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ON THE DIRICHLET PROBLEM FOR QUASILINEAR ELLIPTIC SECOND ORDER EQUATIONS WITH TRIPLE DEGENERACY AND SINGULARITY IN A DOMAIN WITH A BOUNDARY CONICAL POINT

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Abstract. In this article we prove boundedness and Hölder continuity of weak solutions to the Dirichlet problem for a second order quasilinear elliptic equation with triple degeneracy and singularity. In particular, we study equations of the form

$$-\frac{d}{dx_i}(|x|^\tau |u|^q |\nabla u|^{m-2} u_{x_i}) + \frac{a_0|x|^\tau}{(x_{n-1}^2 + x_n^2)^{m/2}} u|u|^{q+m-2} - \mu|x|^\tau u|u|^{q-2}|\nabla u|^m = \\ = f_0(x) - \frac{\partial f_i}{\partial x_i},$$

with $a_0 \geq 0$, $q \geq 0$, $0 \leq \mu < 1$, $1 < m \leq n$, and $\tau > m - n$ in a domain with a boundary conical point. We obtain the exact Hölder exponent of the solution near the conical point.

0. Introduction

Lately many mathematicians have considered nonlinear problems with elliptic degenerate equations (see e.g. [6] and its extensive bibliography). In the present paper, we continue the investigation on the behaviour of solutions of the first boundary value problem for a quasilinear elliptic second order equation with triple degeneracy (see [18]). Namely, we derive an exact Hölder exponent for weak solutions, in a neighbourhood of a conical boundary point, for the Dirichlet problem

$$-\frac{d}{dx_i}(|x|^\tau |u|^q |\nabla u|^{m-2} u_{x_i}) + \frac{a_0|x|^\tau}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}} u|u|^{q+m-2} - \mu|x|^\tau u|u|^{q-2}|\nabla u|^m = \\ = f_0(x) - \frac{\partial f_i}{\partial x_i}, \quad x \in G, \quad (0.1)$$

$$u(x) = 0, \quad x \in \partial G; \quad (0.2)$$

$$a_0 \geq 0, \quad q \geq 0, \quad \mu \geq 0, \quad 1 < m \leq n, \quad \tau > m - n; \quad (0.3)$$

(summation over repeated indices from 1 to n is understood), where G is a n-dimensional *bounded convex* circular cone with the vertex at the origin of coordinates O. We shall construct functions playing a fundamental role in the study of

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the behaviour of solutions to elliptic boundary-value problems in the neighbourhood of irregular boundary points (see [5, 7, 13-15]). The special structure of the solution near a conical point is of particular interest for physical applications ([2, 8, 11]). It can be used also to improve numerical algorithms ([1, 4, 12]). However, the behaviour of solutions near conical boundary points is treated only in special cases (see [5, 13-17]).

Let $\Omega = G \cap S^{n-1}$ be a domain on the unit sphere with smooth boundary $\partial\Omega$. For $x \in \mathbb{R}^n$ we denote the spherical coordinates by $(r, \omega) = (r, \omega_1, \dots, \omega_{n-1})$ with $r = |x|$, $\omega \in \Omega$. We also set $G_0^d = G \cap \{|x| < d\}$ and $\Omega_d = G \cap \{|x| = d\}$ for all $d > 0$.

Let $L_p(G)$ and $W^{k,p}(G), p > 1$ be the usual Lebesgue's and Sobolev's spaces. $W_0^{k,p}(G)$ denotes the space of functions in $W^{k,p}(G)$ that vanish on ∂G in the sense of traces. For k a non-negative integer and τ a real number, we define the space $V_{p,\tau}^k(G)$ as the closure of $C_0^\infty(\overline{G} \setminus \{O\})$ with respect to the norm

$$\|u\|_{V_{p,\tau}^k(G)} = \Big(\int_G \sum_{|\beta|=0}^{k} r^{p(|\beta|-k)+\tau} |D^\beta u|^p dx \Big)^{1/p}.$$

We will denote by $\mathfrak{N}_{m,\tau,q}^1(G)$ the set of functions $u(x) \in V_{m,\tau}^1(G) \cap L_\infty(G)$ such that

$$\int_G \frac{|x|^\tau |u|^{q+m}}{(x_{n-1}^2 + x_n^2)^{m/2}} dx < \infty, \quad q \geq 0,\ \tau > m-n,\ 1 < m \leq n$$

(i.e. the integral is finite).

Through this paper we assume that $f_0(x), f_1(x), \dots, f_n(x)$ are measurable functions such that

$$f_0(x) \in L_p(G), \quad |x|^{-\tau/m} f_i(x) \in L_{\frac{mp}{m-1}}(G), \quad (i = 1, \dots, n), \tag{0.4}$$

where

$$\frac{1}{p} < \frac{m}{n} - \frac{1}{t}, \quad \frac{1}{t} < \frac{m}{n} < 1 + \frac{1}{t} < m, \quad m-n < \tau < \min(m-1; \frac{n}{t}). \tag{0.5}$$

We also set $|f| = (\sum_{i=1}^n f_i^2)^{1/2}$ and $(|u|-k)_+ = \max(|u|-k; 0)$.

1. Boundedness of weak solutions

Definition. A function $u(x)$ is called a weak solution of (0.1)-(0.2), if $u(x)$ belongs to $\mathfrak{N}_{m,\tau,q}^1(G)$ and for all $\phi(x) \in \mathfrak{N}_{m,\tau,q}^1(G)$, it satisfies

$$\int_G \{|x|^\tau |u|^q |\nabla u|^{m-2} u_{x_i} \phi_{x_i} + \frac{a_0 |x|^\tau}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}} u|u|^{q+m-2}\phi - \mu |x|^\tau u|u|^{q-2} |\nabla u|^m \phi\} dx = \int_G \{f_0(x)\phi + f_i \phi_{x_i}\} dx\,.$$

Our goal in this section is to obtain an $L_\infty(G)$-a priori estimate of weak solutions of (0.1)-(0.2). For this end, we need the following statements ([3, Lemma 2.1]).

Lemma 1.1. *Let $\varkappa > 0$, and*

$$\eta(x) = \begin{cases} e^{\varkappa x} - 1 & x \geq 0 \\ -e^{-\varkappa x} + 1 & x < 0 \,. \end{cases}$$

Let a, b be positive constants, and $m > 1$. If $\varkappa > (2b/a) + m$, then

$$a\eta'(x) - b|\eta(x)| \geq \frac{a}{2} e^{\ \varkappa x} \quad \forall x \geq 0 \,, \tag{1.1}$$

$$\eta(x) \geq [\eta(\frac{x}{m})]^m \quad \forall x \geq 0 \,. \tag{1.2}$$

Moreover, there exist $d \geq 0$ and $M > 0$ such that

$$\eta(x) \leq M[\eta(\frac{x}{m})]^m, \quad \eta'(x) \leq M[\eta(\frac{x}{m})]^m \quad \forall x \geq d \,, \tag{1.3}$$

$$|\eta(x)| \geq x, \quad \forall x \in \mathbb{R}. \tag{1.4}$$

The following statement is due to Stampacchia ([9, Lemma B.1]).

Lemma 1.2. *Let α, β, γ, k_0 be real positive numbers, $\gamma > 1$, and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function such that*

$$\psi(l) \leq \frac{\alpha}{(l-k)^\beta}[\psi(k)]^\gamma, \quad \forall l > k \geq k_0.$$

Then $\psi(k_0 + \delta) = 0$, where $\delta^\beta = \alpha\psi(k_0)^{\gamma-1}2^{\beta(\frac{\gamma}{\gamma-1})}$.

We will also use the following properties.

Lemma 1.3. *([6, Examples 1.5, 1.6, p. 29) Let $m^\#$ denote the number associated to m by*

$$\frac{1}{m^\#} = \frac{1}{m}(1 + \frac{1}{t}) - \frac{1}{n} \tag{1.5}$$

and assume that (0.5) hold. Then there exist constants $c_1 > 0$, $c_2 > 0$ (depending only on $\operatorname{meas} G, n, m, t, \tau$) such that

$$\int_G |x|^{\tau-m}|u|^m dx \leq c_1 \int_G |x|^\tau |\nabla u|^m dx, \tag{1.6}$$

$$(\int_G |u|^{m^\#} dx)^{m/m^\#} \leq c_2 \int_G (|x|^{\tau-m}|u|^m + |x|^\tau|\nabla u|^m) dx \tag{1.7}$$

for any $u(x) \in V^1_{m,\tau}(G)$.

Our main statement in this section is as follows.

Theorem 1.4. *Let $u(x)$ be a weak solution of (0.1)-(0.5). Then there exists a constant $M_0 > 0$ depending only on $\operatorname{meas} G$, n, m, τ, μ, q, a_0, $\|f_0(x)\|_{L_p(G)}$, and $\||x|^{-\tau/m}|f(x)|\|_{L_{\frac{mp}{m-1}}(G)}$ such that $||u||_{L^\infty(G)} \leq M_0$.*

Proof. We shall follow the proof of in [3, Theorem 3.1]. Let $A(k) = \{x \in G : |u(x)| > k\}$ and $\chi_{A(k)}$ be the characteristic function for the set $A(k)$. We remark that $A(k+d) \subseteq A(k)$ for all $d > 0$. By setting $\phi(x) = \eta((|u| - k)_+)\chi_{A(k)} \operatorname{sgn} u$ in the definition of weak solution, with η defined by Lemma 1 and $k \geq k_0$ (without less of generality we can assume $k_0 > 1$), we get the inequality

$$\begin{aligned}\int_{A(k)} |x|^\tau |u|^q |\nabla u|^m \eta'((|u|-k)_+) + a_0 \int_{A(k)} \frac{|x|^\tau}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}} |u|^{q+m-1} \eta((|u|-k)_+) \leq \\ \leq \int_{A(k)} |f_0||\eta((|u|-k)_+)| + \mu \int_{A(k)} |x|^\tau |\nabla u|^m |u|^{q-1} |\eta((|u|-k)_+)| + \\ + \int_{A(k)} |f||\nabla u||\eta'((|u|-k)_+)|. \quad (1.8)\end{aligned}$$

By Young's inequality

$$\begin{aligned}|f||\nabla u| = (|x|^{\tau/m}|u|^{q/m}|\nabla u|)(|x|^{-\tau/m}|u|^{-q/m}|f|) \leq \\ \leq \frac{1}{m}|x|^\tau |u|^q |\nabla u|^m + \frac{m-1}{m}|x|^{\frac{-\tau}{m-1}}|u|^{\frac{-q}{m-1}}|f|^{\frac{m}{m-1}}.\end{aligned}$$

Substituting this expression in (1.8), we have

$$\begin{aligned}\int_{A(k)} |x|^\tau |\nabla u|^m [\frac{m-1}{m}|u|^q \eta' - \mu |u|^{q-1}|\eta|] + a_0 \int_{A(k)} \frac{|x|^\tau}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}} |u|^{q+m-1} |\eta| \leq \\ \leq \int_{A(k)} |f_0||\eta| + \frac{m-1}{m} \int_{A(k)} |x|^{\frac{-\tau}{m-1}}|u|^{\frac{-q}{m-1}}|f|^{\frac{m}{m-1}}|\eta'|.\end{aligned}$$

Because $|u| > k$ on $A(k)$, we have

$$\begin{aligned}\int_{A(k)} |x|^\tau |u|^q |\nabla u|^m [\frac{m-1}{m}\eta' - \mu|\eta|] + a_0 \int_{A(k)} \frac{|x|^\tau}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}} |u|^{q+m-1} |\eta| \leq \\ \leq \int_{A(k)} |f_0||\eta| + \frac{m-1}{m} \int_{A(k)} |x|^{\frac{-\tau}{m-1}}|u|^{\frac{-q}{m-1}}|f|^{\frac{m}{m-1}}|\eta'|.\end{aligned}$$

Transforming the first integral on the left-hand side with the respect to (1.1), choosing $\varkappa$ appropriately by setting in Lemma 1.1 $a = \frac{m-1}{m}$, $b = \mu$, i.e. $\varkappa > \frac{2m\mu}{m-1} + m$, we obtain

$$\begin{aligned}\frac{m-1}{2m} k_0^q \int_{A(k)} |x|^\tau |\nabla u|^m \mathrm{e}^{\,\varkappa(|u|-k)_+} + a_0 \int_{A(k)} \frac{|x|^\tau}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}} |u|^{q+m-1} |\eta| \leq \\ \leq \int_{A(k)} |f_0||\eta| + \frac{m-1}{m} k_0^{\frac{-q}{m-1}} \int_{A(k)} |x|^{\frac{-\tau}{m-1}}|f|^{\frac{m}{m-1}}|\eta'|. \quad (1.9)\end{aligned}$$

Setting $w_k(x) = \eta(\frac{(|u|-k)_+}{m})$,

$$\mathrm{e}^{\,\varkappa(|u|-k)_+}|\nabla u|^m = (\mathrm{e}^{\,\frac{\varkappa(|u|-k)_+}{m}}|\nabla u|)^m = (\frac{m}{\varkappa})^m |\nabla w_k|^m.$$

Now, we can rewrite (1.9) with (1.2) and (1.3) as

$$\begin{aligned}\frac{m-1}{2m}(\frac{m}{\varkappa})^m k_0^q \int_{A(k)} |x|^\tau |\nabla w_k|^m + a_0 \int_{A(k)} \frac{|x|^\tau}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}} |u|^{q+m-1} |w_k|^m \leq \\ \leq M \int_{A(k+d)} h(x)|w_k|^m + c_3 \mathrm{e}^{\,\varkappa d} \int_{A(k)\setminus A(k+d)} h(x), \quad (1.10)\end{aligned}$$

where

$$h(x)=|f_0(x)|+|x|^{\frac{-\tau}{m-1}}|f|^{\frac{m}{m-1}}\,. \tag{1.11}$$

By (0.4)-(0.5) we have that $h(x)\in L_p(G)$, where p is such that $\frac{1}{p}<\frac{m}{n}-\frac{1}{t}$. Using Hölder's inequality with exponents p and p', we obtain

$$\int_{A(k+d)} h|w_k|^m \le \|h(x)\|_{L_p(G)}\Big(\int_{A(k+d)}|w_k|^{mp'}\Big)^{1/p'}. \tag{1.12}$$

From $\frac{1}{p}<\frac{m}{n}-\frac{1}{t}$ it follows that $mp'<m^{\#}$, where $m^{\#}$ is defined by (1.5). Let s be a real number such that $mp'<s<m^{\#}$. Using the interpolation inequality

$$\Big(\int_{A(k+d)}|w_k|^{mp'}\Big)^{1/p'}\le\Big(\int_{A(k)}|w_k|^m\Big)^{\theta}\Big(\int_{A(k)}|w_k|^s\Big)^{(1-\theta)m/s}$$

with $\theta\in(0,1)$ which is defined by $\frac{1}{mp'}=\frac{\theta}{m}+\frac{1-\theta}{s}$, by Hölder's inequality with exponents $\frac{m^{\#}}{s}$ and $\frac{m^{\#}}{m^{\#}-s}$ from (1.12) we get

$$\int_{A(k+d)} h|w_k|^m\le c_4\Big(\int_{A(k)}|w_k|^m\Big)^{\theta}\Big(\int_{A(k)}|w_k|^{m^{\#}}\Big)^{(1-\theta)m/m^{\#}}, \tag{1.13}$$

where, $c_4=\|h(x)\|_{L_p(G)}(\operatorname{meas}G)^{m(m^{\#}-s)(1-\theta)/sm^{\#}}$.

Now, using Young's inequality with exponents $1/\theta$ and $1/(1-\theta)$, from (1.13) we obtain

$$\int_{A(k+d)} h|w_k|^m\le\frac{c_5}{\varepsilon^{1/\theta}}\int_{A(k)}|w_k|^m+\varepsilon^{\frac{1}{(1-\theta)}}(1-\theta)\Big(\int_{A(k)}|w_k|^{m^{\#}}\Big)^{\frac{m}{m^{\#}}},\quad\forall\varepsilon>0\,, \tag{1.14}$$

where $c_5=\theta\|h(x)\|^{\frac{1}{\theta}}_{L_p(G)}(\operatorname{meas}G)^{m(m^{\#}-s)(1-\theta)/\theta sm^{\#}}$. Returning to (1.10), by (1.7) from Lemma 1.3 and estimate (1.14) we obtain

$$\begin{aligned}\frac{m-1}{2mc_2}\Big(\frac{m}{\varkappa}\Big)^m k_0^q\Big(\int_{A(k)}|w_k|^{m^{\#}}\Big)^{\frac{m}{m^{\#}}}+a_0\int_{A(k)}\frac{|x|^{\tau}}{(x_{n-1}^2+x_n^2)^{\frac{m}{2}}}|u|^{q+m-1}|w_k|^m\le\\ \le\frac{c_5M}{\varepsilon^{1/\theta}}\int_{A(k)}|w_k|^m+M\varepsilon^{\frac{1}{(1-\theta)}}(1-\theta)\Big(\int_{A(k)}|w_k|^{m^{\#}}\Big)^{\frac{m}{m^{\#}}}+c_3\mathrm{e}^{\ \varkappa d}\int_{A(k)}h,\quad\forall\varepsilon>0.\end{aligned} \tag{1.15}$$

Now we consider two cases: $a_0>0$ and $a_0=0$.

Case 1): $a_0>0$. For this case by (0.5) and $m>\tau$, we have

$$\begin{aligned}\int_{A(k)}|w_k|^m&=\int_{A(k)}\Big[\frac{(x_{n-1}^2+x_n^2)^{\frac{m}{2}}}{|x|^{\tau}}|u|^{-(q+m-1)}\Big]\frac{|x|^{\tau}}{(x_{n-1}^2+x_n^2)^{\frac{m}{2}}}|u|^{q+m-1}|w_k|^m\le\\ &\le k_0^{-(q+m-1)}\int_{A(k)}(|x|^{m-\tau})\frac{|x|^{\tau}}{(x_{n-1}^2+x_n^2)^{\frac{m}{2}}}|u|^{q+m-1}|w_k|^m\le\\ &\le\frac{(\operatorname{diam}G)^{m-\tau}}{k_0^{q+m-1}}\int_{A(k)}\frac{|x|^{\tau}}{(x_{n-1}^2+x_n^2)^{\frac{m}{2}}}|u|^{q+m-1}|w_k|^m,\quad\forall k\ge k_0.\end{aligned}$$

Therefore, from (1.15) we obtain

$$
\begin{aligned}
c_6 k_0^q \Big(\int_{A(k)} |w_k|^{m^\#}\Big)^{\frac{m}{m^\#}} + a_0 \int_{A(k)} \frac{|x|^\tau}{(x_{n-1}^2+x_n^2)^{\frac{m}{2}}} |u|^{q+m-1}|w_k|^m \le \\
\le \frac{c_7 M}{\varepsilon^{1/\theta} k_0^{q+m-1}} \int_{A(k)} \frac{|x|^\tau}{(x_{n-1}^2+x_n^2)^{\frac{m}{2}}} |u|^{q+m-1}|w_k|^m + c_3 \mathrm{e}^{\varkappa d}\int_{A(k)} h + \\
+ M\varepsilon^{\frac{1}{(1-\theta)}}(1-\theta)\Big(\int_{A(k)} |w_k|^{m^\#}\Big)^{m/m^\#}. \quad (1.15_1)
\end{aligned}
$$

Now, we can choose $\varepsilon > 0$ and $k_0 > 0$ such that

$$
k_0^{q+m-1} = \frac{c_7 M}{a_0 \varepsilon^{1/\theta}}, \quad M\varepsilon^{\frac{1}{(1-\theta)}}(1-\theta) \le \frac{c_6 k_0^q}{2};
$$

i.e.,

$$
k_0^{\frac{q+(m-1)\theta}{(1-\theta)}} \ge \frac{2(M)^{\frac{1}{1-\theta}}(1-\theta)}{c_6}\Big(\frac{c_7}{a_0}\Big)^{\frac{\theta}{1-\theta}}; \quad \varepsilon = \Big(\frac{Mc_7}{a_0 k_0^{q+m-1}}\Big)^\theta.
$$

Thus from (1.15_1) for every $k > k_0$ it results

$$
\Big(\int_{A(k)} |w_k|^{m^\#}\Big)^{\frac{m}{m^\#}} \le c_8 \int_{A(k)} h \le c_8 \|h(x)\|_{L_p(G)} \operatorname{meas}^{1-\frac{1}{p}} A(k). \quad (1.16)
$$

Case 2): $a_0 = 0$. In this case from (1.10) by (1.14) we have

$$
\begin{aligned}
k_0^q \int_{A(k)} |x|^\tau |\nabla w_k|^m \le c_9 \int_{A(k+d)} h(x)|w_k|^m + c_{10}\int_{A(k)} h(x) \le \\
\le \frac{c_5 c_9}{\varepsilon^{1/\theta}} \int_{A(k)} |w_k|^m + c_9 \varepsilon^{\frac{1}{(1-\theta)}}(1-\theta)\Big(\int_{A(k)} |w_k|^{m^\#}\Big)^{\frac{m}{m^\#}} + c_{10}\int_{A(k)} h, \quad \forall \varepsilon > 0.
\end{aligned}
$$

Further, by (1.6) and the fact that $m > \tau$, by (0.5) we obtain

$$
\begin{aligned}
\int_{A(k)} |w_k|^m = \int_{A(k)} (|x|^{\tau-m}|w_k|^m)|x|^{m-\tau} \le (\operatorname{diam} G)^{m-\tau}\int_{A(k)} |x|^{\tau-m}|w_k|^m \le \\
\le c_1 (\operatorname{diam} G)^{m-\tau}\int_{A(k)} |x|^\tau |\nabla w_k|^m.
\end{aligned}
$$

From the last two inequalities, for all $\varepsilon > 0$ we have

$$
\begin{aligned}
k_0^q \int_{A(k)} |x|^\tau |\nabla w_k|^m \le c_{11}\varepsilon^{-\frac{1}{\theta}} \int_{A(k)} |x|^\tau |\nabla w_k|^m + \\
+ c_9 \varepsilon^{\frac{1}{(1-\theta)}}(1-\theta)\Big(\int_{A(k)} |w_k|^{m^\#}\Big)^{\frac{m}{m^\#}} + c_{10}\int_{A(k)} h, \quad (1.15_2)
\end{aligned}
$$

Now we set $c_{11}\varepsilon^{-\frac{1}{\theta}} = \frac{1}{2}k_0^q$. Then by virtue of (1.7) for $\forall k > k_0$ we have

$$
\frac{1}{2c_2} k_0^q \Big(\int_{A(k)} |w_k|^{m^\#}\Big)^{\frac{m}{m^\#}} \le c_9 \varepsilon^{\frac{1}{(1-\theta)}}(1-\theta)\Big(\int_{A(k)} |w_k|^{m^\#}\Big)^{\frac{m}{m^\#}} + c_{10}\int_{A(k)} h.
$$

If we choose $c_9\varepsilon^{1/(1-\theta)}(1-\theta) \le \frac{1}{4c_2}k_0^q$, then we get again (1.16). In this case we choose

$$k_0 \ge (4c_2c_9(1-\theta))^{\frac{1-\theta}{q}}(2c_{11})^{\frac{\theta}{q}}; \quad \varepsilon = (\frac{2c_{11}}{k_0^q})^{\theta}.$$

Now we return to (1.16). Let us now take $l > k > k_0$. Due to (1.4) and $|w_k| > \frac{1}{m}(|u|-k)_+$, we have

$$\int_{A(l)} |w_k|^{m^\#} \ge (\frac{l-k}{m})^{m^\#} \operatorname{meas} A(l)\,. \tag{1.17}$$

Combining (1.16) and (1.17), we obtain

$$\operatorname{meas} A(l) \le (\frac{m}{l-k})^{m^\#}(c_8\|h(x)\|_{L_p(G)})^{\frac{m^\#}{m}} \operatorname{meas}^{\frac{m^\#}{m}(1-\frac{1}{p})} A(k), \quad \forall l > k \ge k_0. \tag{1.18}$$

Notice that from (0.5) and (1.5) it follows that $\frac{m^\#}{m}(1-\frac{1}{p}) > 1$.

Lemma 1.2 implies that $\operatorname{meas} A(k_0+\delta) = 0$, where δ depends only on the constants in (1.18). This means that $|u(x)| < k_0 + \delta$ for a.e. $x \in G$. Theorem 1.4 is proved.

2. Hölder continuity of the weak solution.

In this section we prove, that the weak solution of (0.1)-(0.5) is Hölder continuous in a neighbourhood of a conical point.

Theorem 2.1. *Let $u(x)$ be a weak solution of (0.1)-(0.5) with $\tau = 0$, and let d be a positive number. Then in the domain G_0^d, $u(x)$ is Hölder continuous with exponent α depending only on the data assumptions and the domain G.*

Proof. Let M_0 be the number from Theorem 1.4. Let us introduce the cut-off function $\zeta(x) \in C_0^\infty(G_0^\rho)$, $\rho \in (0,d)$:

$$\zeta(x) = \begin{cases} 1, & x \in G_0^{\rho-\sigma\rho}, \quad \sigma \in (0,1) \\ 0, & x \notin G_0^\rho; \end{cases}$$

$$0 < \zeta(x) < 1, \quad x \in G_0^\rho; \quad |\nabla\zeta| \le 1/(\sigma\rho).$$

Let us define the set $A_{k,\rho} := \{x \in G_0^\rho \mid u(x) > k\}$, for all $k \in \mathbb{R}$. Substituting $\phi(x) = \zeta^m(x)\max\{u(x)-k, 0\}$ in the definition of weak solution, we get

$$\int_{A_{k,\rho}} |u|^q\zeta^m|\nabla u|^m + m\int_{A_{k,\rho}} \zeta^{m-1}(x)\zeta_{x_i}u_{x_i}|\nabla u|^{m-2}|u|^q(u-k)+$$
$$+ a_0\int_{A_{k,\rho}} (x_{n-1}^2+x_n^2)^{-\frac{m}{2}}u|u|^{q+m-2}\zeta^m(u-k) = \mu\int_{A_{k,\rho}} |u|^{q-2}u|\nabla u|^m\zeta^m(u-k)+$$
$$+\int_{A_{k,\rho}} f_0\zeta^m(u-k) + m\int_{A_{k,\rho}} \zeta^{m-1}(u-k)f_i\zeta_{x_i} + \int_{A_{k,\rho}} f_iu_{x_i}\zeta^m. \tag{2.1}$$

Using Young's inequality with exponents $m/(m-1)$ and m in the second integral of the right-hand side,

$$|m\zeta^{m-1}\zeta_{x_i}u_{x_i}(u-k)|\nabla u|^{m-2}|u|^q| \le m(|\nabla u|\zeta)^{m-1}((u-k)|\nabla\zeta|)|u|^q \le$$
$$\le (m-1)\varepsilon(|\nabla u|\zeta)^m|u|^q + \frac{1}{m}\varepsilon^{1-m}(u-k)^m|\nabla\zeta|^m|u|^q, \quad \forall\varepsilon > 0.$$

Choosing $\varepsilon = \frac{1}{2(m-1)}$, we obtain

$$\begin{aligned}\frac{1}{2}\int_{A_{k,\rho}} |u|^q\zeta^m|\nabla u|^m + a_0\int_{A_{k,\rho}} (x_{n-1}^2+x_n^2)^{-\frac{m}{2}}u|u|^{q+m-2}\zeta^m(u-k) \leq \\ \leq C(m)\int_{A_{k,\rho}} (u-k)^m|\nabla\zeta|^m|u|^q + \mu\int_{A_{k,\rho}} |u|^{q-1}|\nabla u|^m\zeta^m(u-k)+ \\ +\int_{A_{k,\rho}} f_0\zeta^m(u-k) + m\int_{A_{k,\rho}} \zeta^{m-1}(u-k)f_i\zeta_{x_i} + \int_{A_{k,\rho}} f_iu_{x_i}\zeta^m. \quad (2.2)\end{aligned}$$

Now, we choose $k_0 \geq \frac{1}{2}M_0$, if $\mu = 0$, $\max_{G_0^\rho}(u(x)-k_0) \leq \frac{1}{4\mu}$, if $\mu > 0$, and without lost of generality $k_0 > 1$. Then for every $k \geq k_0$,

$$\mu|u|^{q-1}|\nabla u|^m\zeta^m(u-k)\Big|_{A_{k,\rho}} \leq \mu|u|^{q-1}|\nabla u|^m\zeta^m\max_{G_0^\rho}(u(x)-k) \leq \frac{1}{4}|u|^q\zeta^m|\nabla u|^m. \quad (2.3)$$

Substituting (2.3) in (2.2), we have

$$\begin{aligned}\frac{1}{4}\int_{A_{k,\rho}} |u|^q\zeta^m|\nabla u|^m + a_0\int_{A_{k,\rho}} (x_{n-1}^2+x_n^2)^{-\frac{m}{2}}u|u|^{q+m-2}\zeta^m(u-k) \leq \\ \leq C(m)\int_{A_{k,\rho}} (u-k)^m|\nabla\zeta|^m|u|^q + \int_{A_{k,\rho}} f_0\zeta^m(u-k)+ \\ + m\int_{A_{k,\rho}} \zeta^{m-1}(u-k)f_i\zeta_{x_i} + \int_{A_{k,\rho}} f_iu_{x_i}\zeta^m. \quad (2.4)\end{aligned}$$

Let us now estimate $f_iu_{x_i}$ on $A_{k,\rho}$:

$$f_iu_{x_i} \leq |\nabla u||f| \leq \varepsilon|\nabla u|^m + c(\varepsilon)|f|^{\frac{m}{m-1}} \leq \varepsilon|u|^q|\nabla u|^m + c(\varepsilon)|f|^{\frac{m}{m-1}}, \quad \forall\varepsilon > 0. \quad (2.5)$$

From (2.4) and (2.5) with $\varepsilon = 1/8$ and taking into account that $(u-k) \leq \frac{1}{4\mu}$, if $\mu > 0$, and $(u-k) \leq \frac{1}{2}M_0$, if $\mu = 0$, we get:

$$\begin{aligned}\frac{1}{8}\int_{A_{k,\rho}} |u|^q\zeta^m|\nabla u|^m + a_0\int_{A_{k,\rho}} (x_{n-1}^2+x_n^2)^{-\frac{m}{2}}u|u|^{q+m-2}\zeta^m(u-k) \leq \\ \leq C(m)\int_{A_{k,\rho}} (u-k)^m|\nabla\zeta|^m|u|^q + C(m,M_0,\mu)\int_{A_{k,\rho}} \{|f_0| + |f|^{\frac{m}{m-1}}\}\zeta^m+ \\ + m\int_{A_{k,\rho}} \zeta^{m-1}(u-k)f_i\zeta_{x_i}.\end{aligned}$$

Now using Young's inequality on the last integral,

$$m((u-k)\zeta_{x_i})(f_i\zeta^{m-1}) \leq (u-k)^m|\nabla\zeta|^m + (m-1)|f|^{\frac{m}{m-1}}\zeta^m.$$

It results that

$$\begin{aligned}\frac{1}{8}\int_{A_{k,\rho}} |u|^q\zeta^m|\nabla u|^m + a_0\int_{A_{k,\rho}} (x_{n-1}^2+x_n^2)^{-\frac{m}{2}}u|u|^{q+m-2}\zeta^m(u-k) \leq \\ \leq C(m)\int_{A_{k,\rho}} (u-k)^m|\nabla\zeta|^m|u|^q + C(m,M_0,\mu)\int_{A_{k,\rho}} h\zeta^m,\end{aligned}$$

where we have set $h(x) = |f_0| + |f|^{\frac{m}{m-1}}$ (see (1.11) with $\tau = 0$). We shall strengthen the inequality, if we use the estimate

$$\int_{A_{k,\rho-\sigma\rho}} |u|^q |\nabla u|^m \leq$$
$$\leq C(m) \max_{A_{k,\rho}} |u|^q \int_{A_{k,\rho}} (u-k)^m |\nabla\zeta|^m + C(m, M_0, \mu) \max_{A_{k,\rho}} |u|^q \int_{A_{k,\rho}} h \quad (2.6)$$

From the definition of $\zeta(x)$ it follows that

$$\int_{A_{k,\rho}} (u-k)^m |\nabla\zeta|^m \leq (\sigma\rho)^{-m} \max_{A_{k,\rho}} (u(x)-k_0)^m \operatorname{meas} A_{k,\rho} =$$
$$= (\sigma\rho)^{-m} \max_{A_{k,\rho}} (u(x)-k_0)^m \operatorname{meas}^{\frac{m}{s}} A_{k,\rho} \operatorname{meas}^{1-\frac{m}{s}} A_{k,\rho}, \quad s > n. \quad (2.7)$$

Since $\operatorname{meas} A_{k,\rho} \leq \operatorname{meas} G_0^\rho = \int_{G_0^\rho} dx = \operatorname{meas}\Omega \int_0^\rho r^{n-1} dr = \frac{\rho^n}{n} \operatorname{meas}\Omega$, we can rewrite (2.7) as

$$\int_{A_{k,\rho}} (u-k)^m |\nabla\zeta|^m \leq (\frac{1}{n} \operatorname{meas}\Omega)^{\frac{m}{s}} \sigma^{-m} \rho^{-m(1-\frac{n}{s})} \max_{A_{k,\rho}} (u(x)-k_0)^m \operatorname{meas}^{1-\frac{m}{s}} A_{k,\rho}, \quad (2.8)$$

for $s > n$. Combining (2.6) and (2.8),

$$\int_{A_{k,\rho-\sigma\rho}} |u|^q |\nabla u|^m \leq$$
$$\max_{A_{k,\rho}} |u|^q [\gamma\sigma^{-m} \rho^{-m(1-\frac{n}{s})} \max_{A_{k,\rho}} (u(x)-k)^m \operatorname{meas}^{1-\frac{m}{s}} A_{k,\rho} + C(m, M_0, \mu) \int_{A_{k,\rho}} h], \quad (2.9)$$

where $\gamma = C(m)(\frac{1}{n} \operatorname{meas}\Omega)^{\frac{m}{s}}$, $s > n > m$. By Hölder's inequality,

$$\int_{A_{k,\rho}} h \leq (\int_{A_{k,\rho}} h^{\frac{s}{m}})^{\frac{m}{s}} \operatorname{meas}^{1-\frac{m}{s}} A_{k,\rho} \leq ||h||_{L_{\frac{s}{m}}(G)} \operatorname{meas}^{1-\frac{m}{s}} A_{k,\rho} \leq$$
$$\leq \operatorname{meas}^{\frac{m}{s}-\frac{1}{p}} G ||h||_{L_p(G)} \operatorname{meas}^{1-\frac{m}{s}} A_{k,\rho}, \quad s = \frac{mnt}{mt-1}, \quad (2.10)$$

where p and t is defined by (0.5). From (2.10) and (2.9) it follows that

$$\int_{A_{k,\rho-\sigma\rho}} |u|^q |\nabla u|^m \leq \max_{A_{k,\rho}} |u|^q [\gamma\sigma^{-m} \rho^{-m(1-\frac{n}{s})} \max_{A_{k,\rho}} (u(x)-k)^m + \gamma_1] \operatorname{meas}^{1-\frac{m}{s}} A_{k,\rho}, \quad (2.11)$$

where $\gamma_1 = C(m, M_0, \mu) \operatorname{meas}^{\frac{m}{s}-\frac{1}{p}} G(\|f_0\|_{L_p(G)} + \|f\|_{L_{\frac{mp}{m-1}}(G)})$. This proves that $u(x)$ belongs to the class $\tilde{\mathfrak{B}}_m^1(\overline{G_0^d}, |u|^q, M_0, \gamma, \gamma_1, \frac{1}{s})$ (see [10, §9 chapt. II]), where γ, γ_1 are defined by (2.9), (2.11), and also by [10, Theorem 9.1 chapt. II] $u(x)$ is Hölder continuous with exponent $\alpha > 0$, depending only on μ, q, m, n and domain G. This finishes the proof of Theorem 2.1.

CONSTRUCTION OF THE BARRIER FUNCTION

In this section we construct the barrier function for the homogeneous Dirichlet problem (0.1)-(0.2) with $\tau = 0$ in n-dimensional *infinite convex* circular cone G_0 with the vertex at the origin of coordinates O and its lateral area Γ_0.

$$\mathfrak{L}u \equiv \frac{d}{dx_i}(|u|^q|\nabla u|^{m-2}u_{x_i}) = \frac{a_0}{(x_{n-1}^2+x_n^2)^{\frac{m}{2}}}u|u|^{q+m-2} - \mu u|u|^{q-2}|\nabla u|^m,\ x \in G_0, \tag{3.1}$$

$$u(x) = 0, \quad x \in \Gamma_0 \tag{3.2}$$

$$a_0 \geq 0, \quad 0 \leq \mu < 1, \quad q \geq 0, \quad 2 \leq m \leq n. \tag{3.3}$$

Let us transfer to the spherical coordinates with the pole at the point O.

$$\begin{gathered} x_1 = r\cos\omega_1, \quad x_2 = r\cos\omega_2\sin\omega_1, \\ \ddots \\ x_{n-1} = r\cos\omega_{n-1}\sin\omega_{n-2}\dots\sin\omega_1, \\ x_n = r\sin\omega_{n-1}\dots\sin\omega_1\,, \end{gathered} \tag{3.4}$$

where $0 \leq r = |x| < \infty$, $0 \leq \omega_k \leq \pi$, $k \leq n-2$, $-\frac{\omega_0}{2} \leq \omega_{n-1} \leq \frac{\omega_0}{2}$, $\omega_0 \in (0,\pi)$.

The lateral area Γ_0 may be obtained as the surface of revolution of the angle $x_{n-1} = x_n \cot\frac{\omega_0}{2}$, $0 < \omega_0 < \pi$, around the axis Ox_{n-1}.

$$\Gamma_0 = \big\{x \in \mathbb{R}^n \big|\ x_{n-1}^2 = \big(x_n^2 + \sum_{k=1}^{n-2} x_k^2\big)\cot^2\frac{\omega_0}{2}\big\}. \tag{3.5}$$

Thus we get

$$G_0 = \big\{x \in \mathbb{R}^n \big|\ x_{n-1}^2 > \big(x_n^2 + \sum_{k=1}^{n-2} x_k^2\big)\cot^2\frac{\omega_0}{2}\big\}. \tag{3.6}$$

Lemma 3.1. *Let $q_{n-1} = (\sin\omega_1\dots\sin\omega_{n-2})^2$. Then*

$$q_{n-1}\big|_{\Gamma_0} = 1, \qquad q_{n-1}\big|_{G_0} > \cos^2\frac{\omega_0}{2}.$$

Proof. From (3.4) we have

$$\begin{gathered} x_{n-1}^2 = r^2 q_{n-1}\cos^2\omega, \quad x_n^2 = r^2 q_{n-1}\sin^2\omega \quad \forall\omega \in [-\frac{\omega_0}{2}, \frac{\omega_0}{2}]; \\ x_{n-1}^2 + x_n^2 = r^2 q_{n-1}, \\ \sum_{k=1}^{n-2} x_k^2 = |x|^2 - (x_{n-1}^2 + x_n^2) = r^2(1-q_{n-1}). \end{gathered}$$

Now by (3.5) on Γ_0 we have

$$r^2 q_{n-1}\cos^2\omega = r^2(1 - q_{n-1} + q_{n-1}\sin^2\omega)\cot^2\frac{\omega_0}{2} = r^2(1 - q_{n-1}\cos^2\omega)\cot^2\frac{\omega_0}{2},$$

whence

$$q_{n-1}\frac{\cos^2\omega}{\sin^2\frac{\omega_0}{2}}=\frac{\cos^2\frac{\omega_0}{2}}{\sin^2\frac{\omega_0}{2}}$$

and since $\omega\big|_{\Gamma_0}=\pm\frac{\omega_0}{2}$, we get first assertion of lemma. Similarly by (3.6) in G_0,

$$r^2q_{n-1}\cos^2\omega>r^2(1-q_{n-1}\cos^2\omega)\cot^2\frac{\omega_0}{2},$$

whence $q_{n-1}\cos^2\omega>\cos^2\frac{\omega_0}{2}$ for all $\omega\in(-\frac{\omega_0}{2},\frac{\omega_0}{2})$. This means that second assertion of lemma holds.

We shall seek the solution to Problem (3.1)-(3.2) of the form

$$u=r^\lambda(\sin\omega_1\dots\sin\omega_{n-2})^\lambda\Phi(\omega_{n-1}),\quad \lambda>0 \tag{3.7}$$

with $\Phi(\omega)\ge 0$ and λ, satisfying the following condition

$$\lambda^m(q+m-1+\mu)+\lambda^{m-1}(2-m)>a_0. \tag{*}$$

The differential operator $\mathfrak{L}$ becomes

$$\mathfrak{L}u=\frac{1}{J}\sum_{i=1}^n\frac{d}{d\xi_i}(|u|^q|\nabla u|^{m-2}\frac{J}{H_i^2}\frac{\partial u}{\partial\xi_i}),$$

where

$$\begin{gathered}J=r^{n-1}\sin^{n-2}\omega_1\dots\sin\omega_{n-2};\\ H_1=1,\quad \xi_1=r,\quad \xi_{i+1}=\omega_i;\quad H_{i+1}=r\sqrt{q_i},\quad i=\{\overline{1,n-1}\};\\ q_1=1,\quad q_i=(\sin\omega_1\dots\sin\omega_{i-1})^2,\quad i=\{\overline{2,n-1}\}.\end{gathered}\tag{3.8}$$

Then $\Phi(\omega)$ satisfies the equation

$$\begin{aligned}&\frac{d}{d\omega_{n-1}}[(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}}|\Phi|^q\Phi']+\\ &\quad+\lambda q_{n-1}\Big\{\sum_{k=1}^{n-2}\frac{1}{q_k}\langle[\lambda(q+m-1)+n-m-k+1]\text{ctg }^2\omega_k-\frac{1}{\sin^2\omega_k}\rangle+\\ &\qquad+[\lambda(q+m-1)+n-m]\Big\}\Phi|\Phi|^q(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}}=\\ &\qquad\qquad=a_0\Phi|\Phi|^{q+m-2}-\mu\Phi|\Phi|^{q-2}(\lambda^2\Phi^2+\Phi'^2)^{\frac{m}{2}},\end{aligned}$$

where $\Phi=\Phi(\omega_{n-1})$ and $\omega_{n-1}\in(-\omega_0/2,\omega_0/2)$.

Let us show that

$$\begin{aligned}&\sum_{k=1}^{n-2}\frac{1}{q_k}\langle[\lambda(q+m-1)+n-m-k+1]\text{ctg }^2\omega_k-\frac{1}{\sin^2\omega_k}\rangle=\\ &\qquad=\frac{[\lambda(q+m-1)-m+2]}{(\sin\omega_1\dots\sin\omega_{n-2})^2}-[\lambda(q+m-1)-m+n]\end{aligned}\tag{3.9}$$

For this we use induction on n. In the case $n=3$, (3.9) is obvious. Now we suppose that (3.9) is true for $n-1$, that is

$$\sum_{k=1}^{n-3}\frac{1}{q_k}\langle[\lambda(q+m-1)+n-m-k]\operatorname{ctg}{}^2\omega_k-\frac{1}{\sin^2\omega_k}\rangle = \\ =\frac{[\lambda(q+m-1)-m+2]}{(\sin\omega_1\ldots\sin\omega_{n-3})^2}-[\lambda(q+m-1)-m+n-1]$$

It is easy to calculate that

$$\sum_{k=1}^{n-2}\frac{\operatorname{ctg}{}^2\omega_k}{q_k}=\frac{1}{q_{n-1}}-1. \tag{3.10}$$

Then we have

$$\begin{aligned}
&\sum_{k=1}^{n-2}\frac{1}{q_k}\langle[\lambda(q+m-1)+n-m-k+1]\operatorname{ctg}{}^2\omega_k-\frac{1}{\sin^2\omega_k}\rangle = \\
&=\sum_{k=1}^{n-3}\frac{1}{q_k}\langle[\lambda(q+m-1)+n-m-k+1]\operatorname{ctg}{}^2\omega_k-\frac{1}{\sin^2\omega_k}\rangle+ \\
&+\frac{1}{q_{n-2}}\langle[\lambda(q+m-1)+3-m]\operatorname{ctg}{}^2\omega_{n-2}-\frac{1}{\sin^2\omega_{n-2}}\rangle = \\
&=\sum_{k=1}^{n-2}\frac{\operatorname{ctg}{}^2\omega_k}{q_k}+\frac{1}{q_{n-2}}\langle[\lambda(q+m-1)+2-m]\operatorname{ctg}{}^2\omega_{n-2}-\frac{1}{\sin^2\omega_{n-2}}\rangle+ \\
&+\frac{\lambda(q+m-1)-m+2}{(\sin\omega_1\ldots\sin\omega_{n-3})^2}-[\lambda(q+m-1)-m+n-1] = \\
&=\langle\sum_{k=1}^{n-2}\frac{\operatorname{ctg}{}^2\omega_k}{q_k}-\frac{1}{q_{n-1}}\rangle-[\lambda(q+m-1)-m+n-1]+ \\
&+\frac{1}{q_{n-2}}[\lambda(q+m-1)-m+2](1+\operatorname{ctg}{}^2\omega_{n-2}) \underset{\text{by (3.10)}}{=} \\
&=\frac{1}{q_{n-1}}[\lambda(q+m-1)-m+2]-[\lambda(q+m-1)-m+n],
\end{aligned}$$

Q.E.D.

Now by (3.9), Problem (3.1)-(3.2) for $\Phi(\omega),\omega=\omega_{n-1}$ becomes

$$\frac{d}{d\omega}[(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}}|\Phi|^q\Phi']+\lambda[\lambda(q+m-1)-m+2]\Phi|\Phi|^q(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}} = \\ =a_0\Phi|\Phi|^{q+m-2}-\mu\Phi|\Phi|^{q-2}(\lambda^2\Phi^2+\Phi'^2)^{\frac{m}{2}},\quad \omega\in(-\omega_0/2,\omega_0/2), \tag{3.11}$$

$$\Phi(-\omega_0/2)=\Phi(\omega_0/2)=0. \tag{3.12}$$

By setting $\Phi'/\Phi=y$ we arrive at

$$[(m-1)y^2+\lambda^2](y^2+\lambda^2)^{\frac{m-4}{2}}y'+(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}}+ \\ +\lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}}=a_0,\quad \omega\in(-\omega_0/2,\omega_0/2), \tag{3.13}$$

$$y(0)=0,\quad \lim_{\omega\to\frac{\omega_0}{2}-0}y(\omega)=-\infty. \tag{3.14}$$

We explain (3.14). In fact, from (3.11)-(3.12) it follows easily that $\Phi(-\omega) = \Phi(\omega)$, $\omega \in [-\omega_0/2, \omega_0/2]$, and from this $y(-\omega) = -y(\omega)$, $\omega \in [-\omega_0/2, \omega_0/2]$. Consequently, we have $y(0) = 0$. Further, from (3.13) we obtain

$$\begin{aligned} -[(m-1)y^2 + \lambda^2](y^2+\lambda^2)^{\frac{m-4}{2}} y' = \\ = (m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}} - a_0 = \\ = (y^2+\lambda^2)^{\frac{m-2}{2}}[(m-1+q+\mu)(y^2+\lambda^2) + \lambda(2-m)] - a_0 \geq \\ \geq (y^2+\lambda^2)^{\frac{m-2}{2}}[\lambda^2(m-1+q+\mu) + \lambda(2-m)] - a_0 \geq \\ \geq \lambda^m(m-1+q+\mu) + \lambda^{m-1}(2-m) - a_0 > 0 \end{aligned} \tag{3.15}$$

by (*) and (3.3). Thus it is proved that $y'(\omega) < 0, \omega \in [-\omega_0/2, \omega_0/2]$. Therefore $y(\omega)$ is decreasing function on $[-\omega_0/2, \omega_0/2]$. From this we conclude the last condition of (3.14).

Properties of the function $\Phi(\omega)$. We turn our attention to the properties of the function $\Phi(\omega)$. First of all, notice that the solutions to (3.11)-(3.12) are determined uniquely up to a scalar multiple provided that λ satisfies (*). We will consider the solution normed by the condition

$$\Phi(0) = 1. \tag{3.16}$$

We rewrite the (3.11) in the following form

$$\begin{aligned} -\Phi[(m-1)\Phi'^2 + \lambda^2\Phi^2](\lambda^2\Phi^2 + \Phi'^2)^{\frac{m-4}{2}}\Phi'' = -a_0\Phi^m + (q+\mu)(\lambda^2\Phi^2+\Phi'^2)^{\frac{m}{2}} + \\ + \Phi^2(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-4}{2}}\big\{\lambda[\lambda(m-1)-m+2](\lambda^2\Phi^2+\Phi'^2) + (m-2)\lambda^2\Phi'^2\big\} \end{aligned} \tag{3.17}$$

Now, since $m \geq 2$, by virtue of (*) from the (3.17), it follows that

$$\begin{aligned} -\Phi[(m-1)\Phi'^2 + \lambda^2\Phi^2](\lambda^2\Phi^2+\Phi'^2)^{\frac{m-4}{2}}\Phi'' \geq -a_0\Phi^m + \\ + (\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}}\big\{(q+\mu)(\lambda^2\Phi^2+\Phi'^2) + \lambda[\lambda(m-1)-m+2]\Phi^2\big\} \geq \\ \geq \Phi^m\big\{(q+\mu+m-1)\lambda^m + (2-m)\lambda^{m-1} - a_0\big\} > 0 \end{aligned}$$

(here we take into account that by (*) $(q+\mu+m-1)\lambda^2 + (2-m)\lambda > 0$).

Summarizing the above we obtain the following:

$$\begin{gathered} \Phi(\omega) \geq 0 \quad \forall\omega \in [-\omega_0/2, \omega_0/2]; \qquad \Phi(-\omega_0/2) = \Phi(\omega_0/2) = 0; \\ \Phi(-\omega) = \Phi(\omega) \quad \forall\omega \in [-\omega_0/2, \omega_0/2]; \\ \Phi'(0) = 0; \\ \Phi''(\omega) < 0 \quad \forall\omega \in [-\omega_0/2, \omega_0/2]. \end{gathered} \tag{3.18}$$

Corollary.

$$\max_{[-\omega_0/2, \omega_0/2]} \Phi(\omega) = \Phi(0) = 1 \;\Rightarrow\; 0 \leq \Phi(\omega) \leq 1 \quad \forall\omega \in [-\omega_0/2, \omega_0/2]. \tag{3.19}$$

Now we solve (3.11)-(3.14). Rewriting the (3.13) in the form $y' = g(y,)$ we observe that by (3.15), $g(y) \neq 0$ for all $y \in \mathbb{R}$. Moreover, being rational functions with nonzero denominators $g(y)$ and $g'(y)$ are continuous functions. By the theory of ordinary differential equations the Cauchy's problem (3.13), (3.14) is uniquely solvable in the strip

$$\{(\omega, y)\} \subset [-\frac{\omega_0}{2}, \frac{\omega_0}{2}] \times (-\infty, +\infty).$$

Integrating (3.11)-(3.14), we obtain

$$\begin{gathered}\Phi(\omega) = \exp \int_0^\omega y(\xi) d\xi, \\ \int_0^{-y} \frac{[(m-1)z^2 + \lambda^2](z^2+\lambda^2)^{\frac{m-4}{2}}}{(m-1+q+\mu)(z^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m)(z^2+\lambda^2)^{\frac{m-2}{2}} - a_0} dz = \omega.\end{gathered} \tag{3.20}$$

From this we get in particular that

$$\int_0^{+\infty} \frac{[(m-1)y^2 + \lambda^2](y^2+\lambda^2)^{\frac{m-4}{2}}}{(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}} - a_0} dy = \frac{\omega_0}{2}. \tag{3.21}$$

This expression gives the equation for finding a sharp estimate for the exponent λ in (3.7). For the case $a_0 = 0$ this exponent is calculated explicitly in [18]; we denote this value by λ_0.

Solutions of (3.21). We set

$$\Lambda(\lambda, a_0, y) \equiv \frac{[(m-1)y^2 + \lambda^2](y^2+\lambda^2)^{\frac{m-4}{2}}}{(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}} - a_0}, \tag{3.22}$$

$$\mathfrak{F}(\lambda, a_0, \omega_0) = -\frac{\omega_0}{2} + \int_0^{+\infty} \Lambda(\lambda, a_0, y) dy. \tag{3.23}$$

Then (3.21) takes the form

$$\mathfrak{F}(\lambda, a_0, \omega_0) = 0. \tag{3.24}$$

According to what has been said above

$$\mathfrak{F}(\lambda_0, 0, \omega_0) = 0. \tag{3.25}$$

Direct calculations give

$$\begin{aligned}\frac{\partial \Lambda}{\partial \lambda} = -\lambda \frac{(y^2+\lambda^2)^{\frac{m-6}{2}}}{\left[(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}} - a_0\right]^2} \times \\ \times \Big\{ (y^2+\lambda^2)\left[2(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + (m-2)a_0\right] + \\ + (m-2)y^2\left[4(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + 2\lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}} + (m-4)a_0\right] \Big\}\end{aligned}$$

By (*) with $m \geq 2$, the above expression is less than or equal to

$$
\begin{aligned}
&-2\lambda(m-2)\frac{y^2(y^2+\lambda^2)^{\frac{m-6}{2}}}{\left[(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}}+\lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}}-a_0\right]^2}\times \\
&\quad\times\left\{(y^2+\lambda^2)^{\frac{m-2}{2}}[2(m-1+q+\mu)\lambda^2+(2-m)\lambda]-a_0+\frac{m-2}{2}a_0\right\}< \\
&< -2\lambda(m-2)\frac{y^2(y^2+\lambda^2)^{\frac{m-6}{2}}}{\left[(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}}+\lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}}-a_0\right]^2}\times \\
&\quad\times\left\{(y^2+\lambda^2)^{\frac{m-2}{2}}[(m-1+q+\mu)\lambda^2+(2-m)\lambda]-a_0\right\}< \\
&< -2\lambda(m-2)\frac{y^2(y^2+\lambda^2)^{\frac{m-6}{2}}}{\left[(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}}+\lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}}-a_0\right]^2}\times \\
&\quad\times\left\{(y^2+\lambda^2)^{\frac{m-2}{2}}\lambda^{2-m}a_0-a_0\right\}\leq 0, \qquad (3.26)
\end{aligned}
$$

for all y and λ, a_0 satisfying (*). Similarly for all y, λ, a_0, we have

$$
\frac{\partial\Lambda}{\partial a_0}=\frac{[(m-1)y^2+\lambda^2](y^2+\lambda^2)^{\frac{m-4}{2}}}{\left[(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}}+\lambda(2-m)(y^2+\lambda^2)^{\frac{m-2}{2}}-a_0\right]^2}>0. \qquad (3.27)
$$

Hence, we can apply the Implicit Function Theorem in a neighborhood of the point $(\lambda_0, 0)$. Then (3.24) (and therefore and the (3.21)) determines $\lambda=\lambda(a_0,\omega_0)$ as single-valued **continuous** function of a_0, depending continuously on the parameter ω_0 and having continuous partial derivatives $\frac{\partial\lambda}{\partial a_0}$, $\frac{\partial\lambda}{\partial\omega_0}$. Applying the analytic continuation method, we obtain the solvability of the equation (3.21) for $\forall a_0$, satisfying (*).

Now, we analyze the properties of λ as the function $\lambda(a_0,\omega_0)$. First, from (3.24) we get

$$
\frac{\partial\mathfrak{F}}{\partial\lambda}\frac{\partial\lambda}{\partial a_0}+\frac{\partial\mathfrak{F}}{\partial a_0}=0, \quad \frac{\partial\mathfrak{F}}{\partial\lambda}\frac{\partial\lambda}{\partial\omega_0}+\frac{\partial\mathfrak{F}}{\partial\omega_0}=0;
$$

from this it follows that

$$
\frac{\partial\lambda}{\partial a_0}=-\frac{\left(\frac{\partial\mathfrak{F}}{\partial a_0}\right)}{\left(\frac{\partial\mathfrak{F}}{\partial\lambda}\right)}; \quad \frac{\partial\lambda}{\partial\omega_0}=-\frac{\left(\frac{\partial\mathfrak{F}}{\partial\omega_0}\right)}{\left(\frac{\partial\mathfrak{F}}{\partial\lambda}\right)}. \qquad (3.28)
$$

But by virtue of (3.26), (3.27) we have

$$
\frac{\partial\mathfrak{F}}{\partial a_0}=\int_0^{+\infty}\frac{\partial\Lambda}{\partial a_0}dy>0, \quad \frac{\partial\mathfrak{F}}{\partial\lambda}=\int_0^{+\infty}\frac{\partial\Lambda}{\partial\lambda}dy<0, \quad \frac{\partial\mathfrak{F}}{\partial a_0}=-\frac{1}{2} \qquad \forall(\lambda,a_0). \qquad (3.29)
$$

From (3.28) - (3.29) we obtain

$$
\frac{\partial\lambda}{\partial a_0}>0; \qquad \frac{\partial\lambda}{\partial\omega_0}<0 \qquad \forall a_0, \text{ satisfying } (*). \qquad (3.30)
$$

Thus we derive: *the function* $\lambda(a_0,\omega_0)$ **increases** *with respect to* a_0 *and* **decreases** *with respect to* ω_0.

Multiplying (3.11) by $\Phi(\omega)$ and integrating over $(-\frac{\omega_0}{2}, \frac{\omega_0}{2})$, we have

$$(1-\mu)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^q(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}}\Phi'^2 d\omega = -a_0\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+m}d\omega +$$
$$+ [\lambda^2(m-1+q+\mu)+\lambda(2-m)]\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+2}(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}} d\omega \geq$$
$$\geq \langle \lambda^m(m-1+q+\mu)+\lambda^{m-1}(2-m)-a_0\rangle \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+m}d\omega > 0$$

by virtue of (*). This justifies the condition $0 \leq \mu < 1$ in (3.3).

Lemma 3.2. *Let assumptions (3.3), (*) hold and in addition*

$$q+\mu<1, \qquad if \quad a_0 = 0 \tag{3.31}$$
$$(1-q-\mu)(\lambda m+2-m) > 0, \qquad if \quad a_0 > 0. \tag{3.32}$$

Then we have

$$\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi'|^m d\omega \leq c(a_0, q, \mu, m, \lambda, \omega_0). \tag{3.33}$$

Proof. Dividing (3.11) by $\Phi|\Phi|^{q-2}$, we have

$$\Phi\frac{d}{d\omega}[(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}}\Phi'] + \lambda[\lambda(q+m-1)-m+2]\Phi^2(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}}+$$
$$+ q\Phi'^2(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}} = a_0|\Phi|^m - \mu(\lambda^2\Phi^2+\Phi'^2)^{\frac{m}{2}}.$$

We integrate to obtain

$$(q-1+\mu)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} (\lambda^2\Phi^2+\Phi'^2)^{\frac{m}{2}} d\omega + \lambda(\lambda m+2-m)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi^2(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}} d\omega =$$
$$= a_0\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega. \tag{3.34}$$

At first let $a_0 = 0$. From assumptions and (*) it follows that

$$\lambda m + 2 - m > \lambda(1-q-\mu) > 0,$$

and we have

$$(1-q-\mu)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} (\lambda^2\Phi^2+\Phi'^2)^{\frac{m}{2}} d\omega = \lambda(\lambda m+2-m)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi^2(\lambda^2\Phi^2+\Phi'^2)^{\frac{m-2}{2}} d\omega.$$

Then, applying Young's inequality with $p = \frac{m}{m-2}$, $p' = \frac{m}{2}$, we get :

$$(1-q-\mu)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} (\lambda^2\Phi^2+\Phi'^2)^{\frac{m}{2}} d\omega \leq$$
$$\leq \eta\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} (\lambda^2\Phi^2+\Phi'^2)^{\frac{m}{2}} d\omega + c_\eta\lambda^{\frac{m}{2}}(\lambda m+2-m)^{\frac{m}{2}}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega \qquad \forall\eta>0.$$

From this, choosing $\eta = \frac{1}{2}(1-q-\mu)$ and taking into account (3.16), (3.19), we obtain (3.33).

Now let $a_0 > 0$. If $q+\mu < 1$, then (*) implies that $\lambda(\lambda m + 2 - m) > 0$ and we can rewrite (3.34) in the following way

$$(1-q-\mu)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} (\lambda^2\Phi^2 + \Phi'^2)^{\frac{m}{2}} d\omega + a_0 \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega = \\ = \lambda(\lambda m + 2 - m)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi^2(\lambda^2\Phi^2 + \Phi'^2)^{\frac{m-2}{2}} d\omega.$$

If $q+\mu > 1$ and $\lambda m + 2 - m < 0$, then we rewrite (3.34) as

$$(q-1+\mu)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} (\lambda^2\Phi^2 + \Phi'^2)^{\frac{m}{2}} d\omega = \lambda(m - \lambda m - 2)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi^2(\lambda^2\Phi^2 + \Phi'^2)^{\frac{m-2}{2}} d\omega + \\ + a_0 \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega.$$

In both cases we obtain (3.33) by applying Young's inequality as above.

Finally, if $q+\mu \geq 1$ and $\lambda m + 2 - m \geq 0$, then from (3.34) we have

$$(q-1+\mu)\lambda^m \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega + \lambda^{m-1}(\lambda m + 2 - m)\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega \leq a_0 \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega,$$

what contradicts (*) by $\Phi \geq 0$. The lemma is proved.

From (3.21) and (3.7) we get the function

$$w = (r\sqrt{q_{n-1}})^{\lambda}\Phi(\omega)$$

that will be *a barrier* of our boundary value problem (0.1) – (0.3).

Lemma 3.3. *Let assumptions of lemma 3.2 hold. Then the function $\zeta(|x|)w(x)$ belongs to $\mathfrak{N}^1_{m,0,q}(G_0^d)$, where $\zeta(r) \in C_0^\infty[0,d]$.*

Proof. We must show that

$$I[w] \equiv \int_{G_0^d} (|x|^{-m}|w|^m + |\nabla w|^m + \frac{|w|^{q+m}}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}})dx < \infty. \tag{3.35}$$

Elementary calculations from (3.10) will give

$$|\nabla w|^m = (r\sqrt{q_{n-1}})^{m(\lambda-1)}(\lambda^2\Phi^2 + \Phi'^2)^{m/2}. \tag{3.36}$$

By Lemma 3.1, we have

$$I[w] = \int_{G_0^d} \Big\{ (r\sqrt{q_{n-1}})^{m(\lambda-1)}(\lambda^2\Phi^2 + \Phi'^2)^{\frac{m}{2}} + r^{m(\lambda-1)} q_{n-1}^{m\lambda/2}\Phi^m(\omega) + \\ + r^{m(\lambda-1)+q\lambda} q_{n-1}^{\frac{(m+q)\lambda-m}{2}} \Phi^{m+q}(\omega)\Big\} dx \leq \\ \leq c(\lambda, m, \omega_0)\Big\{ \int_0^d r^{m\lambda+n-1-m} dr \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} (\lambda^2\Phi^2 + \Phi'^2)^{\frac{m}{2}} d\omega + \\ + \int_0^d r^{m\lambda+n-1-m} dr \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi^m d\omega + \int_0^d r^{(m+q)\lambda+n-1-m} dr \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi^{q+m} d\omega \Big\}.$$

By virtue of $m \le n$, (3.19), (3.33) it is clear that $I[w]$ is finite. Thus

$$I[w] \le c(m, \lambda, q, n, \mu, a_0, \omega_0, d).$$

Thus, Lemma 3.3 is proved. Q.E.D.

Example. Let $m = 2$ and consider the Dirichlet problem

$$-\frac{d}{dx_i}(|u|^q u_{x_i}) + \frac{a_0}{x_{n-1}^2 + x_n^2} u|u|^q - \mu u|u|^{q-2}|\nabla u|^2 = 0, \quad x \in G_0,$$
$$u(x) = 0, \quad x \in \Gamma_0,$$

where $a_0 \ge 0$, $0 \le \mu < 1$, $q \ge 0$.

From (3.20), (3.21) we obtain

$$\lambda = \sqrt{\frac{a_0}{1+q+\mu} + (\frac{\pi}{(1+q+\mu)\omega_0})^2} \tag{3.37}$$

and

$$\Phi(\omega) = \left(\cos\frac{\pi\omega}{\omega_0}\right)^{\frac{1}{1+q+\mu}}, \quad \omega \in [-\omega_0/2, \omega_0/2]. \tag{3.38}$$

Thus the solution to the problem above is the function

$$\begin{gathered} u(r,\omega) = r^\lambda (\sin\omega_1 \dots \sin\omega_{n-2})^\lambda \left(\cos\frac{\pi\omega_{n-1}}{\omega_0}\right)^{\frac{1}{1+q+\mu}}, \\ (r,\omega) \in G_0, \quad \omega_{n-1} \in [-\omega_0/2, \omega_0/2], \quad 0 < \omega_0 < \pi, \end{gathered} \tag{3.39}$$

where λ is defined by (3.37).

Now we calculate

$$\Phi'(\omega) = -\frac{\pi}{(1+q+\mu)\omega_0}\left(\cos\frac{\pi\omega}{\omega_0}\right)^{-\frac{q+\mu}{1+q+\mu}} \sin\frac{\pi\omega}{\omega_0}$$

and it is easy to observe that all properties of $\Phi(\omega)$ are fulfilled. Moreover,

$$\begin{aligned} \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi'(\omega)^2 d\omega &= (\frac{\pi}{(1+q+\mu)\omega_0})^2 \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \left(\cos\frac{\pi\omega}{\omega_0}\right)^{-\frac{2(q+\mu)}{1+q+\mu}} \sin^2\frac{\pi\omega}{\omega_0} d\omega = \\ &= \frac{2\pi}{(1+q+\mu)^2\omega_0}\int_0^{\frac{\pi}{2}} (\cos t)^{-\frac{2(q+\mu)}{1+q+\mu}} \sin^2 t dt = \\ &= \frac{2\pi}{(1+q+\mu)^2\omega_0}\int_0^{\frac{\pi}{2}} (\sin t)^{-\frac{2(q+\mu)}{1+q+\mu}} \cos^2 t dt = \\ &= \frac{\pi}{(1+q+\mu)^2\omega_0} \frac{\Gamma(\frac{3}{2})\Gamma\big(\frac{1-q-\mu}{2(1+q+\mu)}\big)}{\Gamma\big(\frac{2+q+\mu}{1+q+\mu}\big)} \end{aligned} \tag{3.40}$$

provided that $q+\mu<1$. This integral is **non-convergent**, if $q+\mu\ge 1$. At the same time for $\forall q>0$ we have

$$\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi(\omega)|^q {\Phi'}^2(\omega)d\omega = (\frac{\pi}{(1+q+\mu)\omega_0})^2 \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \left(\cos\frac{\pi\omega}{\omega_0}\right)^{-\frac{q+2\mu}{1+q+\mu}} \sin^2\frac{\pi\omega}{\omega_0} d\omega =$$
$$= \frac{2\pi}{(1+q+\mu)^2\omega_0}\int_0^{\frac{\pi}{2}} (\sin t)^{-\frac{q+2\mu}{1+q+\mu}} \cos^2 t dt =$$
$$= \frac{\pi}{(1+q+\mu)^2\omega_0}\frac{\Gamma(\frac{3}{2})\Gamma\left(\frac{1-\mu}{2(1+q+\mu)}\right)}{\Gamma\left(\frac{2+\frac{3}{2}q+\mu}{1+q+\mu}\right)}, \quad (3.41)$$

since $\mu<1$.

4. More precise definition of the Hölder exponent.

Now we return to the (0.1)-(0.5) with $\tau=0$. For its weak solutions we make more precise the value of α-Hölder exponent established in the Theorem 2.1. To this end we use the weak comparison principle ([7, §10.4], [13,§3.1]) and the barrier function constructed in §3.

Theorem 4.1. *Let $u(x)$ be a weak solution of (0.1)-(0.5), where*

$$\tau=0, \quad 0\le\mu<1, \quad 2\le m\le n; \quad \textit{and}\; q+\mu<1, \;\textit{if}\; a_0=0. \quad (4.1)$$

Suppose that there exists a nonnegative constant k_1 such that

$$\left|f_0(x)-\sum_{i=1}^n \frac{\partial f_i(x)}{\partial x_i}\right| \le k_1|x|^\beta, \quad (4.2)$$

where

$$\beta > \lambda(q+m-1)-m, \quad (4.3)$$

λ is the positive solution of (3.21) with $\omega_0\in(0,\pi)$ such that (*) is fulfilled and

$$(1-q-\mu)(m\lambda+2-m)>0, \quad \textit{if}\; a_0>0. \quad (4.4)$$

Then $\forall\varepsilon>0$ there exists a constant $c_\varepsilon>0$, depending only on ε, n, m, μ, q, a_0, ω_0 such that

$$|u(x)|\le c_\varepsilon|x|^{\lambda-\varepsilon}. \quad (4.5)$$

Before proving the theorem, we make some transformations and additional investigations. We make the change of variables

$$u=v|v|^{t-1}, \quad t=\frac{m-1}{q+m-1}. \quad (4.6)$$

As result, (0.1)-(0.2) takes the form

$$\begin{aligned} \mathfrak{M}_0 v(x) &= F(x), \quad x\in G, \\ v(x) &= 0, \quad x\in\partial G, \end{aligned} \quad (4.7)$$

where

$$\mathfrak{M}_0 v(x) \equiv -\frac{d}{dx_i}(|\nabla v|^{m-2} v_{x_i}) + \frac{\overline{a}_0}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}} v|v|^{m-2} - \overline{\mu} v^{-1} |\nabla v|^m, \tag{4.8}$$

$$\overline{a}_0 = t^{1-m} a_0, \quad \overline{\mu} = t\mu, \quad F(x) = t^{1-m}\Big(f_0(x) - \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i}\Big). \tag{4.9}$$

Now

$$\begin{gathered} \overline{w} = (r\sqrt{q_{n-1}})^{\overline{\lambda}} \overline{\Phi}(\omega), \\ \overline{\lambda} = \frac{1}{t}\lambda, \quad \overline{\Phi}(\omega) = \Phi^{1/t}(\omega) \end{gathered} \tag{4.10}$$

plays the role of barrier function. Because of (3.11)-(3.14) and (3.21), it is easy to verify that $(\overline{\lambda}, \overline{\Phi}(\omega))$ is the solution to

$$\frac{d}{d\omega}[(\overline{\lambda}^2\overline{\Phi}^2 + \overline{\Phi}'^2)^{\frac{m-2}{2}} \overline{\Phi}'] + \overline{\lambda}[\overline{\lambda}(m-1) - m + 2]\overline{\Phi}(\overline{\lambda}^2\overline{\Phi}^2 + \overline{\Phi}'^2)^{\frac{m-2}{2}} =$$

$$= \overline{a}_0 \overline{\Phi}|\overline{\Phi}|^{m-2} - \overline{\mu}\frac{1}{\overline{\Phi}}(\overline{\lambda}^2\overline{\Phi}^2 + \overline{\Phi}'^2)^{\frac{m}{2}}, \quad \omega \in (-\omega_0/2, \omega_0/2), \tag{4.11}$$

$$\overline{\Phi}(-\omega_0/2) = \overline{\Phi}(\omega_0/2) = 0, \tag{4.12}$$

$$\int_0^{+\infty} \frac{[(m-1)y^2 + \overline{\lambda}^2](y^2 + \overline{\lambda}^2)^{\frac{m-4}{2}}}{(m-1+\overline{\mu})(y^2 + \overline{\lambda}^2)^{\frac{m}{2}} + \overline{\lambda}(2-m)(y^2 + \overline{\lambda}^2)^{\frac{m-2}{2}} - \overline{a}_0}\, dy = \frac{\omega_0}{2}. \tag{4.13}$$

It is obvious that the properties of (λ, Φ), established in § 3 remain valid for $(\overline{\lambda}, \overline{\Phi}(\omega))$. In particular the (*) becomes

$$P_m(\overline{\lambda}) \equiv (m-1+\overline{\mu})\overline{\lambda}^m + (2-m)\overline{\lambda}^{m-1} - \overline{a}_0 > 0. \tag{$\overline{*}$}$$

Now we consider a perturbation of the (4.11)-(4.13). Namely, $\forall \varepsilon \in (0, \pi - \omega_0)$ we consider on the segment $[-\frac{\omega_0+\varepsilon}{2}, \frac{\omega_0+\varepsilon}{2}]$ the problem for $(\lambda_\varepsilon, \Phi_\varepsilon)$:

$$\frac{d}{d\omega}[(\lambda_\varepsilon^2\Phi_\varepsilon^2 + \Phi_\varepsilon'^2)^{\frac{m-2}{2}} \Phi_\varepsilon'] + \lambda_\varepsilon[\lambda_\varepsilon(m-1) - m + 2]\Phi_\varepsilon(\lambda_\varepsilon^2\Phi_\varepsilon^2 + \Phi_\varepsilon'^2)^{\frac{m-2}{2}} =$$

$$= (\overline{a}_0 - \varepsilon)\Phi_\varepsilon|\Phi_\varepsilon|^{m-2} - \overline{\mu}\frac{1}{\Phi_\varepsilon}(\lambda_\varepsilon^2\Phi_\varepsilon^2 + \Phi_\varepsilon'^2)^{\frac{m}{2}}, \quad \omega \in (-\frac{\omega_0+\varepsilon}{2}, \frac{\omega_0+\varepsilon}{2}), \tag{4.11$_\varepsilon$}$$

$$\Phi(-\frac{\omega_0+\varepsilon}{2}) = \Phi(\frac{\omega_0+\varepsilon}{2}) = 0, \tag{4.12$_\varepsilon$}$$

$$\int_0^{+\infty} \frac{[(m-1)y^2 + \lambda_\varepsilon^2](y^2 + \lambda_\varepsilon^2)^{\frac{m-4}{2}}}{(m-1+\overline{\mu})(y^2 + \lambda_\varepsilon^2)^{\frac{m}{2}} + \lambda_\varepsilon(2-m)(y^2 + \lambda_\varepsilon^2)^{\frac{m-2}{2}} + \varepsilon - \overline{a}_0}\, dy = \frac{\omega_0+\varepsilon}{2}. \tag{4.13$_\varepsilon$}$$

The (4.11_ε)-(4.13_ε) is obtained from the (4.11)-(4.13) by change in the last ω_0 by $\omega_0 + \varepsilon$ and $\overline{a}_0$ by $\overline{a}_0 - \varepsilon$. From the monotonicity properties of $\overline{\lambda}(\omega_0, \overline{a}_0)$ established in §3 (see (3.30)), we obtain

$$0 < \lambda_\varepsilon < \overline{\lambda}, \quad \lim_{\varepsilon \to +0} \lambda_\varepsilon = \overline{\lambda}. \tag{4.14}$$

Now we establish lower bounds for the functions q_{n-1} and $\Phi_\varepsilon(\omega)$.

Corollary 4.2. *From Lemma 3.1 it follows that*

$$\sqrt{q_{n-1}}^{(m-1)\lambda_\varepsilon - m} \geq \varkappa_0 = \min\left\{1; \left(\cos\frac{\omega_0}{2}\right)^{(m-1)\lambda_\varepsilon - m}\right\}. \tag{4.15}$$

Lemma 4.3. *There exists $\varepsilon^* > 0$ such that*

$$\Phi_\varepsilon\left(\frac{\omega_0}{2}\right) \geq \frac{\varepsilon}{\omega_0 + \varepsilon} \qquad \forall \varepsilon \in (0, \varepsilon^*). \tag{4.16}$$

Proof. We turn to the $(\overline{*}) : P_m(\overline{\lambda}) > 0$. Since $P_m(\overline{\lambda})$ is a polynomial, by continuity, there exists a δ^*-neighborhood of $\overline{\lambda}$, in which $(\overline{*})$ is satisfied as before, i.e. there exists $\delta^* > 0$ such that $P_m(\lambda) > 0$ for $\forall \lambda$ such that $|\lambda - \overline{\lambda}| < \delta^*$. We choose the number $\delta^* > 0$ in the such way; then

$$P_m(\overline{\lambda} - \delta) > 0 \quad \forall \delta \in (0, \delta^*)\,. \tag{4.17}$$

Recall that $\overline{\lambda}$ solves the (4.13). By (4.14), now for every $\delta \in (0, \delta^*)$ we can put

$$\lambda_\varepsilon = \overline{\lambda} - \delta$$

and solve (4.13_ε) together with this λ_ε with respect to ε; let $\varepsilon(\delta) > 0$ be obtained solution. Since (4.14) is true,

$$\lim_{\delta \to +0} \varepsilon(\delta) = +0.$$

Thus we have the sequence of problems (4.11_ε)-(4.13_ε) with respect to

$$(\lambda_\varepsilon, \Phi_\varepsilon(\omega)) \quad \forall \varepsilon \;\Big|\; 0 < \varepsilon < \min(\varepsilon(\delta); \pi - \omega_0) = \varepsilon^*(\delta), \quad \forall \delta \in (0, \delta^*). \tag{4.18}$$

We consider $\Phi_\varepsilon(\omega)$ with $\forall \varepsilon$ from (4.18). In the same way as (3.18) we verify that

$$\Phi''_\varepsilon(\omega) < 0 \quad \forall \omega \in \left[-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2}\right].$$

But this inequality means that the function $\Phi_\varepsilon(\omega)$ is convex on $[-\frac{\omega_0+\varepsilon}{2}, \frac{\omega_0+\varepsilon}{2}]$, i.e.

$$\Phi_\varepsilon(\alpha_1\omega_1 + \alpha_2\omega_2) \geq \alpha_1\Phi_\varepsilon(\omega_1) + \alpha_2\Phi_\varepsilon(\omega_2) \quad \forall \omega_1, \omega_2 \in \left[-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2}\right];$$
$$\alpha_1 \geq 0,\ \alpha_2 \geq 0 \;\Big|\alpha_1 + \alpha_2 = 1. \tag{4.19}$$

We put $\alpha_1 = \frac{\omega_0}{\varepsilon + \omega_0}$, $\alpha_2 = \frac{\varepsilon}{\varepsilon + \omega_0}$, $\omega_1 = \frac{\varepsilon + \omega_0}{2}$, $\omega_2 = 0$. Then (4.12_ε) we obtain

$$\Phi_\varepsilon\left(\frac{\omega_0}{2}\right) \geq \frac{\varepsilon}{\omega_0 + \varepsilon}\Phi_\varepsilon(0) = \frac{\varepsilon}{\omega_0 + \varepsilon},$$

and this lemma is proved. Q.E.D.

Corollary 4.4.

$$\frac{\varepsilon}{\omega_0+\varepsilon} \le \Phi_\varepsilon(\omega) \le 1 \quad \forall\omega \in [-\omega_0/2, \omega_0/2]; \quad \forall\varepsilon \in (0, \varepsilon^*). \tag{4.20}$$

Proof of the Theorem 4.1. Let $(\lambda_\varepsilon, \Phi_\varepsilon(\omega))$ be the solution of (4.11_ε)-(4.13_ε), with $\varepsilon \in (0, \varepsilon^*)$, where ε^* is defined by (4.18). We set

$$w_\varepsilon(r,\omega) = (r\sqrt{q_{n-1}})^{\lambda_\varepsilon}\Phi_\varepsilon(\omega), \quad \omega \in [-\omega_0/2, \omega_0/2].$$

Elementary calculations show that

$$w_\varepsilon\big|_{\Gamma_0} \ge 0, \tag{4.21}$$

$$w_\varepsilon\big|_{\Omega_d} \ge \frac{\varepsilon}{\omega_0+\varepsilon}\left(d\cos\frac{\omega_0}{2}\right)^{\overline{\lambda}} \tag{4.22}$$

(by virtue of (4.14), Lemma 3.1 and (4.20)). Then for (4.8)-(4.9) we obtain

$$\begin{aligned}\mathfrak{M}_0 w_\varepsilon(r,\omega) = (r\sqrt{q_{n-1}})^{(m-1)\lambda_\varepsilon - m}\Big\{ -\frac{d}{d\omega}[(\lambda_\varepsilon^2\Phi_\varepsilon^2 + {\Phi'}_\varepsilon^2)^{\frac{m-2}{2}}\Phi'_\varepsilon] - \\ - \lambda_\varepsilon[\lambda_\varepsilon(m-1) - m + 2]\Phi_\varepsilon(\lambda_\varepsilon^2\Phi_\varepsilon^2 + {\Phi'}_\varepsilon^2)^{\frac{m-2}{2}} + \overline{a}_0\Phi_\varepsilon^{m-1} - \\ - \overline{\mu}\frac{1}{\Phi_\varepsilon}(\lambda_\varepsilon^2\Phi_\varepsilon^2 + {\Phi'}_\varepsilon^2)^{\frac{m}{2}}\Big\} = \varepsilon(r\sqrt{q_{n-1}})^{(m-1)\lambda_\varepsilon - m}\Phi_\varepsilon^{m-1}\end{aligned}$$

by virtue of (4.11_ε). From (4.14) and Lemma 3.1 we have

$$\sqrt{q_{n-1}}^{(m-1)\lambda_\varepsilon - m} \ge \overline{\varkappa}_0 = \min\left\{1; \left(\cos\frac{\omega_0}{2}\right)^{(m-1)\overline{\lambda}-m}\right\}. \tag{4.23}$$

Taking into account (4.20), we get

$$\mathfrak{M}_0 w_\varepsilon(r,\omega) \ge \frac{\varepsilon^m\overline{\varkappa}_0}{(\omega_0+\varepsilon)^{m-1}} r^{(m-1)\lambda_\varepsilon - m}. \tag{4.24}$$

Now we use the weak comparison principle for (4.7)-(4.9). By definition of weak solution

$$\begin{aligned}Q(v,\phi) \equiv \int_G \Big\{ |\nabla v|^{m-2}v_{x_i}\phi_{x_i} + \frac{\overline{a}_0}{(x_{n-1}^2 + x_n^2)^{\frac{m}{2}}} v|v|^{m-2}\phi - \\ -\overline{\mu}v^{-1}|\nabla v|^m\phi - F(x)\phi\Big\}dx = 0 \quad \forall\phi(x) \in \mathfrak{N}^1_{m,0,0}(G).\end{aligned} \tag{4.25}$$

Now let $\phi(x) \in \mathfrak{N}^1_{m,0,0}(G)$ be such that

$$\phi(x) \ge 0 \quad \forall x \in \overline{G}; \qquad \phi(x) = 0 \quad \forall x \in G \setminus G_0^d.$$

Then $\forall A > 0$,

$$
\begin{aligned}
Q(Aw_\varepsilon, \phi) &= \int_{G_0^d} \phi(x)\Big(\mathfrak{M}_0(Aw_\varepsilon) - F(x)\Big)dx \\
&= \int_{G_0^d} \phi(x)\Big(A^{m-1}\mathfrak{M}_0 w_\varepsilon(x) - F(x)\Big)dx \\
&\geq \int_{G_0^d} \phi(x)\Big\{\Big(\frac{A}{\varepsilon+\omega_0}\Big)^{m-1}\overline{\varkappa}_0\varepsilon^m |x|^{(m-1)\lambda_\varepsilon - m} - k_1 t^{1-m}|x|^\beta\Big\}dx \\
&\geq \Big\{\Big(\frac{A}{\varepsilon+\omega_0}\Big)^{m-1}\overline{\varkappa}_0\varepsilon^m - k_1\Big(\frac{m-1+q}{m-1}\Big)^{m-1}\Big\}\int_{G_0^d}\phi(x)|x|^{(m-1)\lambda_\varepsilon - m}dx \geq 0,
\end{aligned}
\tag{4.26}
$$

if $A > 0$ is chosen sufficiently large,

$$
A \geq \frac{(m-1+q)(\varepsilon+\omega_0)}{\varepsilon(m-1)}\Big(\frac{k_1}{\overline{\varkappa}_0\varepsilon}\Big)^{\frac{1}{m-1}}. \tag{4.27}
$$

To obtain the first inequality of (4.26) we used (4.2), (4.9), and (4.24). And for the second inequality, we used (4.3), (4.6), (4.10), and (4.14).

Furthermore, by the theorem 2.1, $v(x)\Big|_{\Omega_d} \leq c_0 d^\alpha$; therefore by (4.22),

$$
Aw_\varepsilon\Big|_{\Omega_d} \geq \frac{A\varepsilon}{\omega_0+\varepsilon}\Big(d\cos\frac{\omega_0}{2}\Big)^{\overline{\lambda}} \geq v(x)\Big|_{\Omega_d}, \tag{4.28}
$$

if $A > 0$ is chosen such that

$$
A \geq \frac{c_0(\varepsilon+\omega_0)}{\varepsilon\Big(\cos\frac{\omega_0}{2}\Big)^{\overline{\lambda}}} d^{\alpha-\overline{\lambda}}. \tag{4.29}
$$

Thus, if $A > 0$ is chosen according to (4.27), (4.29), then from (4.25), (4.26), (4.28), (4.21) and (4.7) we obtain:

$$
\begin{gathered}
Q(Aw_\varepsilon, \phi) \geq 0, \quad Q(v,\phi) = 0 \quad \text{in} \quad G_0^d; \\
Aw_\varepsilon\Big|_{\partial G_0^d} \geq v\Big|_{\partial G_0^d}.
\end{gathered}
$$

It is easy to verify that rest of the conditions of the weak comparison principle are fulfilled. By this principle we obtain

$$
v(x) \leq Aw_\varepsilon(x), \quad \forall x \in \overline{G_0^d}.
$$

Similarly we can prove that

$$
v(x) \geq -Aw_\varepsilon(x), \quad \forall x \in \overline{G_0^d}.
$$

Thus, finally, we have

$$
|v(x)| \leq Aw_\varepsilon(x) \leq A|x|^{\lambda_\varepsilon}, \quad \forall x \in \overline{G_0^d}. \tag{4.30}
$$

Resubstituting the old variables, by (4.6), (4.10) we obtain from (4.30) the required bound (4.5). Therefore, Theorem 4.1 is proved.

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