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BLOWUP AND ASYMPTOTIC STABILITY OF WEAK SOLUTIONS TO WAVE EQUATIONS WITH NONLINEAR DEGENERATE DAMPING AND SOURCE TERMS

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Abstract. This article concerns the blow-up and asymptotic stability of weak solutions to the wave equation

$$u_{tt} - \Delta u + |u|^k j'(u_t) = |u|^{p-1}u \quad \text{in } \Omega \times (0,T),$$

where $p > 1$ and j' denotes the derivative of a C^1 convex and real value function j. We prove that every weak solution is asymptotically stability, for every m such that $0 < m < 1$, $p < k+m$ and the the initial energy is small; the solutions blows up in finite time, whenever $p > k+m$ and the initial data is positive, but appropriately bounded.

1. Introduction

In this article we study the initial boundary value problem

$$u_{tt} - \Delta u + |u|^k j'(u_t) = |u|^{p-1}u, \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad \text{on } \Gamma \times (0,T), \tag{1.3}$$

where Ω is a bounded domain in R^n with smooth boundary Γ and $j(s)$ is a C^1 convex real function defined on R, and j' denotes the derivative of j [1]. Furthermore, the following assumptions on the convex function j and the parameters k, m, p are imposed throughout the paper.

Assumptions.

(A1) $k, m, p > 0$, and $k < \frac{n}{n-2}$, $p + 1 < \frac{2n}{n-2}$ if $n \geq 3$;

(A2) There exist positive constants C, C_0, C_1 such that for all $s, v \in R$, $j(s) \geq C|s|^{m+1}$, $|j'(s)| \leq C_0|s|^m$, $(j'(s) - j'(v))(s - v) \geq C_1|s - v|^{m+1}$.

The partial differential equation (1.1) is a special case of the prototype evolution equation

$$u_{tt} - \Delta u + Q(x,t,u,u_t) = f(x,u), \tag{1.4}$$

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where the nonlinearities satisfy the structural conditions $vQ(x,t,u,v) \geq 0$,

$$Q(x,t,u,0) = f(x,0) = 0$$

and $f(x,u) \sim |u|^{p-1}u$ for large $|u|$. Various special cases of (1.4) arise in many contexts, for instance, in classical mechanics, fluid dynamics, quantum field theory, see [6] and [18].

A special case of (1.1), is the following well known polynomially damped wave equation studied extensively in the literature(see for instance [11, 17]),

$$u_{tt} - \Delta u + |u|^k |u_t|^{m-1} u_t = |u|^{p-1}u. \tag{1.5}$$

Indeed, by taking $j(s) = \frac{1}{m+1}|s|^{m+1}$ we easily verify that Assumption (A1) and (A2) is satisfied. It is easy to see in this case that equation (1.1) is equivalent to (1.5). It is worth noting that there has been an extensive body of work on the global existence and nonexistence for the equation (1.1) with $k = 0$, see, for example [4]-[9], [12]-[16],[18, 2] and the references therein. One of the pioneering papers in this area was by Lions and Strauss [10]. We also note here the work of Georgiev and Todorova [5] and Levine and Serrin [7].

The situation, however, is different when the damping is degenerate. From the applications point of view degenerate problem of this type arise quite often in specific physical contexts: for example when the friction is modulated by the strain. However, from the mathematical point of view this leads to that some standard arguments to establish the existence of solutions to problem (1.1)-(1.3) is not applicable. These difficulties makes the problem interesting and the analysis more subtle. The problem with degenerate damping has been first addressed in Levine and Serrin [7], where the global nonexistence of solutions was shown for the case $k+m < p$ under several other restrictions imposed on the parameters n, k, m, p and the negative initial energy. However Levine and Serrin [7] provide only negative results (blow up of solutions in finite time if initial energy is negative) without any assurance that a relevant local solutions does indeed exist. Pitts and Rammaha [11] established local and global (when $m + k \geq p$) existence and uniqueness for the case of sub-linear damping, i.e., $m < 1$. In addition, the blow up of solutions (when $m + k < p$ and the negative initial energy) is also proved in [11] for the relevant class of solutions. Barbu, Lasiecka and Rammaha [1, 2] introduced the suitable concept of solution, provided results on the existence and uniqueness of various types of solutions such as generalized solutions, weak solutions, and strong solutions to (1.1)-(1.3). In [2, 3, 11, 17] blow up of the weak or generalized solution was shown if $p > m + k$ and the initial energy is negative. The negativity of the initial energy was used to prove blow up in the above paper [2, 3, 11, 17]. However, the blow up of the solutions for (1.1)-(1.3) in case of positive initial energy has not been discussed, and the asymptotic behavior of the solutions for (1.1)-(1.3) is much less understood. In this paper, following the ideas of "potential well" theory introduced by Payne and Sattinger [12], we extend the results about asymptotic stability and blowup of the solution to (1.1)-(1.3) with $k = 0$ (see, for example, [4, 13, 20, 21, 22] to the problem (1.1)-(1.3) with $k > 0$.

It is worth mentioning here that Levine, Park and Serrin [9] studied the existence and nonexistence of the solution to the quasilinear evolution equation of formally parabolic type, namely

$$Q(t,u,u_t) + A(t,u) = f(t,u). \tag{1.6}$$

The purpose of the paper is, first, to show that the weak solution of the problem (1.1)-(1.3) blow up in the case of positive initial energy $E(0) > 0$ and $p > k + m$, which we do in section 3. The another purpose of this paper is to give an asymptotic stability results of the problem (1.1)-(1.3) with $0 < m < 1$, $p < k + m$, which do in section 4.

The following notation will be used in the sequel:

$$|u|_{s,\Omega} \equiv \|u\|_{H^s(\Omega)}, \quad \|u\|_p \equiv \|u\|_{L^p(\Omega)},$$
$$\|u\| \equiv \|u\|_{L^2(\Omega)}, (u, v) = \int_\Omega u(x)v(x)dx, p^* = \frac{2n}{n-2},$$

where $H^s(\Omega)$ and $L^p(\Omega)$ stands for the classical Sobolev spaces and the Lebesgue spaces, respectively.

2. Preliminaries

In this section we introduce some notations, definitions and some known results which are necessary for the remaining sections of the paper.

Definition 2.1 ([1, 2, 3])**.** We say that u is a weak solution to the problem (1.1)-(1.3) on $[0, T]$ if $u \in C_w([0, T]; H_0^1(\Omega)) \cap C_w^1([0, T], L^2(\Omega)), \Delta u - u_{tt} \in L^2(\Omega \times (0, T)), |u|^k j(u_t) \in L^2(\Omega \times (0, T))$ which satisfies $u(0) = u_0$, $u_t(0) = u_1$ and for all $0 < t \le T$ the following variational equality holds

$$\begin{aligned}
&\int_0^t \int_\Omega (-u_t(s)v_t(s) + \nabla u \nabla v)\, dx\, ds - \int_\Omega u_1 v(0) dx \\
&+ \int_0^t \int_\Omega |u(s)|^k j'(u_t)(s) v(s)\, dx\, ds \\
&= \int_0^t \int_\Omega |u(s)|^{p-1} u(s) v(s)\, dx\, ds
\end{aligned}$$

for all test functions v satisfying $v \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), v(T) = 0$.

Theorem 2.2 ([2])**.** *In addition to Assumption (A1) and (A2) and* $p \le \max\{\frac{p^*}{2}, \frac{p^* m + k}{m+1}\}$*;* $m < 1$ *if* $n = 1, 2$*;* $\frac{k}{p^*} + \frac{m}{2} \le \frac{1}{2}$ *if* $n \ge 3$*. Let* $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$*, then there exists a constant* $T > 0$ *such that the initial boundary problem* (1.1)-(1.3) *has a unique weak solution on* $[0, T]$ *if* $p \le k + m$*.*

Now, we define the energy associated with problem (1.1)-(1.3) by

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}.$$

We see that the energy has the so-called energy identity

$$E(t) + \int_0^t \int_\Omega |u(s)|^k j(u_t)(s) ds = E(0),$$

where

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1}.$$

It is clear that

$$E'(t) = -\int_\Omega |u(s)|^k j(u_t)(s) ds \le 0 \tag{2.1}$$

and $E(t)$ is a non-increasing function in time, then

$$E(t) \geq E(0). \tag{2.2}$$

Finally, we set

$$\begin{gathered}
\lambda_1 = B_1^{-\frac{2}{p-1}}, \quad E_1 = (\frac{1}{2} - \frac{1}{p+1})\lambda_1^{p+1}, \\
\lambda_2 = (\frac{1}{(p+1)B_1^2})^{\frac{1}{p-1}}, \quad E_2 = \frac{p+1}{2}(\frac{1}{2} - \frac{1}{p+1})\lambda_2^{p+1}, \\
\sum\nolimits_1 = \{(\lambda, E) \in R^2, \lambda > \lambda_1, 0 < E < E_1\}, \\
\sum\nolimits_2 = \{(\lambda, E) \in R^2, 0 \leq \lambda < \lambda_2, 0 < E < E_2\},
\end{gathered}$$

where B_1 is the embedding constant (where $H_0^1(\Omega)$ is embedded into $L^{p+1}(\Omega)$). We call $\sum_1$ the unstable set, $\sum_2$ the stable set.

3. Blow-up of the solutions

In this section, we assume that $p > k+m$ and u be a weak solution to (1.1)-(1.3) on the interval $[0, T]$ in the sense of Definition 2.1.

Lemma 3.1. *Let $(\|u_0\|_{p+1}, E(0)) \in \sum_1$, then $E(t) \leq E_0$ for all $t \in [0, T]$, and there exist $\lambda_0 > \lambda_1$ such that $\|u(t)\|_{p+1} \geq \lambda_0 > \lambda_1$ for all $t \in [0, T]$.*

The proof is similar to that of [20, Lemma 1], so we omit it.

Theorem 3.2. *Let $(\|u_0\|_{p+1}, E(0)) \in \sum_1$, $p > k+m$, and u be a weak solution to (1.1)-(1.3) on the interval $[0, T]$ in the sense of Definition 2.1, then T is necessarily finite, i.e. u can not be continued for all $t > 0$.*

Proof. We argue by contradiction. Let $F(t) = \|u(t)\|^2$, $H(t) = E_1 - E(t)$. From (2.1), we have

$$H'(t) = -E'(t) = \int_\Omega |u(t)|^k j(u_t)(t)dx \geq 0. \tag{3.1}$$

Therefore, $H(t)$ is an increasing function, then

$$H(t) \geq H(0) = E_1 - E(0) > 0, \quad t \geq 0. \tag{3.2}$$

Next, by the definition of $E(t)$ and Lemma 3.1,

$$\begin{aligned}
H(t) &\leq E_1 - \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} \\
&\leq E_1 - \frac{1}{2}B_1^{-2}\lambda_1^2 + \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}, \quad t \geq 0.
\end{aligned} \tag{3.3}$$

Hence, since $E_1 - \frac{1}{2}B_1^{-2}\lambda_1^2 = -\frac{1}{p+1}\lambda_1^{p+1} < 0$, we have

$$0 < H(0) \leq H(t) \leq \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}, \quad t \geq 0. \tag{3.4}$$

For simplicity, we denote

$$I(t) = \int_\Omega |u(t)|^k u(t) j'(u_t)(t)dx.$$

By the definition of the solution and the definition of $H(t)$,

$$
\begin{aligned}
\frac{1}{2}F''(t) &= \frac{d}{dt}\int_\Omega u(t)u_t(t)dx = \|u_t(t)\|^2 - \|\nabla u(t)\|^2 + \|u(t)\|_{p+1}^{p+1} - I(t)\\
&= 2\|u_t(t)\|^2 + (1-\frac{2}{p+1})\|u(t)\|_{p+1}^{p+1} - 2E(t) - I(t)\\
&= 2\|u_t(t)\|^2 + (1-\frac{2}{p+1})\|u(t)\|_{p+1}^{p+1} + 2H(t) - 2E_1 - I(t).
\end{aligned}
$$

By Lemma 3.1 again (i.e $\|u(t)\|_{p+1}^{p+1}\lambda_0^{-(p+1)} > 1$, or $E_1\|u(t)\|_{p+1}^{p+1}\lambda_0^{-(p+1)} > E_1$),

$$
\begin{aligned}
\frac{1}{2}F''(t) &\geq 2\|u_t(t)\|^2 + (1-\frac{2}{p+1} - 2E_1\lambda_0^{-(p+1)})\|u(t)\|_{p+1}^{p+1} + 2H(t) - I(t)\\
&= 2\|u_t(t)\|^2 + C_2\|u(t)\|_{p+1}^{p+1} + 2H(t) - I(t),
\end{aligned}
\tag{3.5}
$$

where $C_2 = 1 - \frac{2}{p+1} - 2E_1\lambda_0^{-(p+1)} > 0$, because $\lambda_0 > \lambda_1$ by Lemma 3.1.

Now, to estimate the last term $I(t)$ in (3.5), since $p > k+m$ and Assumption (A1) and (A2) and by applying Holder's inequality and Young's inequality, we obtain

$$
\begin{aligned}
|I(t)| &\leq C_0\int_\Omega |u(t)|^{k+1-\frac{k+m+1}{m+1}}|u(t)|^{\frac{k+m+1}{m+1}}|u_t(t)|^m dx\\
&\leq C_0(\int_\Omega |u(t)|^k|u_t(t)|^{m+1}dx)^{\frac{m}{m+1}}(\int_\Omega |u(t)|^{k+m+1}dx)^{\frac{1}{m+1}}\\
&\leq C_0B_0(H'(t))^{\frac{m}{m+1}}\|u(t)\|_{p+1}^{\frac{k+m+1}{m+1}}\\
&\leq C_0B_0(\frac{1}{\delta}H'(t) + \delta^m\|u(t)\|_{p+1}^{k+m+1}),
\end{aligned}
\tag{3.6}
$$

where δ is a constant to be chosen later, B_0 is the embedding constants from $L^{k+m+1}(\Omega)$ to $L^{p+1}(\Omega)$(since $k+m < p$).

Now, we introduce the auxiliary function

$$
y(t) = H^{1-\alpha}(t) + \epsilon F'(t),
$$

where ϵ is a small positive constant to be fixed later, and $\alpha = \min\{\frac{p-(k+m)}{m(p+1)}, \frac{p-1}{2(p+1)}\}$. Clearly, $0 < \alpha < \frac{1}{2}$. Therefore, (3.5), (3.6) yield

$$
\begin{aligned}
y'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon F''(t)\\
&\geq (1-\alpha)H^{-\alpha}(t)H'(t) + 4\epsilon\|u_t(t)\|^2 + 4\epsilon H(t) + 2\epsilon C_2\|u(t)\|_{p+1}^{p+1} - 2\epsilon I(t)\\
&\geq [(1-\alpha)H^{-\alpha}(t) - \frac{2\epsilon C_0B_0}{\delta}]H'(t) + 4\epsilon\|u_t(t)\|^2 + 4\epsilon H(t)\\
&\quad + 2\epsilon C_2\|u(t)\|_{p+1}^{p+1} - 2C_0B_0\epsilon\delta^m\|u(t)\|_{p+1}^{k+m+1}.
\end{aligned}
$$

Choosing $\delta = (\frac{C_2}{2C_0B_0}\|u(t)\|_{p+1}^{p-k-m})^{\frac{1}{m}}$, then

$$
C_2\epsilon\|u(t)\|_{p+1}^{p+1} - 2C_0B_0\epsilon\delta^m\|u(t)\|_{p+1}^{k+m+1} = 0.
$$

Therefore,

$$
y'(t) \geq [(1-\alpha)H^{-\alpha}(t) - \frac{2\epsilon C_0B_0}{\delta}]H'(t) + 4\epsilon\|u_t(t)\|^2 + 4\epsilon H(t) + \epsilon C_2\|u(t)\|_{p+1}^{p+1}. \tag{3.7}
$$

By (3.4) and the choice δ, then

$$
\begin{aligned}
&(1-\alpha)H^{-\alpha}(t)-\frac{2\epsilon C_0B_0}{\delta}=H^{-\alpha}(t)[1-\alpha-\frac{2\epsilon C_0B_0}{\delta}H^{\alpha}(t)]\\
&\geq H^{-\alpha}(t)[1-\alpha-2^{1+\frac{1}{m}}\epsilon(C_0B_0)^{1+\frac{1}{m}}C_2^{-\frac{1}{m}}(\frac{1}{p+1})^{\alpha}\|u(t)\|_{p+1}^{\frac{k+m-p+\alpha m(p+1)}{m}}].
\end{aligned}
\tag{3.8}
$$

Furthermore, since $\|u(t)\|_{p+1}\geq[(p+1)H(0)]^{\frac{1}{p+1}}$ by (3.4) and α was chosen so that $k+m-p+\alpha m(p+1)\leq 0$, it follows from (3.8) that

$$
\begin{aligned}
&(1-\alpha)H^{-\alpha}(t)-\frac{2\epsilon C_0B_0}{\delta}\\
&\geq H^{-\alpha}(t)[1-\alpha-2^{1+\frac{1}{m}}\epsilon(C_0B_0)^{1+\frac{1}{m}}C_2^{-\frac{1}{m}}(\frac{1}{p+1})^{\frac{p-k+m}{m(p+1)}}(H(0))^{\alpha+\frac{k+m-p}{m(p+1)}}].
\end{aligned}
\tag{3.9}
$$

We choose ϵ sufficiently small such that

$$
1-\alpha-2^{1+\frac{1}{m}}\epsilon(C_0B_0)^{1+\frac{1}{m}}C_2^{-\frac{1}{m}}(\frac{1}{p+1})^{\frac{p-k+m}{m(p+1)}}(H(0))^{\alpha+\frac{k+m-p}{m(p+1}}\geq 0.
\tag{3.10}
$$

Therefore, (3.8)-(3.10) yield

$$
(1-\alpha)H^{-\alpha}(t)-\frac{2\epsilon C_0B_0}{\delta}\geq 0.
\tag{3.11}
$$

Thus, by (3.11) and (3.7), we obtain

$$
y'(t)\geq\epsilon C_3[H(t)+\|u_t(t)\|^2+\|u(t)\|_{p+1}^{p+1}],
\tag{3.12}
$$

where $C_3>0$ is a constant which does not depended on ϵ. In particular, (3.12) shows that $y(t)$ is increasing on $(0,T)$, with

$$
y(t)=H^{1-\alpha}(t)+\epsilon F'(t)\geq H^{1-\alpha}(0)+\epsilon F'(0).
$$

We further choose ϵ sufficiently small such that $y(0)>0$, so $y(t)\geq y(0)>0$ for $t\geq 0$.

Now, let $r=\frac{1}{1-\alpha}$. Since $0<\alpha<\frac{1}{2}$, it is evident that $r>1$. Using Young's inequality again

$$
\begin{aligned}
y^r(t)&\leq 2^{r-1}(H(t)+\epsilon\|u(t)\|^r\|u_t(t)\|^r)\\
&\leq C_4(H(t)+\|u_t(t)\|^2+\|u(t)\|^{\frac{1}{\frac{1}{2}-\alpha}}).
\end{aligned}
\tag{3.13}
$$

By the choice of α, we have $\frac{1}{2}-\alpha>\frac{1}{p+1}$. Now apply the inequality

$$
x^{\sigma}\leq(1+\frac{1}{a})(a+x),\quad x\geq 0,\quad 0\leq\sigma\leq 1,\quad a>0,
$$

and take $x=\|u(t)\|^{p+1},\sigma=\frac{1}{(\frac{1}{2}-\alpha)(p+1)}<1,a=H(0)$, and $d=1+\frac{1}{H(0)}$, we obtain

$$
\|u(t)\|^{\frac{1}{\frac{1}{2}-\alpha}}\leq d(H(0)+\|u(t)\|^{p+1})\leq C_5(H(t)+\|u(t)\|_{p+1}^{p+1}).
\tag{3.14}
$$

Hence, from (3.13) and (3.14) there results

$$
y^r(t)\leq C(H(t)+\|u_t(t)\|^2+\|u(t)\|_{p+1}^{p+1}).
\tag{3.15}
$$

Thus, (3.12) and (3.15) show that

$$
y'(t)\geq C_6y^r(t),\quad t\in[0,T].
$$

Finally, from this inequality and $r=\frac{1}{1-\alpha}>1$, we see that $y(t)=H^{1-\alpha}(t)+\epsilon F'(t)$ blow up in finite time. This completes the proof. $\square$

4. Asymptotic stability of the solutions

To obtain the asymptotic stability of the solution, we start with a series of lemmas. The assumption of Theorem 2.2 will be valid throughout this section.

Lemma 4.1. *If $(\|u_0\|_{p+1}, E(0)) \in \sum_2$, then*
$$(\|u(t)\|_{p+1}, E(t)) \in \sum\nolimits_2, \quad t \ge 0. \tag{4.1}$$
Moreover
$$E(t) \ge \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{4}\|\nabla u(t)\|^2, \qquad t \ge 0. \tag{4.2}$$

Proof. By (2.2) and the embedding theorem, for all $t \ge 0$, there holds
$$\begin{aligned} E_2 > E(0) \ge E(t) &\ge \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{4}\|\nabla u(t)\|^2 + \frac{1}{4}B_1^{-2}\|u(t)\|_{p+1}^2 - \frac{1}{2}\|u(t)\|_{p+1}^{p+1} \\ &\ge \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{4}\|\nabla u(t)\|^2 + g(\|u(t)\|_{p+1}), \end{aligned} \tag{4.3}$$
where $g(\lambda) = \frac{1}{4}B_1^{-2}\lambda^2 - \frac{1}{2}\lambda^{p+1}$, for $\lambda \ge 0$. It is easy to see that $g(\lambda)$ attains its maximum E_2 for $\lambda = \lambda_2$, $g(\lambda)$ is strictly decreasing for $\lambda \ge \lambda_2$ and $g(\lambda) \to -\infty$ as $\lambda \to \infty$. By the continuity of $\|u(t)\|_{p+1}$ and $\lambda(0) = \|u_0\|_{p+1} < \lambda_2$, so that $\lambda(t) < \lambda_2$ for all $t \ge 0$. Also, of course, $E(t) < E_2$ by (4.3). Then, (4.1) holds. To obtain (4.2), it remains to note that $g(\lambda) \ge 0$ whenever $0 \le \lambda < \lambda_2$. Then (4.2) follows at once. ☐

Lemma 4.2. *If $(\|u_0\|_{p+1}, E(0)) \in \sum_2$, then $\|\nabla u(t)\|^2 \ge 2\|u(t)\|_{p+1}^{p+1}$, or*
$$\|\nabla u(t)\|^2 - \|u(t)\|_{p+1}^{p+1} \ge \frac{1}{2}\|\nabla u(t)\|^2. \tag{4.4}$$

Proof. By the embedding theorem
$$\begin{aligned} \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{2}\|u(t)\|_{p+1}^{p+1} &\ge \frac{1}{4}\|\nabla u(t)\|^2 + \frac{1}{4}B_1^{-2}\|u(t)\|_{p+1}^2 - \frac{1}{2}\|u(t)\|_{p+1}^{p+1} \\ &= \frac{1}{4}\|\nabla u(t)\|^2 + g(\|u(t)\|_{p+1}). \end{aligned}$$
Hence (4.4) is true, since $g(\lambda) \ge 0$, if $0 \le \lambda < \lambda_2$ and $0 \le \|u(t)\|_{p+1} < \lambda_2$ by Lemma 4.1. ☐

Lemma 4.3. *If $(\|u_0\|_{p+1}, E(0)) \in \sum_2$, then*
1. *$\|u_t(t)\| \in L^2(0,\infty), H'(t) \in L^1(0,\infty)$*
2. *$\|\nabla u(t)\|, \|u(t)\|_{p+1}, \|u_t(t)\| \le C$.*

Proof. The first result in (1) follows the definition of weak solution. The second result in (1) follows by $H'(t) = -E'(t)$, since $E(t) \ge 0$ for $t \ge 0$ and $H(t) \in AC(0,\infty)$, while (2) follows (4.2) (or(4.3)) and (4.4). ☐

Lemma 4.4. *Let $(\|u_0\|_{p+1}, E(0)) \in \sum_2$ and $E(t) \ge \beta$, where $\beta > 0$ is a constant, then there exists $\alpha = \alpha(\beta) > 0$ such that*
$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 - \|u(t)\|_{p+1}^{p+1} \ge \alpha, \quad t \ge 0. \tag{4.5}$$

Proof. By the definition of $E(t)$ and $E(t) \ge \beta$, we have
$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \ge 2\beta, \quad t \ge 0. \tag{4.6}$$
Now suppose that (4.5) does not hold. From (4.4), there is a sequences $t_n \subset R^+$ such that
$$\|u_t(t_n)\|^2 + \|\nabla u(t_n)\|^2 - \|u(t_n)\|_{p+1}^{p+1} \ge \|u_t(t_n)\|^2 + \frac{1}{2}\|\nabla u(t_n)\|^2 \to 0, (n \to \infty).$$
Then, we get
$$\|u_t(t_n)\|^2 \to 0, \quad \|\nabla u(t_n)\|^2 \to 0, \quad \text{as } n \to \infty.$$
This is contradiction with (4.6). The lemma is proved. $\square$

Theorem 4.5. *Assume the conditions of Theorem 2.2, that* $(\|u_0\|_{p+1}, E(0)) \in \sum_2$, *and that* u *is a weak solution to* (1.1)-(1.3). *Then*
$$\lim_{t\to\infty} E(t) = 0, \lim_{t\to\infty} \|\nabla u(t)\| = 0. \tag{4.7}$$

Proof. Suppose that (4.7) fails, then there exists $\beta > 0$ such that $E(t) \ge \beta$ for all $t \ge 0$ since (2.2) and $E(t) \ge 0$. Multiplying both sides of (1.1) by u, integrating over $[T, t] \times \Omega$ $(0 < T \le t < \infty)$ and integrating by parts with respect to t, we obtain
$$\begin{aligned}(u_t(s), u(s))|_{s=T}^t &= \int_T^t [2\|u_t(s)\|^2 - (\|u_t(s)\|^2 + \|\nabla u(s)\|^2 - \|u(s)\|_{p+1}^{p+1}) \\ &\quad - \int_\Omega |u(s)|^k u(s) j'(u_t)(s) dx] ds \\ &= \int_T^t (I_1 + I_2 + I_3) ds.\end{aligned} \tag{4.8}$$
By (4.2), (2.2) and Lemma 4.3 (1), we have
$$\int_T^t I_1 ds = \int_T^t 2\|u_t(s)\|^2 ds \le 4E^{\frac12}(0)(\int_T^t \|u_t(s)\|^2 ds)^{\frac12}(\int_T^t ds)^{\frac12} \le C_7(\int_T^t ds)^{\frac12}. \tag{4.9}$$
Here and in the following, C_i denotes a positive constant which do not depend on t and T. By Lemma 4.4
$$\int_T^t I_2 ds = -\int_T^t (\|u_t(s)\|^2 + \|\nabla u(s)\|^2 - \|u(s)\|_{p+1}^{p+1}) ds \le -\alpha \int_T^t ds. \tag{4.10}$$
By Holder inequality, Lemma 4.3 (1), Lemma 4.3 (2) and embedding theorem, we have
$$\begin{aligned}\int_T^t I_3 dt &= -\int_T^t \int_\Omega |u(s)|^k u(s) j'(u_t)(s)\, dx\, ds \\ &\le \int_T^t \int_\Omega |u(s)|^{k+1-\frac{k+m+1}{m+1}} |u(s)|^{\frac{k+m+1}{m+1}} |u_t(s)|^m\, dx\, ds \\ &\le (\int_T^t \int_\Omega |u(s)|^k j(u_t)(s)\, dx\, ds)^{\frac{m}{m+1}} (\int_T^t \int_\Omega |u(s)|^{k+m+1}\, dx\, ds)^{\frac{1}{m+1}} \\ &\le C_8(\int_T^t H'(s) ds)^{\frac{m}{m+1}} (\int_T^t \|u(s)\|_{k+m+1}^{k+m+1} ds)^{\frac{1}{m+1}} \\ &\le C_9(\int_T^t \|\nabla u(s)\|^{k+m+1} ds)^{\frac{1}{m+1}} \le C_{10}(\int_T^t ds)^{\frac{1}{m+1}},\end{aligned} \tag{4.11}$$

here we have used the embedding theorem from $H_0^1(\Omega)$ to $L^{k+m}(\Omega)$ since $k+m+1 < \frac{1-m}{2}p^* + m + 1 < p^*$. Then from (4)-(4.11), since $\frac{1}{m+1} > \frac{1}{2}$, we know

$$(u_t(s), u(s))|_{s=T}^t \le C_{11}(\int_T^t ds)^{\frac{1}{m+1}} - \alpha \int_T^t ds. \tag{4.12}$$

On the other hand, from Holder inequality and Lemma 4.3 (2),

$$|(u_t(t), u(t))| \le C_{12}(\|u_t(t)\|^2 + \|\nabla u(t)\|^2) < \infty.$$

In turn, we reach a contradiction with (4.12) for fixing T when $t \to \infty$. Hence, we derive $\lim_{t\to\infty} E(t) = 0$ and $\lim_{t\to\infty} \|\nabla u(t)\|^2 = 0$ by (4.2). This completes the proof. $\square$

Remark 4.6. The set $\sum_2$ is called stable set. It is smaller than the potential well introduced by Payne and Sattinger[12]. Moreover the value λ_2 in this paper can be chosen larger than now but $\lambda_2 < \lambda_1$.

Remark 4.7. The method seems general enough to apply to the generate equation (1.4) with $f(x, u)$ being source term and also let Q and F depending on time but this will be discuss in a future paper.

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A reader informed us that that parts of the introduction were copied from reference [2], without giving the proper credit. Also that the first statement in Lemma 4.3 maybe false; so that Theorem 4.5 has not been proved. The authors agreed to post a new proof, if they succeed in proving the lemma.

Errata: Assumption (A1) should include $p > 1$. Inequality (2.2) should read $E(t) \le E(0)$.