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A singular ODE related to quasilinear elliptic equations *

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Abstract

We consider a quasilinear elliptic problem with the natural growth in the gradient. Existence, non-existence, uniqueness, and qualitative properties of positive solutions are obtained. We consider both weak and strong solutions. All results are based on the study of a suitable singular ODE of the first order. We also introduce a comparison principle for a class of nonlinear integral operators of Volterra type that enables to obtain uniqueness of weak solutions of the quasilinear equation.

0.1 Introduction

In this paper we consider a quasilinear elliptic problem and its spherically symmetric, positive solutions in a ball, both in the weak and strong sense. The main difficulty represents the presence of the natural growth in the gradient on the right-hand side. We study existence, non-existence, uniqueness and qualitative properties of solutions. The quasilinear problem is studied by means of a suitable singular ODE of the first order. To prove uniqueness of weak solutions of quasilinear problem, we use a new type of comparison principle for integral operators of Volterra type, recently introduced by the second author. To our knowledge this seems to be the first uniqueness result for quasilinear elliptic equations with the natural growth in the gradient. Our existence proofs are constructive in the sense that solutions possess explicit integral representation, obtained by means of isoperimetric equalities and monotone rearrangements. Nonexistence results are obtained by constructing an unbounded sequence of subsolutions. The results seem to be new even in the case of $p = 2$.

Relatively simple methods developed in this paper enable numerous generalizations and variations. First, our existence results can be used to study general quasilinear elliptic equations in divergence form on arbitrary open and bounded set Ω, see [15]. These methods can be exploited in the study of quasilinear elliptic systems with strong dependence in the gradient, in the study of biharmonic equations, and even polyharmonic equations. It is also possible to study the corresponding variational inequalities.

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Our basic model is the following class of quasilinear elliptic equations with the natural growth in the gradient:

$$\begin{gathered} -\Delta_p v = \tilde{g}_0 |x|^m + \tilde{f}_0 |\nabla v|^p \quad \text{in } B \setminus \{0\}, \\ v = 0 \quad \text{on } \partial B, \\ v(x) \text{ spherically symmetric and decreasing.} \end{gathered} \tag{1}$$

Here $B = B_R(0)$ is the ball of radius R in $\mathbb{R}^N$, $N \geq 1$, $1 < p < \infty$, $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2} \nabla v)$. We shall also need the conjugate exponent $p' = \frac{p}{p-1}$. Also, we denote $\mathbb{R}^+ = [0, \infty)$. We assume that the constants $\tilde{f}_0$ and $\tilde{g}_0$ are positive real numbers, and $m \in \mathbb{R}$ can also be negative, i.e. the right hand side of (1) may be singular.

We can interpret solutions of equation (1) in three ways. We say that $v(x)$ is

(i) a strong solution of (1) if $v \in C^2(B \setminus \{0\}) \cap C(\overline{B})$;

(ii) a classical solution of (1) if $v \in C^2(\overline{B})$;

(iii) a weak solution of (1) if $v \in W_0^{1,p}(B) \cap L^\infty(B)$, and equation (1) is satisfied in the weak sense:

$$\int_B |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi \, dx = \tilde{g}_0 \int_B |x|^m \varphi(x) \, dx + \tilde{f}_0 \int_B |\nabla v|^p \varphi(x) \, dx \tag{2}$$

for all $\varphi \in W_0^{1,p}(B) \cap L^\infty(B)$.

In Section 1 we show that the study of (1) can be related to the study of the following ordinary differential equation:

$$\begin{gathered} \tfrac{d\omega(t)}{dt} = g_0 \gamma t^{\gamma-1} + f_0 \tfrac{\omega(t)^\delta}{t^\varepsilon}, \quad t \in (0, T), \\ \omega(0) = 0, \end{gathered} \tag{3}$$

where the constants g_0, f_0, δ, γ, T are assumed to be positive, and $\varepsilon \in \mathbb{R}$. Note that for $\varepsilon > 0$ problem (3) is singular. In Section 1.2 we study weak solutions of (1), where we exploit techniques of isoperimetric equalities and monotone rearrangements.

In Section 2 we are interested in finding sufficient conditions on pairs (f_0, g_0) of positive real numbers that ensure existence of at least one solution ω of (3), i.e. of (5). While for $\delta \in (0, 1)$ the problem is solvable for all (f_0, g_0), the case of $\delta > 1$ is strikingly different. Namely, in the latter case we always have an unbounded set of pairs (f_0, g_0) for which the problem is not solvable. More precisely, we obtain two explicit positive constants $C_1 < C_2$ depending only on δ, γ, ε, and T, such that if

$$f_0 \leq \frac{C_1}{g_0^{\delta-1}}$$

then ODE (3) is solvable, while for

$$f_0 \geq \frac{C_2}{g_0^{\delta-1}}$$

problem (3), and even problem (22) below, is not solvable. We know (almost) nothing about solvability of (3) in the case when

$$\frac{C_1}{g_0^{\delta-1}} < f_0 < \frac{C_2}{g_0^{\delta-1}}$$

See Theorems 4, 5, 6 for solvability results related to (3), and Theorem 7 for a nonsolvability result. In Section 2.4 we show a regularity result for solutions of (3) at singular point $t = 0$.

It is interesting that in the case when (f_0, g_0) belongs to existence region described above, we can also prove uniqueness of the solution, but only in the set of the form (7). An important tool in obtaing uniqueness result is played by a pointwise comparison principle for general operators of Volterra type, that we introduce in Theorem 3, see Section 2.1. Its first version has appeared in Pašić [14].

In Section 3 we apply existence and non-existence results from Section 2 to study the problem of existence, qualitative properties, and non-existence of solutions of (1). Using the above results it is easy to obtain two positive constants $\tilde{C}_1 < \tilde{C}_2$ depending only on m, p, N, and R, such that if

$$\tilde{f}_0 \leq \frac{\tilde{C}_1}{\tilde{g}_0^{p'-1}}$$

then PDE (1) possesses a strong solution generated by a solution of the corresponding singular ODE (3), that we call ω-solution. In fact, we obtain that these strong solutions coincide with weak solutions, and furthermore, (1) is uniquely solvable in the weak sense, see Theorem 9. On the other hand, for

$$\tilde{f}_0 \geq \frac{\tilde{C}_2}{\tilde{g}_0^{p'-1}}$$

problem (1) has neither weak nor strong solutions. Again, we know (almost) nothing about solvability of (1) in the case when

$$\frac{\tilde{C}_1}{\tilde{g}_0^{p'-1}} < \tilde{f}_0 < \frac{\tilde{C}_2}{\tilde{g}_0^{p'-1}}$$

Existence results are supplied with constructive proofs, and, as we have said, we have even uniqueness of weak solutions. See our main results in Theorem 8 and Theorem 9 for precise statements. The non-existence result represents a refinement of the corresponding result in Pašić [16]. Aplying regularity result from Section 2.4 we are able to describe the behaviour of solutions of (1) at $x = 0$ and on the boundary of B.

It is worth noting that the phenomenon of having existence and non-existence regions with respect to $(\tilde{f}_0, \tilde{g}_0)$ for (1) is due to the presence of the term $\tilde{f}_0|\nabla v|^p$ with $\tilde{f}_0 > 0$, on the right-hand side. Namely, if we have $\tilde{f}_0 = 0$, we have no more this effect with respect to parameter $\tilde{g}_0$.

1 Connection between ODE and PDE

1.1 Strong solutions

Here we want to describe the connection between strong solutions of quasilinear elliptic problem (1) and solutions of the corresponding singular ODE (3). We obtain solutions of (3) as fixed points of the following singular nonlinear integral operator of Volterra type:

$$K : D(K) \subset C([0,T]) \to C([0,T]), \quad K\varphi(t) = g_0 t^\gamma + f_0 \int_0^t \frac{\varphi(s)^\delta}{s^\varepsilon}\, ds. \tag{4}$$

A domain $D(K)$ will be chosen so that the corresponding fixed point equation

$$\omega \in D(K), \quad \omega = K\omega \tag{5}$$

is solvable. We shall deal with two types of domains. When we apply Banach's contraction method or Schauder's fixed point theorem, then we shall use the domain

$$D(K) = \{\varphi \in C([0,T]) : 0 \le \varphi(t) \le Mt^\gamma\}, \tag{6}$$

with a suitable constant $M > 0$ independent of φ, which ensures that $R(K) \subseteq D(K)$, see Theorems 4 and 5 (by $R(K)$ we denote the range of K). In the case of monotone iterations we take much larger domain, see Theorem 6:

$$D(K) = \{\varphi \in C([0,T]) : \exists M_\varphi \ge 0,\ 0 \le \varphi(t) \le M_\varphi t^\gamma\}. \tag{7}$$

It will be convenient to introduce an auxiliary function $V : [0, |B|] \to \mathbb{R}$, where $|B|$ is the Lebesgue measure of B, such that

$$v(x) = V(C_N|x|^N), \tag{8}$$

where C_N is the volume of the unit ball in $\mathbb{R}^N$. In fact, (8) will have two rôles: if a function $v(x)$ is given, then it will serve to define $V(s)$, and if $V(s)$ is given, it will define $v(x)$. If $V(s)$ is decreasing, then (8) implies that

$$-\Delta_p v = C_N^{p/N} N^p \frac{d}{ds}\left(s^{p(1-1/N)} \left|\frac{dV}{ds}\right|^{p-1}\right) \tag{9}$$

From solutions of ODE to strong solutions of PDE. Let $\omega : [0,T] \to \mathbb{R}$ be a solution of (3). In order to obtain a strong solution of (1) via (8), we define the function

$$V(s) = \int_s^T \frac{\omega(\sigma)^\beta}{\sigma^\alpha}\, d\sigma, \quad T = |B|, \tag{10}$$

with α and β specified below. We shall always have that $0 \le \omega(s) \le Ms^\gamma$ with some $\gamma > 0$ and $M > 0$, so that $V(0) < \infty$ provided $\alpha < \beta\gamma + 1$. Using (10) we obtain

$$-\Delta_p v(x) = C_N^{p/N} N^p \frac{d}{ds}\left(s^{p(1-\frac{1}{N})-\alpha(p-1)}\omega(s)^{\beta(p-1)}\right), \quad s = C_N|x|^N. \tag{11}$$

An easy computation shows that we have the following relation between $\omega(s)$ and $|\nabla v|$, see (8) and (10):

$$\omega(s) = N^{-1/\beta} C_N^{\frac{\alpha-1}{\beta}} |x|^{\frac{N\alpha-N+1}{\beta}} |\nabla v(x)|^{1/\beta}, \quad s = C_N|x|^N. \tag{12}$$

In the following lemma we generate strong solutions of (1) starting from solutions of the corresponding ODE (3), so that the coefficients α, β, γ, δ, ε are defined by $\tilde f_0$, $\tilde g_0$, m, p, N using (14), (15) and (16).

Lemma 1 *Let $\tilde f_0$ and $\tilde g_0$ be given positive real numbers. Assume that $1 < p < \infty$,*

$$m > \max\{-p, -N\}. \tag{13}$$

Let the constants α, β, γ, δ, and ε be defined by

$$\alpha = p'(1 - \frac{1}{N}), \quad \beta = \frac{p'}{p}, \tag{14}$$

and

$$\gamma = 1 + \frac{m}{N}, \quad \delta = p', \quad \varepsilon = p'(1 - \frac{1}{N}), \tag{15}$$

and let

$$g_0 = \frac{\tilde g_0}{C_N^{\frac{m+p}{N}} N^{p-1}(m+N)}, \quad f_0 = \tilde f_0. \tag{16}$$

Then we have $\alpha < \beta\gamma + 1$, $\delta > \frac{\varepsilon-1}{\gamma} + 1$, $\gamma > 0$, and for any solution ω of (3) with $T = |B|$, such that $0 \le \omega(t) \le Mt^\gamma$ for some $M > 0$, we have that the corresponding function $v(x)$ defined by (8) and (10) is a strong solution of quasilinear problem (1). Furthermore, the following relation holds:

$$u'(r) = -\frac{N}{C_N^{\frac{p'}{p}(1-\frac{p}{N})}} r^{-\frac{p'}{p}(N-1)}\, \omega(C_N r^N)^{p'/p}, \tag{17}$$

where $u : [0,R] \to \mathbb{R}$ is defined by $u(r) = v(x)$, $r = |x|$.

PROOF. Note that both the integrability condition $\alpha < \beta\gamma + 1$ of $\sigma^{-\alpha}\omega(\sigma)^{\beta}$ in (10) and the condition $\delta > \frac{\varepsilon-1}{\gamma} + 1$ are equivalent to $m > -p$. Also, since $m > -N$, then $\gamma = 1 + \frac{m}{N} > 0$.

Using (14) we see that (11) reduces to

$$-\Delta_p v = C_N^{p/N} N^p \frac{d\omega}{ds}. \tag{18}$$

Now let us take into account our singular ordinary differential equation (3) with the values of γ, δ, and ε defined in (15). Substituting into (18) and using (12) we obtain that $v(x)$ is a strong solution of (1):

$$\begin{aligned} -\Delta_p v &= C_N^{p/N} N^p [g_0 \gamma s^{\gamma-1} + f_0 s^{-\varepsilon} \omega(s)^{\delta}] \\ &= C_N^{p/N} N^p [g_0 \gamma (C_N|x|^N)^{\gamma-1} + f_0 (C_N|x|^N)^{-\varepsilon} \omega(C_N|x|^N)^{\delta}] \\ &= \tilde{g}_0 |x|^m + \tilde{f}_0 |\nabla v|^p. \end{aligned} \tag{19}$$

Relation (17) follows from (12) and the fact that $|\nabla v(x)| = -u'(r)$. $\diamondsuit$

We say that $v(x)$ is ω-solution of PDE (1), if it is a strong solution which can be obtained as in Lemma 1, using the solution $\omega(t)$ of ODE (3) such that $0 \le \omega(t) \le Mt^{\gamma}$ for some $M > 0$.

Note that our ODE (3) is singular only for $\varepsilon > 0$. For ε as in (15) this condition corresponds to the case when $N \ge 2$ in (1); if $N = 1$ in (1), then $\varepsilon = 0$ in equation (3). On the other hand, the right-hand side of our PDE (1) has singularity at $x = 0$ provided $m < 0$, which means that $\gamma < 1$ in the corresponding ODE (3).

From strong solutions of PDE to solutions of ODE. It is not difficult to show that a strong solution v of (1) generates a solution of a suitable ODE defined by (22), without any initial condition. We seek for solutions contained in the set

$$D^+ = \{\varphi \in C([0,T]) \,:\, \varphi(t) \ge 0,\ \varphi \text{ nondecreasing}\}. \tag{20}$$

Lemma 2 *Let v be a strong solution of (1), where $\tilde{f}_0 > 0$, $\tilde{g}_0 > 0$, and m, p, N satisfy (13). Define $V(s)$ by (8), and let*

$$\omega(s) = s^{p(1-\frac{1}{N})} \left| \frac{dV}{ds} \right|^{p-1}, \quad s \in (0,T), \quad T = |B|. \tag{21}$$

Then ω satisfies the following ODE:

$$\begin{gathered} \tfrac{d\omega(t)}{dt} = g_0 \gamma t^{\gamma-1} + f_0 \tfrac{\omega(t)^{\delta}}{t^{\varepsilon}}, \quad t \in (0,T), \\ \omega \in D^+. \end{gathered} \tag{22}$$

where g_0, f_0 are defined by (16), and constants γ, δ, ε are defined by (15). Also, $V(s)$ can be represented by (10), where α and β are defined by (14).

PROOF. From $V(s)=v(x)$, $s=C_N|x|^N$ we obtain $|\nabla v|=NC_N^{1/N}s^{1-\frac{1}{N}}\left|\frac{dV}{ds}\right|$. Using (1), (18), and (16) it is easy to show that ω satisfies (3). From (21), and using the fact that $V(s)$ is decreasing, we see that $\frac{dV}{ds}=-s^{-\alpha}\omega(s)^{\beta}$. Integrating from s to T we obtain (10). $\diamondsuit$

1.2 Weak solutions

Theorem 2 below shows that every weak solution of (1) is ω-solution of (1). In fact, we consider a more general problem than (1):

$$\begin{gathered}-\Delta_p v=\tilde{g}(|x|,v)+\tilde{f}_0|\nabla v|^p \quad \text{in } B\setminus\{0\},\\ v=0 \quad \text{on } \partial B,\\ v(x) \text{ spherically symmetric and decreasing.}\end{gathered}\tag{23}$$

We define the notion of weak solution of this equation as a function $v\in W_0^{1,p}(B)\cap L^\infty(B)$ satisfying integral identity analogous to (2). In what follows we assume $\tilde{g}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a Carathéodory function such that

$$\int_0^{|B|}\tilde{g}\left(\left(\frac{s}{C_N}\right)^{1/N},\varphi(s)\right)ds<\infty, \quad \forall\varphi\in C([0,|B|]),\ \varphi\ge 0.\tag{24}$$

Theorem 1 *Assume that (24) holds, and let v be a weak solution of (23). Let us define $V(s)$, $s\in[0,T]$, $T=|B|$ by (8) and the function $w:[0,T]\to\mathbb{R}$ by*

$$\omega(s)=\frac{1}{N^pC_N^{p/N}}e^{-\tilde{f}_0V(s)}\int_0^s\tilde{g}\left(\left(\frac{\sigma}{C_N}\right)^{1/N},V(\sigma)\right)e^{\tilde{f}_0V(\sigma)}d\sigma.\tag{25}$$

Then the functions ω and V satisfy the following system of ODE's:

$$\begin{gathered}\frac{d\omega}{ds}=\frac{1}{N^pC_N^{p/N}}\tilde{g}\left(\left(\frac{s}{C_N}\right)^{1/N},V(s)\right)+\tilde{f}_0\frac{\omega(s)^{p'}}{s^{p'(1-\frac{1}{N})}} \quad \text{a.e. } s\in(0,T),\\ \frac{dV}{ds}=-s^{p'(-1+\frac{1}{N})}\omega(s)^{p'/p} \quad \text{a.e. } s\in(0,T)\\ \omega(0)=0, \quad \omega\in AC^+([0,T]),\\ V(T)=0, \quad V\in AC^+([a,T]), \quad \forall a>0\,.\end{gathered}\tag{26}$$

Furthermore, we have

$$\omega(s)\le\frac{e^{\tilde{f}_0V(0)}}{N^pC_N^{p/N}}\int_0^s\tilde{g}\left(\left(\frac{\sigma}{C_N}\right)^{1/N},V(\sigma)\right)d\sigma,\tag{27}$$

and $v\in C^\infty(\overline{B}\setminus\{0\})$. If there exists $M>0$ such that $\omega(s)\le Ms^{1+m/N}$, $s\in[0,T]$, and $m>-p$, then also $v\in C(\overline{B})$, and v is the strong solution of (23).

We have a partial converse: if ω and V are solutions of (26) such that $0\le\omega(s)\le Ms^{1+m/N}$ for some $M>0$, then the corresponding function $v(x)$ defined by (8) is a strong solution of (23).

In the special case of $\tilde{g}(r,\eta)=\tilde{g}_0 r^m$, for some fixed $m\in\mathbb{R}$ and $\tilde{g}_0>0$, we obtain problem (1).

Theorem 2 *Assume that $m>-N$, and let $v(x)$ be a weak solution of (1). If we define $V(s)$, $s\in[0,T]$, $T=|B|$, by (8), then the function $\omega:[0,T]\to\mathbb{R}^+$ defined by*

$$\omega(s)=\frac{\tilde{g}_0}{N^p C_N^{\frac{m+p}{N}}} e^{-\tilde{f}_0 V(s)} \int_0^s \sigma^{\frac{m}{N}} e^{\tilde{f}_0 V(\sigma)} d\sigma, \tag{28}$$

satisfies equation (3), where the constants γ, δ, ε are defined by (15), and f_0, g_0 by (16). Also, we have that for all $s\in[0,T]$

$$\omega(s)\le Ms^{\gamma}, \quad M=g_0\cdot e^{\tilde{f}_0 V(0)}. \tag{29}$$

We have $v\in C^\infty(\overline{B}\setminus\{0\})\cap C(\overline{B})$. If in addition to the above hypotheses we assume $m>-p$, then $v\in C(\overline{B})$ and the weak solution of (1) is also ω-solution.

Proof of Theorem 2. This theorem follows easily from Theorem 1. Concerning the continuity of v on B, note that (29) and (10) imply that $V(s)$ is continuous at $s=0$, and therefore $v(x)$ is continuous at $x=0$. The partial converse can be proved similarly as in Lemma 1. $\diamondsuit$

Before proceeding to proof of Theorem 1 we compare Lemma 2 with the above theorem. On the one hand, it should be noted that the function $\omega(s)$ defined by (28) satisfies the same ODE (3) as the one defined by (21) in Lemma 2. On the other hand, in Theorem 1 we have that weak solutions of (1) yield $w\in AC^+([0,T])$ and the estimate (29), while in Lemma 2 strong solutions yield only $\omega\in D^+$. As we see from Theorem 2, any weak solution of (1) is ω-solution, while for strong solutions this is an open problem.

Before proving Theorem 1 we recall some very well known results on Schwartz symmetrization.

Lemma 3 *Let $v\in W_0^{1,p}(B)\cap L^\infty(B)$ be a spherically symmetric and decreasing function and let $V(s)$, $s\in[0,|B|]$, be defined by (8). Then we have:*

(i) $V(s)=v^*(s)$, where by definition $v^*(s)=|\{t\ge 0 : \mu(t)>s\}|$, $\mu(t)=|\{x\in B : v(x)>t\}|$; also $\mu(s)=V^{-1}(s)$, where V^{-1} denotes the inverse function of V;

(ii) $V\in W^{1,p}_{loc}(0,|B|)\cap C([0,|B|])$, and $V\in AC([a,|B|])$ for all $a>0$.

Proof. (i) follows easily from the fact that v is decreasing. For the proof of (ii) see for example Rakotoson, Temam [17]. $\diamondsuit$

Proof of Theorem 1. We define the function

$$\varphi(x)=e^{\tilde{f}_0 V(C_N|x|^N)} S_{t,h}(v(x)) \tag{30}$$

where $t\in(0,T)$, $h>0$, and

$$S_{t,h}(\tau)=\begin{cases} 0, & \text{for } \tau\le t,\\ \frac{1}{h}(\tau-t), & \text{for } t<\tau\le t+h,\\ 1, & \text{for } \tau>t+h. \end{cases} \tag{31}$$

It is easy to see that $\varphi \in W_0^{1,p}(B) \cap L^\infty(B)$. Now if we test (23) with φ and let $h \to 0$ we obtain

$$-\frac{d}{dt}\int_{\{v>t\}} |\nabla v|^p dx = e^{-\tilde{f}_0 t}\int_{\{v>t\}} \tilde{g}(|x|,v)\, e^{\tilde{f}_0 v(x)} dx. \tag{32}$$

Let us show that

$$-\frac{d}{dt}\int_{\{v>t\}} |\nabla v|^p dx = N^p C_N^{p/N} \mu(t)^{p(1-\frac{1}{N})} |\mu'(t)|^{1-p} \tag{33}$$

$$\int_{\{v>t\}} \tilde{g}(|x|,v)\, e^{\tilde{f}_0 v} dx = \int_0^{\mu(t)} \tilde{g}\left(\left(\frac{\sigma}{C_N}\right)^{1/N}, V(\sigma)\right) e^{\tilde{f}_0 V(\sigma)} d\sigma. \tag{34}$$

(a) To prove relation (33), we start with:

$$-\frac{d}{dt}\int_{\{v>t\}} |\nabla v|^p dx = \lim_{h\to 0}\frac{1}{h}\int_{\{t<v\le t+h\}} |\nabla v|^p dx.$$

Now use (8) and pass to the generalized spherical coordinates:

$$\begin{aligned}
&\int_{\{t<v\le t+h\}} |\nabla v|^p dx \qquad (35)\\
&= C_N^p N^p \int_{\{t<v\le t+h\}} |x|^{p(N-1)} \left|\frac{dV}{ds}(C_N|x|^N)\right|^p dx\\
&= C_N^{p+1} N^{p+1} \int_{\{t<V(C_N r^N)\le t+h\}} r^{p(N-1)} \left|\frac{dV}{ds}(C_N r^N)\right|^p r^{N-1} dr\\
&= N^p C_N^p \int_{\mu(t+h)}^{\mu(t)} s^{p(1-\frac{1}{N})} \left|\frac{dV}{ds}\right|^p ds\\
&= N^p C_N^{p/N} \int_t^{t+h} \mu(\sigma)^{p(1-\frac{1}{N})} |\mu'(\sigma)|^{1-p} d\sigma,
\end{aligned}$$

where we used the change of variables $V(s) = \sigma$, $\frac{dV}{ds} = \frac{1}{\mu'(s)}$, and $\mu^{-1} = V$, see Lemma 3(i). This proves (33). Relation (34) is proved in the same way.

(b) Using (32), (33), (34), and (25) we obtain

$$1 = \mu(t)^{p(-1+\frac{1}{N})} |\mu'(t)|^{p-1} \omega(\mu(t)).$$

This implies

$$\begin{aligned}
\frac{dV}{ds} &= \lim_{h\to 0}\frac{1}{h}\int_{V(s)}^{V(s+h)} 1^{p'/p}\, dt\\
&= \lim_{h\to 0}\frac{1}{h}\int_{V(s)}^{V(s+h)} \mu(t)^{p'(-1+\frac{1}{N})} |\mu'(t)| \omega(\mu(t))^{p'/p} dt\\
&= -\lim_{h\to 0}\frac{1}{h}\int_s^{s+h} \tau^{p'(-1+\frac{1}{N})} \omega(\tau)^{p'/p} d\tau\\
&= -s^{p'(-1+\frac{1}{N})} \omega(s)^{p'/p}.
\end{aligned}$$

Here we have used the change of variables $\mu(t) = \tau$, and the fact that $\mu^{-1}(\tau) = V(\tau)$.

It is now easy to verify that the function $\omega(t)$ defined by (25) satisfies the first ODE in (26). Note that the second ODE in (26) implies that $V(s)$ has the form (10). From this it is easy to conclude that $\omega \in C^\infty((0,T])$, and then also $v \in C^\infty(\overline{B} \setminus \{0\})$. If $\omega(s) \le Ms^\gamma$, then (10), (8) and $m > -p$ imply that $v \in C(\overline{B})$. ♢

2 Singular ODE

2.1 Comparison principle and uniqueness of solutions

Let $(X, \le)$ be a partially ordered set, and assume that $K : D(K) \subset X \to X$ is an operator. We say that u is a *subsolution* of K if

$$u \in D(K) \quad \text{and} \quad u \le Ku. \tag{36}$$

We say that v is a *supersolution* of K if

$$v \in D(K) \quad \text{and} \quad v \ge Kv. \tag{37}$$

Also, the operator K is said to be nondecreasing if for any $\varphi, \psi \in D(K)$ the assumption $\varphi \le \psi$ implies that $K\varphi \le K\psi$.

We say that an operator $K : D(X) \subset X \to X$ has *(weak) comparison property* if for any subsolution u and any supersolution v of K we necessarily have $u \le v$ ($Ku \ge Kv$). Now we formulate the following very simple uniqueness lemma.

Lemma 4 *Let $K : D(K) \subset X \to X$ have (weak) comparison property. Then the fixed point equation $\omega = K\omega$ possesses at most one solution in $D(K)$.*

Proof. Let $\omega_1, \omega_2 \in D(K)$ be such that $\omega_1 = K\omega_1$, $\omega_2 = K\omega_2$. Since ω_1 is a subsolution, and ω_2 is a supersolution of K, then $\omega_1 \le \omega_2$. Similarly $\omega_2 \le \omega_1$, that is $\omega_1 = \omega_2$.

If K has weak comparison property then we obtain $K\omega_1 \le K\omega_2$ and $K\omega_2 \le K\omega_1$, that is $\omega_1 = K\omega_1 = K\omega_2 = \omega_2$. ♢

Let us describe our basic example of operators having (weak) comparison property. Let $X = C([0,T])$ with pointwise partial ordering, and assume that $k(s,\eta) : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (measurable with respect to s for all η and continuous with respect to η for a.e. s) such that the integral operator of Volterra type

$$K : D(K) \subset C([0,T]) \to C([0,T]), \quad K\varphi(t) = \int_0^t k(s,\varphi(s))\, ds \tag{38}$$

is well defined on a nonempty domain $D(K)$. We say that the operator K has property (m) if for all $\varphi, \psi \in D(K)$, for all $a \in [0,T)$ and for all $c \in (a,T)$, there exist $b \in (a,c)$ and $m(a,b,\varphi,\psi) \in [0,1)$ such that

$$\|k(\cdot,\varphi(\cdot)) - k(\cdot,\psi(\cdot))\|_{L^1(a,b)} \le m(a,b,\varphi,\psi) \cdot \|\varphi - \psi\|_{L^\infty(a,b)}. \tag{39}$$

We shall also say that the function $k(s,\eta)$ has property (m).

It is easy to see that our (m)-condition for operator K will be satisfied if we can obtain the last inequality in (39) so that the following property on the coefficient $m(a,b,\varphi,\psi)$ is satisfied:

$$\forall \varphi, \psi \in D(K),\ \forall a \in [0,T),\ \lim_{b \to a+} m(a,b,\varphi,\psi) = 0. \tag{40}$$

This will be the situation in all applications that follow, see Lemma 5 below. In fact, even more general condition suffices for (m)-condition to be fulfilled:

$$\forall \varphi, \psi \in D(K),\ \forall a \in [0,T),\ \liminf_{b \to a+} m(a,b,\varphi,\psi) \in [0,1). \tag{41}$$

However, we do not know any example of operator K satisfying property (41) which does not satisfy (40).

The following comparison principle will play basic rôle in obtaining uniqueness results for the fixed point problem $\omega = K\omega$ using monotone iterations method.

Theorem 3 *(comparison principle) Let $K : D(K) \subset C([0,T]) \to C([0,T])$ be an integral operator of Volterra type given by (38), satisfying property (m), and such that $R(K) \subseteq D(K)$.*

- *(a) If $k(s,\cdot)$ is nondecreasing for a.e. s then the operator K has comparison property, that is, if $u, v \in D(K)$ are such that $u \le Ku$, $v \ge Kv$, then necessarily $u \le v$.*
- *(b) If $k(s,\cdot)$ is nonincreasing for a.e. s then K has weak comparison property, that is, if $u, v \in D(K)$ are such that $u \le Ku$, $v \ge Kv$, then necessarily $K(u) \ge K(v)$.*

In both cases the fixed point equation $\omega = K\omega$ possesses at most one solution in $D(K)$.

Proof. (a) Let u and v be a subsolution and supersolution of K respectively. Denote $\theta = Ku$ and $\omega = Kv$. Then we have $\theta, \omega \in D(K) \cap AC([0,T])$, and $u \le \theta$, $v \ge \omega$ on $[0,T]$. Since $k(s,\cdot)$ is nondecreasing for a.e. s we obtain the following two differential inequalities

$$\begin{aligned} \tfrac{d\theta}{dt} &\le k(t,\theta(t)) \quad a.e.\ t \in [0,T], \\ \tfrac{d\omega}{dt} &\ge k(t,\omega(t)) \quad a.e.\ t \in [0,T]. \end{aligned} \tag{42}$$

Let us first prove that

$$\theta(t) \le \omega(t) \quad \text{for all } t \in [0,T]. \tag{43}$$

It suffices to show that

$$\begin{array}{l} \forall a \in [0,T) \text{ such that } \theta(a)=\omega(a) \text{ there is no } c \in (a,T] \\ \text{such that } \forall t \in (a,c]\ \theta(t) > \omega(t). \end{array} \tag{44}$$

We argue by contradiction. Assume, contrary to (44), that there exists $a \in [0,T)$ such that $\theta(a) = \omega(a)$ and let there exist $c \in (a,T]$ satisfying $\theta(t) > \omega(t)$ for all $t \in (a,c]$. Integrating (42) over $[a,t]$ with $t \in [a,c]$ and using property (m) we obtain

$$\begin{aligned} |\theta(t)-\omega(t)| &= \theta(t)-\omega(t) \le \int_a^t k(s,\theta(s))\,ds - \int_a^t k(s,\omega(s))\,ds \\ &\le \int_a^t |k(s,\theta(s)) - k(s,\omega(s))|\,ds \le \|k(\cdot,\theta(\cdot)) - k(\cdot,\omega(\cdot))\|_{L^1(a,c)} \\ &\le m(a,c,\theta,\omega)\,\|\theta-\omega\|_{L^\infty(a,c)}. \end{aligned}$$

Taking the maximum over $t \in [a,c]$ we obtain $\|\theta-\omega\|_{L^\infty(a,c)} \le m(a,c,\theta,\omega)\,\|\theta-\omega\|_{L^\infty(a,c)}$, and from this $\theta \equiv \omega$ on $[a,c]$, which is a contradiction. This proves (43).

Now using (43), (36), and (37) we have that

$$u \le Ku = \theta \le \omega = Kv \le v \tag{45}$$

The case (b) is treated in the same way as (a). The uniqueness claim follows from Lemma 4. ♢

Note that in case (a) the operator K is monotone, while in case (b) the operator $-K$ is monotone.

It is clear that if a subsolution u and a supersolution v of K are given in advance in the above theorem, then we can replace the condition $R(K) \subseteq D(K)$ with $Ku, Kv \in D(K)$ only.

Now we would like to describe our basic example of operators satisfying property (m). We shall need the following elementary inequality which will be useful in the sequel:

$$|\varphi^\delta - \psi^\delta| \le \delta \max\{\varphi^{\delta-1}, \psi^{\delta-1}\}|\varphi-\psi|, \tag{46}$$

where $\varphi, \psi \ge 0$ and $\delta > 0$. This follows immediately from the mean value theorem applied to $F(t) = t^\delta$, $t \ge 0$.

Lemma 5 *Assume that $\delta > \frac{\varepsilon-1}{\gamma}+1$, $\delta>0$, $\gamma>0$, $\varepsilon \in \mathbb{R}$, $f_0 \in \mathbb{R}$, and $g_0 \in \mathbb{R}$. Let the Volterra type operator K be defined by (4), with its domain $D(K)$ contained in the set $\{\varphi \in C([0,T]) : \exists M_\varphi \ge 0,\ 0 \le \varphi(t) \le M_\varphi t^\gamma\}$. Then the operator K has property (m). In particular, the equation $\omega = K\omega$ possesses at most one solution in $D(K)$.*

Proof. Here we have

$$k(s,\eta)=\gamma g_0 s^{\gamma-1}+f_0\frac{\eta^\delta}{s^\varepsilon}.$$

For any $\varphi,\psi\in D(K)$, $0\le a<b\le T$, we have:

$$\begin{aligned}\|k(\cdot,\varphi(\cdot))-k(\cdot,\psi(\cdot))\|_{L^1(a,b)} &\le |f_0|\int_a^b\frac{|\varphi(s)^\delta-\psi(s)^\delta|}{s^\varepsilon}\,ds\\ &\le |f_0|\delta M^{\delta-1}\int_a^b s^{\gamma(\delta-1)-\varepsilon}|\varphi(s)-\psi(s)|\,ds\\ &\le m(a,b,\varphi,\psi)\|\varphi-\psi\|_{L^\infty(a,b)} \qquad (47)\end{aligned}$$

where we have denoted

$$\begin{aligned}M&=\max\{M_\varphi,M_\psi\},\\ m(a,b,\varphi,\psi)&=\frac{|f_0|\delta M^{\delta-1}}{\gamma(\delta-1)-\varepsilon+1}[b^{\gamma(\delta-1)-\varepsilon+1}-a^{\gamma(\delta-1)-\varepsilon+1}]. \qquad (48)\end{aligned}$$

The condition (40) is clearly satisfied. The uniqueness claim follows from Theorem 3. ♢

The following lemma provides some sufficient conditions for $R(K)\subseteq D(K)$ to hold.

Lemma 6 *Let the operator K be defined by (4), where f_0 and g_0 are given positive real numbers, $\delta\ge\frac{\varepsilon-1}{\gamma}+1$, $\gamma>0$, and $\varepsilon\in\mathbb{R}$. (a) Assume that the domain $D(K)$ is defined by (6), and let $M>0$ be such that*

$$g_0\le M-f_0\frac{M^\delta T^{\gamma(\delta-1)-\varepsilon+1}}{\gamma\delta-\varepsilon+1}. \qquad (49)$$

Then $R(K)\subseteq D(K)$, and the function $v(t)=Mt^\gamma$ is a supersolution of K.
(b) If $D(K)$ is defined by (7), then $R(K)\subseteq D(K)$.

Proof. (a) Let us take any $\varphi\in D(K)$. Then

$$K\varphi(t)\le g_0t^\gamma+f_0M^\delta\int_0^t s^{\gamma\delta-\varepsilon}ds\le g_0t^\gamma+\frac{f_0M^\delta}{\gamma\delta-\varepsilon+1}t^{\gamma\delta-\varepsilon+1}. \qquad (50)$$

Since $\gamma\delta-\varepsilon+1\ge\gamma$ and $t/T\le1$, we have $\left(\frac{t}{T}\right)^{\gamma\delta-\varepsilon+1}\le\left(\frac{t}{T}\right)^\gamma$, and therefore

$$0\le K\varphi(t)\le\left(g_0+f_0\frac{M^\delta T^{\gamma(\delta-1)-\varepsilon+1}}{\gamma\delta-\varepsilon+1}\right)t^\gamma\le Mt^\gamma,$$

that is, $K\varphi\in D(K)$. The proof that $v(t)=Mt^\gamma$ is a supersolution, that is, $Kv\le v$, can be obtained in the same way.

(b) For any $\varphi \in D(K)$ we have in the same way as in (a),

$$0 \le K\varphi(t) \le \left(g_0 + f_0 M_\varphi^\delta \frac{T^{\gamma(\delta-1)-\varepsilon+1}}{\gamma\delta - \varepsilon + 1} \right) t^\gamma.$$

and therefore there exists $M > 0$ such that $K\varphi(t) \le Mt^\gamma$, that is, $K\varphi \in D(K)$. $\diamondsuit$

2.2 Existence and uniqueness of solutions

2.2.1 Contraction method

In this section we present a constructive proof of existence (and uniqueness) of solutions of (3) based on the method of contraction. It will be convenient to introduce the vector space X_γ of all functions $\varphi \in C([0,T])$ such that

$$\|\varphi\|_\gamma = \sup_{t\in(0,T]} \frac{|\varphi(t)|}{t^\gamma} < \infty, \tag{51}$$

where $\gamma > 0$ is given. This norm is equal to $\|\varphi\|_\gamma = \inf\{M > 0 : |\varphi(t)| \le Mt^\gamma\}$. It is not difficult to see that $(X_\gamma, \|\cdot\|_\gamma)$ is a Banach space which is continuously imbedded into $C([0,T])$.

Theorem 4 *Let $\delta \ge \frac{\varepsilon-1}{\gamma} + 1$, $\delta > 0$, $\gamma > 0$, $\varepsilon \in \mathbb{R}$.*

(a) Let $M > 0$ be given, and assume that f_0 and g_0 are positive real numbers satisfying conditions (49) and

$$f_0 < \frac{\gamma\delta - \varepsilon + 1}{\delta M^{\delta-1} T^{\gamma(\delta-1)-\varepsilon+1}}. \tag{52}$$

Then the operator $K : D(K) \to D(K)$ given by (4) and (6) is well defined and X_γ-contractive. There exists a unique $\omega \in D(K)$ such that $\omega = K\omega$. Furthermore, if $\delta > \frac{\varepsilon-1}{\gamma} + 1$, then the solution ω is unique in the set defined by (7).

(b) Let $\delta > 1$. The set of all pairs (f_0, g_0) of positive real numbers for which there exists $M > 0$ satisfying conditions (49) and (52) is described by the following inequality:

$$f_0 < \frac{\gamma\delta - \varepsilon + 1}{\delta T^{\gamma(\delta-1)-\varepsilon+1}(\delta' g_0)^{\delta-1}}, \tag{53}$$

where $\delta' = \frac{\delta}{\delta-1}$. Problem (4) is solvable for all such (f_0, g_0).

Proof. (a) By Lemma 6 we have that $R(K) \subseteq D(K)$. To show that K is contraction, let $\varphi, \psi \in D(K)$. Then using inequality (46) we obtain

$$\frac{1}{t^\gamma}|K\varphi(t) - K\psi(t)| = \frac{f_0}{t^\gamma} \int_0^t \frac{|\varphi(s)^\delta - \psi(s)^\delta|}{s^\varepsilon}\, ds$$

$$\begin{aligned}
&\le \frac{f_0}{t^\gamma}\delta\int_0^t \frac{(Ms^\gamma)^{\delta-1}}{s^{\varepsilon-\gamma}}\cdot\frac{|\varphi(s)-\psi(s)|}{s^\gamma}\,ds \\
&\le \frac{f_0}{t^\gamma}\delta M^{\delta-1}\|\varphi-\psi\|_\gamma\int_0^t s^{\gamma\delta-\varepsilon}ds \qquad (54)\\
&\le c(f_0)\|\varphi-\psi\|_\gamma,
\end{aligned}$$

where

$$c(f_0)=f_0\delta M^{\delta-1}\tfrac{T^{\gamma(\delta-1)-\varepsilon+1}}{\gamma\delta-\varepsilon+1}<1 \tag{55}$$

because of (52). Taking the supremum in (54) over $t\in(0,T]$ we obtain that $\|K\varphi-K\psi\|_\gamma\le c(f_0)\|\varphi-\psi\|_\gamma$, which proves that K is contraction. The claim follows from Banach's fixed point theorem. The uniqueness claim in the set defined by (7) follows from Lemma 5.

(b) Let us fix any $M>0$, and consider the set Q_M of all pairs (f_0,g_0) of positive real numbers satisfying conditions (49) and (52). Owing to claim (a) that we have just proved, it suffices to show that the set

$$\mathcal{M}_b=\bigcup_{M>0}Q_M \tag{56}$$

is described by (53). It is not difficult to see that for any $(f_0,g_0)\in\mathcal{M}_b$ there exists $M>0$ such that

$$g_0=M-f_0\frac{M^\delta T^{\gamma(\delta-1)-\varepsilon+1}}{\gamma\delta-\varepsilon+1} \tag{57}$$

and

$$f_0=\frac{\gamma\delta-\varepsilon+1}{\delta M^{\delta-1}T^{\gamma(\delta-1)-\varepsilon+1}}. \tag{58}$$

Namely, the envelope of the family of lines (57) in (f_0,g_0)-plain is the convex curve defined by (64) below, as we shall see in the proof of Theorem 5. The union of its tangents obviously contains $\mathcal{M}_b$. Substituting (58) into (57) we get $M=\frac{\delta}{\delta-1}g_0$. Inserting this into (58) we obtain

$$f_0=\frac{\gamma\delta-\varepsilon+1}{\delta T^{\gamma(\delta-1)-\varepsilon+1}(\delta' g_0)^{\delta-1}}, \tag{59}$$

which proves the claim. Note that we have strict inequality in (53) because of the strict inequality in (52). Although the sets Q_M are not open, their union $\mathcal{M}_b$ is open. $\diamondsuit$

This result extends the corresponding result in Pašić [14]. Note that we could also have used contraction method in the larger space $C([0,T])$ instead of X_γ, but in this case we obtain a weaker result. Namely in this case we need $\delta>\frac{\varepsilon-1}{\gamma}+1$, and the numerator of the right-hand side of (52) should be $\gamma\delta-\varepsilon+1-\gamma$ instead of $\gamma\delta-\varepsilon+1$.

2.2.2 Schauder's method

In this section we apply Schauder's fixed point theorem to obtain existence result for solutions of our singular ODE (3).

Lemma 7 *Let f_0 and g_0 be positive real numbers. Assume that $\delta > \frac{\varepsilon-1}{\gamma}$, $\gamma > 0$, $\varepsilon \in \mathbb{R}$, and let the operator K be defined by (4), with domain (6) for some fixed $M > 0$. Then then the operator K is compact with respect to uniform topology.*

PROOF. It suffices to show that $R(K)$ is relatively compact in $C([0,T])$. We use Ascoli's theorem. To show equicontinuity of the family $R(K)$ take any $K\varphi \in R(K)$. Then for any a, b such that $0 \le a < b \le T$ we have

$$\begin{aligned} |K\varphi(b) - K\varphi(a)| &\le g_0(b^\gamma - a^\gamma) + f_0 \int_a^b M^\delta s^{\gamma\delta-\varepsilon} ds \\ &\le g_0(b^\gamma - a^\gamma) + \frac{f_0 M^\delta}{\gamma\delta - \varepsilon + 1}[b^{\gamma\delta-\varepsilon+1} - a^{\gamma\delta-\varepsilon+1}]. \quad (60) \end{aligned}$$

The last expression is equal to $h(b) - h(a)$, where $h(t) = g_0 t^\gamma + \frac{f_0 M^\delta}{\gamma\delta-\varepsilon+1} t^{\gamma\delta-\varepsilon+1}$ is uniformly continuous on $[0,T]$. Therefore $|K\varphi(b) - K\varphi(a)|$ tends to zero uniformly as $b-a \to 0$. Also, the the family of functions from $R(K)$ is uniformly bounded, see (50):

$$0 \le K\varphi(t) \le g_0 T^\gamma + \frac{f_0 M^\delta}{\gamma\delta - \varepsilon + 1} T^{\gamma\delta-\varepsilon+1}.$$

This proves that the operator K is compact. ♢

The following theorem shows that (3) is solvable also in the case when we have equality in (53).

Theorem 5 *Assume that $\delta \ge \frac{\varepsilon-1}{\gamma} + 1$, $\delta > 0$, $\gamma > 0$, and $\varepsilon \in \mathbb{R}$.*

(a) Let f_0 and g_0 be positive real numbers and let $M > 0$ satisfy (49). Then there exists at least one $\omega \in D(K)$ such that $\omega = K\omega$, where $D(K)$ is defined by (6).

(b) Assume that also $\delta > 1$. The set of all pairs (f_0, g_0) of positive real numbers for which there exists $M > 0$ satisfying (49) is described by the following inequality:

$$f_0 \le \frac{\gamma\delta - \varepsilon + 1}{\delta T^{\gamma(\delta-1)-\varepsilon+1}(\delta' g_0)^{\delta-1}}. \quad (61)$$

In particular, equation (3) is solvable for all such (f_0, g_0).

(c) If we assume that $\delta < 1$, then for each pair (f_0, g_0) of positive real numbers there exists a solution $\omega \in D(K)$ of the fixed point equation $\omega = K\omega$, where $D(K)$ is defined by (6) with M large enough.

(d) If $\delta > \frac{\varepsilon-1}{\gamma} + 1$, then the solution ω in (a), (b), and (c) is unique in the domain $D(K)$ defined by (7).

PROOF. (a) Note that the domain $D(K)$ is bounded, closed, and convex. By Lemma 7 the operator K is compact, and by Lemma 6 we have $R(K) \subseteq D(K)$. Existence of at least one solution $\omega \in D(K)$ of equation $\omega = K\omega$ follows from Schauder's fixed point theorem. For $\delta > \frac{\varepsilon-1}{\gamma} + 1$ we can give a constructive proof of this result based on the method of monotone iterations, see the remark after the proof.

(b) Let us denote the set of all (f_0, g_0) satisfying (49) by T_M. We have to show that the union of these sets

$$\mathcal{M}_s = \bigcup_{M>0} T_M \tag{62}$$

is described by (61). Consider the family of lines

$$g_0 = M - f_0 \frac{M^\delta T^{\gamma(\delta-1)-\varepsilon+1}}{\gamma\delta - \varepsilon + 1}, \quad M > 0. \tag{63}$$

in the (f_0, g_0)–plane. To find the envelope of this family, we differentiate with respect to M:

$$0 = 1 - f_0 \frac{\delta M^{\delta-1} T^{\gamma(\delta-1)-\varepsilon+1}}{\gamma\delta - \varepsilon + 1}.$$

Eliminating M from the last two equations we obtain after some elementary computation that

$$f_0 = \frac{\gamma\delta - \varepsilon + 1}{\delta T^{\gamma(\delta-1)-\varepsilon+1} (\delta' g_0)^{\delta-1}}. \tag{64}$$

Since the corresponding set of (f_0, g_0) satisfying (61) is convex, we obviously have that it is equal to $\mathcal{M}_s$.

(c) First, let us define $D(K)$ by (6), with M to be chosen later. The operator $K : D(K) \to C([0,T])$ is compact, see Lemma 7. For any $\varphi \in D(K)$ we have

$$\begin{aligned} \frac{K\varphi(t)}{t^\gamma} &= g_0 + \frac{f_0}{t^\gamma} \int_0^t \frac{\varphi(s)^\delta}{s^\varepsilon}\, ds \le g_0 + \frac{f_0 M^\delta}{t^\gamma} \int_0^t s^{\gamma\delta-\varepsilon} ds \\ &\le g_0 + f_0 \frac{T^{\gamma(\delta-1)-\varepsilon+1}}{\gamma\delta - \varepsilon + 1} M^\delta. \end{aligned}$$

If $\delta < 1$, we conclude that the last expression is $\le M$ for M large enough, that is, $K\varphi \in D(K)$. This shows that $R(K) \subset D(K)$. The set $D(K)$ is bounded, closed, and convex in $C([0,T])$, and the existence of a solution follows from Schauder's fixed point theorem.

(d) The uniqueness claim follows from Lemma 5. $\diamondsuit$

Note that the existence (and uniqueness!) result that we obtained via Schauder's theorem is not constructive. However, if $\delta > \frac{\varepsilon-1}{\gamma} + 1$ it is possible to give a constructive proof. Namely, if we take $v(t) = Mt^\gamma$, with M as in (49), then v is a supersolution of K, see Lemma 6, and the corresponding operator K has (m)-property, see Lemma 5. This implies the existence of the solution of $\omega = K\omega$ using the method of monotone iterations in the same way as in the proof of Theorem 6 below.

Note that if we have strict inequality in (61), then we are in the situation of Theorem 4. It is clear that $\overline{\mathcal{M}_b} \subset \mathcal{M}_s$, the closure being taken in $(0,\infty)^2$. Here $\mathcal{M}_b$ and $\mathcal{M}_s$ are defined by (53) and (61).

2.2.3 Monotone iterations

In this section we apply the method of monotone iterations to obtain existence of solutions of (3). First of all, note that since f_0 and g_0 are positive, then the zero function $u = 0$ is a subsolution of integral operator K defined by (4).

Theorem 6 *Assume that $\delta > \frac{\varepsilon-1}{\gamma} + 1$, $\delta \geq \max\{1, \varepsilon/\gamma\}$, $\delta > \varepsilon$, $\gamma > 0$, $\varepsilon \in \mathbb{R}$, and let us define*

$$z(t) = \gamma g_0 e^{h_0 t^{1-\varepsilon/\delta}} \int_0^t s^{\gamma-1} e^{-h_0 s^{1-\varepsilon/\delta}} ds, \quad h_0 = \frac{f_0}{1-\varepsilon/\delta}. \tag{65}$$

If f_0 and g_0 are positive real numbers such that

$$g_0 \leq T^{\frac{\varepsilon}{\delta}-\gamma} e^{-h_0 T^{1-\varepsilon/\delta}}, \tag{66}$$

then $z \in D(K)$, where $D(K)$ is defined by (7), and z is a supersolution of K: $z \geq Kz$.

There exists the unique solution $\omega \in D(K)$ of $\omega = K\omega$. It can be obtained constructively using the method of monotone iterations, and the following estimate holds:

$$g_0 t^\gamma + f_0 g_0^\delta \frac{t^{\gamma\delta-\varepsilon+1}}{\gamma\delta-\varepsilon+1} \leq \omega(t) \leq z(t), \quad \forall t \in [0,T]. \tag{67}$$

Proof. (a) We have that $v(t) \leq C_1 \int_0^t s^{\gamma-1}\, ds$, where C_1 is a positive constant, which implies that $z \in D(K)$. To prove that z is a supersolution of K, note that it satisfies the following linear ODE: $z'(t) = g_0\gamma t^{\gamma-1} + f_0 \frac{z(t)}{t^{\varepsilon/\delta}}$, $z(0) = 0$, that is,

$$z(t) = g_0 t^\gamma + f_0 \int_0^t \frac{z(s)}{s^{\varepsilon/\delta}}\, ds.$$

Therefore, to achieve $z \geq Kz$ it suffices to have

$$\frac{z(s)}{s^{\varepsilon/\delta}} \geq \frac{z(s)^\delta}{s^\varepsilon} = \left(\frac{z(s)}{s^{\varepsilon/\delta}}\right)^\delta, \quad \forall s \in [0,T].$$

Since $\delta \geq 1$, this is equivalent to

$$\frac{z(s)}{s^{\varepsilon/\delta}} \leq 1.$$

Now this inequality can be proved using the following easy consequence of (65)

$$z(t) \leq \gamma g_0 e^{h_0 t^{1-\varepsilon/\delta}} \int_0^t s^{\gamma-1} ds,$$

and (66):

$$\frac{z(t)}{t^{\varepsilon/\delta}} \leq g_0 t^{\gamma-\varepsilon/\delta} e^{h_0 t^{1-\varepsilon/\delta}} \leq g_0 T^{\gamma-\varepsilon/\delta} e^{h_0 T^{1-\varepsilon/\delta}} \leq 1.$$

(b) To prove existence, we proceed with the usual monotone iterations scheme. Let the sequence (u_k) in $D(K)$ be defined by $u_k = Ku_{k-1}$, $u_0 = 0$, where we note that $R(K) \subseteq D(K)$ by Lemma 6. Since $0 \leq z$, K is nondecreasing, and z is a supersolution, it is easy to conclude that the sequence u_k is monotone and bounded in $C([0,T])$:

$$u \leq u_k \leq z, \quad u_{k-1} \leq u_k.$$

Therefore, applying Dominated Convergence Theorem we conclude that there exists $\omega \in L^\infty(0,T)$ such that $u_k \to \omega$ in $L^1(0,T)$ as $k \to \infty$. Now using the continuity property of the integral operator (that is, the Lebesgue Dominated Convergence Theorem again) we easily see that

$$K\omega = K(\lim_{k\to\infty} u_k) = \lim_{k\to\infty} Ku_k = \lim_{k\to\infty} u_{k+1} = \omega,$$

with the limits being taken in $L^1(0,T)$. The integral equality $\omega = K\omega$ implies that $\omega \in C([0,T])$, and the claim is proved. It is clear that $u_k \leq \omega \leq z$ for all k. The estimate (67) is obtained for $k = 2$.

(c) The uniqueness result follows immediately from Lemma 5. $\diamondsuit$

As we have seen in Lemma 6, the operator K possesses another supersolution $z(t) = Mt^\gamma$ for $(f_0, g_0) \in \mathcal{M}_s$. If $\delta > \frac{\varepsilon-1}{\gamma} + 1$, then as in the proof of the preceding theorem, see step (b), we can show that the solution in Theorem 5 can be obtained constructively using monotone iterations. Note that we obtained in fact two solutions of $\omega = K\omega$, using Schauder's method and the method of monotone iterations. They coincide due to Lemma 5, and therefore we have that the a priori estimate (67) holds also for the solution that we have obtained in Theorem 5.

It is clear that the set $\mathcal{M}_s \setminus \mathcal{M}_m$ is always nonempty, where $\mathcal{M}_s$ is defined as the set of all pairs (f_0, g_0) of positive numbers satisfying (61), and $\mathcal{M}_m$ is defined by (66). It is interesting that in some cases the set $\mathcal{M}_m \setminus \mathcal{M}_s$ can also be nonempty. Indeed, let us fix $f_0 > 0$. Denote by g_s the maximum of all g_0 satisfying (61), and denote the right-hand side of (66) by g_m. Then we obtain that

$$\left(\frac{g_s}{g_m}\right)^{\delta-1} = \frac{(\delta-1)^{\delta-1}}{\delta^\delta} (\gamma\delta - \varepsilon + 1) \frac{\exp\left(\frac{\delta-1}{1-\varepsilon/\delta} f_0 T^{1-\varepsilon/\delta}\right)}{f_0 T^{1-\varepsilon/\delta}}.$$

Using the following elementary inequality $e^x \geq ex$ with $x = \frac{\delta-1}{1-\varepsilon/\delta} f_0 T^{1-\varepsilon/\delta}$ (the equality is achieved only for $x = 1$), we get

$$\left(\frac{g_s}{g_m}\right)^{\delta-1} \geq e\left(1 - \frac{1}{\delta}\right)^\delta \frac{\gamma\delta - \varepsilon + 1}{1 - \varepsilon/\delta}$$

Note that $e(1-1/\delta)^\delta < 1$ for all $\delta > 1$. The remaining fraction on the right hand side is > 1, but it can be made as close to 1 as we wish by fixing $\delta > 1$, taking $\gamma > 0$ small enough, and then by taking $\varepsilon \geq 0$ small enough, so that all conditions of Theorem 6 still hold. The above equality condition $x = 1$ is equivalent to $T = \left(\frac{\delta-\varepsilon}{\delta(\delta-1)}\right)^{\delta/(\delta-\varepsilon)}$. Using this T we can therefore achieve that

$$\left(\frac{g_s}{g_m}\right)^{\delta-1} = e\left(1-\frac{1}{\delta}\right)^\delta \frac{\gamma\delta-\varepsilon+1}{1-\varepsilon/\delta} < 1,$$

that is, $g_m > g_s$ in this case.

2.3 Nonexistence of solutions

The aim of this section is to show that problem (3), and even (22), is not solvable provided f_0 and g_0 are sufficiently large.

Theorem 7 *Assume that $\delta > \frac{\varepsilon-1}{\gamma}+1$, $\delta > 1$, $\gamma > 0$, and $\varepsilon \in \mathbb{R}$. Let f_0 and g_0 be positive real numbers. Assume that*

$$f_0 \geq \begin{cases} \dfrac{[\gamma(\delta-1)-\varepsilon+1]\delta^{\delta'}}{(\delta-1)T^{\gamma(\delta-1)-\varepsilon+1}g_0^{\delta-1}} & \text{for } \varepsilon < 1, \\ \dfrac{\gamma\,\delta^{\delta'}}{T^{\gamma(\delta-1)-\varepsilon+1}g_0^{\delta-1}} & \text{for } \varepsilon \geq 1. \end{cases} \tag{68}$$

Then problem (3), and even (22), has no solutions in D^+, see (20). Furthermore, there exists a sequence of subsolutions of K which is unbounded in $C([0,T])$.

To prove this non-existence result, we state two auxiliary propositions.

Proposition 1 *Let $\delta > \frac{\varepsilon-1}{\gamma}+1$, $\delta > 1$, $\gamma > 0$, and $\varepsilon \in \mathbb{R}$, and let us define a sequence of functions $z_m(t)$ inductively by*

$$z_{m+1}(t) = f_0\int_0^t \frac{z_m(s)^\delta}{s^\varepsilon}\,ds, \quad z_0(t) = g_0 t^\gamma. \tag{69}$$

Then for each solution $\omega \in D^+$ of (22) we have

$$\omega(t) \geq \sum_{m=0}^{n} z_m(t), \quad t\in[0,T], \quad \forall n \in \mathbf{N}. \tag{70}$$

Furthermore,

$$z_m(t) = \frac{g_0^{\delta^m} f_0^{\sum_{k=0}^{m-1}\delta^k}\, t^{(1-\varepsilon)\sum_{k=0}^{m-1}\delta^k+\gamma\delta^m}}{\prod_{k=1}^{m}[(1-\varepsilon)\sum_{j=0}^{k-1}\delta^j+\gamma\delta^k]^{\delta^{m-k}}} \geq 0, \tag{71}$$

and $z_m \in D(K)$, with $D(K)$ defined by (7). The function ω_n defined as the right-hand side of (70) is a subsolution of K for each $n = 0, 1, 2\ldots$

Proposition 2 *Let $\delta > \frac{\varepsilon-1}{\gamma}+1$, $\delta>1$, $\gamma>0$, $\varepsilon\in\mathbb{R}$, and let z_m be defined by (71). Assume that condition (68) is fulfilled, and let us define*

$$t^* := \begin{cases} \left(\dfrac{[\gamma(\delta-1)-\varepsilon+1]\delta^{\delta'}}{(\delta-1)f_0 g_0^{\delta-1}}\right)^{1/[\gamma(\delta-1)-\varepsilon+1]} & \text{for } \varepsilon<1, \\ \left(\dfrac{\gamma\,\delta^{\delta'}}{f_0 g_0^{\delta-1}}\right)^{1/[\gamma(\delta-1)-\varepsilon+1]} & \text{for } \varepsilon\ge 1. \end{cases} \tag{72}$$

Then $t^*\le T$ and $\sum_{m=0}^\infty z_m(t)=\infty$ for all $t\in[t^*,T]$.

Proof of Theorem 7. We argue by contradiction. Assume that there exists a solution $\omega\in D^+$ of (3). Using Proposition 1 and inequality (70) we obtain that $\sum_{m=0}^\infty z_m(t)\le\omega(t)<\infty$ for all $t\in[0,T]$ and all m. But this contradicts Proposition 2. Propositions 1 and 2 imply that ω_n is an unbounded sequence of subsolutions of K. $\diamondsuit$

Proof of Proposition 1. We proceed by induction with respect to n. Since $\omega(t)=K\omega(t)\ge g_0t^\gamma$, we can use (3) to obtain

$$\frac{d\omega}{dt}\ge g_0\gamma t^{\gamma-1}+f_0g_0^\delta t^{\gamma\delta-\varepsilon}.$$

Integrating over $(0,t)$, $t\in(0,T]$, we get (recall a well known fact that for ω nondecreasing we have $\omega(t)-\omega(0)\ge\int_0^t\frac{d\omega}{dt}\,dt$):

$$\omega(t)\ge z_0(t)+z_1(t),$$

which proves the claim for $n=1$.

Assume the claim holds for some $n\in\mathbf{N}$ and let us prove that it holds also for $n+1$. From (3), $\omega\in D^+$ and using $(a+b)^\delta\ge a^\delta+b^\delta$, $a\ge0$, $b\ge0$, $\delta>1$, we obtain:

$$\frac{d\omega}{dt}=g_0\gamma t^{\gamma-1}+f_0t^{-\varepsilon}\omega^\delta(t)\ge g_0\gamma t^{\gamma-1}+f_0t^{-\varepsilon}\left(\sum_{m=0}^n z_m(t)\right)^\delta\ge$$

$$\ge g_0\gamma t^{\gamma-1}+f_0t^{-\varepsilon}\sum_{m=0}^n z_m^\delta(t).$$

Integrating the preceding inequality over $(0,t)$, we obtain

$$\omega(t)\ge z_0(t)+\sum_{m=0}^n z_{m+1}(t)=\sum_{m=0}^{n+1}z_m(t),\quad t\in[0,T].$$

This proves (70). Formula (71) can be checked easily by induction using (69). To show that $z_m\in D(K)$, note that $\delta>1$ and $\sum_{k=0}^m\delta^k=\frac{\delta^{m+1}-1}{\delta-1}$ imply:

$$z_m(t)=\frac{g_0^{\delta^m}f_0^{\frac{\delta^m-1}{\delta-1}}t^{\frac{[\gamma(\delta-1)-\varepsilon+1]\delta^m+\varepsilon-1}{\delta-1}}(\delta-1)^{\sum_{k=1}^m\delta^{m-k}}}{\prod_{k=1}^m[(\gamma(\delta-1)-\varepsilon+1)\delta^k+\varepsilon-1]^{\delta^{m-k}}} \tag{73}$$

Since $\gamma(\delta-1)-\varepsilon+1>0$ and $\delta^m>1$, we see that the exponent at t is $\geq\gamma$.

To show that $\omega_n(t)$ are subsolutions of K, note that $\delta>1$ implies:

$$\begin{aligned} K\omega_n(t) &= z_0(t)+f_0\int_0^t \frac{[z_0(s)+z_1(s)+\ldots+z_n(s)]^\delta}{s^\varepsilon}\,ds \\ &\geq z_0(t)+f_0\int_0^t \frac{z_0(s)^\delta+z_1(s)^\delta+\ldots+z_n(s)^\delta}{s^\varepsilon}\,ds \\ &= z_0(t)+z_1(t)+z_2(t)+\ldots+z_{n+1}(t)=\omega_{n+1}(t)\geq\omega_n(t). \end{aligned}$$

$\diamondsuit$

Proof of Proposition 2. (a) Assume that $\varepsilon<1$. Using (73) we have

$$z_m(t)\geq \frac{g_0^{\delta^m} f_0^{\frac{\delta^m-1}{\delta-1}}\, t^{\frac{[\gamma(\delta-1)-\varepsilon+1]\delta^m+\varepsilon-1}{\delta-1}}(\delta-1)^{\sum_{k=1}^m\delta^{m-k}}}{[\gamma(\delta-1)-\varepsilon+1]^{\frac{\delta^m-1}{\delta-1}}\,\delta^{\sum_{k=1}^m k\delta^{m-k}}},$$

where we have used $\sum_{k=1}^m\delta^{m-k}=\frac{\delta^m-1}{\delta-1}$. Now we use

$$\sum_{k=1}^m k\delta^{m-k}=\frac{(2\delta-1)(\delta^m-1)}{(\delta-1)^2}-\frac{\delta^m+m-1}{\delta-1}$$

to obtain

$$\begin{aligned} z_m(t) &\geq \left(\frac{[\gamma(\delta-1)-\varepsilon+1]\delta^{\delta'}t^{\varepsilon-1}}{f_0(\delta-1)}\right)^{\frac{1}{\delta-1}}(\delta^m)^{\frac{1}{\delta-1}} \\ &\quad\times\left(\frac{g_0^{\delta-1}f_0t^{\gamma(\delta-1)-\varepsilon+1}(\delta-1)}{\delta^{\delta'}[\gamma(\delta-1)-\varepsilon+1]}\right)^{\frac{\delta^m}{\delta-1}} \end{aligned}$$

Note that condition (68) implies that $t^*\leq T$. Since the inequality $t\geq t^*$ is equivalent to

$$\frac{g_0^{\delta-1}f_0t^{\gamma(\delta-1)-\varepsilon+1}(\delta-1)}{\delta^{\delta'}[\gamma(\delta-1)-\varepsilon+1]}\geq 1,$$

we obtain $z_m(t)\geq A(t)\cdot(\delta^m)^{\frac{1}{\delta-1}}$, where $A(t)$ does not depend on m. Using $\delta>1$ we have that $\delta^m\to\infty$ as $m\to\infty$, and we conclude that $\sum_{m=0}^\infty z_m(t)=\infty$ as $m\to\infty$ for all $t\in[t^*,T]$.

(b) Assume that $\varepsilon\geq1$. Since $[\gamma(\delta-1)-\varepsilon+1]\delta^k+\varepsilon-1\leq\gamma(\delta-1)\delta^k$, we obtain similarly as in (a):

$$z_m(t)\geq\left(\frac{\gamma\delta^{\delta'}t^{\varepsilon-1}}{f_0}\right)^{\frac{1}{\delta-1}}(\delta^m)^{\frac{1}{\delta-1}}\left(\frac{g_0^{\delta-1}f_0t^{\gamma(\delta-1)-\varepsilon+1}}{\gamma\delta^{\delta'}}\right)^{\frac{\delta^m}{\delta-1}}$$

for all $t\in[0,T]$ and all $m\in\mathbf{N}$. The rest of the proof is analogous to (a). $\diamondsuit$

Comparison of existence and non-existence regions. Let us denote the non-existence set of all (f_0, g_0) satisfying the conditions of Theorem 7 by $\mathcal{M}_n$. Of course, we assume that the constants δ, γ, and ε are fixed. As we already saw, we have existence of solutions of $\omega = K\omega$ for $(f_0, g_0) \in \mathcal{M}_s \cup M_m$, see Theorems 5 and 6. We do not know anything about solvability or nonsolvability of (5) when

$$(f_0, g_0) \in (0, \infty)^2 \setminus (\mathcal{M}_s \cup \mathcal{M}_m \cup \mathcal{M}_n).$$

If we denote the set of all positive real numbers (f_0, g_0) for which (5) is solvable by $\mathcal{S}$, and the set of all (f_0, g_0) for which the fixed point equation is not solvable by $\mathcal{N}$, then we have $\mathcal{M}_s \cup \mathcal{M}_m \subseteq \mathcal{S}$ and $\mathcal{M}_n \subseteq \mathcal{N}$. We do not know whether any of the sets $\mathcal{S}$ or $\mathcal{N}$ is closed or open in $(0, \infty)^2$. However, we conjecture that these two sets are separated by a curve of the form $f_0 = c/g_0^{\delta-1}$, with some constant $c > 0$.

Now we would like to discuss how large is the region in $(0, \infty)^2$ where we do not know anything about existence or non-existence. First, we compare the sets $\mathcal{M}_s$ and $\mathcal{M}_n$. It will be convenient to fix f_0 and see the quotient of the corresponding values of g_n and g_s defined as the minimal possible value of g_0 in (68) and the maximal possible value of g_0 in (61) respectively. It is not difficult to see that both for $\varepsilon < 1$ and $\varepsilon \geq 1$ we obtain the same estimate

$$\frac{g_n}{g_s} \geq \delta' \delta^{\frac{\delta'}{\delta-1}} > e^{\delta'-1}, \tag{74}$$

where $e = 2.71828\ldots$ The last inequality follows from the fact that $\delta^{\delta'}(\delta')^{\delta-1}$ is increasing for $\delta > 1$ and tends to e as $\delta \to 1$. To show the first inequality in (74), let $\varepsilon < 1$. Then

$$\left(\frac{g_n}{g_s}\right)^{\delta-1} = \frac{\gamma(\delta-1) - \varepsilon + 1}{\gamma\delta - \varepsilon + 1}\, \delta^{\delta'}(\delta')^{\delta},$$

and the infimum of the fraction on the right-hand side over $\varepsilon < 1$ is equal to $\frac{\delta-1}{\delta}$. Similarly for $\varepsilon \geq 1$.

Now let us compare the sets $\mathcal{M}_m$ and $\mathcal{M}_n$. Let us fix $f_0 > 0$ again and denote the right-hand side (66) by g_m. We obtain

$$\frac{g_n}{g_m} > (e\,\delta^{\delta'})^{\delta'-1}. \tag{75}$$

Indeed, let $\varepsilon < 1$. Using $e^x \geq ex$ with $x = \frac{\delta-1}{1-\varepsilon/\delta} f_0 T^{1-\varepsilon/\delta}$ we obtain

$$\left(\frac{g_n}{g_m}\right)^{\delta-1} \geq \frac{\gamma(\delta-1) - \varepsilon + 1}{1 - \frac{\varepsilon}{\delta}}\, e\,\delta^{\delta'}.$$

The desired inequality follows using $\gamma > \varepsilon/\delta$. Similarly we can prove that (74) holds also for $\varepsilon \geq 1$.

2.4 Qualitative properties of solutions

The following result will be important in obtaining regularity of solutions of (1), see Proposition 6 below.

Proposition 3 *Assume that* $\delta > \frac{\varepsilon-1}{\gamma}+1$, $\gamma>0$, $\varepsilon\in\mathbb{R}$, *and let* $\omega\in D(K)$ *be such that* $\omega = K\omega$, *where* K *is defined (4) and* $D(K)$ *by (6) or (7). Then*

$$\lim_{t\to 0}\frac{\omega(t)}{t^\gamma} = \lim_{t\to 0}\frac{\omega'(t)}{\gamma\, t^{\gamma-1}} = g_0. \tag{76}$$

We have $\omega\in C^\infty((0,T])$, *and if* $\gamma\ge 1$ *then also* $\omega\in C^1([0,T])$.

Proof. Due to L'Hospital's rule it suffices to prove only the second equality. From (3) we have

$$\frac{\omega'(t)}{\gamma\, t^{\gamma-1}} = g_0 + \frac{\omega(t)^\delta}{t^{\varepsilon+\gamma-1}}.$$

Since $0\le\omega(t)\le Mt^\gamma$ for some $M>0$, the second term on the right-hand side tends to zero as $t\to 0$. The fact that $\omega\in C^\infty((0,T])$ follows easily from $\omega = K\omega$. The continuity of $\omega'(t)$ at $t=0$ for $\gamma\ge 1$ follows immediately from (3). $\diamondsuit$

Now we give a partial answer to the question of continuous dependence of solutions of (3) on $(f_0,g_0)\in\mathcal{M}_s\cup\mathcal{M}_m$, where $\mathcal{M}_s$ is defined by (61) and $\mathcal{M}_m$ by (66). Recall that $\mathcal{M}_b\subset\mathcal{M}_s$.

Proposition 4 *Assume that* $\delta > \frac{\varepsilon-1}{\gamma}+1$, $\delta>0$, $\gamma>0$, $\varepsilon\in\mathbb{R}$. *Assume that* $(f_1,g_1)\in\mathcal{M}_b$, *and let* $\omega_1(t)$ *be the unique solution obtained via Theorem 4. Let* $I(f_1)$ *and* $I(g_1)$ *be closed neighbourhoods of* f_1 *and* g_1 *respectively such that* $I(f_1)\times I(g_1)\subset\mathcal{M}_b$, *and let* $f_0=\max I(f_1)$, $A:=[1-c(f_0)]^{-1}$, *where* $c(f_0)$ *is defined by (55). Then for all* $f_2\in I(f_1)$, $g_2\in I(g_1)$ *we have*

$$\|\omega_1-\omega_2\|_\gamma \le A[\,|g_1-g_2| + |f_1-f_2|B(\omega_1)\,], \tag{77}$$

with

$$B(\omega_1) = \sup_{t\in(0,T]}\frac{1}{t^\gamma}\int_0^T\frac{\omega_1(s)^\delta}{s^\varepsilon}\,ds.$$

In particular, if $f_2\to f_1$ *and* $g_2\to g_1$, *then* $\omega_2\to\omega_1$ *uniformly on* $[0,T]$.

Proof. Since (f_2,g_2) is contained in a convex, open neighbourhood of (f_1,g_1) in $\mathcal{M}_b$, there exists $M>0$ such that $(f_i,g_i)\in Q_M$, see (56). Therefore the corresponding operators K_i, $i=1,2$ have a common domain $D(K)$ defined by (6) with the same $M>0$. Since $\omega_i=K_i\omega_i$, we have for any $t\in[0,T]$ (see the proof of Theorem 4):

$$\begin{aligned}\frac{1}{t^\gamma}|\omega_1(t)-\omega_2(t)| &\le \frac{1}{t^\gamma}|K_1\omega_1-K_2\omega_1| + \|K_2\omega_1-K_2\omega_2\|_\gamma\\ &\le |g_1-g_2| + |f_1-f_2|B(\omega_1) + c(f_2)\|\omega_1-\omega_2\|_\gamma, \quad (78)\\ &\le |g_1-g_2| + |f_1-f_2|B(\omega_1) + c(f_0)\|\omega_1-\omega_2\|_\gamma,\end{aligned}$$

Taking supremum over $t \in (0,T]$ in (78) and using $c(f_0) < 1$ we obtain the claim. $\diamondsuit$

Note that we have established continuous dependence of solutions only in $\mathcal{M}_b$, i.e. in the proper subset of $\mathcal{M}_s$. We do not know anything about continuous dependence of solutions of (3) with respect to $(f_0, g_0) \in (\mathcal{M}_s \cup \mathcal{M}_m) \setminus \mathcal{M}_b$.

Also note that for any given $(f_0, g_0) \in \mathcal{M}_s$ there exist the minimal value of $M > 0$ such that $(f_0, g_0) \in T_M$, see (62). This value, that we denote by M_e, appears in an a priori bound of $\omega(t)$.

Proposition 5 *Assume that $\delta > \frac{\varepsilon-1}{\gamma} + 1$, $\delta > 0$, $\gamma > 0$, and $\varepsilon \in \mathbb{R}$. Let $(f_0, g_0) \in \mathcal{M}_s$, see (62), and let ω be the solution of (4) from Theorem 5. Assume that M_e is the smaller of two positive solutions of equation (63).*

(a) We have the following estimate:

$$0 \le \omega(t) \le M_e t^\gamma. \tag{79}$$

Furthermore, this solution of (3) is unique in the set defined by (7).

(b) If $(f_0, g_0) \in \mathcal{M}_s$, see (62), then

$$|\omega(b) - \omega(a)| \le g_0 |b^\gamma - a^\gamma| + f_0 M_e^\delta \frac{|b^{\gamma\delta-\varepsilon+1} - a^{\gamma\delta-\varepsilon+1}|}{\gamma\delta - \varepsilon + 1}, \quad \forall a, b \in (0,T). \tag{80}$$

In particular,

$$|\omega'(t)| \le g_0 \gamma t^{\gamma-1} + f_0 M_e^\delta t^{\gamma\delta-\varepsilon}. \tag{81}$$

Proof. (a) Note that $\omega \in D(K)$, with $D(K)$ as in (6) and $M = M_e$. The uniqueness claim follows from Lemma 5.

(b) We have that $0 \le \omega(t) \le M_e t^\gamma$, and there holds (49). Hence, similarly as in the proof of equicontinuity in Lemma 7 we get that for all a, $b \in [0,T]$, $a < b$:

$$|\omega(b) - \omega(a)| = |K\omega(b) - K\omega(a)| \le g_0 |b^\gamma - a^\gamma| + \frac{f_0 M_e^\delta}{\gamma\delta - \varepsilon + 1} [\, b^{\gamma\delta-\varepsilon+1} - a^{\gamma\delta-\varepsilon+1} \,] \tag{82}$$

Relation (81) follows if we divide (80) by $|b - a|$ and let $b \to a$. $\diamondsuit$

It is possible to effectively compute M_e. For example, if $\delta = 2$ it is easy to see that

$$M_e = \frac{1 - \sqrt{1 - 4ag_0}}{2a}, \quad a = \frac{f_0 T^{\gamma-\varepsilon+1}}{2\gamma - \varepsilon + 1}. \tag{83}$$

Note that in the case of monotone iterations method (see Theorem 6), that is, when $(f_0, g_0) \in \mathcal{M}_m$, we have uniqueness of the corresponding solution in a much larger domain, which is defined by (7).

3 Quasilinear PDE

3.1 Existence and non-existence of solutions

The main results of this paper are Theorem 8 and Theorem 9 below. Recall that by ω-solutions of (1) we mean strong solutions that can be obtained via ODE (3), as described in Lemma 1.

Theorem 8 *Assume that* $1 < p < \infty$, $m > \max\{-p, -N\}$, *and let* $\tilde{f}_0$ *and* $\tilde{g}_0$ *be positive real numbers.*

(a) If

$$\tilde{f}_0 \le \left(m + 1 + \frac{N}{p'}\right)\left(\frac{m+N}{pR^{m+p}\tilde{g}_0}\right)^{p'-1}, \tag{84}$$

then there exists ω*-solution of (1) which can be obtained constructively using the method of monotone iterations, see the remark after the proof of Theorem 6, and Lemma 1. If we have strict inequality in (84) then the same sequence of monotone iterations can be obtained also by contraction method via Theorem 4 and Lemma 1. Any* ω*-solution of (1) is unique in the set*

$$\{w \in C^2(B \setminus \{0\}) \cap C(\overline{B}) \,:\, \exists M_w > 0,\ |\nabla w(x)| \le M_w |x|^{\frac{p'}{p}(m+1)}\}. \tag{85}$$

and satisfies the following estimate:

$$|\nabla v(x)| \le N C_N^{\frac{(m+p)p'}{Np}} M_e^{p'/p} |x|^{\frac{p'}{p}(m+1)}, \tag{86}$$

where $M_e = M_e(f_0, g_0)$ *is defined in Proposition 5 for* f_0 *and* g_0 *as in (16). In particular, in the case of* $p = 2$ *and* $m = 0$ *we have that*

$$|\nabla v(x)| \le |x| \frac{N+2}{2\tilde{f}_0 R^2}\left(1 - \sqrt{1 - \frac{4\tilde{f}_0\tilde{g}_0 R^2}{N(N+2)}}\right). \tag{87}$$

(b) If $m \ge -1$ *and*

$$\tilde{g}_0 \le N^{p-1} C_N^{\frac{m+p}{N}} (m+N) |B|^{-\frac{m+1}{N}} e^{-N|B|^{1/N}\tilde{f}_0}, \quad |B| = C_N R^N, \tag{88}$$

then there exists ω*-solution of (1) that can be obtained constructively using the method of monotone iterations via Theorem 6 and Lemma 1, and which is unique in the set (85). If* $(\tilde{f}_0, \tilde{g}_0)$ *satisfies also the condition in (a), then the solution in (b) coincides with the one in (a).*

(c) If

$$\tilde{f}_0 \ge \begin{cases} (m+p)(p')^p \left(\dfrac{m+N}{R^{m+p}\tilde{g}_0}\right)^{p'-1} & \text{for } p > N, \\ (m+N)(p')^p \left(\dfrac{m+N}{R^{m+p}\tilde{g}_0}\right)^{p'-1} & \text{for } p \le N, \end{cases} \tag{89}$$

then (1) has neither strong nor weak solutions.

Proof. It suffices to use Lemma 1 together with: (a) Theorem 5, Theorem 4 and Proposition 5, (b) Theorem 6. (c) To show non-existence, we proceed by contradiction. Assume that v is a strong or weak solution of (1). Using Lemma 2 or Theorem 2 respectively, in both cases we obtain a solution $\omega \in D^+$ of (22). This contradicts Theorem 7.

Coincidence of solutions in (a) and (b) follows easily from the fact that the uniqueness domain in (b) equals the uniqueness domain in (a). $\diamondsuit$

In Theorem 9 we will show that ω-solutions, whose existence we have proved in the above theorem, see (a) and (b), are in fact unique weak solutions of (1).

It is natural to define the sets $\tilde{\mathcal{M}}_s$, $\tilde{\mathcal{M}}_m$, and $\tilde{\mathcal{M}}_n$ of all $(\tilde{f}_0, \tilde{g}_0) \in (0,\infty)^2$ satisfying conditions in (a), (b), and (c) respectively. If

$$(\tilde{f}_0, \tilde{g}_0) \in (0,\infty)^2 \setminus (\tilde{\mathcal{M}}_s \cup \tilde{\mathcal{M}}_m \cup \tilde{\mathcal{M}}_n),$$

we do not know anything about existence or non-existence of weak and strong solutions of (1). Also, let us denote by $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{N}}$ the existence and non-existence sets of all $(\tilde{f}_0, \tilde{g}_0)$ in $(0,\infty)^2$. We do not know anything about geometrical properties of these sets, except that

$$\tilde{\mathcal{S}} \supseteq \tilde{\mathcal{M}}_s \cup \tilde{\mathcal{M}}_m, \quad \tilde{\mathcal{N}} \supseteq \tilde{\mathcal{M}}_n.$$

3.2 Qualitative properties of solutions

First we study the behaviour of the gradient of ω-solutions of (1) near the origin $x = 0$ and near the boundary of ball B.

Lemma 8 *Let $m > \max\{-p, -N\}$, and let $v(x)$ be ω-solution of (1). Let us define $u(r) = v(x)$, $r = |x|$. Then*

(a) $v \in C^\infty(\overline{B} \setminus \{0\}) \cap C(\overline{B})$;

(b)

$$\lim_{r\to 0} \frac{u'(r)}{r^{\frac{p'}{p}(m+1)}} = -\left(\frac{\tilde{g}_0}{m+N}\right)^{p'/p} \tag{90}$$

$$\lim_{r\to 0} \frac{u''(r)}{r^{\frac{p'}{p}(m-p+2)}} = -\frac{m+1}{p-1}\left(\frac{\tilde{g}_0}{m+N}\right)^{p'/p}, \tag{91}$$

(c) if $(\tilde{f}_0, \tilde{g}_0) \in \tilde{\mathcal{M}}_s$, see (84), then

$$|u'(R)| \le D_1 R^{\frac{p'}{p}(m+1)} + D_2 R^{\frac{2p'}{p}(m+1)+1}, \tag{92}$$

where

$$D_1 = \frac{\tilde{g}_0 N^2 M_e^{p'-2}}{m+N} C_N^{\frac{m(p'-2)+p'-p}{N}}, \; D_2 = \frac{\tilde{f}_0 p N^2 M_e^{2(p'-1)}}{p'(2m+N+1)+p} C_N^{\frac{p'}{p}\frac{m+p}{N}+\frac{p'(m+1)-m}{N}}.$$

Here $M_e = M_e(f_0, g_0)$ is defined in Proposition 5 with f_0, g_0 from (16).

Proof. (a) This follows easily from (8), (10) and Proposition 3.

(b) It suffices to use (17). The first relation is immediate, while the second relation follows after differentiating (17) and using Proposition 3 with $t = C_N r^N$. Indeed, we have that $\frac{\omega(t)}{t^\gamma} \to g_0$, $\frac{\omega'(t)}{\gamma\, t^{\gamma-1}} \to g_0$ as $t \to 0$. Denoting $c = -NC_N^{\frac{p'}{p}(\frac{p}{N}-1)}$ and $\gamma = 1 + \frac{m}{N}$ we obtain after an easy computation that

$$\begin{aligned}\frac{u''(r)}{c\, r^{\frac{p'}{p}(m-p+2)}} &= -(N-1)C_N^{\frac{\gamma p'}{p}}\frac{p'}{p}\left[\frac{\omega(C_N r^N)}{(C_N r^N)^\gamma}\right]^{p'/p} + \\ &\quad +\gamma N C_N^{\frac{p'\gamma}{p}}\frac{p'}{p}\left[\frac{\omega(C_N r^N)}{(C_N r^N)^\gamma}\right]^{\frac{p'}{p}-1}\frac{\omega'(C_N r^N)}{\gamma(C_N r^N)^{\gamma-1}} \\ &\to -(N-1)C_N^{\frac{\gamma p'}{p}}\frac{p'}{p}g_0^{p'/p} + \gamma N C_N^{\frac{p'\gamma}{p}}\frac{p'}{p}g_0^{\frac{p'}{p}-1}g_0 \\ &= \frac{p'}{p}g_0^{p'/p}C_N^{\gamma p'/p}(\gamma N - N + 1) \quad \text{as } r \to 0.\end{aligned}$$

Now we use (16) to obtain the desired result.

(c) To prove (92) we first use (17):

$$r^{\frac{p'}{p}(N-1)}|u'(r)| = |c| \cdot \omega(C_N r^N)^{p'/p}.$$

Now we differentiate this equality with respect to r:

$$\frac{d}{dr}(r^{\frac{p'}{p}(N-1)}|u'(r)|) = |c|\frac{p'}{p}\omega(C_N r^N)^{\frac{p'}{p}-1}\omega'(C_N r^N) \cdot C_N\, N\, r^{N-1},$$

and then use estimates $0 \le \omega(t) \le Mt^\gamma$ and (81). The desired inequality follows after a short computation upon integration over $(0, R)$:

$$R^{\frac{p'}{p}(N-1)}|u'(R)| - \lim_{r\to 0} r^{\frac{p'}{p}(N-1)}|u'(r)| = \int_0^R [E_1 r^{\frac{p'}{p}(m+N)-1} + E_2 r^{\frac{p'}{p}(2m+N+1)}]dr,$$

where E_1, E_2 are positive constants depending on m, p, N, $\tilde{g}_0$, and $\tilde{f}_0$. Note that the function under the integral sign is integrable since $m > \max\{-N, -p\}$ implies that the exponents at r are > -1. We also need (90) to obtain that

$$\lim_{r\to 0} r^{\frac{p'}{p}(N-1)}|u'(r)| = |A| \lim_{r\to 0} r^{\frac{p'}{p}(m+N)} = 0,$$

where A is the right-hand side of (90). ♢

A priori bound (92) is a refinement of (86) for $|x| = R$. Relations (90) and (91) immediately imply the following qualitative properties of the gradient of ω-solutions of (1) at $x = 0$.

Proposition 6 *Assume that $m > \max\{-p, -N\}$, and let $v(x)$ be ω-solution of (1). Then we have the following regularity results at $x = 0$:*

(a) If $m < -1$ then $\lim_{r\to 0} u'(r) = -\infty$. In particular, $v \notin C^1(\overline{B})$.

(b) If $m = -1$ then

$$\lim_{r\to 0} u'(r) = -\left(\frac{\tilde{g}_0}{m+N}\right)^{p'/p} \tag{93}$$

As in case (a), we have $v \notin C^1(\overline{B})$.

(c) If $-1 < m < p-2$, then

$$\lim_{r\to 0} u'(r) = 0, \quad \lim_{r\to 0} u''(r) = -\infty. \tag{94}$$

In particular, $v \in C^1(\overline{B})$ and $v \notin C^2(B)$.

(d) If $m \geq p-2$, then $\lim_{r\to 0} u'(r) = 0$ and

$$\lim_{r\to 0} u''(r) = \begin{cases} -\frac{m+1}{p-1}\left(\frac{\tilde{g}_0}{m+N}\right)^{p'/p} & \text{for } m = p-2, \\ 0 & \text{for } m > p-2. \end{cases} \tag{95}$$

In particular, v is classical solution, $v \in C^2(\overline{B})$.

The above proposition shows that we have precise information on the gradient of ω-solutions at $x = 0$. With larger values of m we can have even more regularity of ω-solutions, and it is possible to study Hölder continuity as well, see [19]. On the other hand, we are not able to obtain precise information about $v(0)$. We can obtain only upper and lower estimates for $v(0)$.

Proposition 7 *(a priori estimate of $v(0)$) (a) Let $m > \max\{-p, -N\}$ and let $v(x)$ be ω-solution of (1) corresponding to $(\tilde{f}_0, \tilde{g}_0) \in \tilde{\mathcal{M}}_s$, see Theorem 5, and let $M_e = M_e(f_0, g_0)$ be defined as in Proposition 5, where f_0, g_0 are given by (16). Then*

$$v(0) \leq N\frac{p-1}{m+p} \cdot C_N^{\frac{m+p}{N}\cdot\frac{p'}{p}} R^{\frac{p'}{p}(m+p)} M_e^{p'/p}. \tag{96}$$

In particular, for $p = 2$ and $m = 0$ we have

$$v(0) \leq \frac{N+2}{4\tilde{f}_0}\left(1 - \sqrt{1 - \frac{4\tilde{f}_0\tilde{g}_0 R^2}{N(N+2)}}\right) \tag{97}$$

(b) For any weak solution $v(x)$ of (1) we have the following lower bound:

$$v(0) \geq \begin{cases} \dfrac{1}{(2p)^{p'}}\left(\dfrac{R^{m+p}\tilde{g}_0}{2^N - 1}\right)^{p'-1} & \text{for } m < 0, \\ \left(\dfrac{c(m,p,N)R^{m+p}\tilde{g}_0}{p^p}\right)^{p'-1} & \text{for } m \geq 0, \end{cases} \tag{98}$$

where

$$c(m,p,N)=\sup_{t\in(0,\frac{1}{2})}\frac{t^p[(1-t)^N-t^N](1-t)^m}{1+t^N-(1-t)^N}. \tag{99}$$

If $m>\max\{-p,-N\}$ then the above estimate holds also for any ω-solution of (1).

In particular, when $p=2$, $N=2$, $m\geq 0$, we obtain the following lower bound for weak solutions of (1):

$$v(0)\geq \frac{1}{4}c(m)R^{m+2}\tilde{g}_0, \tag{100}$$

where

$$c(m)=\frac{1}{2}t(1-2t)(1-t)^m,\quad t=\frac{m+5-\sqrt{m^2+2m+9}}{4(m+2)}.$$

If in addition to this we assume that $m=0$, we obtain $v(0)\geq\frac{1}{64}R^2\tilde{g}_0$.

(c) If $m\geq -1$ and $(\tilde{f}_0,\tilde{g}_0)\in\tilde{\mathcal{M}}_m$, see (88), then for the corresponding ω-solution $v(x)$ of (1) we have

$$\begin{aligned} c(p)\frac{p-1}{m+p}\Bigg[\left(\frac{\tilde{g}_0}{m+N}\right)^{p'/p}R^{p'(m+1)-m}+&\\ +\left(\frac{\tilde{f}_0\tilde{g}_0^{p'}}{[p'(m+1)+N](m+N)^{p'}}\right)^{p'/p}(p')^{-1}R^{p'[p'(m+1)-m]}\Bigg]&\leq \\ \leq v(0)\leq\frac{p-1}{m+p}\left(\frac{\tilde{g}_0\cdot e^{\tilde{f}_0N|B|^{1/N}}R^{m+p}}{m+N}\right)^{p'/p}&, \end{aligned} \tag{101}$$

where $c(p)=1$ for $p\in(1,2)$ and $c(p)=2^{p'-2}$ for $p\geq 2$. The same lower bound holds also for $(\tilde{f}_0,\tilde{g}_0)\in\tilde{\mathcal{M}}_s$, see (84).

PROOF. (a) We obtain the upper bound using (17) and $v(0)\leq\int_0^R|u'(r)|\,dr$.

(b) This lower bound is obtained using lower oscillation estimate for general quasilinear elliptic problems studied in [9]. In the case of $m<0$ the estimate follows immediately from [9, Corollary 12]. If $m\geq 0$, we use the following version of oscillation estimate, see [9, Theorem 9]:

$$\operatorname*{osc}_{A_r} v\geq\frac{1}{p^{p'}}\left(\frac{r^p|A|}{|A_r\setminus A|}\right)^{p'/p}\operatorname*{ess\,inf}_{x\in A}(\tilde{g}_0|x|^m)^{p'/p}, \tag{102}$$

for any open subset $A\subset B$ and $r>0$ such that $A_r\subset B$, where A_r denotes open r-neighbourhood of A. Note that $\operatorname{osc}_B v=v(0)$. We consider the family of sub-rings A of B such that $A_r=B\setminus\{0\}$, $r\in(0,\frac{R}{2})$. Then substituting $t=\frac{r}{R}$

we obtain

$$\begin{aligned} v(0) &\geq \frac{\tilde{g}_0^{p'/p}}{p^{p'}} \sup_{r\in(0,\frac{R}{2})} \left(\frac{r^p[(R-r)^N - r^N]}{R^N + r^N - (R-r)^N} \right)^{p'/p} (R-r)^{mp'/p} \\ &= \frac{\tilde{g}_0^{p'/p}}{p^{p'}} \left(\sup_{r\in(0,\frac{R}{2})} \frac{r^p[(R-r)^N - r^N](R-r)^m}{R^N + r^N - (R-r)^N} \right)^{p'/p} \\ &= \frac{\tilde{g}_0^{p'/p}}{p^{p'}} R^{\frac{p'}{p}(m+p)} \left(\sup_{t\in(0,\frac{1}{2})} \frac{t^p[(1-t)^N - t^N](1-t)^m}{1 + t^N - (1-t)^N} \right)^{p'/p} . \end{aligned} \tag{103}$$

It will be shown in Proposition 11 that for $m > \max\{-p, -N\}$ any ω-solution is also weak solution.

(c) The claim follows from estimate (67) in Theorem 6 and from Lemma 1, similarly as in (a); see also the remark after the proof of Theorem 6. We use the fact that $(a+b)^q \geq d(q)(a^q + b^q)$, where $d(q) = 2^{q-1}$ for $q \in (0,1)$, $d(q) = 1$ for $q \geq 1$, $a \geq 0$, $b \geq 0$, with $q = p'/p$. $\diamondsuit$

Note that in particular, under the assumptions of (a), we have that for any ω-solution $v(x)$ of (1) we have

$$\text{if } R \to 0 \text{ then } v(0) \to 0. \tag{104}$$

Property (b) implies that for any weak solution $v(x)$ we have

$$\text{if } R \to \infty \text{ or } \tilde{g}_0 \to \infty, \text{ then } v(0) \to \infty. \tag{105}$$

Under the assumptions of (c) we have that all these implications hold for ω-solutions. Furthermore, from (101) we also see that

$$\text{if } \tilde{f}_0 \tilde{g}_0^{p'} \to \infty \text{ then } v(0) \to \infty. \tag{106}$$

It is possible to obtain even better constant than $c(m,p,N)$ in (98), if in the proof we consider the family of sub-annuli A such that $A_r \subset B \setminus \{0\}$, and not only $A_r = B \setminus \{0\}$.

In formulating our problem (1) we imposed the condition that $v(x)$ be decreasing. Let us consider the corresponding problem without this condition:

$$\begin{gathered} -\Delta_p v = \tilde{g}_0 |x|^m + \tilde{f}_0 |\nabla v|^p, \\ v|_{\partial B} = 0\,, \quad v \in C^2(B \setminus \{0\}) \cap C(\overline{B}), \\ v(x) \text{ spherically symmetric.} \end{gathered} \tag{107}$$

The following proposition shows that if $v(x)$ is a strong solution of (107) which belongs to $C^1(B)$, or $p > 2$, then $u(r)$ is necessarily decreasing, where $u(r) = v(x)$, $r = |x|$.

Proposition 8 *Let $\tilde f_0$ and $\tilde g_0$ be positive real numbers. Let $v(x)$ be a strong solution of (1).*

(a) Then $u|_{(0,R)}$ can have at most one local maximum r_0, and $u(r)$ is decreasing on (r_0,R).

(b) If $v\in C^1(B)$, then $u(r)$ is decreasing on $[0,R]$.

(c) If $p>2$ then $u(r)$ is decreasing on $[0,R]$.

Proof. (a) Let us define $V(s)$, $s\in(0,|B|)$ by $V(C_Nr^N)=u(r)$. It suffices to show that if $r_0\in[0,R]$ is such that $V'(s_0)=0$, $s_0=C_Nr_0^N$, then $V'(s)<0$ for $s=C_Nr^N\in(s_0,C_NR^N)$. Let $x\in B$ be such that $s=C_N|x|^N$ for some given $s>s_0$. We have

$$\begin{aligned} -N^pC_N^{\frac{p}{N}}\frac{d}{ds}\left(s^{p(1-\frac1N)}\left|\frac{dV}{ds}\right|^{p-2}\frac{dV}{ds}\right) &= -\Delta_p v=\tilde g_0|x|^m+\tilde f_0|\nabla v|^p\\ &\ge \tilde g_0\left(\frac{s}{C_N}\right)^{\frac{m}{N}}. \end{aligned} \tag{108}$$

Integrating from s_0 to s we obtain

$$-s^{p(1-\frac1N)}\left|\frac{dV}{ds}(s)\right|^{p-2}\frac{dV}{ds}(s)+s_0^{p(1-\frac1N)}\left|\frac{dV}{ds}(s_0)\right|^{p-2}\frac{dV}{ds}(s_0)>0.$$

Since $\frac{dV}{ds}(s_0)=0$ we arrive to $\frac{dV}{ds}<0$, which proves that $\frac{du}{dr}<0$ for all $r\in(r_0,R)$, see (8).

(b) If $v\in C^1(B)$, then $u'(0)=0$, and we can proceed as in (a) with $r_0=0$.

(c) Assume that $u(r)$ is not decreasing. Then there exists $r_0\in(0,R)$ such that $u'(r_0)=0$. From (108) we obtain

$$p(1-\frac1N)s^{p(1-\frac1N)-1}|V'(s)|^{p-2}V'(s)+(p-1)s^{p(1-\frac1N)}|V'(s)|^{p-2}V''(s)\le -c\cdot s^{m/N},$$

where $c>0$. Substituting $s=s_0$ we obtain a contradiction: $0\le -c\cdot s_0^{m/N}$.

Assume that u is decreasing on $(0,a)$, but is not decreasing on $[a,R]$. Then there exists a point $r_0\in[r_0,R)$ such that $u'(r_0)=0$, which implies a contradiction in the same way as above. ♢

In (92) we obtained an upper estimate of $|u'(R)|$, that is, of the outward normal derivative on the boundary of B for any ω-solution $v(x)$ of (1). Now we want to obtain the lower bound of $|u'(R)|$ for any solution of (1).

Proposition 9 *Let $m>-N$. For any strong solution $v(x)$ of (1) such that $u'(r)\to 0$ as $r\to 0$ we have the following lower bound:*

$$|u'(R)|\ge\left(\frac{\tilde g_0R^{m+1}}{m+N}\right)^{p'-1}. \tag{109}$$

In particular, if $m>-1$ this estimate holds for any ω-solution of (1).

PROOF. Let $s = C_N r^N$, and let us integrate (108) from 0 to $|B|$. We obtain

$$N^p C_N^{\frac{p}{N}} |B|^{p(1-\frac{1}{N})} |V'(|B|)|^{p-1} \geq \frac{N\tilde{g}_0 |B|^{\frac{m+N}{N}}}{C_N^{m/N}(m+N)}.$$

On the other hand, after differentiating $V(C_N r^N) = u(r)$ with respect to r, we get $V'(|B|) = \frac{u'(R)}{NC_N R^{N-1}}$, and the result follows easily. The claim for ω-solutions follows from Proposition 6. $\diamondsuit$

We can also state a continuous dependence result for ω-solutions of (1), which follows easily from Theorem 8(a) and Lemma 1.

Proposition 10 *Assume that $1 < p < \infty$, $m > \max\{-p, -N\}$ and $(\tilde{f}_1, \tilde{g}_1) \in \tilde{\mathcal{M}}_b$, where $\tilde{\mathcal{M}}_b$ is defined by (56). Let $I(\tilde{f}_1)$ and $I(\tilde{g}_1)$ be closed neighbourhoods of $\tilde{f}_1$ and $\tilde{g}_1$ respectively, $f_0 = \max I(f_1)$ and $g_0 = \max I(g_1)$, where f_1, g_1 and the corresponding intervals $I(f_1)$, $I(g_1)$ are defined by (16), $A > 0$, $B(\omega_1)$ as in Proposition 4. If $\tilde{f}_2 \in I(\tilde{f}_1)$ and $\tilde{g}_2 \in I(\tilde{g}_1)$, then for the corresponding ω-solutions v_1, v_2 from Theorem 8(a) we have that for all $r \in [0, R]$:*

$$\begin{aligned} |u_1'(r) - u_2'(r)| &\leq C\, r^{\frac{p'}{p}(m+1)} [\,|g_1 - g_2| + B(\omega_1)|f_1 - f_2|\,], && (110) \\ |u_1(r) - u_2(r)| &\leq C\, \frac{R^{p'(m+1)-m} - r^{p'(m+1)-m}}{p'(m+1)-m} \\ &\quad \times [\,|g_1 - g_2| + B(\omega_1)|f_1 - f_2|\,], && (111) \end{aligned}$$

where

$$C = (p'-1) N C_N^{\frac{m+p}{N(p-1)}} M_e^{p'-2} A. \qquad (112)$$

Here f's, g's, u's, and ω_1 are defined analogously as in Lemma 1, and $M_e = M_e(f_0, g_0)$, see Proposition 5, with f_0 and g_0 as in Proposition 4. In particular, if $\tilde{f}_2 \to \tilde{f}_1$ and $\tilde{g}_2 \to \tilde{g}_1$, then $v_2 \to v_1$ in $C^1(\overline{B})$. If $p = 2$, then $v_2 \to v_1$ in $C^2(\overline{B})$.

PROOF. (a) We have

$$|u_1'(r) - u_2'(r)| = c r^{-(N-1)\frac{p'}{p}} |\omega_1(C_N r^N)^{\frac{p'}{p}} - \omega_2(C_N r^N)^{\frac{p'}{p}}|, \qquad (113)$$

where $c = NC_N^{\frac{p'}{p}(\frac{p}{N}-1)}$. We can use (46) and $\omega(t) \leq Mt^\gamma$, $t = C_N r^N$, $\gamma = 1 + \frac{m}{N}$, together with

$$|\omega_1(t) - \omega_2(t)| \leq t^\gamma A[\,|g_1 - g_2| + B(\omega_1)|f_1 - f_2|\,],$$

see Proposition 4. (113).

(b) To prove (111) note that the zero boundary condition implies

$$|u_1(r) - u_2(r)| \leq \int_r^R |u_1'(\rho) - u_2'(\rho)|\, d\rho.$$

It suffices to use (110).

(c) The convergence in $C^2(\overline{B})$ for $p = 2$ follows from standard L^2-regularity for elliptic equations. $\diamondsuit$

3.3 Uniqueness of weak solutions

Here we study the problem of existence of unique weak solution of (1).

Proposition 11 *Assume that $m > \max\{-p, -N\}$. (a) Then any ω-solution of (1) is weak solution, and conversely, any weak solution of (1) is ω-solution. (b) There exists at most one weak solution of (1). In particular, under conditions (a) or (b) of Theorem 8 problem (1) possesses the unique weak solution.*

PROOF. (a1) Let us show that the pointwise derivative $\frac{\partial v}{\partial x_i}$ of ω-solution v is also the weak derivative. First, since v is continuous, it is integrable on B. We can write

$$\int_B v\frac{\partial\varphi}{\partial x_i}\,dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} v\frac{\partial\varphi}{\partial x_i}\,dx,$$

where $\varphi \in C_0^\infty(\Omega)$. Using Green's formula we have that

$$\int_{\Omega_\varepsilon} v\cdot\frac{\partial\varphi}{\partial x_i}\,dx = -\int_{\Omega_\varepsilon}\frac{\partial v}{\partial x_i}\cdot\varphi\,dx + \int_{S_\varepsilon} v\,\varphi\,\nu_i\,dS,$$

where ν is the outward unit normal vector at x, $|x| = \varepsilon$, with respect to domain $\Omega_\varepsilon = B\setminus B_\varepsilon(0)$, and S_ε is the inner bounding sphere of Ω_ε whose radius is ε. The last integral tends to zero as $\varepsilon\to 0$ since v is bounded. From this we can easily see that

$$\int_B v\cdot\frac{\partial\varphi}{\partial x_i} = -\int_B\frac{\partial v}{\partial x_i}\cdot\varphi\,dx,$$

i.e. the pointwise derivative of v is also the weak derivative of v.

(a2) Let us prove that the ω-solution v of (1) is also weak solution. Since v is of class C^∞ on Ω_ε, see Lemma 8(c), we have that it satisfies (1) pointwise on Ω_ε. This together with Green's formula yields:

$$\begin{aligned}\tilde g_0\int_{\Omega_\varepsilon}|x|^m\varphi\,dx + \tilde f_0\int_{\Omega_\varepsilon}|\nabla v|^p\varphi\,dx &= -\int_{\Omega_\varepsilon}\Delta_p v\,\varphi\,dx = \\ = \int_{\Omega_\varepsilon}|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx &- \int_{S_\varepsilon}\sum_{i=1}^N|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\,\varphi\,\nu_i\,dS.\end{aligned}$$

To show that the last integral tends to zero we use the fact that there exists a constant $C > 0$ such that

$$|u'(r)| \le C\cdot r^{(m+1)\frac{p'}{p}}$$

for all $r\in[0,R]$, see Lemma 8(b). Therefore the last integral does not exceed

$$C\int_{S_\varepsilon}|\nabla v|^{p-1}dS \le C|u'(\varepsilon)|^{p-1}\cdot\varepsilon^{N-1} \le C\cdot\varepsilon^{m+N}\to 0 \quad\text{as } \varepsilon\to 0, \tag{114}$$

where $C > 0$ is a generic constant. Passing to the limit in the above integral equality we obtain that $-\Delta_p v = \tilde g_0|x|^m + \tilde f_0|\nabla v|^p$ in the weak sense.

It remains to check that $v \in W_0^{1,p}(B)$, that is,

$$\int_B |\nabla v|^p dx \le C \int_0^R r^{(m+1)p'+N-1} dr < \infty.$$

This is equivalent to $(m+1)p' + N > 0$, and this inequality follows easily from $m > \max\{-p, -N\}$.

(b) Since any weak solution is ω-solution, uniqueness of weak solutions follows from Lemma 5. $\diamondsuit$

Theorem 9 *Assume that* $1 < p < \infty$*,* $m > \max\{-p, -N\}$*, and*

$$(\tilde{f}_0, \tilde{g}_0) \in \tilde{\mathcal{M}}_s \cup \tilde{\mathcal{M}}_n,$$

where $\tilde{\mathcal{M}}_s$ *and* $\tilde{\mathcal{M}}_n$ *are subsets of* $(0, \infty)^2$ *defined by (84) and (88) respectively. Then there exists a unique weak solution* $v \in W_0^{1,p}(B) \cap L^\infty(B)$ *of (1). Furthermore, we have* $v \in C^\infty(\overline{B} \setminus \{0\}) \cap C(\overline{B})$*, and* v *is* ω*-solution. It has all qualitative properties described in Section 3.2.*

Proof. By Proposition 11 we know that for $m > \max\{-p, -N\}$ any ω-solution is also weak solution of (1). Therefore it suffices to show that there exists a unique ω-solution. The existence of ω-solutions has been proved in Theorem 8(a) and (b). Now by Theorem 2 we know that weak solutions of (1) are in fact ω-solutions. Assume that there exist two different weak solutions v_1 and v_2 of (1). Then this implies the existence of two different functions ω_1 and ω_2 both satisfying equation $\omega = K\omega$, where K is defined by (4) on domain (7). The fact that $\omega_1 \ne \omega_1$ follows easily from (8). But this contradicts Proposition 11. This proves unique solvability of (1). For regularity of v see Lemma 8(a). $\diamondsuit$

As we see from Theorem 2 and Proposition 11, the notions of weak solution and ω-solution coincide provided $m > \max\{-p, -N\}$.

Using a slight modification it is also possible to obtain existence and uniqueness results for (1) with a weaker notion of strong solution: $v \in C^2(B \setminus \{0\})$ instead of $v \in C^2(B) \cap C(\overline{B})$, that is, for solutions allowing singularity at $0 \in B$.

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