APPLICATION OF THE LEFSCHETZ-HOPF FIXED POINT THEOREM TO CLOSED ORIENTABLE SURFACES

THESIS

Presented to the Graduate Council of
Texas State University–San Marcos
in Partial Fulfillment
of the Requirements

for the Degree

Master of SCIENCE

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San Marcos, Texas
December 2007

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CHAPTER I

INTRODUCTION

In this thesis, we give certain sufficient conditions for a continuous function from a closed orientable surface into itself to have a fixed point. To accomplish this, we will apply the Lefschetz-Hopf fixed point theorem, which relies on the computation of traces of a continuous function's induced endomorphisms on the surface's homology groups. We thereby arrive at the following two theorems.

Thorem 2..1. Let $f : \mathbb{T}_1 \to \mathbb{T}_1$ be a map. Write $m = \text{tr } f_2$, and suppose $x, y \in \mathbb{Z}$. Then f has a fixed point if any of the following conditions are met.

- **Case 1.** $m = 0$ and $\lambda_1 + \lambda_2 \neq 1$.
- **Case 2.1.** $m = \det f_1 = \lambda_1\lambda_2$, $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$.
- **Case 2.2.** $[f_1] = \begin{pmatrix} \lambda_1 & \lambda_1 \\ 1-\lambda_1 & \lambda_2 \end{pmatrix}$, $m = \det f_1$, $\lambda_1 \neq 1$ and $\text{tr } f_1 \neq 1$.
- **Case 2.3.** $[f_1] = \begin{pmatrix} \lambda_1 & y \\ y & \lambda_1 \end{pmatrix}$, $m = \det f_1$, and $\lambda_1 \pm y \neq 1$.
- **Case 2.4.** $[f_1] = \begin{pmatrix} x & (-1)^k \\ (-1)^k & xy \end{pmatrix}$, $m = \det f_1$, and $y \neq 1$.

Theorem 3..3. Let f be an orientation-preserving self-map of $\mathbb{T}_g$, where $g \in \mathbb{N}$. If for each $1 \leq \imath \leq g$, $f_1(a_\imath) = a_\jmath$ and $f_1(b_\imath) = b_\jmath$ for some $\jmath \neq \imath$, then f has a fixed point.

We then examine whether these conditions are realizable, providing examples of classes of maps which can be shown to have fixed points.

In Chapter **II**, we discuss the preliminaries needed to understand the main result of this thesis. Notation for basic concepts from algebra and topology are established in Section 1.. Definitions for homology theory are given in Section 2., where the Eilenberg-Steenrod axioms and the Barratt-Whitehead lemma are also stated. Section 3. presents the idea of a closed orientable surface, paying closest attention to orientable surfaces. The classification theorem for closed surfaces without boundary closes the section. In Section 4., the homology groups of all closed orientable surfaces without boundary are computed.

Our main results, Theorem 2..1 and Theorem 3..3, are proven in Chapter **III**, which is divided into three sections. The first dispenses with the (simple) case of the sphere, and gives the well-known result that a map $S^2 \to S^2$ has a fixed point if it is not homotopic to the antipodal map $\mathbf{x} \mapsto -\mathbf{x}$. The second and third sections state conditions for a map from a torus of genus $g = 1$ and $g > 1$, respectively, to have a fixed point. We prove in those sections that the conditions given are indeed sufficient.

The final chapter demonstrates that the conditions of Chapter **III** are realizable by certain maps. We do this for the torus of genus 1 chiefly by providing examples of maps which realize the conditions given in Theorem 2..1. The case where $g > 1$ is dealt with in Section 2., where we discuss the constraints placed on maps in Theorem 3..3, our main result for this case.

CHAPTER II

PRELIMINARIES

This chapter gives the algebraic machinery and topological background material needed in the sequel. Once general notation has been established, we give the standard axioms for a homology theory and explain how these axioms will be applied in this thesis. In this section on surfaces, we introduce a few elementary examples, including the Möbius strip and the Klein bottle, and we state the classification theorem for closed surfaces without boundary. We will then be prepared to compute the homology groups of all such surfaces up to homeomorphism.

1. CONVENTIONS AND NOTATION

Throughout this document, boldface type will indicate that a term or concept is being defined. When an idea is introduced without formal definition, we typeset it in italics.

Definitions needed for the homology of surfaces may be found in any introductory text on algebraic topology. The references [26], [24], [7], and [18] are examples, arranged in order of increasing accessibility. Definitions from point set topology may be found in [7] and [9].

The symbol $\oslash$ represents the *empty set*. We write $\mathbb{Z}$ for the set of *integers*, and $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$ for the set of *natural numbers*. The set $\mathbb{Z}^+ \cup \{0\}$ will be denoted by $\mathbb{N}_0$.

Let (X, T_X) and (Y, T_Y) be two topological spaces, and let $f : X \to Y$ be a function. If f is continuous with respect to the topologies T_X and T_Y, we call f a **map from X into Y**. If in addition $Y = X$, we call f a **self-map of X**. The **identity map** of a space X is the self-map of X defined by $x \mapsto x$. If $A \subset X$, the **inclusion map from A into X** is defined by $a \mapsto a$, and we replace the usual arrow with a hooked arrow, as in $\imath : A \hookrightarrow X$. The set of fixed points of a self-map f of X will be denoted

$$\mathrm{Fix}(f) = \{x \in X : f(x) = x\}.$$

If X and Y are two topological spaces, the notation $X \cong Y$ means that the spaces are homeomorphic.

If G and G' are two groups, the notation $G \cong G'$ indicates that the groups are isomorphic. If φ is an injective homomorphism from a group A into group B, we say that the image of φ is an **isomorphic embedding of A into B**, noting that $A \cong \varphi(A)$, since a function is by definition surjective onto its image.

The **(external) direct sum** of two abelian groups A and B will be written as

$$A \oplus B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Let A, B, C, D be four abelian groups. If f is a homomorphism from A to C, and g is a homomorphism from B to D, we define a homomorphism $f \oplus g : A \oplus B \to C \oplus D$

by

$$(f \oplus g)(a, b) = (f(a), g(b))$$

for each $(a, b) \in A \oplus B$. If $D = C$, we define a homomorphism $f - g : A \oplus B \to C$ by

$$(f - g)(a, b) = f(a) - g(b)$$

for each $(a, b) \in A \oplus B$.

The **trace** of an $n \times n$ matrix A with diagonal entries $\lambda_\imath$, $1 \leq \imath \leq n$ is defined as $\operatorname{tr} A = \sum_{\imath=1}^{n} \lambda_\imath$. In this thesis, **determinants** will only be computed for 2×2 matrices of integers, *e.g.* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that we may define $\det A = ad - bc$. We shall also need the well-known result that the expression $|\det A| = |(a, b, 0) \times (c, d, 0)|$ is the area of a parallelogram in the xy-plane whose sides are the vectors $a\mathbf{i} + b\mathbf{j}$ and $c\mathbf{i} + d\mathbf{j}$, where $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ are the standard unit vectors (see [21], I §7).

We write $\left(u_\imath^\jmath\right)$ for a square matrix, where the subscripted index indicates the column and the superscripted index indicates the row of the entry $u_\imath^\jmath$. Let R be a ring, and suppose M is a free R-module generated by $g_\imath \in M$, $1 \leq \imath \leq s$. The **rank** of M is defined as s. A linear transformation $\varphi : M \to M$ can be represented by an $s \times s$ matrix $\left(u_\imath^\jmath\right)$ with entries in R in the sense that, for any $m_\imath \in R$, $1 \leq \imath \leq s$,

$$\varphi\left(\sum_{\imath=1}^{s} m_\imath g_\imath\right) = \sum_{\imath=1}^{s} m_\imath \varphi(g_\imath)$$

is the dot product of the vector $(m_1, \ldots, m_s)$ with the result of the matrix multiplication

$$\begin{pmatrix} u_1^1 & \dots & u_1^s \\ \vdots & \ddots & \vdots \\ u_s^1 & \dots & u_s^s \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_s \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^s u_1^j g_1 \\ \vdots \\ \sum_{j=1}^s u_s^j g_s \end{pmatrix}.$$

We write $[\varphi]$ for the matrix representing φ.

If X is a topological space, and $A \subset X$, we denote the *interior* of A by A° and the *closure* of A by $\overline{A}$. The *frontier* (or *boundary*) of A will be denoted by Fr A and the symbol ∂ will be reserved for other purposes [9].

Unless otherwise noted, the remaining definitions in this section appear in [26].

A **topological pair** is an ordered pair (X, A), where X is a topological space, and A is a subspace of X. A map $(X, A) \to (Y, B)$ between topological pairs is a map $f : X \to Y$ such that $f(A) \subset B$.

Let (X, A) and (Y, B) be two topological pairs. Let f_0, $f_1 : (X, A) \to (Y, B)$ be two maps. A **homotopy** $(X, A) \to (Y, B)$ from f_0 to f_1 is a continuous function $H : X \times I \to Y$ such that

- $H(X \times I) \subset Y$,
- $H(A \times I) \subset B$,
- $H(x, 0) = f_0(x)$ for each $x \in X$, and
- $H(x, 1) = f_1(x)$ for each $x \in X$.

If there exists such a map F, we say f_0 is **homotopic (in Y)** to f_1, and we write $F : f_0 \simeq f_1$.

Let X and Y be two topological spaces. X and Y are said to be **homotopy equivalent** if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and

$f \circ g \simeq 1_Y$.

Let A and B be two subsets of a topological space X. We say that B is **(continuously) deformable into A (over B)** if the identity map of B is homotopic in X to a map of B into A [9].

Let A be a subspace of a topological space X. A continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$ is called a **retraction** of X onto A. We call A a **retract** of X if there exists a retraction of X onto A.

A **deformation retraction** of X onto A is a map $H : X \times I \to X$ such that $H(x, 0) = x$ for each $x \in X$, $H(x, 1) \in A$ for each $x \in X$, and $H(a, 1) = a$ for each $a \in A$. If $H(a, t) = a$ for each $(a, t) \in A \times I$, then H is called a **strong deformation retraction**. We call A a **(strong) deformation retract** of X if there is a (strong) deformation retraction from X onto A.

2. HOMOLOGY THEORIES

Loosely speaking, a *functor on the category of topological spaces* is a symbol F which denotes two functions, one that maps each space X to an "object" $F(X)$, and one that maps each continuous function $f : X \to Y$ between two spaces to an "arrow" $F(f) : F(X) \to F(Y)$ between objects [5].[1]

The method of applying a functor to spaces and maps between them is fundamental in the subject of algebraic topology. Typically one uses a functor to translate a difficult, perhaps intractable topological problem into an algebraic problem which can be solved more easily. In the early pages of his classic text on the subject [15], Greenberg explains how the famous fixed point theorem of Brouwer can be proved with little more than a pair of arrow diagrams (one for maps between spaces, and one for the corresponding homomorphisms between groups) and an appeal to functorial properties.[2] In the present work, we shall reduce the question of whether or not certain classes of self-maps of surfaces have fixed points to the problem of finding values which make a multivariate polynomial with integer coefficients not equal to 0.

The casual reader may prefer to skim the remainder of this section, which is highly technical. However, it should be emphasized that the purpose of the admittedly formidable machinery needed for homology theory is to clarify the underlying issues which a generally stated inquiry may obscure. When used effectively, the machinery has the effect of drawing out what is essential.

Let G', G, G'' be three abelian groups, and let $\alpha : G' \longrightarrow G$ and $\beta : G \longrightarrow G''$ be homomorphisms. We call the diagram

$$G' \xrightarrow{\alpha} G \xrightarrow{\beta} G''$$

a **sequence of three groups**, and we say that the sequence is **exact at** G if Image $\alpha =$ Kernel β. A collection $\{G_n, \alpha_n\}$ of abelian groups G_n and homomorphisms $\alpha_n : G_n \longrightarrow G_{n-1}$ indexed by the integers will be called a **(long) exact sequence** if every sequence of three consecutive groups is exact at its middle group.

A **graded abelian group** $C = \{C_n\}$ is a collection of abelian groups C_n indexed by the integers. Let C and D be two graded abelian groups. A **homomorphism of degree** $e \in \mathbb{Z}$ from C to D is a collection $f = \{f_n\}$ of homomorphisms $f_n : C_n \longrightarrow D_{n+e}$. If f is a homomorphism of degree -1 from C to C, and $f_{n-1} \circ f_n$ is the zero homomorphism for each $n \in \mathbb{Z}$, we write $Z_n(C_n) =$ Kernel (∂_n) and $B_n(C_n) =$ Image (∂_{n+1}). The elements of $Z_n(C_n)$ are called n-**cycles**, and the elements of

$B_n(C_n)$ are called n-**boundaries**.

The following definition appears as in [26]. Let $\mathcal{T}$ denote the category of topological pairs and maps and let $\mathcal{G}$ denote the category of graded abelian groups and homomorphisms of degree 0. An **homology theory** is an ordered pair (H, ∂), where H is a covariant functor from $\mathcal{T}$ to $\mathcal{G}$, and ∂ is a natural transformation of degree -1 from the functor H on (X, A) to the functor H on $(A, \oslash)$, provided that the Eilenberg-Steenrod axioms are satisfied.

The Eilenberg-Steenrod axioms

Axiom 2..1. (Homotopy) If f_0, $f_1 : (X, A) \to (Y, B)$ are homotopic, then $H(f_0) = H(f_1) : H(X, A) \to H(Y, B)$.

Axiom 2..2. (Exactness) For any pair (X, A) with inclusion maps $\imath : A \hookrightarrow X$ and $\jmath : X \hookrightarrow (X, A)$, there is an exact sequence

$$\cdots \xrightarrow{\partial_{q+1}(X,A)} H_q(A) \xrightarrow{H_q(\imath)} H_q(X) \xrightarrow{H_q(\jmath)} H_q(X, A) \xrightarrow{\partial_q(X,A)} H_{q-1}(A) \xrightarrow{H_{q-1}(\imath)} \cdots .$$

Axiom 2..3. (Excision) For any pair (X, A), if U is an open subset of X such that $\overline{U} \subset A$, then the **excision map** $\jmath : (X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism $H(\jmath) : H(X - U, A - U) \cong H(X, A)$.

Axiom 2..4. (Dimension) If P is a space consisting of one point, then $H_0(P) = \mathbb{Z}$, and $H_q(P) = 0$ for each $q \neq 0$.

If $f : (X, \oslash) \to (X, \oslash)$ is a map, we denote the function $H_n(f) : H_n(X, \oslash) \to H_n(X, \oslash)$ by f_n and call the collection $f_* = \{f_n\}$ the **homology homomorphism**

induced by f. We will only use the **absolute homology groups**, *i.e.* homology groups for pairs (X, A) with $A = \oslash$. Henceforth, we will not distinguish between the pair $(X, \oslash)$ and the space X.

In order to carry out calculations of the particular homology groups associated with a given space, it is often valuable to have at one's disposal explicit formulas for the homomorphisms between the groups to be computed. For the exact sequences we employ in this thesis, the next lemma (stated as it appears in appears in [15]) provides such formulas.

Lemma 2..5. (Barratt-Whitehead) Given a diagram of R-modules and homomorphisms in which all rectangles commute and rows are exact

$$\begin{tikzcd}
{} \arrow[r] & C_{i+1} \arrow[r] \arrow[d, "\gamma_{i+1}"] & A_i \arrow[r, "f_i"] \arrow[d, "\alpha_i"] & B_i \arrow[r, "g_i"] \arrow[d, "\beta_i"] & C_i \arrow[r, "h_i"] \arrow[d, "\gamma_i"] & A_{i-1} \arrow[r, "f_i"] \arrow[d, "\alpha_{i-1}"] & B_{i-1} \arrow[r, "g_i"] \arrow[d, "\beta_{i-1}"] & C_{i-1} \arrow[r] \arrow[d, "\gamma_{i-1}"] & {} \\
{} \arrow[r] & C'_{i+1} \arrow[r] & A'_i \arrow[r, "f'_i"'] & B'_i \arrow[r, "g'_i"'] & C'_i \arrow[r, "h'_i"'] & A'_{i-1} \arrow[r, "f'_i"'] & B'_{i-1} \arrow[r, "g'_i"'] & C'_{i-1} \arrow[r] & {}
\end{tikzcd}$$

if the γ_i are isomorphisms, then there is a long exact sequence

$$\longrightarrow A_i \xrightarrow{\varphi_i} A'_i \oplus B_i \xrightarrow{\psi_i} B'_i \xrightarrow{\Gamma_i} A_{i-1} \longrightarrow$$

where

$$\varphi_i = (\alpha_i \oplus f_i)\Delta,$$

$$\psi_i = \nabla(-f_i \oplus \beta_i),$$

$$\Gamma_i = h_i \gamma^{-1} g'_i,$$

and $\Delta(a) = (a, a)$, $\nabla(x, y) = x + y$.

The sequence whose existence is assured by the next theorem is called a **Mayer-Vietoris sequence**. It will be our tool of choice when we compute the homology groups of specific surfaces.

Theorem 2..6. (Mayer-Vietoris) Let X_1, X_2 be subspaces of a topological space X such that $X = X_1 \cup X_2$. Then there is an exact sequence

$$\cdots \to H_{n+1}X \xrightarrow{\partial_n} H_n(X_1 \cap X_2) \xrightarrow{\imath_n} H_nX_1 \oplus H_nX_2 \xrightarrow{\jmath_n} H_nX \xrightarrow{\partial_{n-1}} H_{n-1}(X_1 \cap X_2) \to \dots$$

where

$$\imath_n = \imath'_n \oplus \imath''_n,$$

$$\jmath_n = \imath'_n - \imath''_n,$$

and $\imath' : X_1 \cap X_2 \hookrightarrow X_1$, $\imath'' : X_1 \cap X_2 \hookrightarrow X_2$ are the inclusion maps.

Proof. This result follows from the Excision Axiom via the Barratt-Whitehead lemma. See [26], [15]. □

Finally, we introduce the easiest method for computing homology groups. When a space X has a strong deformation retract A whose homology groups are known, the following theorem says that X and A have the same homology groups.

Theorem 2..7.

1. Homotopy equivalent spaces have isomorphic homology groups.

2. If A is a strong deformation retract of a space X, then A and X are homotopy equivalent.

Proof. See [24], §19. $\square$

3. CLOSED SURFACES

We now introduce the object of our study, the closed orientable surfaces without boundary, and additionally some of their more familiar (non-orientable) relatives. The goal of this section is to provide the needed background for the statement of the classification theorem for closed surfaces without boundary, with which the section concludes.

An n-**manifold** is a Hausdorff topological space X such that every point in X has a neighborhood homeomorphic to the open n-dimensional ball

$$B^n = \left\{ (x_k)_{k=1}^n :\in \mathbb{R}^n : \sqrt{\sum_{k=1}^n x_k^2} < 1 \right\}$$

in $\mathbb{R}^n$ under the usual topology.[3] We define a **(closed) surface (without boundary)** to be a compact connected 2-manifold.[4] The typical example of a surface is the sphere

$$S^2 = \{\mathbf{x} \in \mathbb{R}^3 : d(\mathbf{x}, \mathbf{0}) = 1\}.$$

Another familar example is the (hollow) torus, which we denote by $\mathbb{T}_1$. (Note that our notation for the torus is not standard. The usual notation for a locally n-dimensional torus is $\mathbb{T}^n$. As we will only be dealing with 2-manifolds, we omit the superscripted local dimension, and reserve the subscript for the number of holes in the surface

$\mathbb{T}_g$.)

A **crosscap** is a surface $\mathbb{M}$ homeomorphic to the image of the function $f : [0, 2\pi) \times [-1, 1] \to \mathbb{R}^3$ defined by

$$\begin{aligned}
f(u,v) &= (x(u,v), y(u,v), z(u,v)),\\
x(u,v) &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u,\\
y(u,v) &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u,\\
z(u,v) &= \frac{v}{2}\sin\frac{u}{2}.
\end{aligned}$$

We call the image of f the **Möbius strip**. Inhabitants of a locally 3-dimensional universe can construct this important figure with a twisted strip of paper and a piece of tape as shown in Figure **II**.1.

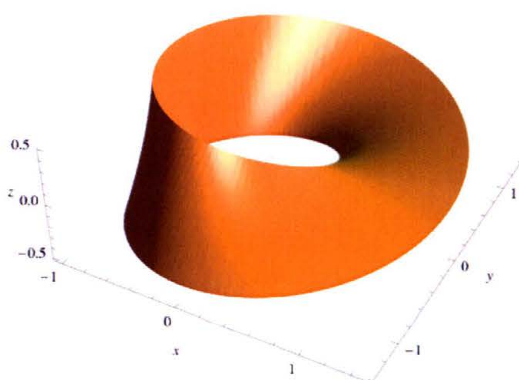

Figure II.1: *The Möbius strip.*

The significance of the crosscap for topologists is that it appears as a subspace in all nonorientable surfaces, including the so-called **Klein bottle** $\mathbb{K}$. In some sense, $\mathbb{K}$ is a twisted cylinder glued to itself along its bases. We show its three-dimensional projection (Figure **II**.2, left). Given that the surface does not actually intersect itself, the Klein bottle is plainly the union of two crosscaps (Figure **II**.2, right).

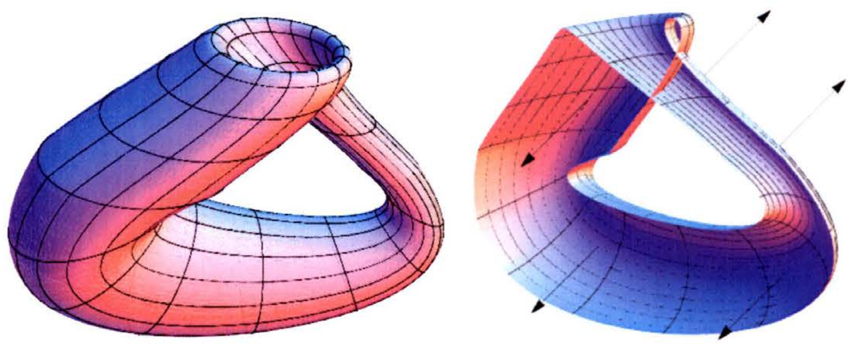

Figure II.2: *A projection of the Klein bottle in $\mathbb{R}^3$ (left), and a cutaway of that projection (right).*

Like the Möbius strip, the Klein bottle does not enclose a cavity, and a normal vector traveling around the surface in the manner shown in Figure **II**.2 flips upon returning to its starting point. Water poured into the opening at its top would exit the apparent cavity in the "bottle" after flowing halfway around the embedded Möbius strip. A surface is called **nonorientable** if it contains a crosscap as a subspace, and **orientable** otherwise.

If we identify every pair of antipodal points along the frontier of a disk, we obtain a space known as the **(real) projective plane** $\mathbb{R}P^2$. In projective geometry, the projective plane is conceived as the set of all lines in $\mathbb{R}^3$ through the origin. Since such a line ℓ intersects the frontier of the unit ball centered at the origin in exactly two antipodal points if ℓ lies in the xy-plane, and in exactly one point otherwise, there is a one-to-one correspondence between such lines and the equivalence classes of the relation

$$g(x_1, y_1) \sim g(x_2, y_2) \Leftrightarrow (x_1, x_2) = (y_1, y_2) \text{ or } (x_1, x_2) = -(y_1, y_2)$$

defined for all points $g(x_1, y_1)$, $g(x_2, y_2)$ on the upper half of the unit sphere, where g is the function defined by $g(x, y) = \sqrt{1 - x^2 - y^2}$ for each $(x, y) \in \overline{B^2}$.

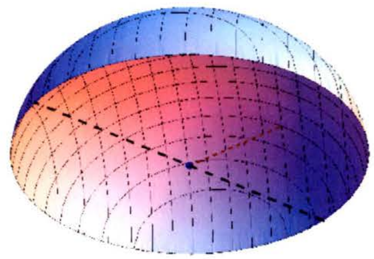

Figure II.3: *The geometric construction of the real projective plane.*

The realization of the projective plane typically employed by topologists is the identifcation space[5]

$$\mathbb{R}P^2 = \overline{B^2} \Big/ \{(x_1, x_2) \sim -(x_1, x_2) : (x_1, x_2) \in \mathrm{Fr}\ B^2)\}.$$

New surfaces can be constructed from the aforementioned surfaces (sphere, torus, crosscap, Klein bottle, projective plane) by means of a certain "attaching" operation which we now define. If S_1 and S_2 are two surfaces, and $D_1 \subset S_1$, $D_2 \subset S_2$ are two regions homeomorphic to the closed disk $\overline{B^2}$, then the **connected sum** of the two surfaces is defined as the quotient space

$$S_1 \# S_2 = \Big[(S_1 - D_1^\circ) \coprod (S_2 - D_2^\circ)\Big] \Big/ \{x \sim h(x) : x \in \mathrm{Fr}\ D_1\}$$

where $h : \mathrm{Fr}\ D_1 \to \mathrm{Fr}\ D_2$ is a homeomorphism and $\coprod$ denotes the operation of topological sum.[6]

As an illustration, consider the connected sum $\mathbb{T}_1 \# \mathbb{T}_1$ of two tori (Figure **II**.4).

Observe that the operation $\#$ is both commutative and associative over the set of homeomorphism classes of topological spaces, and that S^2 is its identity, since $S^2 \# \Sigma \cong \Sigma$ for any surface Σ.

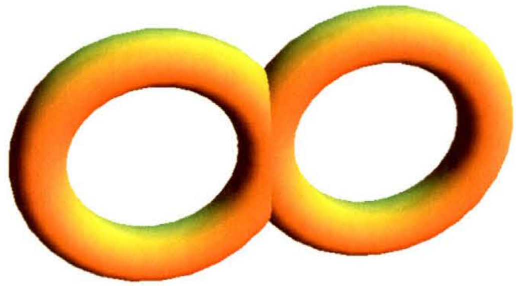

Figure II.4: $\mathbb{T}_1 \# \mathbb{T}_1$.

It can be shown that every orientable surface is homeomorphic either to a sphere or to a connected sum $\mathbb{T}_g = \#_{i=1}^{g} \mathbb{T}_1$ of tori. More generally, we have the following famous classification theorem:

Theorem 3..1. (Classification of closed surfaces without boundary) Every (closed) surface (without boundary) is homeomorphic either to a sphere, to a connected sum of tori, or to a connected sum of projective planes.

Proof. See [22], pp. 10-29. □

We may now define the **genus** of an orientable surface X to be 0 if $X \cong S^2$, and the (finite) number of tori whose connected sum is homeomorphic to X otherwise. In the next section, we use the classification theorem to classify the homology groups of all orientable surfaces.

4. HOMOLOGY OF ORIENTABLE SURFACES

There are many different homology theories, all of which satisfy the Eilenberg-Steenrod axioms.[7] Moreover, if we fix a coefficient ring, the standard homology theories are equivalent for surfaces.[8] We are therefore entitled, for example, to apply results about homology groups of surfaces stated in terms of one homology theory to the homology groups of another.

In this section, we take cellular homology with integer coefficients as our homology theory, and compute the homology groups of the sphere S^2 and the tori $\mathbb{T}_g$ for $g \in \mathbb{N}$.

Theorem 4..1. $H_0(X) = \mathbb{Z}$ for a surface X.

Proof. The rank of the 0th homology group of a space X for which there exists a cellular decomposition is the number of connected components of X (for a proof of this, see [24], §7). Since a surface is connected, $H_0(X) = \mathbb{Z}$. □

Theorem 4..2. The homology groups of the sphere S^2 are

$$H_k(S^2) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let α denote the subspace of $\mathbb{R}^2$ whose underlying set is $I \times I$. Then the subspace

$$A = \{\mathbf{x} \in \mathbb{R}^3 : d(\mathbf{x}, \mathbf{0}) = 1\}$$

of $\mathbb{R}^3$ is homeomorphic to the identification space

$$\alpha/\mathrm{Fr}\ \alpha = \alpha\,/\{\mathbf{x} \sim \mathbf{y} : \mathbf{x}, \mathbf{y} \in \mathrm{Fr}\ \alpha\}\ .$$

Let S^2 denote the latter space, and write $p : A \to S^2$ for the projection $\mathbf{x} \mapsto [\mathbf{x}]_\sim$. Then the 1-cell $p(\mathrm{Fr}\ \alpha)$ and the 2-cell $p(\alpha)$ comprise a cellular decomposition of S^2. (We will later refer to this decomposition as the **"purse-string" decomposition**, since identifying the frontier of a cell to a point may be intuitively understood as pulling a purse-string tight. This analogy is found in Kinsey [18].) Since there are no 1-cells, $p(\alpha)$ is a cycle, and also $H_1(S^2) = 0$. Since there are no 3-cells, $p(\alpha)$ is not a boundary, so $H_2(S^2)$ is the infinite cyclic group generated by $p(\alpha)$. The result now follows by Theorem 4..1. $\square$

A bit more groundwork will be required to determine the homology groups of a torus than was needed in the case of the sphere. We provide a series of lemmas which will aid in the computations.

Lemma 4..3. The homology groups of the circle S^1 are

$$H_k(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Similar to that of the previous lemma. $\square$

Let X and Y be two connected manifolds, let $x_0 \in X$, and let $y_0 \in Y$. The **wedge** of X and Y is the space

$$X \vee Y = X \coprod Y \Big/ \{x_0 \sim y_0\}.$$

Lemma 4..4. (Wedge axiom) Let X_1 and X_2 be two connected manifolds. Then $H_k(X_1 \vee X_2) = H_k(X_1) \oplus H_k(X_2)$ for each $k \in \mathbb{N}$.

Proof. Standard. See [1], Ch. 5. $\square$

Lemma 4..5. $H_1(S^1 \vee S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Immediate from the previous two lemmas. □

Lemma 4..6. A one-point space is a strong deformation retract of the closed disk $\overline{B^2}$.

Proof.

$$H(\mathbf{x}, t) = (1 - t)\mathbf{x}$$

is the required homotopy from $\overline{B^2}$ to $\{\mathbf{0}\}$, since $H(\mathbf{x}, 0) = \mathbf{x}$ and $H(\mathbf{x}, 1) = \mathbf{0}$ for each $\mathbf{x} \in \overline{B^2}$. □

Impose the cellular decomposition of the unit square $I \times I$ (pictured below) into four points, four edges, and one 2-cell. Identify the opposite edges via the equivalence relation whose nondegenerate relations are

$$\begin{aligned} (x, 0) &\sim (x, 1), \qquad \forall\, x \in I, \\ (0, y) &\sim (1, y), \qquad \forall\, y \in I, \end{aligned}$$

so that the resulting space is a torus. Denote $\mathbb{T}_1 = (I \times I)/\!\sim$. Orienting α counter-clockwise, a and b as pictured, and P arbitrarily, we obtain the cellular decomposition of $\mathbb{T}_1$ specified by the following schematic diagram, which is called a **gluing diagram**.

For the remainder of this section, let $\mathbb{T}_1 = (I \times I)/\!\sim$ be endowed with the cellular decomposition given above.

Lemma 4..7. Let $D \subset \mathbb{T}_1$ be homeomorphic to the open disk B^2. A figure eight is a deformation retract of $\mathbb{T}_1 - D$.

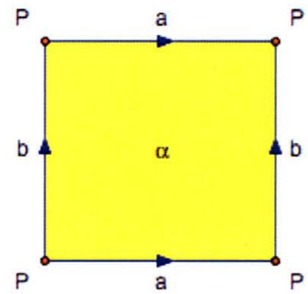

Figure II.5: *Gluing diagram and cellular decomposition for* $\mathbb{T}_1$.

Proof. Without loss of generality, assume D is the image of a square under the identification map $p : I \times I \to \mathbb{T}_1$. The homotopy H determined by Figure **II**.6 deforms each radial line segment to a point on the frontier of $I \times I$.

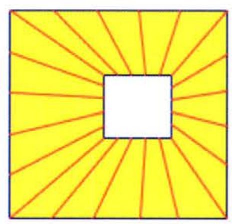

Figure II.6: *Strong deformation retract of the punctured torus onto Fr* $I \times I$.

Since the image of Fr $(I \times I)$ under p is a figure eight, then H is the required homotopy. ∎

Figure II.7: *The image of* $H(x,t)$ *for four values of* t *as* t *increases.*

Lemma 4..8. $H_2(\mathbb{T}_1) \cong \mathbb{Z}$.

Proof. Let $X_1 \subset \mathbb{T}_1$ denote the smaller of the closed squares indicated in the following gluing diagram, and denote $X_2 = \mathbb{T}_1 - X_1^\circ$.

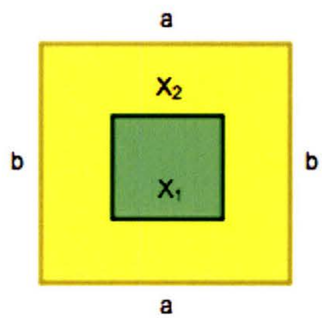

Figure II.8: *X_1 is the small square (green) along with its frontier (dark green), and $X_2 \supset a \cup b \cup Fr\ X_1$.*

Then $X_1 \cap X_2 \cong S^1$, and since homeomorphic spaces have isomorphic homology groups (see [14] or [24]), Lemma 4..3 yields $H_1(X_1 \cap X_2) \cong \mathbb{Z}$.

Let

$$
\begin{aligned}
0 \rightarrow & H_2(X_1 \cap X_2) \xrightarrow{i_2} H_2(X_1) \oplus H_2(X_2) \xrightarrow{j_2} H_2(X_1 \cup X_2) \\
\xrightarrow{\partial_2} & H_1(X_1 \cap X_2) \xrightarrow{i_1} H_1(X_1) \oplus H_1(X_2) \xrightarrow{j_1} H_1(X_1 \cup X_2) \\
\xrightarrow{\partial_1} & H_0(X_1 \cap X_2) \xrightarrow{i_0} H_0(X_1) \oplus H_0(X_2) \xrightarrow{j_0} H_0(X_1 \cup X_2) \rightarrow 0
\end{aligned}
$$

be an exact sequence.

From Theorem 2..7 and Lemmas 4..6 and 4..7, it follows that $H_1(X_1) = H_2(X_1) = 0$ and $H_2(X_2) = 0$. Then Image $j_2 = 0$, so ∂_2 is an injection by exactness at $H_2(X_1 \cup X_2)$. Then Kernel i_1 = Image ∂_2 is an isomorphic embedding of $H_2(X_1 \cup X_2)$ into $H_1(X_1 \cap X_2)$.

Let $\imath' : X_1 \cap X_2 \hookrightarrow X_1$ and $\imath'' : X_1 \cap X_2 \hookrightarrow X_2$ be the inclusion maps. Then

$$\imath_1 = \imath_1' \oplus \imath_1'',$$
$$\jmath_1 = \imath_1' - \imath_1'',$$

by Theorem 2..6. Then $\imath_1' = 0$, since $H_1(X_1) = 0$.

To prove that

$$H_2(X_1 \cup X_2) \cong H_1(X_1 \cap X_2),$$

it suffices to show that $i_1'' = 0$, so that $\imath_1 = 0$. Let z be a cycle in $X_1 \cap X_2$ whose homology class generates $H_1(X_1 \cap X_2)$, noting that $X_1 \cap X_2 \cong S^1$. By assigning to $X_1 \cup X_2$ an appropriate cellular decomposition, we may ensure that $z = \imath''(z)$ is a chain in $C_1(X_1 \cup X_2)$, and it follows that the expression $\imath_1''([z])$ is defined. Then $\imath_1''([z])$ is homologous to 0 in $X_1 \cup X_2$, since z can be deformed in $X_1 \cup X_2$ to the boundary $a + b - a - b = 0$ as was shown in the proof of Lemma 4..7. Then $\imath_1''$ is the zero homomorphism, so $\imath_1 = 0$, since $[z]$ generates $H_1(X_1 \cap X_2)$. Thus $H_2(X_1 \cup X_2) \cong$ Kernel $i_1 = H_1(X_1 \cap X_2) \cong \mathbb{Z}$. $\square$

Lemma 4..9. $H_1(\mathbb{T}_1) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Let X_1, X_2, $\imath_0$, etc., be as in the previous theorem. Endow $X_1 \cap X_2 \cong S^1$ with a cellular decomposition whose elements are a single point P and a single 1-cell a. Let $\imath' : X_1 \cap X_2 \hookrightarrow X_1$ and $\imath'' : X_1 \cap X_2 \hookrightarrow X_2$ be the inclusion maps. Writing $[P]$ for the homology class of P, we have $\imath_0([P]) = i_0'([P]) \oplus \imath_0''([P]) = ([P], [P]) \neq 0$ by Theorem 2..6, so $[P]$ is not in the kernel of i_0. Since there are no other 0-cells in $X_1 \cap X_2$, the kernel of $\imath_0$ is trivial. Hence $\imath_0$ is an injection. Then ∂_1 is the zero homomorphism, so $\jmath_1$ is a surjection. Since i_1 is the zero homomorphism, it follows that j_1 is an injection, and hence $H_1(\mathbb{T}_1) \cong H_1(X_1) \oplus H_1(X_2) \cong H_1(X_2)$.

By Theorem 2..7, since $S^1 \vee S^1$ is a deformation retract of X_2, then $H_1(X_2) \cong \mathbb{Z} \oplus \mathbb{Z}$. $\square$

We can compute the homology groups for a torus of arbitrary genus $g \in \mathbb{N}$ by induction, using the previous result as a base case.

Lemma 4..10. $H_1(\mathbb{T}_g) \cong \bigoplus_{i=1}^{g}(\mathbb{Z} \oplus \mathbb{Z})$ for each $g \in \mathbb{N}$.

Proof. Let $g \in \mathbb{N}$. Assume as an induction hypothesis that $H_1(\mathbb{T}_g) \cong \oplus_{i=1}^{g}(\mathbb{Z} \oplus \mathbb{Z})$. Abusing notation,[9] we write

$$X_1 = \mathbb{T}_1 - D_1^\circ \subset \mathbb{T}_1 \# \mathbb{T}_g,$$
$$X_2 = \mathbb{T}_g - D_2^\circ \subset \mathbb{T}_1 \# \mathbb{T}_g,$$

where $D_1 \subset \mathbb{T}_1$ and $D_2 \subset \mathbb{T}_g$ are each homeomorphic to the closed disk $\overline{B^2}$. (The topological sum construction saves us from having to require that $D_1 \cap D_2 = \oslash$ when $g = 1$.) Then $X_1 \cup X_2 = \mathbb{T}_1 \# \mathbb{T}_g$. Then $H_1(X_1) \oplus H_1(X_2) \cong \bigoplus_{i=1}^{g+1}(\mathbb{Z} \oplus \mathbb{Z})$, since $H_1(X_1) \cong H_1(S^1 \vee S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let

$$\begin{aligned}
0 \to & H_2(X_1 \cap X_2) \xrightarrow{i_2} H_2(X_1) \oplus H_2(X_2) \xrightarrow{j_2} H_2(X_1 \cup X_2) \\
\xrightarrow{\partial_2} & H_1(X_1 \cap X_2) \xrightarrow{i_1} H_1(X_1) \oplus H_1(X_2) \xrightarrow{j_1} H_1(X_1 \cup X_2) \\
\xrightarrow{\partial_1} & H_0(X_1 \cap X_2) \xrightarrow{i_0} H_0(X_1) \oplus H_0(X_2) \xrightarrow{j_0} H_0(X_1 \cup X_2) \to 0
\end{aligned}$$

be an exact sequence. To prove that $H_1(X_1 \cup X_2) \cong H_1(X_1) \oplus H_1(X_2)$, it suffices to show that *(i)* i_0 is an injection (so that j_1 is a surjection) and *(ii)* $i_1 = 0$ (so that j_1 is an injection).

Ad (i). As above, each of the inclusion maps $i' : X_1 \cap X_2 \hookrightarrow X_1$ and $i'' : X_1 \cap X_2 \hookrightarrow X_2$ sends $[P] \mapsto [P]$. Thus Kernel $i_0 = 0$.

Ad (ii). Endow $\mathbb{T}_g$ with a cellular decomposition that contains exactly one 2-cell α (we will exhibit such a decomposition in Section 3.). Since the generating circle a may be deformed in $X_1 \cup X_2$ to the frontier of α (see figures below[10]), it follows that $i_1([a])$ is homologous to the chain $\partial[\alpha]$, where α is the single 2-cell in the decomposition of $\mathbb{T}_{g+1}$.[11] Therefore, a is a boundary in $X_1 \cup X_2$. Then i_1 is the zero homomorphism, since there are no 1-cells besides a in the decomposition of $X_1 \cap X_2$.

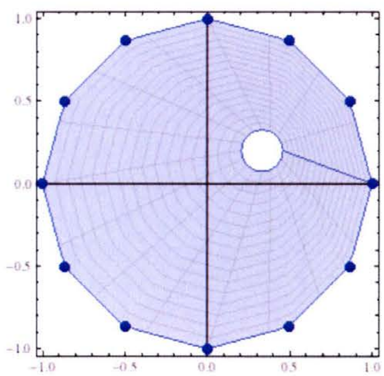

Figure II.9: *The circle a along which $\mathbb{T}_1$ and $\mathbb{T}_g$ were attached is deformable over $\mathbb{T}_g$ to the 1-skeleton of $\mathbb{T}_g$.*

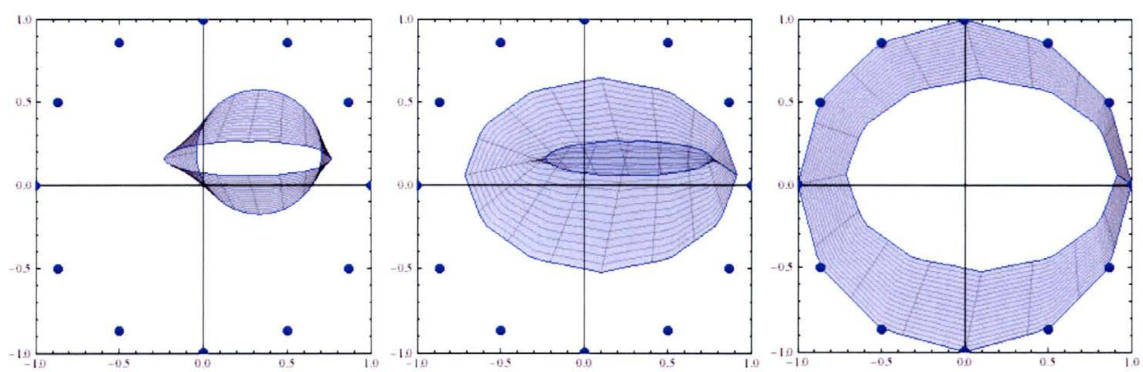

Figure II.10: *The circle a may be so deformed regardless of its orientation. From left to right, the three figures show a homotopy $H(s,t) : \mathbb{T}_g \times I \to \mathbb{T}_g$ $0 \leq t \leq 1/3$, for $1/3 \leq t \leq 2/3$, and for $2/3 \leq t \leq 1$, respectively.*

Thus $H_1(\mathbb{T}_{g+1}) \cong H_1(X_1) \oplus H_1(X_2) \cong \bigoplus_{\imath=1}^{g+1}(\mathbb{Z} \oplus \mathbb{Z})$. Taking Lemma 4..9 as a base case, the result follows by induction. □

Theorem 4..11. $H_2(\mathbb{T}_g) \cong \mathbb{Z}$.

Proof. As above, write

$$X_1 = \mathbb{T}_1 - D_1^\circ \subset \mathbb{T}_1 \# \mathbb{T}_g,$$
$$X_2 = \mathbb{T}_g - D_2^\circ \subset \mathbb{T}_1 \# \mathbb{T}_g,$$

where $\overline{B^2} \cong D_1 \subset \mathbb{T}_1$ and $\overline{B^2} \cong D_2 \subset \mathbb{T}_g$, and let $\imath' : X_1 \cap X_2 \hookrightarrow X_1$ and $\imath'' : X_1 \cap X_2 \hookrightarrow X_2$ be the inclusion maps. Consider the exact sequence

$$H_2(X_1) \oplus H_2(X_2) \xrightarrow{\jmath_2} H_2(X_1 \cup X_2) \xrightarrow{\partial_2} H_1(X_1 \cap X_2) \xrightarrow{\imath_1} H_1(X_1) \oplus H_1(X_2).$$

The homomorphism ∂_2 is an injection, since $H_2(X_1) \oplus H_2(X_2) = 0$, so $H_2(X_1 \cup X_2) \cong \text{Image}\, \partial_2 = \text{Kernel}\, i_1$. As above, $\imath_1' = i_1'' = 0$, so $\imath_1 = \imath_1' \oplus \imath_1'' = 0$. Now Kernel $\imath_1 \cong H_1(X_1 \cap X_2) \cong H_1(S^1) \cong \mathbb{Z}$ gives $H_2(X_1 \cup X_2) \cong \mathbb{Z}$. □

Corollary 4..12. Let X be an orientable surface of genus $g \in \mathbb{N}_0$. Then $H_0(X) = H_2(X) = \mathbb{Z}$, and

$$H_1 X \cong \begin{cases} \bigoplus_{\imath=1}^{2g} \mathbb{Z} & \text{if } g \in \mathbb{N}, \\ 0 & \text{if } g = 0. \end{cases}$$

Proof. By the classification theorem for surfaces, X is homeomorphic either to the sphere or to the torus of genus g. The result then follows from the results of this section. □

Having computed the homology groups of the sphere and the tori $\mathbb{T}_g$ for $g \in \mathbb{N}$, it is now possible to apply the Lefschetz-Hopf theorem to any given self-map of an orientable surface.

CHAPTER III

SUFFICIENT CONDITIONS FOR Fix$(f) \neq \oslash$

The main results of this thesis will now be given. Sufficient conditions for a self-map of a closed orientable surface to have a fixed point will be deduced and presented in this chapter. We deal with each class of orientable surfaces separately, taking the sphere S^2 first, then the torus of genus 1, and then all other locally 2-dimensional tori.

Our sufficient conditions for a self-map to have a fixed point derive from the Lefschetz-Hopf fixed point theorem. This result relies on the fact that each homology homomorphism has an associated matrix representation. If f is a self-map of an orientable surface, the alternating sum of the traces of the homology homomorphisms $f_\imath$ is called the **Lefschetz number** $\Lambda(f)$ of f. That is,

$$\Lambda(f) = \sum_{\imath=0}^{2} (-1)^\imath \mathrm{tr}\ f_\imath.$$

Theorem 0..13. (Lefschetz-Hopf fixed point theorem) Let X be a surface. Let $f : X \to X$ be continuous. If $\Lambda(f) \neq 0$, then f has a fixed point.

1. THE SPHERE

Every self-map of S^2 not homotopic to the antipodal map has a fixed point. We have a quite general condition for a self-map of the sphere S^2 to have a fixed point. That this well-known condition is sufficient follows from the fact that the Brouwer degree of a map f, which records how many times f "wraps" the sphere around itself, can be determined solely on the basis of whether f is homotopic to the antipodal map $\mathbf{x} \to -\mathbf{x}$.

The sphere S^2 has $H_2(S^2) = \mathbb{Z}$, and since the "purse-string" decomposition has no 1-cells, then $H_1(S^2) = 0$. Then the Lefschetz number of any given self-map f of S^2 is

$$\Lambda(f) = 1 + \text{tr } f_2 = 1 + m$$

for some integer m, so $\Lambda(f)$ is 0 only if $m = -1$. But by the subsequent theorem, $m = -1$ if and only if f is homotopic to the antipodal map.

The **(Brouwer) degree** of a map $f : S^2 \to S^2$ is the integer

$$m = \text{tr } f_2.$$

Thus $f_2(\alpha) = m \cdot \alpha$, where α denotes the generator of $H_2(S^2)$.

Theorem 1..1. Two maps $S^2 \to S^2$ have the same degree if and only if they are homotopic.

Proof. See [14], §9. □

We immediately have

Corollary 1..2. Every self-map of S^2 not homotopic to the antipodal map has a fixed point.

2. THE TORUS OF GENUS 1

Throughout this section, we adopt the following standing assumptions. The torus of genus 1 will be realized as the product space $\mathbb{T}_1 = S^1 \times S^1$, where

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

For the sake of readability, we will write θ for the element $e^{\imath\theta} \in S^1$ and

$$\mathbb{T}_1 = \{(\theta_1, \theta_2) : 0 \leq \theta_k < 2\pi \text{ for } k = 1, 2\}.$$

Suppose f is a self-map of $\mathbb{T}_1$. We may assume that f is cellular without loss of generality. We write $m = \text{tr } f_2$ for the entry in the 1×1 matrix representing f_2, and

$$\begin{pmatrix} \lambda_1 & r \\ s & \lambda_2 \end{pmatrix}$$

for the matrix representing f_1.

Theorem 2..1. Let $f : \mathbb{T}_1 \to \mathbb{T}_1$ be a map. Suppose $x, y \in \mathbb{Z}$. Then f has a fixed point if any of the following conditions are met.

- **Case 1.** $m = 0$ and $\lambda_1 + \lambda_2 \neq 1$.
- **Case 2.1.** $m = \det f_1 = \lambda_1\lambda_2$, $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$.
- **Case 2.2.** $[f_1] = \begin{pmatrix} \lambda_1 & \lambda_1 \\ 1-\lambda_1 & \lambda_2 \end{pmatrix}$, $m = \det f_1$, $\lambda_1 \neq 1$ and $\operatorname{tr} f_1 \neq 1$.
- **Case 2.3.** $[f_1] = \begin{pmatrix} \lambda_1 & y \\ y & \lambda_1 \end{pmatrix}$, $m = \det f_1$, and $\lambda_1 \pm y \neq 1$.
- **Case 2.4.** $[f_1] = \begin{pmatrix} x & (-1)^k \\ (-1)^k & xy \end{pmatrix}$, $m = \det f_1$, and $y \neq 1$.

Proof.

Case 1. $m = 0$.

Suppose $m = 0$. In general, we have

$$\Lambda(f) = 1 - (\lambda_1 + \lambda_2) \neq 0$$

whenever $\operatorname{tr} f_1 \neq 1$. As a special case, observe that $\Lambda(f) \neq 0$ whenever both of λ_1 and λ_2 are nonpositive.

A priori, we cannot conclude anything about the individual values of λ_1, λ_2, r, and s from the fact that $m = 0$. This fact will be explored in the chapter on realizability to follow.

On the other hand, if $m \neq 0$, we have $\Lambda(f) \neq 0$ whenever $\lambda_1 + \lambda_2 \neq 1 + m$, in which case f has a fixed point.

In each of the subcases 2.1, 2.2, 2.3 and 2.4, we assume $m = \det\ f_1$. In general, $\Lambda(f) \neq 0$ whenever $1 + \det f_1 \neq \operatorname{tr} f_1$.

Case 2.1.

If $m = \det\ f_1$ and $rs = 0$, then $m = \lambda_1\lambda_2$, and

$$\begin{aligned}\Lambda(f) &= 1 - (\lambda_1 + \lambda_2) + \lambda_1\lambda_2 \\ &= (1 - \lambda_1)(1 - \lambda_2),\end{aligned}$$

so $\operatorname{Fix}(f) \neq \emptyset$ whenever $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$.

If $\lambda_1\lambda_2 = rs$, then $m = 0$, and Case 1 obtains.

Case 2.2.

If $m = \det\ f_1$, $r = \lambda_1$, and $s = 1 - \lambda_1$, then

$$\begin{aligned}\Lambda(f) &= 1 - (\lambda_1 + \lambda_2) + \lambda_1\lambda_2 - \lambda_1(1 - \lambda_1) \\ &= (\lambda_1 - 1)(\lambda_1 + \lambda_2 - 1)\end{aligned}$$

is nonzero whenever $\lambda_1 \neq 1$ and $\operatorname{tr} f_1 \neq 1$. It follows that $\operatorname{Fix}(f) \neq \emptyset$.

Case 2.3.

If the matrix representing f_1 is $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$, then

$$\begin{aligned}\Lambda(f) &= 1 - 2x + x^2 - y^2 \\ &= (x^2 - y^2) - (x - y) - (x + y) + 1 \\ &= [(x - y) - 1]\,[(x + y) - 1]\end{aligned}$$

is nonzero whenever $x \pm y \neq 1$, in which case f has a fixed point.

Case 2.4.

When $r = s = 1$ or $r = s = -1$, we have

$$\Lambda(f) = -\mathrm{tr}\ f_1 + \lambda_1 \lambda_2.$$

In particular, if the diagonal entries of the matrix representing f_1 have a common (nonzero) integer factor x, *e.g.* $\begin{pmatrix} x & 1 \\ 1 & xy \end{pmatrix}$, then

$$\begin{aligned} \Lambda(f) &= -(x + xy) + x^2 y \\ &= x\left[xy - (y + 1)\right], \end{aligned}$$

is nonzero, since $xy = y + 1$ is absurd for any choice of integers x, y with $y > 1$.

□

3. THE TORUS OF GENUS g GREATER THAN 1

As the student of elementary calculus knows, the two-dimensional torus is easy to parametrize as a map g of the unit square. The topologist's method of expressing the seamlessness of the torus along $g(\mathrm{Fr}\ (I \times I))$ is to identify opposite edges of the square appropriately, as shown in Section 4.. The analogous construction of $\mathbb{T}_g$ as an identification space generalizes well for arbitrary genus g. We sacrifice the feeling of comfort which a global coordinate system provides, but this is no great loss, since any parametrization of $\mathbb{T}_g$ obtained by methods of analytic geometry may be expected to

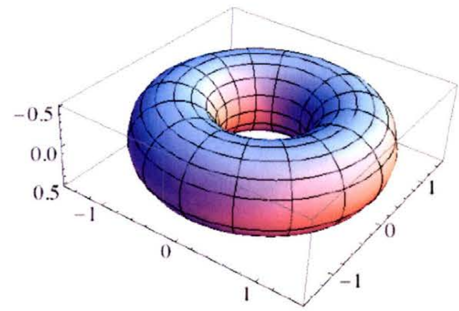

Figure III.1: *The two-dimensional torus encountered in elementary calculus.*

be horrendously complicated.

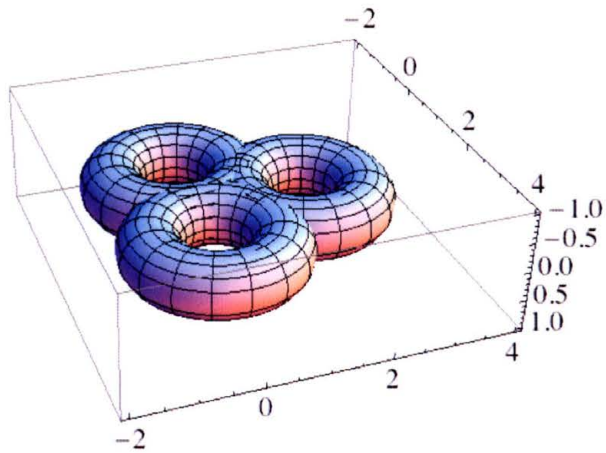

Figure III.2: *A realization of $\mathbb{T}_3$ in Euclidean space as a subset of the union of three tori.*

We construct the torus of genus $g \in \mathbb{N}$ from a convex polygon α of $4g$ sides, called the **fundamental polygon**.[12] Label the edges according to the schema

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1},$$

where adjacent symbols denote adjacent edges. Orient the edges labeled a_i or b_i in one direction, say clockwise, and each edge labeled a_i^{-1} or b_i^{-1} in the opposite direction. Identify each a_i and a_i^{-1} according to the edges' orientations, and similarly for b_i and

b_i^{-1}. Then in the identification space, which we denote by $\mathbb{T}_g$, each vertex of the

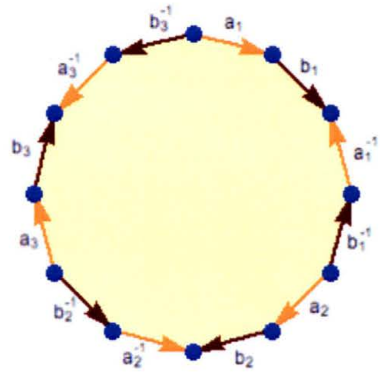

Figure III.3: *A realization of $\mathbb{T}_g$ for $g = 3$ by the method of topological identification.*

polygon represents the same point P. Moreover,

$$\{P\} \cup \{a_i : 1 \leq i \leq g\} \cup \{b_i : 1 \leq i \leq g\} \cup \{\alpha\}$$

is a cellular decomposition of $\mathbb{T}_g$.

Proceeding formally, we observe that

$$\begin{aligned} \Lambda(f) &= 1 - \operatorname{tr} \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_{2g} \end{pmatrix} + m \\ &= 1 - \sum_{j=1}^{2g} x_j + m, \end{aligned}$$

so that the expression on the right side contains the $2g+1$ unknowns $x_1, \ldots, x_{2g}$, m.

As it stands, the problem of determining when $\Lambda(f) \neq 0$ by considering the polynomial $\Lambda(f)$ appears intractable. Some simplifying assumptions will have to be made.

Theorem 3..1. If tr f_1 and m are either both odd or both even, then f has a fixed point.

Proof. Trivial. If tr $f_1 \equiv m \pmod 2$, then $\Lambda(f)$ is odd, and hence nonzero. □

Theorem 3..2. Any self-map f of $\mathbb{T}_g$ for $g \geq 2$ which is homotopic to the identity has $\mathrm{Fix}(f) \neq \emptyset$.

Proof. By Theorem 4..12, rank $H_0(\mathbb{T}_g) =$ rank $H_2(\mathbb{T}_g) = 1$ and rank $H_1(\mathbb{T}_g) = 2g$. If $g \geq 2$, then

$$\begin{aligned} \Lambda(f) &= \text{rank } H_0(\mathbb{T}_g) - \text{rank } H_1(\mathbb{T}_g) + \text{rank } H_2(\mathbb{T}_g) \\ &= 2 - 2g < 0, \end{aligned}$$

in which case f has a fixed point. □

Theorem 3..3. Let f be an orientation-preserving self-map of $\mathbb{T}_g$, where $g \in \mathbb{N}$. If for each $1 \leq \imath \leq g$, $f_1(a_\imath) = a_\jmath$ and $f_1(b_\imath) = b_\jmath$ for some $\jmath \neq \imath$, then f has a fixed point.

Proof. For each $1 \leq i \leq g$, the coefficient of $a_\imath$ in the expression $f_1(a_\imath)$ is 0, and similarly for each $b_\imath$. Thus the matrix representing f_1 contains all zeros on its diagonal, so tr $f_1 = 0$. Since f is orientation-preserving, it follows that $m > 0$, so $\Lambda(f) = 1 + m \neq 0$. □

CHAPTER IV

REALIZABILITY OF CONDITIONS

In this chapter, we show that there do exist functions which satisfy the conditions of our main results, Theorem 2..1 and Theorem 3..3. We begin by considering particular classes of maps which show that $\Lambda(f)$ is not homotopy invariant when $\Lambda(f) = 0$. We then provide examples of maps which confirm that the conditions in Theorem 2..1 are realizable in Section 1.. Finally, in Section 2., we determine a class of self-maps of the torus of arbitrary genus to which Theorem 3..3 may be applied.

1. THE TORUS OF GENUS 1

Write a, b for the two generators of $H_1(\mathbb{T}_1)$, and α for the generator of $H_2(\mathbb{T}_1)$. For the identity map $1_{\mathbb{T}_1}$, we have

$$a \mapsto a,$$

$$b \mapsto b,$$

$$\alpha \mapsto \alpha,$$

so that $\Lambda(1_{\mathbb{T}_1}) = 1 - 2 + 1 = 0$. Although the identity fixes every point in $\mathbb{T}_1$, we cannot conclude that every map *homotopic* to the identity has a fixed point. To see this, consider a map $g : \mathbb{T}_1 \to \mathbb{T}_1$ which rotates every point latitudinally around the center of the torus by a small angle, and a map g' which "curls" every point around a longitudinal circle which passes through the central hole, say,

$$g : (\theta_1, \theta_2) \mapsto (\theta_1 + \pi/4, \theta_2),$$
$$g' : (\theta_1, \theta_2) \mapsto (\theta_1, \theta_2 + \pi/4).$$

Clearly, neither g nor g' has a fixed point. Since homotopic maps induce identical endomorphisms on the homology groups (see [14]), and hence have identical Lefschetz numbers, these two functions demonstrate that the property of having a fixed point is not a homotopy invariant for maps with $\Lambda(f) = 0$. Another example is the map $(\theta_1, \theta_2) \mapsto (\omega + \theta_1, -\theta_2)$ for fixed $0 < \omega < 2\pi$, which is homotopic to a reflection in a plane bisecting the torus, and hence sends $\alpha \mapsto -\alpha$.

The next elementary example illustrates the fact that composing a map with a rotation by a small angle does not eliminate all fixed points when $\Lambda(f) \neq 0$, *i.e.* when the condition for the Lefschetz-Hopf theorem is satisfied. Suppose $h : \mathbb{T}_1 \to \mathbb{T}_1$ is a map which sends

$$a \mapsto -a,$$
$$b \mapsto -b,$$
$$\alpha \mapsto \alpha.$$

Then tr $h_1 \neq 2$, and $\Lambda(h) = 1 - (-2) + 1 \neq 0$. Take h to be the cellular map defined by

$$(\theta_1, \theta_2) \mapsto (-\theta_1, -\theta_2).$$

Then the map $\hat{h} : \mathbb{T}_1 \to \mathbb{T}_1$ defined by

$$(\theta_1, \theta_2) \mapsto (\omega_1 - \theta_1, -\theta_2)$$

for fixed $\omega_1 \in (0, 2\pi)$ is obviously homotopic to h (the map $H : \mathbb{T}_1 \times I \to \mathbb{T}_1$ defined by $((\theta_1, \theta_2), t) \mapsto (t\omega_1 - \theta_2, -\theta_2)$ is a homotopy $h \simeq \hat{h}$), and $(\omega_1/2, 0)$ is a fixed point of $\hat{h}$.

We now explore the realizability of each condition enumerated in Theorem 2..1. As above, take $\mathbb{T}_1 = S^1 \times S^1$ and, if f is the self-map of $\mathbb{T}_1$ under consideration, denote $m = \text{tr } f_2$,

$$[f_1] = \begin{pmatrix} \lambda_1 & r \\ s & \lambda_2 \end{pmatrix}.$$

Case 1. $m = 0$.

The projection of $\mathbb{T}_1 = S^1 \times S^1$ onto either factor $S^1 \subset \mathbb{C}$ shows that a continuous function may map the torus surjectively onto either generating circle. Composing the projection with exponentiation $\theta \mapsto \lambda\theta$ for fixed $\lambda \in \mathbb{N}$ (recall our notation θ for a complex number $e^{\imath\theta}$ in S^1), we obtain a continuous surjection f such that f_1 maps a generating circle to any positive multiple of the circle we like. Since $f_2(\alpha) = 0$, and since one of the matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$$

represents f_1, we either have $\Lambda(f) = 1 \neq 0$ or $\Lambda(f) = 1 - \lambda$. In the latter case, f has a fixed point whenever $\lambda \neq 1$.

The following example illustrates that there is a continuous surjection from $\mathbb{T}_1$ onto its 1-skeleton (*i.e.* the union of the two generating circles) which assigns any integers we choose to λ_1 and λ_2. Let p denote the projection from $S^1 \times S^1$ onto its first factor S^1. Suppose $u : S^1 \to \mathbb{R}^2$ is a continuous surjection from the circle onto a figure eight in the plane. Then u is continuous (to verify this, observe that for any "$\times$"-shaped open neighborhood N of the double point of the figure eight, there is an arc of the circle which is mapped into N). Let g be a homeomorphism g from the figure eight $u(S^1) = u \circ p(\mathbb{T}_1)$ onto $a \cup b$. Then the map $f = g \circ u \circ p$ is continuous. Without loss of generality, we may suppose u wraps around either loop of the figure eight as many times as we please. If $\lambda_1 + \lambda_2 \neq 1$, then f has a fixed point.

Case 2.1.

The maps

$$(\theta_1, \theta_2) \mapsto (3\theta_1, 3\theta_2),$$

$$(\theta_1, \theta_2) \mapsto (-3\theta_1, -3\theta_2)$$

are 9-to-1, and correspond to a wrapping of each generating circle around itself three times. Since the signs of $\lambda_1 = \pm 3$ and $\lambda_2 = \pm 3$ are the same for each map, it follows that $m > 0$, so $m = \det f_1 = 9$ in either case.

Case 2.2.

A map of this class is given by

$$f : (\theta_1, \theta_2) \mapsto \begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix} (\theta_1, \theta_2).$$

Note that, for this map, tr $f_1 = 4$ and $m = 9$.

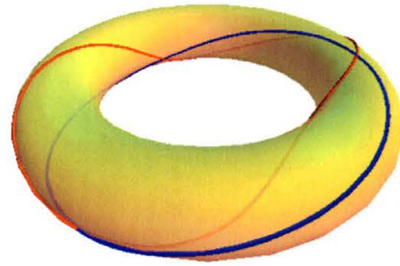

Figure IV.1: *f wraps the two circles a and b (shown in red and blue) around the torus.*

Winding b the same number of times in the opposite direction around the tube results in a change in the sign of λ_2, and a rather different map; call it g. We now have a 3-to-1 map with tr $g_1 = 4$ and $m = 3$. Furthermore, while $f(a)$ and $f(b)$ have three points of intersection, we have $|g(a) \cap g(b)| = 1$.

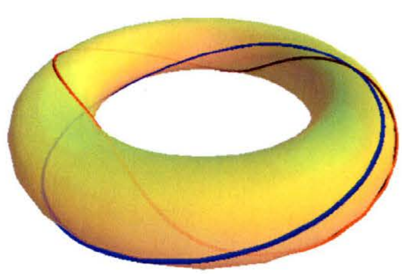

Figure IV.2: *g_1 is represented by the same matrix as f_1 except that the sign of the $2, 2$ entry has been reversed.*

On the other hand, we have $\Lambda(g) = 0$ for the map

$$g : (\theta_1, \theta_2) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (\theta_1, \theta_2).$$

Case 2.3.

Consider the map

$$f : (\theta_1, \theta_2) \mapsto \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} (\theta_1, \theta_2).$$

whose restriction to the 1-skeleton $a \cup b$ of $\mathbb{T}_1$ is pictured below.

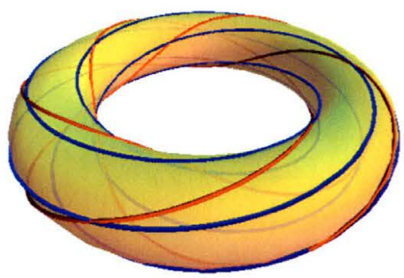

Figure IV.3: *The map* $f(\theta_1, \theta_2) = (3\theta_1 + 5\theta_2, 5\theta_1 + 3\theta_2)$ *realizes the conditions of Case 2.3.*

A representative of a homotopy class of maps for which $\Lambda(g) = 0$ is

$$g : (\theta_1, \theta_2) \mapsto \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} (\theta_1, \theta_2).$$

Observe that $g(t_1, t_1) = (t_1, t_1)$ for any $t_1 \in [0, 2\pi)$.

Case 2.4.

The map

$$f : (\theta_1, \theta_2) \mapsto (3\theta_1 + \theta_2, \theta_1 + 6\theta_2)$$

$$\lambda_1 = 3,\ y = 2$$

is realizable, and consequently has a fixed point.

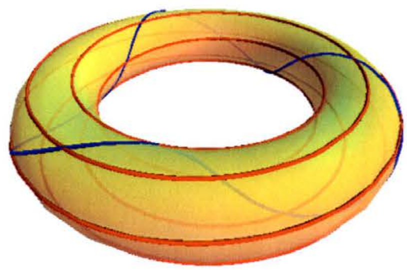

Figure IV.4: *The map* $f(\theta_1, \theta_2) = (3\theta_1 + \theta_2, \theta_1 + 6\theta_2)$ *realizes the conditions of Case* 2.4.

Note that the condition $\Lambda(f) \neq 0$ is not necessary for a map to have a fixed point. For instance, the map

$$g : (\theta_1, \theta_2) \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} (\theta_1, \theta_2).$$

has $\Lambda(g) = 0$, and we have a fixed point $g(t_1, -t_1) = (t_1, -t_1)$.

2. THE TORUS OF GENUS g GREATER THAN 1

Observe that permuting the generating circles of the first homology groups determines a set of functions $\mathbb{T}_g \to \mathbb{T}_g$ which map each generating circle to its image under the permutation.

Let $a_\imath$ and $b_\imath$, $1 \leq \imath \leq g$, denote the two generating circles of each torus in the connected sum $\mathbb{T}_g = \#_{\imath=1}^{g}\mathbb{T}_1$. An iteration of the permutation with cycle decomposition

$$(a_1, a_2, \ldots, a_g)(b_1, b_2, \ldots, b_g)$$

corresponds to a class of maps that send each $a_\imath$ to $a_{\imath+1}$ and each $b_\imath$ to $b_{\imath+1}$, where the indices are to be understood mod g. Any continuous map which induces such a permutation of the edges of the fundamental polygon has a fixed point if the hypothesis of Theorem 3..3 is satisfied. It can be shown using theory beyond the scope of this thesis that certain permutations cannot be realized by a continuous map.

NOTES

[1]To be brief, a **category** $\mathcal{C}$ may be defined as an ordered pair $(\mathcal{O}_\mathcal{C}, \mathcal{M}_\mathcal{C})$, where $\mathcal{O}_\mathcal{C}$ is a class of elements called **objects**, and $\mathcal{M}_\mathcal{C} = \{\hom(X,Y) : X, Y \in \mathcal{O}_\mathcal{C}\}$, where $\hom(X,Y)$ is a set of elements $f : X \to Y$ called **arrows with domain** X **and range** Y, provided that the following three **functorial properties** are satisfied. *(i)* For every ordered triple (X,Y,Z) of objects, there is a function that assigns to each ordered pair of arrows $(f : X \to Y, g : Y \to Z)$ an arrow $gf : X \to Z$ called the **composite** of f and g. *(ii)* If and $h : Z \to W$ then $h(gf) = (hg)f : X \to W$. *(iii)* For every $Y \in \mathcal{O}_\mathcal{C}$, there is an arrow $1_Y : Y \to Y$ such that if $f : X \to Y$ then $1_Y f = f$, and if $h : Y \to Z$, then $h1_Y = h$. A **covariant functor** F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is two functions $F : \mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{D}$ and $F : \mathcal{M}_\mathcal{C} \to \mathcal{M}_\mathcal{D}$ denoted by the same symbol such that *(i)* $F(\mathrm{id}_X) = 1_{F(X)}$ and *(ii)* $F(gf) = F(g)F(f)$. If F and G are two covariant functors from a category $\mathcal{C}$ to a category $\mathcal{D}$, a **natural transformation** η from F to G is a function $\mathcal{M}_\mathcal{C} \to \mathcal{M}_\mathcal{D}$ such that $\eta(Y)F(f) = G(f)\eta(X)$ for each $f : X \to Y$ in $\mathcal{O}_\mathcal{C}$. See [26] and [5].

[2]Brouwer's theorem states that a continuous function from the closed n-ball into itself has a fixed point. Neither the original nor the modern functorial proof is constructive, but constructive methods for finding such fixed points exist: see, for instance, [13] and [27].

[3]We ought to mention the idea of an *n-manifold with boundary*, a slight generalization of a manifold. Its defining property when $n = 2$ states that every point has a neighborhood homeomorphic to a basic open set in the subspace of the Euclidean plane which consists of all points with nonnegative vertical coordinates. Such spaces, called *surfaces with boundary*, are not considered in the current work, and so we write

surface where, strictly speaking, we mean *surface without boundary.*

[4]Some authors do not require a surface to be compact, and call a compact connected 2-manifold a **closed surface**.

[5]If (X, T_X) is a topological space, and $\sim$ is an equivalence relation on X, the **quotient set** $X/\!\sim$ is the set of all equivalence classes $[x]_\sim$ of points in X. The corresponding **identification space** is the ordered pair $(X/\!\sim, T_p)$, where $T_p = \{G \subset Y : p^{-1}(G) \in T_X\}$ is the **identification topology**, and p is the map $X \to X/\!\sim$ defined by $x \mapsto [x]_\sim$. [9]

[6] It can be shown that the connected sum of two surfaces is well-defined up to topological invariance. That is, $S_1 \# S_2$ and $S_1' \# S_2'$ are homeomorphic whenever S_1, S_1' are two homeomorphic spaces and S_2, S_2' are two homeomorphic spaces. Furthermore, the result is independent of choice of the homeomorphism h and the disks D_1 and D_2.

[7]Some standard theories include simplicial homology with integer coefficients [18] (in which the basic geometric building blocks are homeomorphic images of oriented simplices, *i.e.* points, line segments, triangles, tetrahedra, and their higher-dimensional analogues), cellular homology for CW complexes [24] (in which the building blocks are continuous images of n-balls), singular homology [15] (in which the building blocks are maps of simplices), homology with rational coefficients [14] (in which cycles whose integral multiples are boundaries vanish), and mod 2 simplicial homology [2] (in which the orientation of simplices is discarded).

[8]That is, there is a natural isomorphism between the graded groups resulting from any two competing homology theories.

[9]That is, we take it as understood that X_1 and X_2 are actually subsets of the

identification space $\mathbb{T}_1 \# \mathbb{T}_g$.

[10]In section §2.3, we explain the relationship between the polygon pictured and $\mathbb{T}_g$.

[11]Of course, $-\partial[\alpha]$ is also a boundary homologous to $i_1(a)$, and only one of the boundaries $\pm\partial[\alpha]$ has the same orientation as $\imath_1(a)$, but we find it worthwhile to visualize the deformation in either case.

[12]The fundamental polygon is typically realized as a regular polygon, but this constraint is inessential for our purposes, since we are concerned only with the topology of $\mathbb{T}_g$, and not its geometry.

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VITA

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