y_2'(u)\, du.$$

Integrating by parts again with $f(u) = P_1(u,t)$, $g(u) = y_2(u)$, we have

$$\int_t^s P_{2,1}(u,t)\ y_3'(u)\, du = P_{2,1}(s,t)\ y_3(s) - P_1(s,t)\ y_2(s) + z_1(s) - z_1(t)\,. \tag{3.6}$$

This equation implies

$$z_1(t) = z_1(s) - P_1(s,t)y_2(s) + P_{2,1}(s,t)y_3(s) - \int_t^s P_{2,1}(u,t)y_3'(u)\, du, \tag{3.7}$$

for $s > t \geq t_2$. From (3.7) in regard to (3.5), (e) and the third equation of (1.1), we get

$$z_1(t) \geq \int_t^s KP_{2,1}(u,t)p_3(u)y_1(h(u))\, du, \quad s > t \geq t_2. \tag{3.8}$$

Since $z_1(t) \leq y_1(t)$ for $t \geq t_2$, it follows that

$$z_1(h(t)) \leq y_1(h(t)) \quad \text{for} \quad t \geq t_3, \tag{3.9}$$

where $t_3 \geq t_2$ is sufficiently large. Combining (3.8) and (3.9) we have

$$z_1(t) \geq \int_t^s KP_{2,1}(u,t)p_3(u)z_1(h(u))\,du, \quad s > t \geq t_3.$$

Putting $s = h^{-1}(t)$ and using the monotonicity of $z_1(h(u))$ from the previous inequality we obtain

$$\begin{gathered} z_1(t) \geq z_1(t)\int_t^{h^{-1}(t)} KP_{2,1}(u,t)p_3(u)\,du, \quad t \geq t_3; \\ 1 \geq \int_t^{h^{-1}(t)} KP_{2,1}(u,t)p_3(u)\,du, \quad t \geq t_3, \end{gathered}$$

which contradicts (3.3) and $N_1^+ = \emptyset$.

(II) Let $y \in N_2^-$ on $[t_2, \infty)$. In this case

$$y_1(t) > 0, \quad z_1(t) < 0, \quad y_2(t) < 0, \quad y_3(t) > 0 \quad \text{for} \quad t \geq t_2. \tag{3.10}$$

Integrating $\int_t^s P_2(u,t)y_3'(u)\,du$ by parts we derive the integral identity

$$y_2(t) = y_2(s) - P_2(s,t)y_3(s) + \int_t^s P_2(u,t)y_3'(u)\,du, \quad s > t \geq t_2. \tag{3.11}$$

From (3.11) with regard to (3.10), (e) and the third equation of (1.1) we get

$$y_2(t) \leq -\int_t^s KP_2(u,t)p_3(u)y_1(h(u))\,du, \quad s > t \geq t_2. \tag{3.12}$$

Because $z_1(t) > -a(t)y_1(g(t))$ for $t \geq t_2$ it follows

$$z_1(g^{-1}(h(t))) > -a(g^{-1}(h(t)))y_1(h(t));$$

$$-y_1(h(t)) < \frac{z_1(g^{-1}(h(t)))}{a(g^{-1}(h(t)))} \quad \text{for } t \geq t_2. \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$y_2(t) \leq \int_t^s \frac{KP_2(u,t)p_3(u)z_1(g^{-1}(h(u)))\,du}{a(g^{-1}(h(u)))}, \quad s > t \geq t_2.$$

Multiplying the last inequality by $p_1(t)$, using the first equation of (1.1) and the monotonicity of $z_1(g^{-1}(h(t)))$ we get

$$z_1'(t) \leq \Big[p_1(t)\int_t^s \frac{KP_2(u,t)p_3(u)\,du}{a(g^{-1}(h(u)))}\Big] z_1(g^{-1}(h(t))), \quad s > t \geq t_2.$$

Let $s = \alpha(t)$ and so

$$z_1'(t) - \Big[p_1(t)\int_t^{\alpha(t)} \frac{KP_2(u,t)p_3(u)\,du}{a(g^{-1}(h(u)))}\Big] z_1(g^{-1}(h(t))) \leq 0, \quad t \geq t_2.$$

By Lemma 2.6 and condition (3.4), the last inequality has no eventually negative solution, which is a contradiction and $N_2^- = \emptyset$. The proof is complete. □

Theorem 3.2. *Suppose that*

$$1 < \lambda_\star \leq a(t) \leq c, \quad \textit{for } t \geq t_0 \textit{ and some constants } \lambda_\star,\ c; \tag{3.14}$$

$$t < g(t) < h(t) \quad \textit{for } t \geq t_0; \tag{3.15}$$

$$t < \alpha(t), \quad \textit{where } \alpha : [t_0, \infty) \to \mathbb{R} \textit{ is a continuous function} \tag{3.16}$$

and (3.4) holds. Then for every non-oscillatory solution, $y \in W$ of (1.1) *with* $y_1(t)$ *bounded, we have* $\lim_{t\to\infty} y_i(t) = 0$*,* $i = 1, 2, 3$*.*

Proof. Let $y \in W$ be a non-oscillatory solution of (1.1) with $y_1(t)$ bounded. From Lemma 2.4 we have $y \in N_1^+ \cup N_2^-$ on $[t_2, \infty)$. Without loss of generality we may suppose that $y_1(t)$ is positive for $t \ge t_2$.

(I) Let $y \in N_1^+$ on $[t_2, \infty)$. In this case (3.5) holds. By Lemma 2.5 it follows that $\lim_{t\to\infty} y_1(t) = 0$. We prove that $\lim_{t\to\infty} y_2(t) = \lim_{t\to\infty} y_3(t) = 0$ indirectly.

Let $\lim_{t\to\infty} y_2(t) = -S$, $0 < S = \text{const}$. Then

$$y_2(t) \le -S, \quad t \ge t_2. \tag{3.17}$$

Integrating the first equation of (1.1) from t_2 to t and using (3.17) we get

$$z_1(t) - z_1(t_2) \le -S \int_{t_2}^{t} p_1(s)\, ds, \quad t \ge t_2. \tag{3.18}$$

From this inequality and (c) we have $\lim_{t\to\infty} z_1(t) = -\infty$ which contradicts $z_1(t) > 0$ for $t \ge t_2$ and so $\lim_{t\to\infty} y_2(t) = 0$.

Let $\lim_{t\to\infty} y_3(t) = P$, $0 < P = \text{const}$. Then

$$y_3(t) \ge P, \quad t \ge t_2. \tag{3.19}$$

Integrating the second equation of (1.1) from t_2 to t and using (3.19) we get

$$y_2(t) - y_2(t_2) \ge P \int_{t_2}^{t} p_2(s)\, ds, \quad t \ge t_2. \tag{3.20}$$

From (3.20) and (c) we have $\lim_{t\to\infty} y_2(t) = \infty$ and that contradicts $y_2(t) < 0$ for $t \ge t_2$ and so $\lim_{t\to\infty} y_3(t) = 0$.

(II) Let $y \in N_2^-$ on $[t_2, \infty)$. Analogously as in the case (II) of the proof of Theorem 3.1 we can show that $N_2^- = \emptyset$. The proof is complete. □

Example. Consider the system

$$\begin{aligned}
\big[y_1(t) - 2y_1(3t)\big]' &= t y_2(t),\\
y_2'(t) &= t y_3(t),\\
y_3'(t) &= -45 t^{-5} y_1(9t), \quad t \ge t_0 > 0.
\end{aligned} \tag{3.21}$$

In this example $a(t) = 2$, $g(t) = 3t$, $h(t) = 9t$, $p_1(t) = p_2(t) = t$, $p_3(t) = 45t^{-5}$, $f(t) = t$, $K = 1$, $P_2(u, v) = \frac{1}{2}(u^2 - v^2)$. We chose $\alpha(t) = 2t$ and calculate the condition (3.4) as follows

$$\liminf_{t\to\infty} \frac{45}{4} \int_t^{3t} v \int_v^{2v} (u^2 - v^2) u^{-5}\, du\, dv = \frac{405}{256} \ln 3.$$

All conditions of Theorem 3.2 are satisfied. Then for every non-oscillatory solution $y \in W$ of (3.21) with $y_1(t)$ bounded, it holds

$$\lim_{t\to\infty} y_1(t) = \lim_{t\to\infty} y_2(t) = \lim_{t\to\infty} y_3(t) = 0.$$

For instance functions

$$y_1(t) = \frac{1}{t}, \quad y_2(t) = \frac{-1}{3t^3}, \quad y_3(t) = \frac{1}{t^5}, \quad t \ge t_0$$

are such a kind of solutions.

Theorem 3.3. *Suppose that*

$$1 < \lambda_\star \le a(t), \quad t \ge t_0 \text{ for some constant } \lambda_\star; \tag{3.22}$$

$$\limsup_{t\to\infty} \int_{h^{-1}(g(t))}^{t} \frac{KP_{1,2}(t,u)p_3(u)\,du}{a(g^{-1}(h(u)))} > 1\,. \tag{3.23}$$

and (3.4), (3.15) and (3.16) hold. Then for every non-oscillatory solution $y \in W$ of (1.1)*, it holds* $\lim_{t\to\infty} y_i(t) = 0$*,* $i = 1,2,3$.

Proof. Let $y \in W$ be a non-oscillatory solution of (1.1). From (2.7) we have $y \in N_1^+ \cup N_3^+ \cup N_2^- \cup N_3^-$ on $[t_2,\infty)$. Without loss of generality we may suppose that $y_1(t)$ is positive for $t \ge t_2$.

(I) Let $y \in N_1^+$ on $[t_2,\infty)$. In this case (3.5) holds. By Lemma 2.5 it follows that $\lim_{t\to\infty} y_1(t) = 0$. We prove, that $\lim_{t\to\infty} y_2(t) = \lim_{t\to\infty} y_3(t) = 0$ indirectly analogously as in the case (I) of the proof of Theorem 3.2.

(II) Let $y \in N_3^+$ on $[t_2,\infty)$. In this case

$$y_1(t) > 0, \quad z_1(t) > 0, \quad y_2(t) > 0, \quad y_3(t) > 0 \quad \text{for } t \ge t_2. \tag{3.24}$$

In this case,

$$y_2(t) \ge M, \quad 0 < M = \text{const.} \quad \text{for } t \ge t_2.$$

Integrating the first equation of (1.1) from s to t and using the last inequality we get

$$z_1(t) - z_1(s) \ge M \int_s^t p_1(u)du, \quad t > s \ge t_2. \tag{3.25}$$

From (3.25) and (c) we have $\lim_{t\to\infty} z_1(t) = \infty$ and the function $z_1(t)$ is unbounded. From (1.2), (3.15), (3.22) we have

$$y_1(t) > a(t)y_1(g(t)) > y_1(g(t)),$$

which implies that $y_1(t)$ is bounded, but $z_1(t) < y_1(t)$ which is a contradiction. $N_3^+ = \emptyset$.

(III) Let $y \in N_2^-$ on $[t_2,\infty)$. Analogously as in the case (II) of the proof of Theorem 3.1 we can show that $N_2^- = \emptyset$.

(IV) Let $y \in N_3^-$ on $[t_2,\infty)$. In this case

$$y_1(t) > 0, \quad z_1(t) < 0, \quad y_2(t) < 0, \quad y_3(t) < 0 \quad \text{for } t \ge t_2. \tag{3.26}$$

By interchanging the order of integrating in $P_{1,2}(t,u)$, we have

$$\int_s^t P_{1,2}(t,u)y_3'(u)\,du = \int_s^t \Big(\int_u^t p_2(x)\int_x^t p_1(v)\,dv\,dx\Big)y_3'(u)\,du$$

and integrating $\int_s^t P_{1,2}(t,u)y_3'(u)\,du$ by parts with $f(u) = \int_u^t p_2(x)\int_x^t p_1(v)\,dv\,dx$, $g(u) = y_3(u)$, we get

$$\int_s^t P_{1,2}(t,u)y_3'(u)\,du = -P_{1,2}(t,s)y_3(s) + \int_s^t P_1(t,u)y_2'(u)\,du.$$

Integrating by parts again with $f(u) = P_1(t,u)$, $g(u) = y_2(u)$, one gets

$$\int_s^t P_{1,2}(t,u)y_3'(u)\,du = -P_{1,2}(t,s)y_3(s) - P_1(t,s)y_2(s) - z_1(s) + z_1(t).$$

From the equation about, we derive the integral identity

$$z_1(t) = z_1(s) + P_1(t,s)y_2(s) + P_{1,2}(t,s)y_3(s) + \int_s^t P_{1,2}(t,u)y_3'(u)\,du, \tag{3.27}$$

for $t > s \ge t_2$. From (3.27) in regard to (3.26), (e) and the third equation of (1.1) we get

$$-z_1(t) \ge \int_s^t KP_{1,2}(t,u)p_3(u)y_1(h(u))\,du, \quad t > s \ge t_2. \tag{3.28}$$

Since $z_1(t) \ge -a(t)y_1(g(t))$ for $t \ge t_2$ it follows that

$$y_1(g(t)) \ge \frac{z_1(t)}{-a(t)} \quad \text{for } t \ge t_2.$$

From the above inequality we have

$$y_1(h(t)) \ge \frac{z_1(g^{-1}(h(t)))}{-a(g^{-1}(h(t)))}, \quad t \ge t_2. \tag{3.29}$$

Combining (3.28) and (3.29) we have

$$-z_1(t) \ge \int_s^t \frac{-KP_{1,2}(t,u)p_3(u)z_1(g^{-1}(h(u)))\,du}{a(g^{-1}(h(u)))}, \quad t > s \ge t_2.$$

Putting $s = h^{-1}(g(t))$ and using the monotonicity of $z_1(g^{-1}(h(u)))$ from the last inequality we get

$$-z_1(t) \ge -z_1(t)\int_{h^{-1}(g(t))}^t \frac{KP_{1,2}(t,u)p_3(u)\,du}{a(g^{-1}(h(u)))}, \quad t \ge t_3\,,$$

where $t_3 \ge t_2$ is sufficiently large and

$$1 \ge \int_{h^{-1}(g(t))}^t \frac{KP_{1,2}(t,u)p_3(u)\,du}{a(g^{-1}(h(u)))}, \quad t \ge t_3,$$

which contradicts (3.23) and $N_3^- = \emptyset$. The proof is complete. $\square$

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