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H-CONVERGENCE FOR EQUATIONS DEPENDING ON MONOTONE OPERATORS IN CARNOT GROUPS

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Abstract. This article presents some results related to the convergence of solutions and momenta of Dirichlet problems for sequences of monotone operators in the sub-Riemannian framework of Carnot groups.

1. Introduction

The term *H-convergence* was coined by François Murat and Luc Tartar in the 70's and it is addressed to differential operators. Tartar[31, 32] reported applications of the H-convergence to many different frameworks covering, among other things, the case involving monotone operators (see Definition 2.5) of the form

$$\mathcal{A}(u) = -\operatorname{div}(A(x, \nabla u)),$$

where A is a Carathéodory function satisfying uniformly ellipticity and continuous conditions, in the setting of Hilbert spaces. See [32, Chapter 11] for details and [29, Chapter 2.3] for a general discussion about this topic.

In recent years, this theory found numerous applications in literature, such as homogenization. We refer the interested reader to [3, 4, 5, 6, 7, 9, 11, 12, 13, 15, 16, 20, 28, 30] for details. In particular, De Arcangelis and Serra Cassano [14] extended into the setting of Banach spaces the original Murat and Tartar H-compactness theorem, working with weights. A linear counterpart of this study, in Carnot groups, was faced up by Baldi, Franchi, Tchou and Tesi [1, 2, 21]. This environment has become of particular interest for analysis and PDEs over the previous decades, see e.g. [10, 17, 18, 25, 27].

The class of linear operators considered in [1, 2, 21] is made of *matrix-valued measurable functions*, that is, operators of the form

$$\mathcal{A}(u) = -\operatorname{div}_{\mathbb{G}}(A(x)\nabla_{\mathbb{G}} u), \tag{1.1}$$

where A is a $(m \times m)$-matrix-valued measurable function and $\nabla_{\mathbb{G}}$ and $\operatorname{div}_{\mathbb{G}}$ are, respectively, the intrinsic gradient and the intrinsic divergence (see Definition 2.2 for details). We remind that a definition of *intrinsic curl*, $\operatorname{curl}_{\mathbb{G}}$, can be found in [2, Section 5]. The key tool in [1, 2, 21] was an extension to Carnot groups of Murat and Tartar' *Div-curl lemma* [32, Lemma 7.2], namely [2, Theorem 5.1].

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Motivated by the previous results, in this paper we look for extensions to Carnot groups, in the general setting of Banach spaces, of the original result of Murat and Tartar [32, Theorem 11.2] and we provide a H-compactness theorem for (nonlinear) monotone operators, working with operators of the form

$$\mathcal{A}(u) = -\operatorname{div}_{\mathbb{G}}(A(x, \nabla_{\mathbb{G}} u)) \tag{1.2}$$

for a given $A \in \mathcal{M}(\alpha, \beta; \Omega)$. The class $\mathcal{M}(\alpha, \beta; \Omega)$ is defined as follows.

Definition 1.1. Let $\Omega \subset \mathbb{G}$ be open, $2 \leq p < \infty$ and $\alpha \leq \beta$ be positive constants. We define $\mathcal{M}(\alpha, \beta; \Omega)$ the class of Carathéodory functions $A : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ such that

(i) $A(x, 0) = 0$;
(ii) $\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \alpha |\xi - \eta|^p$;
(iii) $|A(x, \xi) - A(x, \eta)| \leq \beta \left[1 + |\xi|^p + |\eta|^p\right]^{\frac{p-2}{p}} |\xi - \eta|$

for every $\xi, \eta \in \mathbb{R}^m$ and a.e. $x \in \Omega$.

The main result of this article is the following theorem.

Theorem 1.2. *Let $\Omega \subset \mathbb{G}$ be open, connected and bounded, $2 \leq p < \infty$, $\alpha \leq \beta$ positive constants and let $(A^n)_n \subset \mathcal{M}(\alpha, \beta; \Omega)$. Then, up to subsequences, there exists $A^{\text{eff}} \in \mathcal{M}(\alpha, \beta; \Omega)$ such that*

$$(A^n)_n \ H\text{-converges to } A^{\text{eff}}.$$

We would like to stress that, for $p = 2$, Theorem 1.2 generalizes several previous results. For instance, if the Carnot groups $\mathbb{G}$ is the Euclidean space $\mathbb{R}^n$, then Theorem 1.2 immediately gives [32, Theorem 11.2]. Moreover, in the sub-Riemannian framework of Carnot groups, if we restrict to operators (1.1), then Theorem 1.2 generalizes both [21, Theorem 4.4], if $\mathbb{G}$ is the first Heisenberg group, [1, Theorem 6.4], if $\mathbb{G}$ is a general Heisenberg group and [2, Theorem 5.4], in any Carnot group.

The structure of this article is the following one: in Section 2, we give the definitions of Carnot groups and the functional setting required throughout the paper. In Section 3, we study the main properties of the class of monotone operators we are interested in and, in Section 4, after defining a proper notion of H-convergence (see Definition 4.1), we prove Theorem 1.2.

2. Preliminaries

2.1. **Carnot groups.** Let us recall just few definitions concerning Carnot groups. We refer the interested reader to [8].

Definition 2.1. A Carnot group $\mathbb{G}$ of step k is a connected, simply connected and nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a step k stratification, that is, there exist $V_1, \dots, V_k$ linear subspaces of $\mathfrak{g}$, usually called layers, such that

(i) $\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$;
(ii) $[V_1, V_i] = V_{i+1}$ for any $i < k$, where $[V_1, V_i]$ is the sub-algebra of $\mathfrak{g}$ generated by the commutation $[X, Y]$, with $X \in V_1, Y \in V_i$;
(iii) $V_k \neq \{0\}$ and $V_i = \{0\}$ for any $i > k$, where 0 is the identity element of $\mathfrak{g}$.

Typical examples of Carnot groups are the Euclidean space, the only *Abelian* Carnot group of step 1 and the Heisenberg group, a Carnot group of step 2.

It is clear from Definition 2.1, that the first layer V_1 plays the role of generator of the algebra $\mathfrak{g}$, by commutation. For this reason, we refer to V_1 as the *horizontal layer*, while the other layers V_i, $1 < i \leq k$, are called *vertical layers*.

We can define two different dimensions on $\mathbb{G}$: the *topological dimension*, which is its dimension as Lie group, i.e.,

$$\dim(\mathbb{G}) = \dim(\mathfrak{g}) = \sum_{i=1}^{k} m_i,$$

where $m_i := \dim(V_i)$ for any i, and the *homogeneous dimension*, defined by

$$Q := \sum_{i=1}^{k} i\ m_i.$$

Let us notice that, when $\mathbb{G}$ is not $\mathbb{R}^n$, the homogeneous dimension of $\mathbb{G}$ is always bigger than the topological one. In the sequel, we denote $m := m_1$, for simplicity.

2.2. **Functional setting.** Through the paper, $(X_1, \dots, X_m)$ denotes a basis of the horizontal layer V_1, $|\Omega|$ the Lebesgue measure of any set $\Omega \subset \mathbb{G}$ and, if $\xi, \eta \in \mathbb{R}^m$, we denote by $|\xi|$ and $\langle \xi, \eta \rangle$ the Euclidean norm and the scalar product, respectively. The subbundle of the tangent bundle $T\mathbb{G}$, which is spanned by the vector fields $X_1, \dots, X_m$, is called the *horizontal bundle* and is denoted by $H\mathbb{G}$. Each section Φ of $H\mathbb{G}$ is called *horizontal sections* and is identified with canonical coordinates with respect to the moving frame, by a function $\Phi = (\Phi_1, \dots, \Phi_m) : \mathbb{G} \to \mathbb{R}^m$.

Definition 2.2. Let $u \in L^1_{\rm loc}(\mathbb{G})$, let $X_i u$ exist in sense of distributions, and assume $X_i \Phi_i \in L^1_{\rm loc}(\mathbb{G})$ for $i = 1, \dots, m$. We define the *intrinsic gradient* of u and the *intrinsic divergence* of Φ, respectively, as

$$\nabla_{\mathbb{G}} u := \sum_{j=1}^{m} (X_j u) X_j = (X_1 u, \dots, X_m u), \quad \operatorname{div}_{\mathbb{G}}(\Phi) := \sum_{i=1}^{m} X_i \Phi_i.$$

Definition 2.3. For $1 \leq p < \infty$ we define

$$W^{1,p}_{\mathbb{G}}(\Omega) := t\{u \in L^p(\Omega) : X_j u \in L^p(\Omega) \text{ for } j = 1, \dots, m\},$$

endowed with its natural norm, $W^{1,p}_{\mathbb{G},0}(\Omega)$ the closure of $\mathbf{C}^\infty_c(\Omega) \cap W^{1,p}_{\mathbb{G}}(\Omega)$ in $W^{1,p}_{\mathbb{G}}(\Omega)$ and $W^{-1,p'}_{\mathbb{G}}(\Omega)$ the dual space of $W^{1,p}_{\mathbb{G},0}(\Omega)$. Notice that, if Ω is bounded, then

$$\|u\|^p_{W^{1,p}_{\mathbb{G},0}(\Omega)} := \int_\Omega |\nabla_{\mathbb{G}} u|^p \, dx$$

defines an equivalent norm on $W^{1,p}_{\mathbb{G},0}(\Omega)$ (see [24, Section 2] and, for more details, [23, 26]). Finally, we denote $L^p(\Omega, H\mathbb{G})$ the set of measurable sections $\Phi \in L^p(\Omega)^m$.

Proposition 2.4 ([19, Corollary 4.14])**.** *If $1 < p < \infty$, then $W^{1,p}_{\mathbb{G}}(\Omega)$ is independent of the choice of the basis $(X_1, \dots, X_m)$.*

2.3. **Monotone operators.** Let us recall the definition of monotone operators. See, for instance, [22] for more details.

Definition 2.5 ([22, Definitions 1.1–1.3, Chapter III])**.** Let V be a reflexive Banach space, V^* its dual space and let $\mathcal{A} : V \to V^*$ be a mapping. We say that

- $\mathcal{A}$ is *monotone*, if
$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{V^* \times V} \geq 0 \quad \text{for all } u, v \in V;$$

- $\mathcal{A}$ is *strictly-monotone*, if it is monotone and
$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle_{V^*\times V} = 0 \quad \text{implies } u = v\,;$$
- $\mathcal{A}$ is *coercive*, if there exists an element $v \in V$ such that
$$\frac{\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle_{V^*\times V}}{\|u - v\|_V} \to \infty \quad \text{as } \|u\|_V \to \infty\,;$$
- $\mathcal{A}$ is *continuous on finite dimensional subspaces* of V if, for any finite dimensional subspace M of V, the restriction of $\mathcal{A}$ to M is weakly continuous, namely, if $\mathcal{A} : M \to V^*$ is weakly continuous.

Operator (1.2) is strictly-monotone, in sense of Definition 1.1. The following result will be crucial later on.

Theorem 2.6 ([22, Corollary 1.8, Chapter III])**.** *Let X be a Banach space, let K be a closed, nonempty and convex subset of X and let $A : K \to X^*$ be monotone, coercive and continuous on finite dimensional subspaces of K. Then, there exists $u \in K$ such that*
$$\langle A(u), v - u\rangle_{X^*\times K} \geq 0$$
for any $v \in K$.

3. Existence results for equations driven by monotone operators

Let $\Omega \subset \mathbb{G}$ be open, connected and bounded, $2 \leq p < \infty$, $V = W^{1,p}_{\mathbb{G},0}(\Omega)$ and $V^* = W^{-1,p'}_{\mathbb{G}}(\Omega)$. Moreover, let $\mathcal{A} : V \to V^*$ be as in (1.2).

Proposition 3.1. *Let $A \in \mathcal{M}(\alpha, \beta; \Omega)$. Then, for every $f \in V^*$ there exists a unique (weak) solution $u \in V$ of*
$$-\operatorname{div}_{\mathbb{G}}(A(\cdot, \nabla_{\mathbb{G}} u)) = f \quad \text{in } \Omega\,, \tag{3.1}$$
i.e.,
$$\int_\Omega \langle A(x, \nabla_{\mathbb{G}} u), \nabla_{\mathbb{G}}\varphi\rangle dx = \int_\Omega f\,\varphi\, dx \quad \forall \varphi \in \mathbf{C}^\infty_c(\Omega). \tag{3.2}$$

Remark 3.2. By standard approximation arguments, (3.2) holds for every $\varphi \in V$.

Proof of Proposition 3.1. Let $f \in V^*$ and let $\mathcal{B} : V \to V^*$ be defined by
$$\langle \mathcal{B}(u), v\rangle_{V^*\times V} := \int_\Omega \Big(\langle A(x, \nabla_{\mathbb{G}} u), \nabla_{\mathbb{G}} v\rangle - f\, v\Big)\, dx \quad \forall u, v \in V.$$
Let us show that $\mathcal{B}$ is strictly-monotone, coercive and continuous on any finite dimensional subspace of V. To obtain the weak continuity on finite dimensional Banach spaces, it is enough to prove that $\mathcal{B}$ is strongly continuous in the whole space V.

Fix $u, v \in V$. Then, by Definition 1.1 (ii)
$$\langle \mathcal{B}(u) - \mathcal{B}(v), u - v\rangle_{V^*\times V} \geq \alpha \|u - v\|^p_V \geq 0\,,$$
$$\frac{\langle \mathcal{B}(u) - \mathcal{B}(v), u - v\rangle_{V^*\times V}}{\|u - v\|_V} \geq \alpha \|u - v\|^{p-1}_V.$$
Let $(u_n)_n$ be strongly convergent to u in V. By Hölder's inequality, we have
$$\langle \mathcal{B}(u_n) - \mathcal{B}(u), u_n - u\rangle_{V^*\times V} \leq \|A(\cdot, \nabla_{\mathbb{G}} u_n) - A(\cdot, \nabla_{\mathbb{G}} u)\|_{L^{p'}(\Omega, H\mathbb{G})} \|u_n - u\|_V.$$

Notice that $(A(\cdot,\nabla_{\mathbb{G}}u_n))_n$ strongly converges to $A(\cdot,\nabla_{\mathbb{G}}u)$ in $L^{p'}(\Omega,H\mathbb{G})$ since, by Definition 1.1 (iii) and Hölder's inequality

$$\begin{aligned}
&\|A(\cdot,\nabla_{\mathbb{G}}u_n)-A(\cdot,\nabla_{\mathbb{G}}u)\|^{p'}_{L^{p'}(\Omega,H\mathbb{G})}\\
&\leq \beta^{p'}\int_\Omega [1+|\nabla_{\mathbb{G}}u_n|^p+|\nabla_{\mathbb{G}}u|^p]^{\frac{p-2}{p-1}}\,|\nabla_{\mathbb{G}}u_n-\nabla_{\mathbb{G}}u|^{p'}\,dx\\
&\leq \beta^{p'}\Big(\int_\Omega [1+|\nabla_{\mathbb{G}}u_n|^p+|\nabla_{\mathbb{G}}u|^p]\,dx\Big)^{\frac{p-2}{p-1}}\Big(\int_\Omega |\nabla_{\mathbb{G}}u_n-\nabla_{\mathbb{G}}u|^p\,dx\Big)^{\frac{p'}{p}}\\
&= \beta^{p'}\,[|\Omega|+\|u_n\|_V^p+\|u\|_V^p]^{\frac{p-2}{p}p'}\,\|u_n-u\|_V^{p'}.
\end{aligned}$$

Moreover, by Theorem 2.6, there exists $u\in V$ such that

$$\langle \mathcal{B}(u),v-u\rangle_{V^*\times V}\geq 0\quad \forall v\in V \tag{3.3}$$

and, choosing $v_1:=u+\varphi$ and $v_2:=u-\varphi$, we obtain

$$\langle \mathcal{B}(u),\varphi\rangle_{V^*\times V}=0\quad \forall \varphi\in V.$$

Then, u satisfies (3.2).

Finally, if $u,v\in V$ are weak solutions of (3.1) then, by Remark 3.2 (choosing $\varphi=u-v\in V$) and by Definition 1.1 (ii)

$$0=\int_\Omega \langle A(x,\nabla_{\mathbb{G}}u)-A(x,\nabla_{\mathbb{G}}v),\nabla_{\mathbb{G}}u-\nabla_{\mathbb{G}}v\rangle dx\geq \alpha\|u-v\|_V^p\geq 0\,,$$

that is, the solution of (3.1) is unique. $\square$

As a direct consequence of Proposition 3.1, $\mathcal{A}$ is continuous and invertible in V. We conclude this section providing useful estimates.

Proposition 3.3. *Let $A\in\mathcal{M}(\alpha,\beta;\Omega)$, let $\mathcal{A}$ be as in* (1.2) *and let $\mathcal{A}^{-1}$ be its inverse operator. Then*

(a) $\langle \mathcal{A}(u)-\mathcal{A}(v),u-v\rangle_{V^*\times V}\geq \alpha\|u-v\|_V^p$;
(b) $\|\mathcal{A}^{-1}(f)-\mathcal{A}^{-1}(g)\|_V^p\leq (\frac{1}{\alpha})^{p'}\|f-g\|_{V^*}^{p'}$;
(c) $\|\mathcal{A}(u)-\mathcal{A}(v)\|_{V^*}\leq \beta[|\Omega|+\|u\|_V^p+\|v\|_V^p]^{\frac{p-2}{p}}\|u-v\|_V$

for any $u,v\in V$ and for any $f,g\in V^*$.

Proof. Fix $u,v\in V$ and $f,g\in V^*$ such that

$$\mathcal{A}(u)=f\quad\text{and}\quad \mathcal{A}(v)=g\quad\text{in }\Omega\,.$$

Notice that (a) directly follows from Definition 1.1 (ii). Moreover, recalling that

$$\langle \mathcal{A}(u)-\mathcal{A}(v),u-v\rangle_{V^*\times V}\leq \|\mathcal{A}(u)-\mathcal{A}(v)\|_{V^*}\|u-v\|_V\quad \forall u,v\in V,$$

and applying (a), with $u=\mathcal{A}^{-1}(f)$ and $v=\mathcal{A}^{-1}(g)$, we obtain

$$\alpha\|\mathcal{A}^{-1}(f)-\mathcal{A}^{-1}(g)\|_V^p\leq \|f-g\|_{V^*}\|\mathcal{A}^{-1}(f)-\mathcal{A}^{-1}(g)\|_V,$$

which implies (b).

Finally, by Definition 1.1 (iii),

$$\|A(\cdot,\nabla_{\mathbb{G}}u)-A(\cdot,\nabla_{\mathbb{G}}v)\|_{L^{p'}(\Omega,H\mathbb{G})}\leq \beta\,[|\Omega|+\|u\|_V^p+\|v\|_V^p]^{\frac{p-2}{p}}\,\|u-v\|_V,$$

i.e.,

$$\begin{aligned}
\langle \mathcal{A}(u)-\mathcal{A}(v),u-v\rangle_{V^*\times V}&\leq \|A(\cdot,\nabla_{\mathbb{G}}u)-A(\cdot,\nabla_{\mathbb{G}}v)\|_{L^{p'}(\Omega,H\mathbb{G})}\|u-v\|_V\\
&\leq \beta[|\Omega|+\|u\|_V^p+\|v\|_V^p]^{\frac{p-2}{p}}\|u-v\|_V^2.
\end{aligned}$$

Then, (c) follows by the definition of $\|\cdot\|_{V^*}$. $\square$

4. H-convergence and Div-curl lemma

The following statement of H-convergence is a natural adaptation of the original definition of Murat and Tartar in our context.

Definition 4.1. Let $A^n \in \mathcal{M}(\alpha,\beta;\Omega)$ and let $A^{\mathrm{eff}} \in \mathcal{M}(\alpha',\beta';\Omega)$, for some $\alpha \le \beta$ and $\alpha' \le \beta'$ positive constants. Fix $f \in W^{-1,p'}_{\mathbb{G}}(\Omega)$ and let $u_n, u_\infty \in W^{1,p}_{\mathbb{G},0}(\Omega)$ be, respectively, weak solutions of

$$-\operatorname{div}_{\mathbb{G}}(A^n(\cdot,\nabla_{\mathbb{G}}u)) = f \quad \text{in } \Omega$$
$$-\operatorname{div}_{\mathbb{G}}(A^{\mathrm{eff}}(\cdot,\nabla_{\mathbb{G}}u)) = f \quad \text{in } \Omega\,.$$

We say that $(A^n)_n$ H-converges to A^{eff} if, as $n\to\infty$,

$$u_n \to u_\infty \text{ weakly in } W^{1,p}_{\mathbb{G},0}(\Omega) \text{ (convergence of solutions)}$$

and

$$A^n(\cdot,\nabla_{\mathbb{G}}u_n) \to A^{\mathrm{eff}}(\cdot,\nabla_{\mathbb{G}}u_\infty) \text{ weakly in } L^{p'}(\Omega,H\mathbb{G}) \text{ (convergence of momenta).}$$

Before proving Theorem 1.2, we need two preliminary results.

Lemma 4.2. *Let $A^n \in \mathcal{M}(\alpha,\beta;\Omega)$ and define $\mathcal{A}_n : W^{1,p}_{\mathbb{G},0}(\Omega) \to W^{-1,p'}_{\mathbb{G}}(\Omega)$ as*

$$\mathcal{A}_n(u) := -\operatorname{div}_{\mathbb{G}}(A^n(\cdot,\nabla_{\mathbb{G}}u)) \quad \text{in } \Omega\,.$$

Then, there exist a continuous and invertible operator $\mathcal{A}_\infty : W^{1,p}_{\mathbb{G},0}(\Omega) \to W^{-1,p'}_{\mathbb{G}}(\Omega)$ and a subsequence $(\mathcal{A}_m)_m$ of $(\mathcal{A}_n)_n$, such that

$$\mathcal{A}_m^{-1}(f) \to \mathcal{A}_\infty^{-1}(f) \quad \text{weakly in } W^{1,p}_{\mathbb{G},0}(\Omega)$$

for every $f \in W^{-1,p'}_{\mathbb{G}}(\Omega)$.

Proof. For the sake of simplicity, let us denote $V = W^{1,p}_{\mathbb{G},0}(\Omega)$ and $V^* = W^{-1,p'}_{\mathbb{G}}(\Omega)$. We divide the proof of the lemma into three steps.

Step 1. Let X be a fixed countable and dense subset of V^*. We show that, for any fixed $f \in X$, the sequence of solutions of

$$\mathcal{A}_n(u) = f \quad \text{in } \Omega \tag{4.1}$$

weakly converges, up to subsequences, in V. Moreover, we provide an upper-bound for its limit, in terms of f.

Fix $f \in X$. Then, by Proposition 3.1, there exists $u_n \in V$, weak solution of (4.1), that is, $u_n = \mathcal{A}_n^{-1}(f)$ for any $n \in \mathbb{N}$. Moreover, by Proposition 3.3 (b)

$$\|u_n\|_V \le \Big(\frac{1}{\alpha}\Big)^{\frac{1}{p-1}} \|f\|_{V^*}^{\frac{1}{p-1}},$$

i.e., $(u_n)_n$ is bounded in V, reflexive Banach space and, therefore, there exist $u_\infty(f) \in V$ and $(u_m)_m$, diagonal subsequence of $(u_n)_n$, such that

$$u_m \to u_\infty(f) \quad \text{weakly in } V.$$

Notice that, by the lower semicontinuity of the norm and by Proposition 3.3(a),

$$\langle f, u_\infty\rangle_{V^*\times V} = \lim_{m\to\infty} \langle \mathcal{A}_m(u_m), u_m\rangle_{V^*\times V} \ge \alpha \liminf_{m\to\infty} \|u_m\|_V^p \ge \alpha \|u_\infty\|_V^p$$

and, since

$$\langle f, u_\infty\rangle_{V^*\times V} \le \|f\|_{V^*}\|u_\infty\|_V,$$

it follows that

$$\|u_\infty\|_V \le \Big(\frac{1}{\alpha}\Big)^{\frac{1}{p-1}}\|f\|_{V^*}^{\frac{1}{p-1}}.$$

Step 2. Define $S : X \to V$ as

$$S(f) := \lim_{m\to\infty} \mathcal{A}_m^{-1}(f) \quad \text{for any } f \in X.$$

Let us show that S can be extended to the whole space V^*. Since X is countable and dense in V^*, it is sufficient to show that S is continuous in $(X, \|\cdot\|_{V^*})$.

Fix $f, g \in X$. Then, by Proposition 3.3(b),

$$\|\mathcal{A}_m^{-1}(f) - \mathcal{A}_m^{-1}(g)\|_V \le \Big(\frac{1}{\alpha}\Big)^{\frac{1}{p-1}}\|f-g\|_{V^*}^{\frac{1}{p-1}} \quad \forall m \in \mathbb{N}$$

and, passing to the limit, by the lower semicontinuity of the norm, we obtain

$$\|S(f) - S(g)\|_V \le \liminf_{m\to\infty}\|\mathcal{A}_m^{-1}(f) - \mathcal{A}_m^{-1}(g)\|_V \le \Big(\frac{1}{\alpha}\Big)^{\frac{1}{p-1}}\|f-g\|_{V^*}^{\frac{1}{p-1}}.$$

For the sake of completeness, the extension of S to $V^* \setminus X$ is defined as

$$S(f) := \lim_{n\to\infty} S(f_n)$$

for any $f \in V^*$ and $(f_n)_n \subset X$ such that $f_n \to f$ in V^*.

Step 3. Let us finally prove that, as a consequence of Theorem 2.6, S is invertible in V^*. To this aim, we show that S is monotone and coercive in V^*. Fix $f, g \in V^*$. Then, by Proposition 3.3(a),

$$\begin{aligned}
\langle S(f) - S(g), f-g\rangle_{V\times V^*} &= \lim_{m\to\infty}\langle \mathcal{A}_m^{-1}(f) - \mathcal{A}_m^{-1}(g), f-g\rangle_{V\times V^*}\\
&= \lim_{m\to\infty}\langle \mathcal{A}_m(u_m) - \mathcal{A}_m(v_m), u_m - v_m\rangle_{V^*\times V}\\
&\ge \alpha \lim_{m\to\infty}\|u_m - v_m\|_V^p \ge 0\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\|\mathcal{A}_m(u_m) - \mathcal{A}_m(v_m)\|_{V^*}^p\\
&\le \beta^p\,[|\Omega| + \|u_m\|_V^p + \|v_m\|_V^p]^{p-2}\,\|u_m - v_m\|_V^p\\
&\le \frac{\beta^p}{\alpha}\,[|\Omega| + \|u_m\|_V^p + \|v_m\|_V^p]^{p-2}\,\langle \mathcal{A}_m(u_m) - \mathcal{A}_m(v_m), u_m - v_m\rangle_{V^*\times V}\\
&\le \frac{\beta^p}{\alpha}\Big[|\Omega| + \Big(\frac{1}{\alpha}\Big)^{p'}\|f\|_{V^*}^{p'} + \Big(\frac{1}{\alpha}\Big)^{p'}\|g\|_{V^*}^{p'}\Big]^{p-2}\langle \mathcal{A}_m^{-1}(f) - \mathcal{A}_m^{-1}(g), f-g\rangle_{V\times V^*}\,.
\end{aligned}$$

Passing to the limit,

$$\begin{aligned}
&\|f-g\|_{V^*}^p\\
&\le \frac{\beta^p}{\alpha}\Big[|\Omega| + \Big(\frac{1}{\alpha}\Big)^{p'}\|f\|_{V^*}^{p'} + \Big(\frac{1}{\alpha}\Big)^{p'}\|g\|_{V^*}^{p'}\Big]^{p-2}\langle S(f) - S(g), f-g\rangle_{V\times V^*}.
\end{aligned}$$

We obtain the conclusion, defining $\mathcal{A}_\infty := S^{-1} : V \to V^*$. $\square$

Lemma 4.3. *Let $\mathcal{A}_n$ be as in the Lemma 4.2. Then, for any $f \in W_{\mathbb{G}}^{-1,p'}(\Omega)$, there exists a continuous operator $M : W_{\mathbb{G}}^{-1,p'}(\Omega) \to L^{p'}(\Omega, H\mathbb{G})$ such that, up to subsequences*

$$A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)) \to M(f) \quad \textit{weakly in } L^{p'}(\Omega, H\mathbb{G}).$$

Proof. Let X be a countable and dense subspace of $L^{p'}(\Omega, H\mathbb{G})$ and let $f \in X$. Then, by Definition 1.1(iii) and Hölder's inequality

$$\begin{aligned}\int_\Omega |A^n(x, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f))|^{p'}\,dx &\le \beta^{p'} \int_\Omega [1 + |\nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)|^p]^{\frac{p-2}{p-1}} |\nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)|^{p'}\,dx \\ &\le \beta^{p'} [|\Omega| + \|\mathcal{A}_n^{-1}(f)\|_V^p]^{\frac{p-2}{p}p'} \|\mathcal{A}_n^{-1}(f)\|_V^{p'},\end{aligned}$$

i.e.,

$$\|A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f))\|_{L^{p'}(\Omega,H\mathbb{G})} \le \beta[|\Omega| + \|\mathcal{A}_n^{-1}(f)\|_V^p]^{\frac{p-2}{p}} \|\mathcal{A}_n^{-1}(f)\|_V$$

and, by Proposition 3.3,

$$\|A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f))\|_{L^{p'}(\Omega,H\mathbb{G})} \le \frac{\beta}{\alpha^{\frac{1}{p-1}}} \big[|\Omega| + \big(\frac{1}{\alpha}\big)^{p'} \|f\|_{V^*}^{p'}\big]^{\frac{p-2}{p}} \|f\|_{V^*}^{\frac{1}{p-1}}.$$

Therefore, $(A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)))_n$ is bounded in $L^{p'}(\Omega, H\mathbb{G})$ and, by the countability of X, there exists a diagonal subsequence of $(A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)))_n$ weakly convergent to $M = M(f)$ in $L^{p'}(\Omega, H\mathbb{G})$.

We define $M : X \to L^{p'}(\Omega, H\mathbb{G})$ as

$$M(f) := \lim_{m\to\infty} A^m(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_m^{-1}(f)) \quad \text{for any } f \in X.$$

If $f, g \in X$, then, by Proposition 3.3,

$$\begin{aligned}&\|A^m(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_m^{-1}(f)) - A^m(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_m^{-1}(g))\|_{L^{p'}(\Omega,H\mathbb{G})} \\ &\le \frac{\beta}{\alpha^{\frac{1}{p-1}}} \big[|\Omega| + \big(\frac{1}{\alpha}\big)^{p'} \|f\|_{V^*}^{p'} + \big(\frac{1}{\alpha}\big)^{p'} \|g\|_{V^*}^{p'}\big]^{\frac{p-2}{p}} \|f-g\|_{V^*}^{\frac{1}{p-1}}.\end{aligned}$$

Therefore, by the lower semicontinuity of the norm, M can be extended to the whole space V^*, and the thesis follows. □

We recall now the statement of *Div-curl lemma*, in the framework of Carnot groups, given by Baldi, Franchi, Tchou and Tesi [2].

Theorem 4.4 ([2, Theorem 5.1])**.** *Let $\Omega \subset \mathbb{G}$ be an open set and let $p, q > 1$ be a Hölder's conjugate pair. Moreover, following the notations of* [2], *if $\sigma \in \mathcal{I}_0^2$, let $a(\sigma) > 1$ and $b > 1$ be such that*

$$a(\sigma) > \frac{Qp}{Q + (\sigma-1)p} \quad \textit{and} \quad b > \frac{Qq}{Q+q}.$$

Finally, let $E^n, E \in L^p_{\rm loc}(\Omega, H\mathbb{G})$ and $D^n, D \in L^q_{\rm loc}(\Omega, H\mathbb{G})$ be such that

(i) *$E^n \to E$ weakly in $L^p_{\rm loc}(\Omega, H\mathbb{G})$;*
(ii) *$D^n \to D$ weakly in $L^q_{\rm loc}(\Omega, H\mathbb{G})$;*
(iii) *the components of $(\operatorname{curl}_{\mathbb{G}} E^n)_n$ of weight σ are bounded in $L^{a(\sigma)}_{\rm loc}(\Omega, H\mathbb{G})$;*
(iv) *$(\operatorname{div}_{\mathbb{G}} D^n)_n$ is bounded in $L^b_{\rm loc}(\Omega, H\mathbb{G})$.*

Then $\langle D^n, E^n\rangle \to \langle D, E\rangle$ in $\mathcal{D}'(\Omega)$, i.e.,

$$\int_\Omega \langle D^n(x), E^n(x)\rangle \varphi(x)\,dx \to \int_\Omega \langle D(x), E(x)\rangle \varphi(x)\,dx \quad \textit{for any } \varphi \in \mathcal{D}(\Omega).$$

Proof of Theorem 1.2. We denote $\mathcal{A}_\infty$ and M the operators defined in Lemma 4.2 and Lemma 4.3, and define

$$C := M \circ \mathcal{A}_\infty : W^{1,p}_{\mathbb{G},0}(\Omega) \to L^{p'}(\Omega, H\mathbb{G}) .$$

Let us show the existence of $A^{\rm eff} \in \mathcal{M}(\alpha, \beta; \Omega)$ such that

$$C(u) = A^{\rm eff}(x, \nabla_{\mathbb{G}} \mathcal{A}_\infty^{-1}(f))$$

for any $f \in W^{-1,p'}_{\mathbb{G}}(\Omega)$ and for any $u \in W^{1,p}_{\mathbb{G},0}(\Omega)$ such that

$$\mathcal{A}_\infty(u) = f \quad \text{a.e. } x \in \Omega . \tag{4.2}$$

Fix $f \in W^{-1,p'}_{\mathbb{G}}(\Omega)$ and ω open set such that $\overline{\omega} \subset \Omega$. For any $v \in W^{1,p}_{\mathbb{G},0}(\Omega)$, weak solution of (3.1), we define the Carathéodory function $A^{\rm eff} : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ as

$$A^{\rm eff}(x, \xi) := C(v) \quad \text{if } \nabla_{\mathbb{G}} v(x) = \xi \quad \text{ a.e. } x \text{ in } \omega .$$

Let us show that

$$A^{\rm eff}(x, \xi_1) = A^{\rm eff}(x, \xi_2) \quad \text{a.e. } x \text{ in } \omega_1 \cap \omega_2 \tag{4.3}$$

for any $\xi_1 = \xi_2 \in \mathbb{R}^m$ and for any ω_1, ω_2 open sets such that $\overline{\omega_1}, \overline{\omega_2} \subset \Omega$.

We fix $\varphi_1, \varphi_2 \in \mathbf{C}^1_c(\Omega)$ such that $\varphi_i|_{\omega_i} = 1$ for $i = 1, 2$, and let $(v_{1,n})_n \subset W^{1,p}_{\mathbb{G},0}(\Omega)$ and $(v_{2,n})_n \subset W^{1,p}_{\mathbb{G},0}(\Omega)$ be, respectively, weakly convergent, up to subsequences, to

$$\begin{gathered} v_{1,\infty}(x) = \varphi_1(x) \, \langle \xi_1, \pi(x) \rangle \\ v_{2,\infty}(x) = \varphi_2(x) \, \langle \xi_2, \pi(x) \rangle, \end{gathered} \tag{4.4}$$

where $\pi(x) := (x_1, \dots, x_m)$ for every $x = (x_1, \dots, x_n) \in \Omega$. Moreover, define

$$\begin{aligned} D_i^n &:= A^n(\cdot, \nabla_{\mathbb{G}} v_{i,n}) \in L^{p'}(\Omega, H\mathbb{G}) \\ E_i^n &:= \nabla_{\mathbb{G}} v_{i,n} \in L^p(\Omega, H\mathbb{G}) \end{aligned}$$

and fix $f_1, f_2 \in W^{-1,p'}_{\mathbb{G}}(\Omega)$ such that

$$f_1 = -\operatorname{div}_{\mathbb{G}}(C(v_{1,\infty})), \quad f_2 = -\operatorname{div}_{\mathbb{G}}(C(v_{2,\infty})) \quad \text{in } \Omega .$$

By (4.4), it holds that

$$\begin{gathered} \nabla_{\mathbb{G}} v_{1,\infty} = \xi_1 \quad \text{in } \omega_1 \\ \nabla_{\mathbb{G}} v_{2,\infty} = \xi_2 \quad \text{in } \omega_2 . \end{gathered} \tag{4.5}$$

Notice that $\operatorname{curl}_{\mathbb{G}}(E_i^n) = 0$, for any $n \in \mathbb{N}$ and $i = 1, 2$. Moreover, there exist $(D_i^m)_m$, $(E_i^m)_m$, diagonal subsequences of $(D_i^n)_n$ and $(E_i^n)_n$ and $D_i \in L^{p'}(\Omega, H\mathbb{G})$ and $E_i \in L^p(\Omega, H\mathbb{G})$, $i = 1, 2$, such that

$$\begin{aligned} D_i^m &\to D_i \text{ weakly in } L^{p'}(\Omega, H\mathbb{G}) \\ E_i^m &\to E_i \text{ weakly in } L^p(\Omega, H\mathbb{G}). \end{aligned}$$

Therefore, by (4.5), by Lemma 4.2, Lemma 4.3, and by Theorem 4.4 (where a is each value grater than 1, which satisfies the hypotheses of the theorem, and $b = p'$), it follows that

$$\begin{aligned} &\int_\Omega \langle A^m(x, \nabla_{\mathbb{G}} v_{2,m}) - A^m(x, \nabla_{\mathbb{G}} v_{1,m}), \nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m} \rangle \varphi(x) \, dx \\ &\to \int_\Omega \langle A^{\rm eff}(x, \xi_2) - A^{\rm eff}(x, \xi_1), \xi_2 - \xi_1 \rangle \varphi(x) \, dx \end{aligned} \tag{4.6}$$

for any $\varphi \in \mathcal{D}(\omega_1 \cap \omega_2)$.

Fix $\varphi \geq 0$ and notice that, by Definition 1.1(ii), it holds that

$$\begin{aligned}&\int_\Omega \langle A^m(x,\nabla_{\mathbb{G}} v_{2,m}) - A^m(x,\nabla_{\mathbb{G}} v_{1,m}), \nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}\rangle \varphi(x)\,dx \\ &\geq \alpha \int_\Omega |\nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}|^p \,\varphi(x)\,dx\,.\end{aligned} \tag{4.7}$$

Then, by (4.5), (4.6) and (4.7) and Fatou's lemma,

$$\begin{aligned}&\int_\Omega \langle A^{\rm eff}(x,\xi_2) - A^{\rm eff}(x,\xi_1), \xi_2-\xi_1\rangle \varphi(x) dx \\ &\geq \liminf_{m\to\infty} \alpha \int_\Omega |\nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}|^p \varphi(x) dx \\ &\geq \alpha \int_\Omega |\nabla_{\mathbb{G}} v_{2,\infty} - \nabla_{\mathbb{G}} v_{1,\infty}|^p \varphi(x)\,dx \\ &= \alpha \int_\Omega |\xi_2-\xi_1|^p \varphi(x)\,dx\,.\end{aligned} \tag{4.8}$$

Moreover, since by Definition 1.1(iii)

$$\begin{aligned}&\int_\Omega |\nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}|^p \varphi(x)\,dx \\ &\geq \frac{1}{\beta^p} \int_\Omega \big[1 + |\nabla_{\mathbb{G}} v_{2,m}|^p + |\nabla_{\mathbb{G}} v_{1,m}|^p\big]^{2-p} \\ &\quad\times |A^m(x,\nabla_{\mathbb{G}} v_{2,m}) - A^m(x,\nabla_{\mathbb{G}} v_{1,m})|^p \varphi(x) dx\,,\end{aligned} \tag{4.9}$$

then, by (4.5), (4.6), (4.7) and (4.9), and Fatou's lemma,

$$\begin{aligned}&\int_\Omega \langle A^{\rm eff}(x,\xi_2) - A^{\rm eff}(x,\xi_1), \xi_2-\xi_1\rangle \varphi(x)\,dx \\ &\geq \frac{\alpha}{\beta^p} \int_\Omega [1+|\xi_2|^p+|\xi_1|^p]^{2-p} |A^{\rm eff}(x,\xi_2) - A^{\rm eff}(x,\xi_1)|^p \varphi(x)\,dx\,.\end{aligned} \tag{4.10}$$

Varying φ in $\mathcal{D}(\omega_1\cap\omega_2)$, (4.8) and (4.10) give

$$\begin{gathered}\langle A^{\rm eff}(x,\xi_2) - A^{\rm eff}(x,\xi_1), \xi_2-\xi_1\rangle \geq \alpha\,|\xi_2-\xi_1|^p\,, \\ \langle A^{\rm eff}(x,\xi_2) - A^{\rm eff}(x,\xi_1), \xi_2-\xi_1\rangle \\ \geq \frac{\alpha}{\beta^p} [1+|\xi_2|^p+|\xi_1|^p]^{2-p} |A^{\rm eff}(x,\xi_2) - A^{\rm eff}(x,\xi_1)|^p\end{gathered}$$

a.e. $x\in\omega_1\cap\omega_2$.

If $\xi_1=\xi_2$, we obtain (4.3), and if $\xi_1\neq\xi_2$, then $A^{\rm eff}$ satisfies Definition 1.1(ii).

Moreover, by Definition (1.1)(iii), by (4.5) and Fatou's lemma,

$$\begin{aligned}&\int_\Omega |\xi_2-\xi_1|^p \,\varphi(x) dx \\ &\geq \liminf_{m\to\infty} \int_\Omega |\nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}|^p \varphi(x)\,dx \\ &\geq \liminf_{m\to\infty} \frac{1}{\beta^p} \int_\Omega \big[1 + |\nabla_{\mathbb{G}} v_{2,m}|^p + |\nabla_{\mathbb{G}} v_{1,m}|^p\big]^{2-p} \\ &\quad\times |A^m(x,\nabla_{\mathbb{G}} v_{2,m}) - A^m(x,\nabla_{\mathbb{G}} v_{1,m})|^p \,\varphi(x)\,dx \\ &\geq \frac{1}{\beta^p} \int_\Omega [1+|\xi_2|^p+|\xi_1|^p]^{2-p} |A^{\rm eff}(x,\xi_2) - A^{\rm eff}(x,\xi_1)|^p \,\varphi(x) dx\end{aligned}$$

and, varying φ in $\mathcal{D}(\omega_1 \cap \omega_2)$, $A^{\rm eff}$ satisfies Definition 1.1(iii).

Let $u_n \in W^{1,p}_{\mathbb{G},0}(\Omega)$ be the (unique) weak solution of (4.1), relative to $f = 0$. Since $A^n(\cdot, 0) = 0$ a.e. in Ω by Definition 1.1(i), then $u_n = 0$ a.e. in Ω and, by Lemma 4.2 and Lemma 4.3, up to subsequences

$$0 = A^n(x, \nabla_{\mathbb{G}} u_n) \to A^{\rm eff}(x, 0) \quad \text{weakly in } L^{p'}(\Omega, H\mathbb{G}).$$

Then, $A^{\rm eff}$ satisfies also Definition 1.1 (i) and, therefore

$$A^{\rm eff} \in \mathcal{M}(\alpha, \beta; \Omega)\,.$$

To conclude the proof of the theorem, we show that

$$C(u_\infty) = A^{\rm eff}(x, \nabla_{\mathbb{G}} u_\infty) \quad \text{a.e. } x \in \Omega\,. \tag{4.11}$$

Let $u_\infty \in W^{1,p}_{\mathbb{G},0}(\Omega)$ be the (unique) weak solution of (4.2), let $(u_m)_m$ be weakly convergent to u_∞ in $W^{1,p}_{\mathbb{G},0}(\Omega)$ and define $D_2^m = A^m(x, \nabla_{\mathbb{G}} u_m)$ and $E_2^m = \nabla_{\mathbb{G}} u_m$.

Then, by Theorem 4.4,

$$\begin{aligned}
&\int_\Omega \langle A^m(x, \nabla_{\mathbb{G}} u_m) - A^m(x, \nabla_{\mathbb{G}} v_{1,m}), \nabla_{\mathbb{G}} u_m - \nabla_{\mathbb{G}} v_{1,m}\rangle \varphi(x)\,dx \\
&\to \int_\Omega \langle C(u_\infty) - A^{\rm eff}(x, \xi_1), \nabla_{\mathbb{G}} u_\infty - \xi_1\rangle \varphi(x)\,dx
\end{aligned}$$

for any $\varphi \in \mathcal{D}(\omega_1)$ and, following the same techniques of the first part of the proof,

$$\begin{aligned}
&\langle C(u_\infty) - A^{\rm eff}(x, \xi_1), \nabla_{\mathbb{G}} u_\infty - \xi_1\rangle \ge \alpha |\nabla_{\mathbb{G}} u_\infty - \xi_1|^p\,, \\
&\langle C(u_\infty) - A^{\rm eff}(x, \xi_1), \nabla_{\mathbb{G}} u_\infty - \xi_1\rangle \\
&\ge \frac{\alpha}{\beta^p}[1 + |\nabla_{\mathbb{G}} u_\infty|^p + |\xi_1|^p]^{2-p} |C(u_\infty) - A^{\rm eff}(x, \xi_1)|^p;
\end{aligned}$$

that is,

$$|C(u_\infty) - A^{\rm eff}(x, \xi_1)| \le \beta [1 + |\nabla_{\mathbb{G}} u_\infty|^p + |\xi_1|^p]^{\frac{p-2}{p}} |\nabla_{\mathbb{G}} u_\infty - \xi_1| \quad \text{a.e. } x \in \omega_1.$$

Finally, varying $\varphi \in \mathcal{D}(\omega_1)$ and $\xi_1 \in \mathbb{R}^m$, we obtain (4.11). $\square$

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