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NEW APPROACH TO STREAMING SEMIGROUPS WITH DISSIPATIVE BOUNDARY CONDITIONS

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Abstract. This paper concerns the generation of a C_0-semigroup by a streaming operator with general dissipative boundary conditions. Here, we give a third approach based on the construction of the generated semigroup without using the Hille-Yosida's Theorem. The first approach, based on the Hille-Yosida's Theorem, was given by Dautray [8], Protopopescu [9] and Voigt [10]. The second approach, based on the characteristic method, was given by Beals [1] and Protopopescu [9].

1. Introduction

In this paper, we are concerned by generation Theorem and the explicit expression of the generated semigroup of the streaming operator T_K defined by

$$\begin{gathered} T_K\varphi(x,v) = -v\cdot\nabla_x\varphi(x,v), \quad \text{on the domain} \\ D(T_K) = \{\varphi\in W^p_-(\Omega) : \gamma_-\varphi = K\gamma_+\varphi\} \end{gathered} \tag{1.1}$$

where $(x,v)\in\Omega = X\times V$ with $X\subset\mathbb{R}^n$ is a smoothly bounded open subset and $d\mu$ is a Radon measure on $\mathbb{R}^n$ with support V. The traces $\gamma_+\varphi = \varphi_{|\Gamma_+}$ and $\gamma_-\varphi = \varphi_{|\Gamma_-}$ present respectively the outgoing and the incoming particles fluxes and K is a bounded linear operator between the traces spaces $L^p(\Gamma_+)$ and $L^p(\Gamma_-)$ (see the next section for more explanations).

In the phase space $\Omega = X\times V$, the function $\varphi(x,v)$ presents the density of all particles (neutrons, photons, molecules of gas,...) having, at the time $t=0$, the position $x\in X$ with the directional velocity $v\in V$. The boundary conditions $\gamma_-\varphi = K\gamma_+\varphi$ included in the domain $D(T_K)$ generalize naturally all well-known boundary conditions such as (vacuum, reflection, specular, periodic,...). For the convenience of reader and more explanations, we refer for instance to [1], [9, Chapter XI and XII], [8, Chapter 21] and [10].

The existence of a strongly continuous semigroup generated by the streaming operator has been investigated by several authors and several important results have been cleared. When $\|K\|<1$, the first approach, based on the characteristic method, has been used in [1] and [9, Theorem 4.3, p.386]. For the same case (i.e. $\|K\|<1$), the second approach, based on the Hille-Yosida's Theorem, has been used in [9, Theorem 2.2, p.410] [8, Theorem 3, p.1118] and [10, Theorem 4.3, p.66].

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The motivation, of the present work, is to give the third approach when $\|K\| < 1$ without using the Hille-Yosida's Theorem or characteristic method. This approach is concerned by two steps. The first one is devoted to the construction of a C_0-semigroup given in the Proposition 3.5. In the second step, we show that T_K given by the relation (1.1) is the infinitesimal generator of this semigroup in the Theorem 4.2.

To obtain our objective, we use our technics successfully applied in [3][4]. We point out that the present work is new and gives the explicit expression of the generated semigroup and all of this result doesn't hold for the case $\|K\| \geq 1$. In [2], we give another and different treatment for the last case (i.e., $\|K\| \geq 1$).

2. Setting of the problem

We consider the Banach space $L^p(\Omega)$ $(1 \leq p < \infty)$ with its natural norm

$$\|\varphi\|_p = \Big[\int_\Omega |\varphi(x,v)|^p dx d\mu\Big]^{1/p}, \tag{2.1}$$

where $\Omega = X \times V$ with $X \subset \mathbb{R}^n$ be a smoothly bounded open subset and $d\mu$ be a Radon measure on $\mathbb{R}^n$ with support V. We also consider the partial Sobolev space

$$W^p(\Omega) = \big\{\varphi \in L^p(\Omega),\ \ v \cdot \nabla_x \varphi \in L^p(\Omega)\big\},$$

with the norm $\|\varphi\|_{W^p(\Omega)} = \big[\|\varphi\|_p^p + \|v \cdot \nabla_x \varphi\|_p^p\big]^{1/p}$. We set $n(x)$ the outer unit normal at $x \in \partial X$, where ∂X is the boundary of X equipped with the measure of surface $d\gamma$. We denote

$$\begin{aligned}
\Gamma\ &= \partial X \times V,\\
\Gamma_0 &= \{(x,v) \in \Gamma,\ \ v \cdot n(x) = 0\},\\
\Gamma_+ &= \{(x,v) \in \Gamma,\ \ v \cdot n(x) > 0\},\\
\Gamma_- &= \{(x,v) \in \Gamma,\ \ v \cdot n(x) < 0\},
\end{aligned}$$

and suppose that $d\gamma(\Gamma_0) = 0$. For $(x,v) \in \Omega$, the time which a particle starting at x with velocity $-v$ needs until it reaches the boundary ∂X of X is denoted by

$$t(x,v) = \inf\{t > 0 : x - tv \notin X\}.$$

Similarly, if $(x,v) \in \Gamma_+$ we set

$$\tau(x,v) = \inf\{t > 0 : x - tv \notin X\}.$$

We also consider the trace spaces $L^p(\Gamma_\pm)$ equipped with the norms

$$\|\varphi\|_{L^p(\Gamma_\pm)} = \Big[\int_{\Gamma_\pm} |\varphi(x,v)|^p d\xi\Big]^{1/p},$$

where $d\xi = |v \cdot n(x)| d\gamma d\mu$. In this context we define the trace applications by

$$\gamma_\pm \ \ : \ \ \varphi \to \varphi|_{\Gamma_\pm},$$

and the Banach spaces

$$\begin{aligned}
W^p_-(\Omega) &= \{\varphi \in W^p(\Omega),\ \ \gamma_-\varphi \in L^p(\Gamma_-)\}\\
W^p_+(\Omega) &= \{\varphi \in W^p(\Omega),\ \ \gamma_+\varphi \in L^p(\Gamma_+)\}.
\end{aligned}$$

Note that, by [5], [6], we have $W_-^p(\Omega) = W_+^p(\Omega)$. Finally, we consider the boundary operator

$$K \in \mathcal{L}(L^p(\Gamma_+), \ \ L^p(\Gamma_-)), \tag{2.2}$$

and we set $\|K\| := \|K\|_{\mathcal{L}(L^p(\Gamma_+),L^p(\Gamma_-))}$ for the rest of this article.

Lemma 2.1 ([8, Theorem 2, pp.1087])**.** *The operator T_0 defined by*

$$\begin{gathered} T_0\varphi(x,v) = -v\cdot\nabla_x\varphi(x,v), \quad \text{on the domain} \\ D(T_0) = \{\varphi \in W^p(\Omega), \ \ \gamma_-\varphi = 0\} \end{gathered} \tag{2.3}$$

generates, on $L^p(\Omega)$, the C_0-semigroup $\{U_0(t)\}_{t\geq 0}$ of contractions given by

$$U_0(t)\varphi(x,v) = \chi(t - t(x,v))\varphi(x - tv, v),$$

where

$$\chi(t-t(x,v)) = \begin{cases} 1 & \text{if } t - t(x,v) \leq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

We complete this section by the following Lemma that we will need later.

Lemma 2.2. *The following applications are contunous:*

1. $\gamma_+ : D(T_0) \to L^p(\Gamma_+)$;
2. $\gamma_+(\lambda - T_0)^{-1} : L^p(\Omega) \to L^p(\Gamma_+)$, *for all* $\lambda > 0$;
3. $t \geq 0 \to \gamma_+ U_0(t)\varphi \in L^p(\Gamma_+)$, *for all* $\varphi \in D(T_0)$.

Proof. (1). For all $\varphi \in D(T_0)$, we have $\gamma_-\varphi = 0 \in L^p(\Gamma_-)$ and thus the Green's formula holds on $D(T_0)$. Using the relation

$$\operatorname{sgn} u|u|^{p-1} v\cdot\nabla_x u = \frac{1}{p} v\cdot\nabla_x|u|^p, \tag{2.5}$$

we obtain

$$\begin{aligned} -\int_\Omega \left[\operatorname{sgn}\varphi|\varphi|^{p-1}T_0\varphi\right](x,v)dxd\mu &= \frac{1}{p}\int_{X\times V} v\cdot\nabla_x(|\varphi|^p)(x,v)dxd\mu \\ &= \frac{1}{p}\int_{\Gamma_+}|\gamma_+\varphi(x,v)|^p d\xi - \frac{1}{p}\int_{\Gamma-}|\gamma_-\varphi(x,v)|^p d\xi \\ &= \frac{1}{p}\|\gamma_+\varphi\|^p_{L^p(\Gamma_+)}, \end{aligned} \tag{2.6}$$

which implies, by Hölder's inequality, that

$$\begin{aligned} \|\gamma_+\varphi\|^p_{L^p(\Gamma_+)} &\leq p\int_\Omega |\varphi(x,v)|^{p-1}|T_0\varphi(x,v)|dxd\mu \\ &\leq p\Big[\int_\Omega |\varphi(x,v)|^{q(p-1)}dxd\mu\Big]^{\frac{1}{q}}\Big[\int_\Omega |T_0\varphi(x,v)|^p dxd\mu\Big]^{1/p} \\ &\leq p\|\varphi\|_p^{p/q}\|T_0\varphi\|_p \end{aligned}$$

where $q \geq 1$ is the conjugate of $p \geq 1$ (i.e. $p^{-1} + q^{-1} = 1$). Now, the Young's formula gives us

$$\|\gamma_+\varphi\|^p_{L^p(\Gamma_+)} \leq p\big[\frac{1}{q}\|\varphi\|_p^p + \frac{1}{p}\|T_0\varphi\|_p^p\big] \leq \max\{\frac{p}{q}, 1\}\|\varphi\|^p_{D(T_0)}$$

which prove the continuity.

(2). Let $\lambda > 0$. We know from the previous Lemma that, for all $g \in L^p(\Omega)$, the function $\varphi = (\lambda - T_0)^{-1}g \in D(T_0)$ is the unique solution of the equation $\lambda\varphi = T_0\varphi + g$. Multiplying this last equation by $\operatorname{sgn}\varphi|\varphi|^{p-1}$ and using the relation (2.5) we obtain

$$\begin{aligned}\lambda \int_\Omega |\varphi|^p(x,v)dxd\mu &= \int_\Omega \left[\operatorname{sgn}\varphi|\varphi|^{p-1}T_0\varphi\right](x,v)dxd\mu \\ &\quad + \int_\Omega \left[\operatorname{sgn}\varphi|\varphi|^{p-1}\right](x,v)g(x,v)dxd\mu\end{aligned}$$

which implies, by the relation (2.6), that

$$\lambda\|\varphi\|_p^p = -\frac{1}{p}\|\gamma_+\varphi\|^p_{L^p(\Gamma_+)} + \int_\Omega \left[\operatorname{sgn}\varphi|\varphi|^{p-1}g\right](x,v)g(x,v)dxd\mu$$

and therefore

$$\|\gamma_+\varphi\|^p_{L^p(\Gamma_+)} \le p\int_\Omega |\varphi|^{p-1}(x,v)|g|(x,v)dxd\mu.$$

The Höder's inequality and $\|\varphi\|_p = \|(\lambda - T_0)^{-1}g\|_p \le (\|g\|_p/\lambda)$ which follows from the contractiveness of the semigroup $\{U_0(t)\}_{t\ge 0}$ in the previous Lemma, infer that

$$\|\gamma_+\varphi\|^p_{L^p(\Gamma_+)} \le p\|\varphi\|_p^{\frac{p}{q}}\|g\|_p \le \frac{p}{\lambda^{\frac{p}{q}}}\|g\|_p^{\frac{p}{q}}\|g\|_p = \frac{p}{\lambda^{\frac{p}{q}}}\|g\|_p^p.$$

Thus

$$\|\gamma_+(\lambda - T_0)^{-1}g\|_{L^p(\Gamma_+)} \le \Big[\frac{p}{\lambda^{\frac{p}{q}}}\Big]^{1/p}\|g\|_p$$

for all $g \in L^p(\Omega)$. The second statement is proved.
(3). Let $h > 0$. For all $\varphi \in D(T_0)$ we have

$$\begin{aligned}\|U_0(h)\varphi - \varphi\|_{D(T_0)} &= \left[\|U_0(h)\varphi - \varphi\|_p^p + \|T_0\left[U_0(h)\varphi - \varphi\right]\|_p^p\right]^{1/p} \\ &= \left[\|U_0(h)\varphi - \varphi\|_p^p + \|U_0(h)T_0\varphi - T_0\varphi\|_p^p\right]^{1/p}\end{aligned}$$

which implies

$$\lim_{h\searrow 0}\|U_0(h)\varphi - \varphi\|_{D(T_0)} = 0$$

and therefore the continuity at $t = 0_+$ follows. Now, the continuity at $t > 0$ follows from the previous relation and the fact that $\{U_0(t)\}_{t\ge 0}$ is a semigroup (i.e., $U_0(t+s) = U_0(t)U_0(s)$). $\square$

3. Construction of the semigroup

In this section we give the expression of the generated semigroup $\{U_K(t)\}_{t\ge 0}$ only for the case $\|K\| < 1$. In order to show the proposition 3.5 which is the main result of this section, we are going to show some preparatory Lemmas.

Lemma 3.1. *The Cauchy's problem*

$$\begin{gathered}\frac{du}{dt} + v\cdot\nabla_x u = 0, \quad (t,x,v) \in (0,\infty)\times\Omega; \\ \gamma_- u = f_- \in L^p(\mathbb{R}_+, L^p(\Gamma_-)); \\ u(0) = f_0 \in L^p(\Omega),\end{gathered} \tag{3.1}$$

admits an unique solution $u = u(t,x,v) = u(t)(x,v)$. Furthermore, for all $t \geq 0$, we have

$$\|u(t)\|_p^p + \int_0^t \|\gamma_+ u(s)\|^p_{L^p(\Gamma_+)} ds = \int_0^t \|f_-(s)\|^p_{L^p(\Gamma_-)} ds + \|f_0\|_p^p. \tag{3.2}$$

Proof. Let $f_- \in L^p(\mathbb{R}_+, L^p(\Gamma_-))$ and $f_0 \in L^p(\Omega)$. First. The existence of the solutions u of the Cauchy's problem (3.1) is guaranteed by [8, pp.1124] and it is given by

$$u(t,x,v) = f_-(t - t(x,v), x - t(x,v)v, v) + U_0(t)\varphi(x,v). \tag{3.3}$$

Next. If u and u' are two solution of the Cauchy's problem (3.1), then $w = u - u'$ is solution of the Cauchy's problem (3.1) with $f_- = 0, f_0 = 0$ which implies that $w = 0$, by the relation (3.2), and therefore $u = u'$.

Multiplying the first equation of the Cauchy's problem (3.1) by $\operatorname{sgn} u|u|^{p-1}$, using the relation (2.5) and integrating on Ω, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d\|u(t)\|_p^p}{dt} &= \frac{1}{p}\int_{\Gamma_-} |\gamma_- u(t,x,v)|^p d\xi - \frac{1}{p}\int_{\Gamma_+} |\gamma_+ u(t,x,v)|^p d\xi \\
&= \frac{1}{p}\int_{\Gamma_-} |f_-(t,x,v)|^p d\xi - \frac{1}{p}\int_{\Gamma_+} |\gamma_+ u(t,x,v)|^p d\xi \\
&= \frac{1}{p}\|f_-(t)\|^p_{L^p(\Gamma_-)} - \frac{1}{p}\|\gamma_+ u(t)\|^p_{L^p(\Gamma_+)}
\end{aligned}$$

which implies, by integration with respect to t, that

$$\|u(t)\|_p^p - \|f_0\|_p^p = \int_0^t \|f_-(s)\|^p_{L^p(\Gamma_-)} ds - \int_0^t \|\gamma_+ u(s)\|^p_{L^p(\Gamma_+)} ds$$

and completes the proof. □

Remark 3.2. In the sequel, we use the fact that all expression on the form of the relation (3.3) is automatically solution of the Cauchy's problem (3.1).

In the sequel, when it is necessary, we implicitly define by zero all function outside their domain (for instance $f_\varphi(\cdot)$ in the next Lemma or the the operator-value functions $A_K(\cdot)$ in the Lemma 3.4)

Lemma 3.3. *Let $\|K\| < 1$. For all $\varphi \in L^p(\Omega)$, the equation*

$$f(t) = V_K(t)\varphi + H_K f(t), \tag{3.4}$$

has an unique solution $f_\varphi \in L^p(\mathbb{R}_+, L^p(\Gamma_-))$, where

$$V_K(t)\varphi = K\left[\gamma_+ U_0(t)\varphi\right], \tag{3.5}$$

$$H_K f(t,x,v) = K\left[\gamma_+ u(t)\right](x,v), \tag{3.6}$$

with

$$u(t,x,v) = \xi(t - t(x,v)) f(t - t(x,v), x - t(x,v)v, v), \tag{3.7}$$

$$\xi(t - t(x,v)) = 1 - \chi(t - t(x,v)) = \begin{cases} 1 & \text{if } t - t(x,v) \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

where χ is given by (2.4). Furthermore, the application

$$\varphi \in L^p(\Omega) \to f_\varphi \in L^p(\mathbb{R}_+, L^p(\Gamma_-)) \tag{3.9}$$

is linear and bounded satisfying

$$\|f_\varphi\|_{L^p(\mathbb{R}_+,L^p(\Gamma_-))} \leq \frac{\|K\|}{(1-\|K\|)}\|\varphi\|_p. \tag{3.10}$$

Proof. Let $\varphi \in L^p(\Omega)$. Using the boundedness of K we get that

$$\begin{aligned}
\|V_K(\cdot)\varphi\|^p_{L^p(\mathbb{R}_+,L^p(\Gamma_-))} &= \lim_{t\to\infty}\int_0^t \|V_K(s)\varphi\|^p_{L^p(\Gamma_-)}ds \\
&= \lim_{t\to\infty}\int_0^t \|K\left[\gamma_+U_0(s)\varphi\right]\|^p_{L^p(\Gamma_-)}ds \\
&\leq \|K\|^p \lim_{t\to\infty}\int_0^t \|\gamma_+U_0(s)\varphi\|^p_{L^p(\Gamma_+)}ds.
\end{aligned}$$

As the function $u(t,x,v) = U_0(t)\varphi(x,v)$ is solution of Cauchy's problem (3.1) with $f_- = 0, f_0 = \varphi$, then the relation (3.2) infers that

$$\|V_K(\cdot)\varphi\|_{L^p(\mathbb{R}_+,L^p(\Gamma_-))} \leq \|K\|\|\varphi\|_p \tag{3.11}$$

and therefore $V_K(\cdot)\varphi \in L^p(\mathbb{R}_+, L^p(\Gamma_-))$.

Let $f \in L^p(\mathbb{R}_+, L^p(\Gamma_-))$. Using the boundedness of K, we get

$$\begin{aligned}
\|H_K f\|^p_{L^p(\mathbb{R}_+,L^p(\Gamma_-))} &= \lim_{t\to\infty}\int_0^t \|H_K f(s)\|^p_p ds \\
&= \lim_{t\to\infty}\int_0^t \|K\left[\gamma_+u(s)\right]\|^p_{L^p(\Gamma_-)}ds \\
&\leq \|K\|^p \lim_{t\to\infty}\int_0^t \|\gamma_+u(s)\|^p_{L^p(\Gamma_+)}ds.
\end{aligned}$$

As $u = u(t,x,v)$ given by the relation (3.7) is solution of Cauchy's problem (3.1) with $f_- = f, f_0 = 0$, then the relation (3.2) infers that

$$\|H_K f\|^p_{L^p(\mathbb{R}_+,L^p(\Gamma_-))} \leq \|K\|^p \lim_{t\to\infty}\int_0^t \|f(s)\|^p_{L^p(\Gamma_-)}ds \leq \|K\|^p\|f\|^p_{L^p(\mathbb{R}_+,L^p(\Gamma_-))}.$$

Thus we have

$$\|H_K f\|_{L^p(\mathbb{R}_+,L^p(\Gamma_-))} \leq \|K\|\|f\|_{L^p(\mathbb{R}_+,L^p(\Gamma_-))} \tag{3.12}$$

and therefore $H_K f \in L^p(\mathbb{R}_+, L^p(\Gamma_-))$.

Since $\|K\| < 1$, for all $\varphi \in L^p(\Omega)$, the equation (3.4) admits an unique solution $f_\varphi \in L^p(\mathbb{R}_+, L^p(\Gamma_-))$ given by

$$f_\varphi = (I - H_K)^{-1}V_K(\cdot)\varphi$$

which implies the linearity of the application (3.9). Finally, using the relations (3.11) and (3.12) we get

$$\|f\|_{L^p(\mathbb{R}_+,L^p(\Gamma_-))} \leq \|K\|\|\varphi\|_p + \|K\|\|f\|_{L^p(\mathbb{R}_+,L^p(\Gamma_-))}$$

Now, the relation (3.10) follows. □

In the following Lemma, we give the second part of the semigroup given in the Proposition 3.5.

Lemma 3.4. *Let $\|K\| < 1$. For all $t \geq 0$, the operator $A_K(t)$ given by*
$$A_K(t)\varphi(x,v) = \xi(t - t(x,v))f_\varphi(t - t(x,v), x - t(x,v)v, v)$$
is a linear and bounded from $L^p(\Omega)$ into itself, where f_φ is given in the previous Lemma. Furthermore, for all $\varphi \in L^p(\Omega)$, we have
(1) $A_K(0) = 0$ *and* $\lim_{t\searrow 0} \|A_K(t)\varphi\|_p = 0$;
(2) $t \geq 0 \to A_K(t)\varphi$ *is continuous.*

Proof. (1). Let $t \geq 0$ and $\varphi \in L^p(\Omega)$. As $u(t,x,v) = A_K(t)\varphi(x,v)$ is solution of the Cauchy's problem $\mathrm{P}(f_- = f_\varphi, f_0 = 0)$ with $f_\varphi \in L^p(\mathbb{R}_+, L^p(\Gamma_-))$, then the relation (3.2) infers that
$$\|A_K(t)\varphi\|_p^p \leq \int_0^t \|f_\varphi(s)\|^p_{L^p(\Gamma_-)} ds \leq \|f_\varphi\|^p_{L^p(\mathbb{R}_+, L^p(\Gamma_-))},$$
which implies $A(0) = 0$, $A_K(t)\varphi \in L^p(\Omega)$ and $\lim_{t\searrow 0} \|A_K(t)\varphi\|_p = 0$. Furthermore, the previous relation together the relation (3.10) imply the boundedness of the operator $A_K(t)$ from $L^p(\Omega)$ into itself.

(2). Let $t \geq 0$ and $\varphi \in L^p(\Omega)$. For all $h > 0$, defining the function $u_h(t) = A_K(t+h)\varphi - A_K(t)\varphi$. As u_h is the solution of the Cauchy's problem (3.1) with $f_- = f_\varphi(\cdot + h) - f_\varphi, f_0 = A_K(h)\varphi$ and $f_\varphi(\cdot + h) - f_\varphi \in L^p(\mathbb{R}_+, L^p(\Omega))$ and $f_0 = A_K(h)\varphi \in L^p(\Omega)$, the relation (3.2) infers
$$\|A_K(t+h)\varphi - A_K(t)\varphi\|_p^p \leq \int_0^t \|f_\varphi(s+h) - f_\varphi(s)\|^p_{L^p(\Gamma_-)} ds + \|A_K(h)\varphi\|_p^p.$$
However, the operator V_K defined by the relation (3.5) satisfies
$$V_K(t+h)\varphi = K\left[\gamma_+ U_0(t+h)\varphi\right] = K\left[\gamma_+ U_0(t)U_0(h)\varphi\right] = V_K(t)\left[U_0(h)\varphi\right].$$
Thus, by uniqueness of the solution of the equation (3.4) we get that $f_\varphi(t+h) = f_{U_0(h)\varphi}(t)$ and therefore
$$\|A_K(t+h)\varphi - A_K(t)\varphi\|_p^p \leq \int_0^t \|f_{U_0(h)\varphi}(s) - f_\varphi(s)\|^p_{L^p(\Gamma_-)} ds + \|A_K(h)\varphi\|_p^p.$$
Using the linearity of the application $\varphi \to f_\varphi$ in the previous Lemma and the relation (3.10) we finally obtain
$$\begin{aligned}\|A_K(t+h)\varphi - A_K(t)\varphi\|_p^p &\leq \int_0^t \|f_{U_0(h)\varphi-\varphi}(s)\|^p_{L^p(\Gamma_-)} ds + \|A_K(h)\varphi\|_p^p \\ &\leq \|f_{U_0(h)\varphi-\varphi}\|^p_{L^p(\mathbb{R}_+, L^p(\Gamma_-))} + \|A_K(h)\varphi\|_p^p \\ &\leq \big[\frac{\|K\|}{(1-\|K\|)}\big]^p \|U_0(h)\varphi - \varphi\|_p^p + \|A_K(h)\varphi\|_p^p.\end{aligned}$$
Now the required continuity follows from the first part and the fact that $\{U_0(t)\}_{t\geq 0}$ is a semigroup. $\square$

The following Proposition is devoted to the explicit expression of a semigroup which will be, in the Theorem 4.2, the generated semigroup by the streaming operator given by the relation (1.1).

Proposition 3.5. *If $\|K\| < 1$, then the family $\{U_K(t)\}_{t\geq 0}$ defined by*
$$U_K(t) = U_0(t) + A_K(t), \quad t \geq 0, \tag{3.13}$$

is a C_0-semigroup on $L^p(\Omega)$. Furthermore, for all $\varphi \in L^p(\Omega)$, we have

$$U_K(t)\varphi(x,v) = U_0(t)(x,v) + \xi(t-t(x,v))K\left[\gamma_+ U_K(t-t(x,v))\varphi\right](x-t(x,v)v,v) \tag{3.14}$$

for all $t \geq 0$ and a.e. $(x,v) \in \Omega$.

Proof. Let $t \geq 0$ and $\varphi \in L^p(\Omega)$. Note that from the lemmas 2.1 and 3.4 the operator $U_K(t)$ is linear and bounded from $L^p(\Omega)$ into itself, $U_K(0) = U_0(0) + A_K(0) = 0 + I = I$ (I is the identity operator of $L^p(\Omega)$) and

$$\lim_{t\searrow 0} \|U_K(t)\varphi - \varphi\|_p \leq \lim_{t\searrow 0} \|U_0(t)\varphi - \varphi\|_p + \lim_{t\searrow 0} \|A_K(t)\varphi\|_p = 0.$$

Now, let us shown the formula (3.14) and let $\varphi \in D(T_0)$. First. Using the definition of the operator $A_K(t)$ in the previous Lemma and all of the notations of the Lemma 3.3 we get that

$$\gamma_- A_K(t)\varphi - K\left[\gamma_+ A_K(t)\varphi\right] = f_\varphi(t) - K\left[\gamma_+ A_K(t)\varphi\right] = K\left[\gamma_+ U_0(t)\varphi\right]$$

a.e. $t \geq 0$. As, the first point of the Lemma 2.2 and the boundedness of the operator K imply the continuity of the application $t \geq 0 \to K\left[\gamma_+ U_0(t)\varphi\right] \in L^p(\Gamma_-)$, then the previous equality holds for all $t \geq 0$.
Next. The previous relation and the fact that $U_0(t)\varphi \in D(T_0)$ imply

$$\begin{aligned}
\gamma_- U_K(t)\varphi - K\gamma_+ U_K(t)\varphi &= \gamma_- A_K(t)\varphi - K\gamma_+ U_K(t)\varphi \\
&= \gamma_- A_K(t)\varphi - K\gamma_+\left[U_0(t)\varphi + A_K(t)\varphi\right] \\
&= \gamma_- A_K(t)\varphi - K\gamma_+\left[U_0(t)\varphi\right] - K\gamma_+\left[A_K(t)\varphi\right] \\
&= K\gamma_+\left[U_0(t)\varphi\right] - K\gamma_+\left[U_0(t)\varphi\right] \\
&= 0
\end{aligned}$$

for all $t \geq 0$, which implies $\gamma_- U_K(t)\varphi = K\gamma_+ U_K(t)\varphi$ and therefore

$$\gamma_- A_K(t-t(x,v))\varphi(x-t(x,v)v,v) = K\left[\gamma_+ U_K(t-t(x,v))\varphi\right](x-t(x,v)v,v)$$

for all $t \geq 0$ and a.e. $(x,v) \in \Omega$. From other hand, the definition of $A_K(t)$ in the Lemma 3.4 infers that $\gamma_- A_K(t)\varphi(x,v) = f_\varphi(t,x,v)$ which implies

$$\begin{aligned}
\gamma_- A_K(t-t(x,v))\varphi(x-t(x,v)v,v) &= f_\varphi(t-t(x,v), x-t(x,v)v, v) \\
&= A_K(t)\varphi(x,v)
\end{aligned}$$

a.e. $(x,v) \in \Omega$ and for all $t \geq 0$ because of the second point of the previous Lemma. Now, the two previous relation gives us

$$A_K(t)\varphi(x,v) = K\left[\gamma_+ U_K(t-t(x,v))\varphi\right](x-t(x,v)v,v)$$

for all $t \geq 0$ and a.e. $(x,v) \in \Omega$. Replacing this relation in the relation (3.13), we obtain the relation (3.14) which holds on $L^p(\Omega)$ because of the density of $D(T_0)$ in $L^p(\Omega)$.

Now, in order to order to finish the proof, we have to show that

$$G_K(t,t') := U_K(t)U_K(t') - U_K(t+t') = 0$$

for all $t, t' \geq 0$. Using the relation (3.14), a simple calculation shows

$$G_K(t,t') = A_K(t)U_K(t') + \left(U_0(t)A_K(t') - A_K(t+t')\right)$$

Let $\varphi \in D(T_0)$. Using the definition of $A_K(t)$ and $U_K(t)$ we easily get

$$\begin{aligned} G_K(t,t')\varphi(x,v) &= \xi(t-t(x,v))K\left[\gamma_+ U_K(t-t(x,v))U_K(t')\varphi\right](x-t(x,v)v,v) \\ &\quad + \left[\chi(t-t(x,v))\xi(t+t'-t(x,v)) - \xi(t+t'-t(x,v))\right] \\ &\quad \times K\left[\gamma_+ U_K(t+t'-t(x,v),t')\varphi\right](x-t(x,v)v,v). \end{aligned}$$

Now the definition of χ and ξ implies

$$G_K(t,t')\varphi(x,v) = \xi(t-t(x,v))K\left[\gamma_+ G_K(t-t(x,v),t')\varphi\right](x-t(x,v)v,v).$$

As $u_{t'}(t) = G_K(t,t')\varphi$ is solution of the following Cauchy's problem (3.1) with $f_- = K[\gamma_+ G_K(\cdot,t')\varphi]$, $f_0 = 0$ and $f_- \in \mathcal{R}(K) \subset L^p(\Gamma_-)$, then the relation (3.2) and the boundedness of K infer that

$$\begin{aligned} \int_0^t \|\gamma_+ G(s,t')\varphi\|^p_{L^p(\Gamma_+)} ds &\le \int_0^t \|K\left[\gamma_+ G_K(s,t')\varphi\right]\|^p_{L^p(\Gamma_-)} ds \\ &\le \|K\|^p \int_0^t \|\gamma_+ G_K(s,t')\varphi\|^p_{L^p(\Gamma_+)} ds \end{aligned}$$

which implies

$$\int_0^t \|\gamma_+ G(s,t')\varphi\|_{L^p(\Gamma_+)} ds = 0$$

because of $\|K\| < 1$. From other hand, the relation (3.2) gives us

$$\begin{aligned} \|G_K(t,t')\varphi\|_p^p &\le \int_0^t \|K\left[\gamma_+ G_K(s,t')\varphi\right]\|^p_{L^p(\Gamma_-)} ds \\ &\le \|K\|^p \int_0^t \|\gamma_+ G_K(s,t')\varphi\|^p_{L^p(\Gamma_+)} ds. \end{aligned}$$

Now the two previous relations and the density of $D(T_0)$ in $L^p(\Omega)$ imply that $G_K(t,t') = 0$ for all $t, t' \ge 0$ and thus $\{U_K(t)\}_{t\ge 0}$ is a strongly continuous semigroup on $L^p(\Omega)$. The proof is complete. $\square$

Now, let us calculate the resolvent operator of the generator of the semigroup $\{U_K(t)\}_{t\ge 0}$ given in the previous Proposition. But, before to state this result, recall that the Albedo operator A associate to the following problem

$$\begin{aligned} &v \cdot \nabla_x u = 0, \quad \text{on } \Omega \\ &\gamma_- u = \psi \in L^p(\Gamma_-), \end{aligned}$$

is defined by $A\psi(x,v) = \psi(x-\tau(x,v)v,v)$. Furthermore, a simple calculation shows that $\|A\psi\|_{L^p(\Gamma_+)} = \|\psi\|_{L^p(\Gamma_-)}$ for all $\psi \in L^p(\Gamma_-)$ and thus $\|A\|_{\mathcal{L}(L^p(\Gamma_-),L^p(\Gamma_+))} = 1$.

Proposition 3.6. *Let $\|K\| < 1$ and suppose that $(B_K, D(B_K))$ is the generator of the semigroup $\{U_K(t)\}_{t\ge 0}$. Then, for all $\lambda > 0$, the resolvent of B_K is linear and bounded operator from $L^p(\Omega)$ into itself given by*

$$\begin{aligned} (\lambda - B_K)^{-1} g(x,v) &= (\lambda - T_0)^{-1} g(x,v) + \epsilon_\lambda(x,v) \\ &\quad \times \left[K(I-K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1} g\right](x-t(x,v)v,v) \end{aligned} \tag{3.15}$$

where I is the identity operator of $L^p(\Gamma_+)$, $\epsilon_\lambda(x,v) = e^{-\lambda t(x,v)}$ and $K_\lambda = [\gamma_+ \epsilon_\lambda] AK$ with A is the Albedo operator.

Proof. For all $\lambda > 0$ and all $\varphi \in D(T_0)$, we have
$$\|K_\lambda\|_{\mathcal{L}(L^p(\Gamma_+))} \le \|A\|_{\mathcal{L}(L^p(\Gamma_-),L^p(\Gamma_+))}\|K\|_{\mathcal{L}(L^p(\Gamma_+),L^p(\Gamma_-))} < 1 \tag{3.16}$$
which implies that the operators K_λ and $(I - K_\lambda)^{-1}$ belong to $\mathcal{L}(L^p(\Gamma_+))$ and therefore the relation (3.15) admits a sense because of the second point of the Lemma 2.2.

Let $\lambda > 0$ and $\varphi \in D(T_0)$. First, for a.e. $(x, v) \in \Gamma_+$, the relation (3.14) gives us
$$\begin{aligned}\gamma_+U_0(t)\varphi(x,v) =&\gamma_+U_K(t)\varphi(x,v)\\ &- \xi(t-\tau(x,v))K\left[\gamma_+U_K(t-\tau(x,v))\varphi\right](x-\tau(x,v)v,v)\\ =&\gamma_+U_K(t)\varphi(x,v)\\ &- \xi(t-\tau(x,v))AK\left[\gamma_+U_K(t-\tau(x,v))\varphi\right](x,v)\end{aligned}$$
for all $t \ge 0$ because the second point of the Lemma 2.2. Next, the first point of the Lemma 2.2 and the previous relation imply that
$$\begin{aligned}&\gamma_+(\lambda - T_0)^{-1}\varphi(x,v)\\ &= \gamma_+\left[\int_0^\infty e^{-\lambda t}U_0(t)\varphi dt\right](x,v)\\ &= \int_0^\infty e^{-\lambda t}\gamma_+U_0(t)\varphi(x,v)dt\\ &= \int_0^\infty e^{-\lambda t}\gamma_+U_K(t)\varphi(x,v)dt\\ &\quad - \int_0^\infty e^{-\lambda t}\xi(t-\tau(x,v))K\big[\gamma_+U_K(t-\tau(x,v))\varphi\big](x-\tau(x,v)v,v)dt.\end{aligned}$$
The change of variable $s = t - \tau(x, v)$ infers
$$\begin{aligned}&\gamma_+(\lambda - T_0)^{-1}\varphi(x,v)\\ &= \int_0^\infty e^{-\lambda t}\gamma_+U_K(t)\varphi(x,v)dt\\ &\quad - \int_0^\infty e^{-\lambda s}\left[(\gamma_+\epsilon_\lambda)K\left[\gamma_+U_K(s)\varphi\right]\right](x-\tau(x,v)v,v)ds\\ &= \int_0^\infty e^{-\lambda t}\gamma_+U_K(t)\varphi(x,v)dt\\ &\quad - (\gamma_+\epsilon_\lambda)A\Big[\int_0^\infty e^{-\lambda s}K\left[\gamma_+U_K(s)\varphi\right]ds\Big](x,v)\\ &= \int_0^\infty e^{-\lambda t}\gamma_+U_K(t)\varphi(x,v)dt - (\gamma_+\epsilon_\lambda)AK\left[\int_0^\infty e^{-\lambda s}\left[\gamma_+U_K(s)\varphi\right]ds\right](x,v)\\ &= \int_0^\infty e^{-\lambda t}\gamma_+U_K(t)\varphi(x,v)dt - K_\lambda\int_0^\infty e^{-\lambda s}\gamma_+U_K(s)\varphi(x,v)ds,\end{aligned}$$
where A is the Albedo operator. Using the boundedness of K_λ we get
$$\begin{aligned}\gamma_+(\lambda - T_0)^{-1}\varphi &= \int_0^\infty e^{-\lambda t}\gamma_+U_K(t)\varphi dt - K_\lambda\int_0^\infty e^{-\lambda s}\gamma_+U_K(s)\varphi ds\\ &= (I - K_\lambda)\int_0^\infty e^{-\lambda t}\gamma_+U_K(t)\varphi dt.\end{aligned}$$

The relation (3.16), implies the invertibility of $(I - K_\lambda)$ and thus

$$\int_0^\infty e^{-\lambda t}\gamma_+ U_K(t)\varphi dt = (I - K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1}\varphi. \tag{3.17}$$

On the other hand, using the relation (3.14), we get

$$\begin{aligned}
&\left[(\lambda - B_K)^{-1} - (\lambda - T_0)^{-1}\right]\varphi(x,v)\\
&= \int_0^\infty e^{-\lambda t}\left[U_K(t)\varphi - U_0(t)\varphi\right](x,v)dt\\
&= \int_0^\infty \xi(t - t(x,v))e^{-\lambda t}K\left[\gamma_+ U_K(t - t(x,v))\varphi\right](x - t(x,v)v, v)dt.
\end{aligned}$$

The change of variable $s = t - t(x,v)$ and the boundedness of K infer

$$\begin{aligned}
&\left[(\lambda - B_K)^{-1} - (\lambda - T_0)^{-1}\right]\varphi(x,v)\\
&= \epsilon_\lambda(x,v)\int_0^\infty e^{-\lambda s}K\left[\gamma_+ U_K(s)\varphi\right](x - t(x,v)v, v)ds\\
&= \epsilon_\lambda K\Big[\int_0^\infty e^{-\lambda s}\gamma_+ U_K(s)\varphi ds\Big](x - t(x,v)v, v).
\end{aligned}$$

Combining, the last relation with the relation (3.17) and the density of $D(T_0)$ in $L^p(\Omega)$, we obtain

$$(\lambda - B_K)^{-1}\varphi - (\lambda - T_0)^{-1}\varphi = \epsilon_\lambda K(I - K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1}\varphi$$

for all $\varphi \in L^p(\Omega)$. The proof is complete. □

4. Generation Theorem

In this section, we state the main generation Theorem where we prove that operator T_K given by the relation (1.1) is well the generator of the semigroup $\{U_K(t)\}_{t\geq 0}$.

Lemma 4.1. *Suppose that $\|K\| < 1$. If $\lambda > 0$, then we have $\lambda \in \rho(T_K)$ and*

$$(\lambda - B_K)^{-1} = (\lambda - T_K)^{-1}. \tag{4.1}$$

Proof. Let $\lambda > 0$, $g \in L^p(\Omega)$ and $\varphi = (\lambda - B_K)^{-1}g \in D(B_K) \subset L^p(\Omega)$. Using the relation (3.15), a simple calculation of derivative give us

$$\begin{aligned}
v\cdot\nabla_x\varphi(x,v) &= v\cdot\nabla_x(\lambda - T_0)^{-1}g(x,v)\\
&\quad + v\cdot\nabla_x\left[\epsilon_\lambda(x,v)\left[K(I - K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1}g\right](x - t(x,v)v, v)\right]\\
&= g(x,v) - \lambda(\lambda - T_0)^{-1}g(x,v)\\
&\quad - \lambda\epsilon_\lambda(x,v)\left[K(I - K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1}g\right](x - t(x,v)v, v)\\
&\quad + \epsilon_\lambda(x,v)v\cdot\nabla_x\left[K(I - K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1}g\right](x - t(x,v)v, v).
\end{aligned}$$

The last term vanishes and we can write $v\cdot\nabla_x\varphi = g - \lambda\varphi$ which implies

$$\|v\cdot\nabla_x\varphi\|_p = \|g - \lambda\varphi\|_p \leq \|g\|_p + \lambda\|\varphi\|_p < \infty$$

and therefore $\varphi = (\lambda - B_K)^{-1}g \in W^p(\Omega)$. Furthermore, we trivially have

$$\gamma_-(\lambda - B_K)^{-1}g = K(I - K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1}g \in L^p(\Gamma_-)$$

because the rang of K is such that $\mathcal{R}(K) \subset L^p(\Gamma_-)$. Thus we have $(\lambda - B_K)^{-1}g \in W_-^p(\Omega)$. But, using all of the notations of the Proposition 3.6, we get

$$\begin{aligned}
\gamma_-(\lambda - B_K)^{-1}g &= K(I - K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1}g \\
&= K\left[K_\lambda(I - K_\lambda)^{-1} + I\right]\gamma_+(\lambda - T_0)^{-1}g \\
&= K\left[K_\lambda(I - K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1}g + \gamma_+(\lambda - T_0)^{-1}g\right] \\
&= K\left[(\gamma_+\epsilon_\lambda)AK(I - K_\lambda)^{-1}\gamma_+(\lambda - T_0)^{-1}g + \gamma_+(\lambda - T_0)^{-1}g\right] \\
&= K\gamma_+(\lambda - B_K)^{-1}g
\end{aligned}$$

which implies $(\lambda - B_K)^{-1}g \in D(T_K)$ and therefore $\varphi = (\lambda - B_K)^{-1}g$ is the solution of $(\lambda - T_K)\varphi = g$. The arbitrary of $g \in L^p(\Omega)$ infers that $\lambda \in \rho(T_K)$ and the invertibility of the operator $(\lambda - T_K)$. $\square$

Now, we are able to state the main result of this work as follows.

Theorem 4.2. *If $\|K\| < 1$, then the operator T_K defined by the relation* (1.1)*, i.e,*

$$T_K\varphi(x,v) = -v \cdot \nabla_x\varphi(x,v), \quad \text{on the domain}$$
$$D(T_K) = \{\varphi \in W^p(\Omega),\ \gamma_\pm\varphi \in L^p(\Gamma_\pm), \quad \gamma_-\varphi = K\gamma_+\varphi\}$$

generates, on $L^p(\Omega)$, the strongly continuous semigroup $\{U_K(t)\}_{t\ge0}$ satisfying

$$\begin{aligned}
U_K(t)\varphi(x,v) &= U_0(t)(x,v) \\
&\quad + \xi(t - t(x,v))K\left[\gamma_+U_K(t - t(x,v))\varphi\right](x - t(x,v)v, v)
\end{aligned} \tag{4.2}$$

for all $t \ge 0$ and a.e. $(x,v) \in \Omega$ and all $\varphi \in L^p(\Omega)$. Furthermore,

$$\|U_K(t)\|_{\mathcal{L}(L^p(\Omega))} \le 1, \quad t \ge 0. \tag{4.3}$$

Proof. The existence of the semigroup $\{U_K(t)\}_{t\ge0}$ and the relation (4.2) are guaranteed by the Proposition 3.5. Now, let us show the identity $B_K = T_K$.

First. If $\varphi \in D(T_K)$, then the relation (4.1) infers

$$\varphi = (\lambda - T_K)^{-1}(\lambda - T_K)\varphi = (\lambda - B_K)^{-1}(\lambda - T_K)\varphi$$

which implies that $\varphi \in D(B_K) = \mathcal{R}((\lambda - B_K)^{-1}(\lambda - T_K))$ and therefore $D(T_K) \subset D(B_K)$. Inversely, if $\varphi \in D(B_K)$, the relation (4.1) also infers

$$\varphi = (\lambda - B_K)^{-1}(\lambda - B_K)\varphi = (\lambda - T_K)^{-1}(\lambda - B_K)\varphi$$

which implies that $\varphi \in D(T_K) = \mathcal{R}((\lambda - T_K)^{-1}(\lambda - B_K))$ and therefore $D(B_K) \subset D(T_K)$. Thus, $D(T_K) = D(B_K)$.

Next. Since the relation (4.1) we get, for all $\varphi \in D(T_K) = D(B_K)$, that $(\lambda - T_K)\varphi = (\lambda - B_K)\varphi$ which implies that $B_K\varphi = T_K\varphi$. Thus we have $B_K = T_K$.

To complete the proof, let us show the relation (4.3). Let $\varphi \in D(B_K) = D(T_K) \subset L^p(\Omega)$. As $u(t) = U_K(t)\varphi$ is the solution of the following Cauchy's problem (3.1) with $f_- = K[\gamma_+U_K(\cdot)\varphi], f_0 = \varphi$, applying the relation (3.2) together with the boundedness of the operator K we get, for all $t \ge 0$, that

$$\begin{aligned}
\|U_K(t)\varphi\|_p^p - \|\varphi\|_p^p &= \int_0^t \|K\gamma_+U_K(s)\varphi\|_{L^p(\Gamma_-)}^p ds - \int_0^t \|\gamma_+U_K(s)\varphi(s)\|_{L^p(\Gamma_+)}^p ds \\
&\le [\|K\|^p - 1]\int_0^t \|\gamma_+U_K(s)\varphi(s)\|_{L^p(\Gamma_+)}^p ds,
\end{aligned}$$

which implies $\|U_K(t)\varphi\|_p \le \|\varphi\|_p$ for all $t \ge 0$ because of $\|K\| < 1$. Now, the density of $D(B_K)$ in $L^p(\Omega)$ archives the proof. $\square$

Remark 4.3. First, we can positively close the conjecture [10, p 103]. Next, it is clear that the previous Theorem and all of result of this work are based on the fact that $\|K\| < 1$ and we cannot apply these results for the case $\|K\| \geq 1$. In [2], we give others and different proofs to obtain the same main objective of the present work, but for the case $\|K\| \geq 1$.

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