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DIRICHLET-NEUMANN BRACKETING FOR BOUNDARY-VALUE PROBLEMS ON GRAPHS

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Abstract. We consider the spectral structure of second order boundary-value problems on graphs. A variational formulation for boundary-value problems on graphs is given. As a consequence we can formulate an analogue of Dirichlet-Neumann bracketing for boundary-value problems on graphs. This in turn gives rise to eigenvalue and eigenfunction asymptotic approximations.

1. Introduction

Let G be an oriented graph with finitely many edges, each of finite length. We consider the second-order differential equation

$$ly := -\frac{d^2y}{dx^2} + q(x)y = \lambda y, \tag{1.1}$$

on G, where q is real valued and essentially bounded on G. At the vertices or nodes of G we impose formally self-adjoint boundary conditions, see [3] for more details regarding the self-adjointness of boundary conditions.

We give a variational formulation for a class of self-adjoint boundary-value problems on graphs, Lemma 3.1, and hence a max-min principle for Sturm-Liouville boundary-value problems on directed graphs, Theorem 4.1. In turn, this enables us to develop one of our two main results, an analogue of Dirchlet-Neumann bracketing for the eigenvalues of the boundary-value problem, in Corollary 5.1. Corollary 5.1 forms a theoretical structure for our second main result, Theorem 6.1, in which spectral asymptotics are found. In amongst the above noted results we show the differential operator associated with the boundary-value problem to be lower semi-bounded, see Theorem 2.1.

It should be noted that, a self-adjoint boundary-value problem on a graph is not necessarily regular in the sense of [10] and [11]. For the case of regular boundary conditions many of the results in this paper are well known. In fact many of the more important classes of boundary conditions for boundary-value problems on graphs fail to meet these regularity conditions, for example "Kirchhoff" boundary conditions.

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In parallel to the variational aspects of boundary-value problems on graphs studied here and on trees in [15], the work of Pokornyi and Pryadiev, and Pokornyi, Pryadiev and Al-Obeid, in [12] and [13], should be noted for the extension of Sturmian oscillation theory to second order operators on graphs. The idea of approximating the behaviour of eigenfunctions and eigenvalues for a boundary-value problem on a graph by the behaviour of associated problems on the individual edges, used here, appeared previously in [16].

An extensive survey of the physical systems giving rise to boundary-value problems on graphs can be found in [9] and the bibliography thereof. Second order boundary-value problems on finite graphs arise naturally in quantum mechanics and circuit theory, [2, 6]. Multi-point boundary-value problems and periodic boundary-value problems can be considered as particular cases of boundary-value problems on graphs, [4].

In Section 2 the boundary-value problem, which forms the topic of this paper, is stated and allowable boundary conditions discussed. An operator formulation is given along with definitions of the various function spaces used in the paper. Following this, we show the operator associated with the boundary-value problem to be lower semibounded.

A variational reformulation of the boundary-value problem is given in Section 3. This leads to a max-min characterization of the eigenvalues of the boundary-value problem and hence to a type of Dirichlet-Neumann bracketing of the eigenvalues. For the analogue in the case of partial differential equations we refer the reader to [5].

2. Preliminaries

Let G denote a directed graph with a finite number of edges, say K, each of finite length and having the path-length metric. Each edge, e_i, of length say l_i can thus be considered as the interval $[0, l_i]$, where 0 is identified with the initial point of e_i and l_i with the terminal point.

The following classes of function spaces will be used in this paper, the first three of which are Hilbert spaces when given Sobolev norms:

$$\begin{aligned}
\mathcal{L}^2(G) &:= \bigoplus_{i=1}^{K} \mathcal{L}^2(0, l_i), \\
\mathcal{H}^m(G) &:= \bigoplus_{i=1}^{K} \mathcal{H}^m(0, l_i), \quad m = 0, 1, 2, \dots, \\
\mathcal{H}_o^m(G) &:= \bigoplus_{i=1}^{K} \mathcal{H}_o^m(0, l_i), \quad m = 0, 1, 2, \dots, \\
\mathcal{C}^\omega(G) &:= \bigoplus_{i=1}^{K} \mathcal{C}^\omega(0, l_i), \quad \omega = \infty, 0, 1, 2, \dots, \\
\mathcal{C}_o^\omega(G) &:= \bigoplus_{i=1}^{K} \mathcal{C}_o^\omega(0, l_i), \quad \omega = \infty, 0, 1, 2, \dots .
\end{aligned}$$

The inner product on $\mathcal{H}^m(G)$, denoted $(\cdot,\cdot)_m$, is defined by

$$(f,g)_m := \sum_{i=1}^{K}\sum_{j=0}^{m}\int_0^{l_i} f|_{e_i}^{(j)}\ \bar{g}|_{e_i}^{(j)}\ dt =: \sum_{j=0}^{m}\int_G f^{(j)}\ \bar{g}^{(j)}\ dt. \tag{2.1}$$

The inner products on $\mathcal{L}^2(G)$ and $\mathcal{H}_o^m(G)$ follow from noting that $\mathcal{L}^2(G) = \mathcal{H}^0(G)$ and $\mathcal{H}_o^m(G) \subset \mathcal{H}^m(G)$. For brevity we will write $(\cdot,\cdot) = (\cdot,\cdot)_0$, $\|f\|_m^2 = (f,f)_m$ and $\|f\| = \|f\|_0$. Reasoning componentwise we obtain immediate analogues of both Rellich's Theorem, [19, page 114], and the Sobolev Embedding Theorem, [19, page 107]. In particular the embedding of $\mathcal{H}^m(G)$ in $\mathcal{H}^n(G)$ for $n < m$ is a compact map and for each $f \in \mathcal{H}^m(G)$ there exists $g \in \mathcal{C}^n(G)$ such that $f^{(k)} = g^{(k)}$ a.e. on G for all $k = 0,\dots,n$. Making this identification, we have

$$\mathcal{H}^m(G) \subset \mathcal{C}^n(G), \quad n < m,$$

and there exists a constant $C(G,m) > 0$ such that

$$\sup_G |f^{(k)}| \le C(G,m)\|f\|_m \quad \text{for all } f \in \mathcal{H}^m(G),\ k < m.$$

Since Rellich's Theorem holds, the abstract Ehrling Lemma, [19, page 114] applies and yields a concrete Ehrling Lemma, i.e. for each $\epsilon > 0$ there exists a constant $C(G,m) > 0$ such that

$$\|f\|_{m-1} \le \epsilon\|f\|_m + C(G,m)\|f\|_0 \quad \text{for all } f \in \mathcal{H}^m(G).$$

The differential equation (1.1) on the graph G can now be considered as the system of equations

$$-\frac{d^2y_i}{dx^2} + q_i(x)y_i = \lambda y_i, \quad x \in [0,l_i],\ i = 1,\dots,K, \tag{2.2}$$

where q_i and y_i denote $q|_{e_i}$ and $y|_{e_i}$.

The boundary conditions at the node ν are specified in terms of the values of y and y' at ν on each of the incident edges. In particular if the edges which start at ν are $e_i, i \in \Lambda_s(\nu)$ and the edges which end at ν are $e_i, i \in \Lambda_e(\nu)$ then the boundary conditions at ν can be expressed as

$$\sum_{j\in\Lambda_s(\nu)} \left[\alpha_{ij}y_j + \beta_{ij}{y'}_j\right](0) + \sum_{j\in\Lambda_e(\nu)} \left[\gamma_{ij}y_j + \delta_{ij}{y'}_j\right](l_j) = 0, \quad i = 1,\dots,N(\nu), \tag{2.3}$$

where $N(\nu)$ is the number of linearly independent boundary conditions at node ν. For formally self-adjoint boundary conditions $N(\nu) = \sharp(\Lambda_s(\nu)) + \sharp(\Lambda_e(\nu))$ and $\sum_\nu N(\nu) = 2K$, see [3, 11] for more details.

Let $\alpha_{ij} = 0 = \beta_{ij}$ for $i = 1,\dots,N(\nu)$ and $j \notin \Lambda_s(\nu)$ and similarly let $\gamma_{ij} = 0 = \delta_{ij}$ for $i = 1,\dots,N(\nu)$ and $j \notin \Lambda_e(\nu)$. The boundary conditions (2.3) considered over all nodes ν, after possible relabelling, may thus be written as

$$\sum_{j=1}^{K} \left[\alpha_{ij}y_j + \beta_{ij}{y'}_j\right](0) + \sum_{j=1}^{K} \left[\gamma_{ij}y_j + \delta_{ij}{y'}_j\right](l_j) = 0, \quad i = 1,\dots,2K, \tag{2.4}$$

where $2K$ is the total number of linearly independent boundary conditions. It should be noted that the complete geometry of the graph G (other than the number of and length of the edges) is encapsulated in the boundary conditions.

The boundary-value problem (2.2)-(2.3) on G can be formulated as an operator eigenvalue problem in $\mathcal{L}^2(G)$, [1, 3, 14], for the closed densely defined operator

$$Lf := -f'' + qf \tag{2.5}$$

with domain

$$\mathcal{D}(L) = \{ f \mid f, f' \in AC, Lf \in \mathcal{L}^2(G),\ f \text{ obeying (2.3) } \}. \tag{2.6}$$

The formal self-adjointness of (2.2)-(2.3) ensures that L is a closed densely defined self-adjoint operator in $\mathcal{L}^2(G)$, see [8, 11, 17].

Theorem 2.1. *The operator L is lower semibounded.*

Proof. From [17, page 247, Corollary 2] as L is self adjoint, we need only show that L is lower semibounded on $\mathcal{C}_o^\infty(G)$. Let $f \in \mathcal{C}_o^\infty(G)$. Then

$$(Lf, f) = \int_G (-f''\bar{f} + q|f|^2)\ dx = \int_G (|f'|^2 + q|f|^2)\ dx \geq -\|f\|^2 \operatorname{ess\,sup} |q|.$$

$\square$

3. Variational Formulation

In this section we give an $\mathcal{H}^1(G)$, variational formulation for the boundary-value problem (2.2)-(2.3) or equivalently for the eigenvalue problem associated with the operator L defined in (2.5)-(2.6). For details in the setting of partial differential equations we refer the reader to [5]. The variational formulation gives rise to a max-min characterization of the eigenvalues and eigenfunctions of the boundary-value problem, developed in the next section. We conclude the section by proving that the $\mathcal{H}^1(G)$ eigenfunctions are in fact regular, i.e. are in $\mathcal{H}^2(G)$.

Without loss of generality, we assume the boundary conditions (2.4) to be in the form

$$\sum_{j=1}^{K} [\alpha_{ij} y_j(0) + \gamma_{ij} y_j(l_j)] = 0, \quad i = 1, \dots, J, \tag{3.1}$$

$$\sum_{j=1}^{K} [\alpha_{ij} y_j(0) + \beta_{ij} y_j'(0) + \gamma_{ij} y_j(l_j) + \delta_{ij} y_j'(l_j)] = 0, \quad i = J+1, \dots, 2K, \tag{3.2}$$

where we take $y_i := y|_{e_i}$. Here all possible Dirichlet-like terms are in (3.1), i.e. if (3.2) is written in matrix form then Gauss-Jordan reduction will not allow any pure Dirichlet conditions linearly independent of (3.1) to be extracted.

Let $F(x, y)$ to be the sesquilinear form given by

$$F(x, y) := \int_{\partial G} f x \overline{y}\, d\sigma + \int_G (x'\overline{y}' + xq\overline{y})\, dt, \tag{3.3}$$

with domain $\mathcal{D}(F) = \{ y \in \mathcal{H}^1(G) \mid y \text{ obeys (3.1)} \}$, where

$$\int_{\partial G} y\, d\sigma := \sum_{i=1}^{K} [y_i(l_i) - y_i(0)] = \int_G y'\, dt.$$

Definition. We say that the boundary conditions on a graph are co-normal with respect to l if there exists f defined on ∂G, such that $x \in \mathcal{D}(F)$ has

$$\int_{\partial G} f x \overline{y}\, d\sigma = \int_{\partial G} x' \overline{y}\, d\sigma, \quad \text{for all} \quad y \in \mathcal{D}(F)$$

if and only if x obeys (3.2).

We remark that co-normal boundary conditions on a graph correspond in nature to co-normal (non-oblique) boundary conditions for elliptic partial differential operators.

Most physically interesting boundary conditions on graphs fall into the co-normal category. In particular, 'Kirchhoff', Dirichlet, Neumann and periodic boundary conditions are all co-normal, but this class does not include all self-adjoint boundary-value problems on graphs. For example consider a single loop, i.e. the interval $[0, 1]$ where the boundary conditions at 0 and at 1 are connected as follows $y(0) = y'(1)$ and $y(1) = -y'(0)$. These boundary conditions give a self-adjoint boundary-value problem with non co-normal boundary conditions.

The following lemma shows that a function is a variational eigenfunction if and only if it is a classical eigenfunction.

Lemma 3.1. *Suppose that (3.1)-(3.2) are co-normal boundary conditions with respect to l of (1.1). Then $u \in \mathcal{D}(F)$ satisfies $F(u, v) = \lambda(u, v)$ for all $v \in \mathcal{D}(F)$ if and only if $u \in \mathcal{H}^2(G)$ and u obeys (1.1), (3.1)-(3.2).*

Proof. Assume that $u \in \mathcal{H}^2(G)$ and u obeys (1.1), (3.1)-(3.2). Then for each $v \in \mathcal{D}(F)$

$$\begin{aligned} F(u, v) &= \int_{\partial G} f u \overline{v}\, d\sigma + \int_G (u'\overline{v}' + q u \overline{v})\, dt \\ &= \int_{\partial G} f u \overline{v}\, d\sigma + \int_G ((u'\overline{v})' - u''\overline{v} + q u \overline{v})\, dt \\ &= \int_{\partial G} f u \overline{v}\, d\sigma + \int_G (u'\overline{v})'\, dt + \lambda(u, v) \\ &= \int_{\partial G} (f u + u')\overline{v}\, d\sigma + \lambda(u, v). \end{aligned}$$

The assumption that (3.1)-(3.2) are co-normal boundary conditions with respect to l gives that $u \in \mathcal{D}(F)$ and

$$\int_{\partial G} (f u + u')\overline{v}\, d\sigma = 0, \quad \text{for all } v \in \mathcal{D}(F),$$

completing the proof this in case.

Now assume $u \in \mathcal{D}(F)$ satisfies $F(u, v) = \lambda(u, v)$ for all $v \in \mathcal{D}(F)$. As $\mathcal{C}_o^\infty(G) \subset \mathcal{D}(F)$, it follows that

$$F(u, v) = \lambda(u, v), \quad \text{for all } v \in \mathcal{C}_0^\infty(G).$$

Hence $F(u, \cdot)$ can be extended to a continuous linear functional on $\mathcal{L}^2(G)$. In particular this gives that

$$\partial u' \in \mathcal{L}^2(G) \subset \mathcal{L}^1_{\text{loc}}(G)$$

where ∂ denotes the distributional derivative. Then, by [14, Theorem 1.6, page 44], $u' \in AC$ and $u'' \in \mathcal{L}^1_{\text{loc}}(G)$ allowing integration by parts. Thus

$$lu = -u'' + qu \in \mathcal{L}^1_{\text{loc}}(G)$$

and consequently $lu = \lambda u \in \mathcal{L}^2(G)$. Now $q \in \mathcal{L}^\infty(G)$ and $\mathcal{D}(F) \subset \mathcal{L}^2(G)$, giving $u, u'' \in \mathcal{L}^2(G)$ and hence $u \in \mathcal{H}^2(G)$.

The definition of $\mathcal{D}(F)$ ensures that (3.1) holds. Integration by parts gives

$$\int_{\partial G} (fu + u')\bar{y}\, d\sigma = 0, \quad \text{for all } y \in \mathcal{D}(F),$$

which, from the definition of f and the constraints on the class of boundary conditions allowed, is equivalent to u obeying (3.2). $\square$

4. Max-Min Property

In this section we give a maximum-minimum characterization for the eigenvalues of boundary-value problems on graphs. We refer the reader to [5, page 406] and [18] where boundary-value problems for partial differential operators are considered, and analogous results for such eigenvalues developed.

In the following theorem $\{v_0, \dots, v_{n-1}\}^\perp$ will denote the orthogonal complement in $\mathcal{L}^2(G)$ of $\{v_0, \dots, v_{n-1}\}$. In addition, as is customary, it will be assumed that the eigenvalues, λ_n, are listed in increasing order and repeated according to multiplicity, and that the eigenfunctions, y_n, are chosen so as to form a complete orthonormal family in $\mathcal{L}^2(G)$. In this case it is easily verified that $F(y_i, y_j) = \lambda_i \delta_{i,j}$.

Theorem 4.1. *For $v_j \in \mathcal{L}^2(G), j = 0, 1, \dots,$ let*

$$d_n(v_0, \dots, v_{n-1}) = \inf \left\{ \frac{F(\varphi,\varphi)}{||\varphi||^2} \;\middle|\; \varphi \in \{v_0, \dots, v_{n-1}\}^\perp \cap D(F) \setminus \{0\} \right\}. \tag{4.1}$$

Then

$$\lambda_n = \sup \{d_n(v_0, \dots, v_{n-1}) \mid v_0, \dots, v_{n-1} \in \mathcal{L}^2(G)\}, \quad \textit{for } n = 0, 1, \dots,$$

and this maximum-minimum is attained for $\varphi = y_n$ and $v_i = y_i$, $i = 0, \dots, n-1$.

Proof. Let $v_0, \dots, v_{n-1} \in \mathcal{L}^2(G)$. As $\operatorname{span}\{y_0, \dots, y_n\}$ is $n+1$ dimensional and $\operatorname{span}\{v_0, \dots, v_{n-1}\}$ is at most n dimensional there exists φ in $\operatorname{span}\{y_0, \dots, y_n\} \setminus \{0\}$ having

$$(\varphi, v_i) = 0, \quad \text{for all} \quad i = 0, \dots, n-1.$$

In particular, this ensures that $\varphi \in \mathcal{D}(F)$ as each y_i is in $\mathcal{D}(F)$.

Denote $\varphi = \sum_{k=0}^n c_k y_k$, then

$$F(\varphi,\varphi) = \sum_{i,k=0}^n c_i \bar{c}_k F(y_i, y_k) = \sum_{i,k=0}^n c_i \bar{c}_k \lambda_i \delta_{i,j} = \sum_{i=0}^n |c_i|^2 \lambda_i \le \lambda_n \sum_{i=0}^n |c_i|^2 = \lambda_n \|\varphi\|^2,$$

thus showing that

$$d_n(v_0, \dots, v_{n-1}) \le \lambda_n \quad \text{for all } v_0, \dots, v_{n-1} \in \mathcal{L}^2(G).$$

For brevity denote

$$m := \sup \{d_n(v_0, \dots, v_{n-1}) \,|\, v_0, \dots, v_{n-1} \in \mathcal{L}^2(G)\}.$$

The above reasoning has shown that $m \le \lambda_n$.

In order to complete the proof we require that there exists $\varphi \in \mathcal{D}(F)$ with $\|\varphi\| = 1$ and $(\varphi, v_i) = 0$ for all $i = 0, \dots, n-1$ such that $F(\varphi, \varphi) = d_n(v_0, \dots, v_{n-1})$. From the definition of $d_n(v_0, \dots, v_{n-1})$, there exists a sequence $(u_k) \subset \mathcal{D}(F)$ with $\|u_k\| = 1$ and $(u_k, v_i) = 0$ for all $i = 0, \dots, n-1$ and $k \in \mathbb{N}$ such that

$$\lim_{k \to \infty} F(u_k, u_k) = d_n(v_0, \dots, v_{n-1}).$$

As $\mathcal{H}^1(G)$ is compactly embedded in $\mathcal{L}^2(G)$, see [14, page 64], there exists a subsequence of (u_k), which we again denote by (u_k) which converges in $\mathcal{L}^2(G)$ to say u with $\|u\| = 1$.

To show that $u \in \mathcal{H}^1(G)$ we need only show that the distribution ∂u is in $\mathcal{L}^2(G)$. For each $\psi \in \mathcal{C}_o^\infty(G) \subset \mathcal{D}(F)$,

$$\partial u(\psi) = -\int_G u\psi' \, dt = -\lim_{k\to\infty} \int_G u_k \psi' \, dt = \lim_{k\to\infty} \int_G u_k' \psi \, dt.$$

Thus

$$|\partial u(\psi)| \le \limsup \|u_k'\| \|\psi\| \le [d_n(v_0, \dots, v_{n-1}) + \operatorname{ess\,sup} |q|]^{1/2} \|\psi\|$$

and ∂u can be extended to a continuous linear functional on $\mathcal{L}^2(G)$. By the Riesz-Fischer Theorem this gives $\partial u \in \mathcal{L}^2(G)$ and then by [14, Theorem 1.6, page 44], $u \in AC$ with

$$\|u'\|^2 \le d_n(v_0, \dots, v_{n-1}) + \operatorname{ess\,sup} |q|.$$

Thus $u \in \mathcal{H}^1(G)$ and as

$$(u_k', \psi) = -(u_k, \psi') \to -(u, \psi') = (u', \psi), \quad \text{for all } \psi \in \mathcal{C}_o^\infty(G),$$

it follows, by [1] applied componentwise, that $u_k \to u$ in $\mathcal{H}^1(G)$.

Hence there exists $u \in \mathcal{D}(F) \cap \{v_0, \dots, v_{n-1}\}^\perp \backslash \{0\}$ with $F(u, u) = d_n(v_0, \dots, v_{n-1})$ and $\|u\| = 1$. We now show that such a u is an eigenfunction of (1.1), (3.1)-(3.2) with eigenvalue $\lambda = d_n(v_0, \dots, v_{n-1})$.

Let

$$J(\varphi, \epsilon) = \frac{F(u + \epsilon\varphi)}{\|u + \epsilon\varphi\|^2} \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(G), \epsilon \in \mathbb{R}.$$

Differentiation with respect to ϵ of $J(\varphi, \epsilon)$ gives

$$0 = \frac{\partial}{\partial\epsilon} J(\varphi, \epsilon)|_{\epsilon=0} = 2\Re[F(\varphi, u) - d_n(v_0, \dots, v_{n-1})(\varphi, u)],$$

for all $\varphi \in \mathcal{C}_0^\infty(G)$. Thus u is a variational eigenfunction with eigenvalue $\lambda = d_n(v_0, \dots, v_{n-1})$. Lemma 3.1 now gives that u is in $\mathcal{H}^2(G)$, obeys boundary conditions (3.1)-(3.2) and equation (1.1) with $\lambda = d_n(v_0, \dots, v_{n-1})$, making u and eigenfunction of (1.1), (3.1)-(3.2) with eigenvalue λ.

In the case of $n = 0$, d_0 does not depend on any v_i and d_0 is an eigenvalue having $m = d_0 \le \lambda_0$. Thus, in this case, $m = d_0 = \lambda_0$.

In general we have shown $d_n(v_0, \dots, v_{n-1})$ to be an eigenvalue less than or equal to λ_n and $m \le \lambda_n$. But if $v_i = y_i, i = 0, \dots, n-1$, then for u to be orthogonal to $v_0, \dots, v_{n-1}$ and an eigenfunction to an eigenvalue, μ, less than or equal to λ_n forces $\mu = \lambda_n$ and u to be in the eigenspace of λ_n and orthogonal to $y_0, \dots, y_{n-1}$. $\square$

5. Eigenvalue Bracketing

If the boundary conditions (2.4) are replaced by the Dirichlet condition $y = 0$ at each node of G, i.e.

$$y_i(l_i) = 0 \quad \text{and} \quad y_i(0) = 0, \quad i = 1, \dots, K, \tag{5.1}$$

then the graph G becomes disconnected with each edge e_i becoming a component sub-graph, G_i, with Dirichlet boundary conditions at its two nodes (ends). The boundary-value problem on each sub-graph G_i is equivalent to a Sturm-Liouville boundary-value problem on a compact interval with Dirichlet boundary conditions.

Denote by $A(\lambda)$ the number of eigenvalues less than λ, counted according to multiplicity, of (1.1), (3.1)-(3.2). Let $A^D(\lambda)$ be the number of eigenvalues less than λ of (1.1) but with (3.1)-(3.2) replaced by Dirichlet boundary conditions as discussed above, and let $A_j^D(\lambda)$ be the number of eigenvalues less than λ of (1.1) on G_j with Dirichlet boundary conditions. Then

$$\sum_{j=1}^{K} A_j^D(\lambda) = A^D(\lambda), \quad \lambda \in \mathbb{R}.$$

Denote by λ_n^D the eigenvalues of (1.1) with Dirichlet boundary conditions, as discussed above.

Consider the boundary-value problem (1.1), (3.1)-(3.2) with the boundary conditions (3.1)-(3.2) replaced by the non-Dirichlet conditions

$$y_i'(l_i) = f(l_i)y_i(l_i) \quad \text{and} \quad y_i'(0) = f(0)y_i(0), \quad i = 1, \dots, K \tag{5.2}$$

where f is given in (3.3), then, as in the Dirichlet case above, G decomposes into a union of disconnected graphs $G_1, \dots, G_K$. Let λ_n^N denote the eigenvalues of (1.1), (5.2) and $A^N(\lambda)$ the number of eigenvalues less than λ counted according to multiplicity.

Let $A_i^N(\lambda)$ denote the number of eigenvalues less than λ of (1.1) on G_i with boundary conditions

$$y_i'(l_i) = f(l_i)y_i(l_i) \quad \text{and} \quad y_i'(0) = f(0)y_i(0).$$

Then

$$\sum_{i=1}^{K} A_i^N(\lambda) = A^N(\lambda).$$

In the case of co-normal boundary conditions, Theorem 4.1 has as a consequence that the spectral counting functions defined above are related by

$$\sum_{i=1}^{K} A_i^D(\lambda) = A^D(\lambda) \le A(\lambda) \le A^N(\lambda) = \sum_{i=1}^{K} A_i^N(\lambda), \quad \lambda \in \mathbb{R}, \tag{5.3}$$

and hence the eigenvalues are ordered by

$$\lambda_n^N \le \lambda_n \le \lambda_n^D, \quad n = 0, 1, \dots. \tag{5.4}$$

These results are the content of the following corollary to Theorem 4.1 which gives an analogue of [5, pages 407-410] for graphs.

Corollary 5.1. *If the boundary conditions (3.1)-(3.2) are co-normal with respect to l, then the the spectral counting functions for (1.1), (3.1)-(3.2) and the related boundary-value problems with the Dirichlet and non-Dirichlet boundary conditions given in (5.1) and (5.2) are related by (5.3) and their spectra are related by (5.4).*

Proof. Denote by F_D the restriction of F to $\mathcal{H}_o^1(G)$ and by F_N the continuous extension (with respect to the $\mathcal{H}^1(G)$ norm) of F to $\mathcal{H}^1(G)$. As $\mathcal{H}_o^1(G) \subset \mathcal{D}(F) \subset$

$\mathcal{H}^1(G)$ it follows that

$$
\begin{aligned}
&\left\{ \frac{F_D(\varphi,\varphi)}{\|\varphi\|^2} \;\middle|\; \varphi \in \{v_0,\dots,v_{n-1}\}^\perp \cap \mathcal{H}^1_o(G)\backslash\{0\} \right\} \\
&\subset \left\{ \frac{F(\varphi,\varphi)}{\|\varphi\|^2} \;\middle|\; \varphi \in \{v_0,\dots,v_{n-1}\}^\perp \cap \mathcal{D}(F)\backslash\{0\} \right\} \\
&\subset \left\{ \frac{F_N(\varphi,\varphi)}{\|\varphi\|^2} \;\middle|\; \varphi \in \{v_0,\dots,v_{n-1}\}^\perp \cap \mathcal{H}^1(G)\backslash\{0\} \right\}.
\end{aligned}
$$

Taking infima gives

$$
\begin{aligned}
&d_n^D(v_0,\dots,v_{n-1}) := \inf \left\{ \frac{F_D(\varphi,\varphi)}{\|\varphi\|^2} \;\middle|\; \varphi \in \{v_0,\dots,v_{n-1}\}^\perp \cap \mathcal{H}^1_o(G)\backslash\{0\} \right\} \\
&\geq d_n(v_0,\dots,v_{n-1}) \\
&\geq \inf \left\{ \frac{F_N(\varphi,\varphi)}{\|\varphi\|^2} \;\middle|\; \varphi \in \{v_0,\dots,v_{n-1}\}^\perp \cap \mathcal{H}^1(G)\backslash\{0\} \right\} =: d_n^N(v_0,\dots,v_{n-1}).
\end{aligned}
$$

Theorem 4.1 now gives

$$
\begin{aligned}
\lambda_n^D &= \sup\{d_n^D(v_0,\dots,v_{n-1}) \mid v_0,\dots,v_{n-1} \in \mathcal{L}^2(G)\} \\
&\geq \lambda_n = \sup\{d_n(v_0,\dots,v_{n-1}) \mid v_0,\dots,v_{n-1} \in \mathcal{L}^2(G)\} \\
&\geq \sup\{d_n^N(v_0,\dots,v_{n-1}) \mid v_0,\dots,v_{n-1} \in \mathcal{L}^2(G)\} = \lambda_n^N
\end{aligned}
$$

from which the claims of the theorem follow directly. □

6. Eigenvalue Asymptotics

The results of the previous section provide a means by which to approximate the spectrum of a boundary-value problem (2.2) when boundary conditions (2.4) are of co-normal type, by considering the spectrum of a finite family of Sturm-Liouville problems on bounded intervals with separated boundary conditions. Sturm-Liouville problems on bounded intervals with separated boundary conditions have been extensively studied, and consequently eigenvalue approximations for such problems are well known, see [7]. These eigenvalue approximations in turn provide information about the spectral counting function for each Sturm-Liouville problem. Corollary 5.1 can now be applied, giving bounds on the spectral counting function for the original boundary-value problem on the graph, from which eigenvalue asymptotics can be deduced.

Theorem 6.1. *Let G be a compact graph with finitely many nodes. If the boundary-value problem (2.2), (2.4) has co-normal boundary conditions, then its eigenvalues obey the asymptotic development*

$$\lambda_n = \frac{n^2\pi^2}{L^2} + O(n), \quad \text{as } n \to \infty,$$

and its spectral counting function has asymptotic approximation

$$A(\lambda) = \frac{L\sqrt{\lambda}}{\pi} + O(1), \quad \text{as } \lambda \to \infty,$$

where $L = \sum_{i=1}^K l_i$ is the total length of the graph.

Proof. In this proof we use the notation of Section 4. If we denote by $\lambda_n^{D,i}, n = 0, 1, \dots$ the eigenvalues of l operating on the graph G_i with Dirichlet conditions at both ends, then [7, Theorem A4] gives that

$$\lambda_n^{D,i} = \frac{(n+1)^2\pi^2}{l_i^2} + O(1), \quad n = 0, 1, \dots,$$

and consequently as $\lambda \to \infty$ we obtain

$$A_i^D(\lambda) \geq \frac{l_i\sqrt{\lambda - c_i^D}}{\pi} - 1, \tag{6.1}$$

for some constant $c_i^D > 0$.

Similarly, if we denote by $\lambda_n^{N,i}, n = 0, 1, \dots$ the eigenvalues of l operating on the graph G_i with the separated boundary conditions given in (5.2), then [7, Theorem A4] gives that

$$\lambda_n^{N,i} = \frac{n^2\pi^2}{l_i^2} + O(1), \quad n = 0, 1, \dots,$$

and consequently for large λ

$$A_i^N(\lambda) \leq \frac{l_i\sqrt{\lambda + c_i^N}}{\pi} + 1, \tag{6.2}$$

for some constant $c_i^N > 0$.

Taking $c = \max_{i=1,\dots,K}\{c_i^D, c_i^N\}$, equations (6.1) and (6.2) remain valid with c_i^D and c_i^N replaced by c. Thus (6.1) and (6.2) yield

$$\frac{l_i\sqrt{\lambda - c}}{\pi} - 1 \leq A_i^D(\lambda) \leq A_i^N(\lambda) \leq \frac{l_i\sqrt{\lambda + c}}{\pi} + 1, \quad \text{as} \quad \lambda \to \infty. \tag{6.3}$$

Corollary 5.1, equation (5.3), can now be combined with (6.3) to give

$$\frac{L\sqrt{\lambda - c}}{\pi} - K \leq \sum_{i=1}^{K} A_i^D(\lambda) \leq A(\lambda) \leq \sum_{i=1}^{K} A_i^N(\lambda) \leq \frac{L\sqrt{\lambda + c}}{\pi} + K.$$

This can be rewritten as

$$A(\lambda) = \frac{L\sqrt{\lambda}}{\pi} + O(1), \quad \text{as } \lambda \to \infty.$$

Solving the asymptotic equation $A(\lambda) = n$ as both λ and n tend to infinity gives

$$\sqrt{\lambda_n} = \frac{n\pi}{L} + \delta_n$$

where $\delta_n = O(1)$, from which the stated eigenvalue asymptotic approximation follows directly. □

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References

[1] S. Agmon; *Lectures on elliptic boundary-value problems*, Princeton, 1965.

[2] J. Avron; Adiabatic quantum transport in multiply connected systems, *Reviews of Modern Physics.*, **60** (1988), 873-915.

[3] R. Carlson; Adjoint and self-adjoint differential operators on graphs, *Electronic J. Differential Equations*, **1998** (1998), No. 06, 1-10.

[4] E. A. Coddington, N. Levinson; *Theory of ordinary differential equations* , McGraw-Hill, New York, 1955.

[5] R. Courant, D. Hilbert; *Methods of Mathematical Physics Volume I*, Wiley-Interscience, New York, 1989.

[6] N. Gerasimenko, B. Pavlov; Scattering problems on non-compact graphs, *Theoretical and Mathematical Physics*, **74** (1988), 230-240.

[7] H. Hochstadt; On Inverse Problems Associated with Sturm-Liouville Operators, *J. Differential Equations*, **17** (1975), 220-235.

[8] V. Hutson, J.S. Pym; *Applications of Functional Analysis and Operator Theory*, Academic Press, 1980.

[9] P. Kuchment; Graph models for waves in thin structures, *Waves Random Media*, **12** (2002), R1-R24.

[10] J. Locker; *Spectral Theory of Non-Self-Adjoint Two-Point Differential Operators*, American Mathematical Society, 2000.

[11] M. A. Naimark; *Linear differential operators, part I*, Frederick Ungar Publishing Co., New York, 1967.

[12] Yu. V. Pokornyi, V. L. Pryadiev; Some problems of the qualitative Sturm-Liouville theory on a spatial network, *Russian Math. Surveys*, **59**:3 (2004), 515-552.

[13] Yu. V. Pokornyi, V. L. Pryadiev, A. Al-Obeid; On the oscillation of the spectrum of a boundary-value problem on a graph, *Math. Notes*, **60** (1997), 351-353.

[14] R. E. Showalter; *Hilbert Space Methods for Partial Differential Equations*, Electron. J. Diff. Eqns., Monograph 01, 2001.

[15] M. Solomyak; On the spectrum of the Laplacian on regular metric trees, *Waves Random Media*, **14** (2004), S155-S171.

[16] J. von Below; Sturm-Liouville eigenvalue problems on networks, *Math. Methods Appl. Sci.*, **10** (1988), 383-395.

[17] J. Weidmann; *Linear Operators in Hilbert Spaces*, Springer-Verlag, 1980.

[18] H. F. Weinberger; *Variational methods for eigenvalue approximation*, Society for Industrial and Applied Mathematics, Philadelphia, 1974.

[19] J. Wloka; *Partial Differential Equations*, Cambridge University Press, 1987.

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