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The bowed narrow plate model *

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Abstract

The derivation of a narrow plate model that accommodates shearing, torsional, and bowing effects is presented. The resulting system has mathematical and computational advantages since it is in the form of a system of differential equations depending on only one spatial variable. A validation of the model against frequency data observed in laboratory experiments is presented. The models may be easily combined to form more complicated structures that are hinged along all or portions of their junction boundaries or are coupled differentiably as through the insertion of dowels between the narrow plates. Computational examples are presented to illustrate the types of deformations possible by coupling these models.

1 Introduction

In this paper, we consider a class of models for elastic structures, so-called narrow plate models [6,7], of an intermediate nature between beams and plates. Of particular interest are models that include linear shearing effects. Additionally, linear torsion and quadratic bending terms reflecting narrowness of the body are included. In [6, 7] we introduced models of this type, whose derivation was based on the energy form for the Mindlin-Timoshenko plate equations, by imposing restrictions to account for the geometry. The resulting dynamic model involves a system of linear symmetric hyperbolic equations in two independent variables, time and the longitudinal beam coordinate x, which are a special case of the Mindlin-Timoshenko plate equations [5]. Alternatively, they may be viewed as an extension of the familiar Timoshenko beam system, as described in [1] for example, to include torsional vibrations. The Mindlin-Timoshenko system is itself a generalization of the Kirchhoff model, modified to include shearing and rotatory inertia [10].

In previous work, validation and parameter identification have been carried out on the affine model in which the structure is sufficiently narrow that only linear torsional deformations across the width occur. In this work we present a validation of the so-called bowed model in which the width of the narrow plate is sufficiently large that parabolic deformations may occur across the width.

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The inclusion of such terms is significant in that it enables us to model more complicated structures that are coupled in a variety of ways. Moreover, the mathematical justification of coupling conditions and point loads are simplified since the resulting models involve only one spatial dimension. This approach enables us to model more complicated structures with relatively simple systems. Finally, we give a model expressing the deformation of a structure associated with a point force.

In Section 2, we present the bowed narrow plate model along with a brief discussion of the well-posedness of solutions. In Section 3, we give a numerical formulation of the problem and a validation of the model using natural frequency data collected in the laboratory. In Section 4, we present several examples of models of structures composed of narrow plates coupled in various ways continuously and differentiably along sides and also at various points along the plates.

2 Derivation of the bowed narrow plate equations

We begin by giving the stress-strain relations obtained under the following assumption, cf [2].

(E) The gradient of the deformation is small so that products of derivatives of the deformation are neglected.

Assumption (E) results in the following linear stress-strain relations, using the standard notation in [2].

$$\begin{aligned}\sigma_{11} &= \frac{E}{(1+\mu)(1-2\mu)}[(1-\mu)\epsilon_{11}+\mu\epsilon_{22}+\mu\epsilon_{33}]\\ \sigma_{22} &= \frac{E}{(1+\mu)(1-2\mu)}[\mu\epsilon_{11}+(1-\mu)\epsilon_{22}+\mu\epsilon_{33}]\\ \sigma_{33} &= \frac{E}{(1+\mu)(1-2\mu)}[\mu\epsilon_{11}+\mu\epsilon_{22}+(1-\mu)\epsilon_{33}]\\ &\sigma_{12}=G\epsilon_{12},\quad \sigma_{13}=G\epsilon_{13},\quad \sigma_{23}=G\epsilon_{23}\end{aligned} \tag{2.1)(i}$$

where E is the Young's modulus, μ is the Poisson's ratio, and $G = 2E/(1+\mu)$ is the shear modulus.

We suppose that the body occupies an open domain Ω in $\mathbb{R}^3$ given by

$$\Omega = \{(x,y,z) : 0 \le x \le L,\ -k(x) \le y \le k(x),\ \text{and } -h(x,y) \le z \le h(x,y)\}.$$

To facilitate our analysis, we assume that

(i) The functions h and k are bounded nonnegative piecewise continuous with finitely many jump discontinuities.

(ii) For each $x \in [0, L]$ the mapping $y \mapsto h(x, y)$ of $(-k(x), k(x))$ into $\mathbb{R}$ is an even function.

The underlying assumptions for the linear plate approximation are

(P1) Normal stresses in the z-direction are absorbed into the body force

(P2) No stretching or shearing of the neutral surface occurs.

Under assumption (P1), σ_{33} is set to zero and the resulting relation is used to eliminate ϵ_{33} from the expressions for σ_{11} and σ_{22}. In this case from (2.1)(i), one obtains

$$\begin{aligned}\sigma_{11} &= \frac{E}{1-\mu^2}[\epsilon_{11} + \mu\epsilon_{22}]\,, \\ \sigma_{22} &= \frac{E}{1-\mu^2}[\mu\epsilon_{11} + \epsilon_{22}].\end{aligned} \tag{2.1)(ii}$$

The assumption (P2) implies there are no geometric nonlinearities resulting from large deformations that would result, for example, in von Karman-type plate models, [10]. The displacements in the x, y, and z directions are given by the functions

$$U = U(x, y, z, t), \quad V = V(x, y, z, t), \quad W = W(x, y, z, t)\,.$$

The strain-displacement relations under the assumption (E) are given by

$$\begin{gathered}\epsilon_{11} = \frac{\partial U}{\partial x}, \quad \epsilon_{12} = \frac{1}{2}(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}), \\ \epsilon_{13} = \frac{1}{2}(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}), \quad \epsilon_{22} = \frac{\partial V}{\partial y}, \\ \epsilon_{23} = \frac{1}{2}(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z})\,.\end{gathered} \tag{2.2}$$

Assumptions (E), (P1), and (P2) imply the resulting model being linear. The Mindlin-Timoshenko model assumes that the displacements U, V, and W can be expressed as

$$U(x, y, z, t) = zu(x, y, t), \quad V(x, y, z, t) = zv(x, y, t), \quad W(x, y, z, t) = w(x, y, t)\,.$$

In this work the displacements are further specialized by assuming

$$\begin{aligned}U(x, y, z, t) &= z[\phi_0(x, t) + y\phi_1(x, t)], \\ V(x, y, z, t) &= z[\psi_0(x, t) + y\psi_1(x, t)], \\ W(x, y, z, t) &= w_0(x, t) + yw_1(x, t) + y^2 w_2(x, t)\,.\end{aligned} \tag{2.3}$$

These assumptions are motivated by our desire to develop a model of an elastic body that includes shearing as well as torsion and bending across the width.

Substitution of equations (2.3) into equations (2.1)(ii) and (2.2) allows us to express the strain and the stress in terms of the displacement functions ϕ_0, ϕ_1, ψ_0, ψ_1, w_0, w_1, and w_2 to obtain

$$\begin{gathered}
\epsilon_{11} = z[\phi_{0x} + y\phi_{1x}], \quad \epsilon_{12} = \frac{z}{2}[(\phi_1 + \psi_{0x}) + y\psi_{1x}],\\
\epsilon_{13} = \frac{1}{2}[(\phi_0 + w_{0x}) + y(\phi_{1x} + w_{1x}) + y^2 w_{2x}],\\
\epsilon_{22} = z\psi_1\,, \quad \epsilon_{23} = \frac{1}{2}[(\psi_0 + w_1) + y(\psi_1 + 2w_2)],
\end{gathered}$$

and

$$\begin{gathered}
\sigma_{11} = \frac{zE}{1-\mu^2}[(\phi_{0x} + \mu\psi_1) + y\phi_{1x}],\\
\sigma_{12} = \frac{zG}{2}[(\phi_1 + \psi_{0x}) + y\psi_{1x}],\\
\sigma_{13} = \frac{G}{2}[(\phi_0 + w_{0x}) + y(\phi_1 + w_{1x}) + y^2 w_{2x}],\\
\sigma_{22} = \frac{zE}{1-\mu^2}[(\mu\phi_{0x} + \psi_1) + y\mu\phi_{1x}],\\
\sigma_{23} = \frac{G}{2}[(\psi_0 + w_1) + y(\psi_1 + 2w_2)],
\end{gathered}$$

We next formulate the potential energy due to strain as the quadratic functional

$$\begin{aligned}
\mathcal{V}(t) \;=\; & \frac{1}{2}\int_0^L \int_{-k(x)}^{k(x)} \int_{-h(x,y)}^{h(x,y)} \Big\{ \frac{z^2 E}{1-\mu^2}[(\phi_{0x} + \mu\psi_1) + y\phi_{1x}][\phi_{0x} + y\phi_{1x}]\\
& + \frac{z^2 E}{1-\mu^2}[(\mu\phi_{0x} + \psi_1) + y\mu\phi_{1x}]\psi_1 + z^2\frac{G}{4}[(\phi_1 + \psi_{0x}) + y\psi_{1x}]^2\\
& + \frac{G}{4}[(\phi_0 + w_{0x}) + y(\phi_1 + w_{1x}) + y^2 w_{2x}]^2 \qquad (2.4)\\
& + \frac{G}{4}[(\psi_0 + w_1) + y(\psi_1 + 2w_2)]^2 \Big\}\, dz\, dy\, dx\,.
\end{aligned}$$

Define the following functions

$$\begin{gathered}
K_0(x) = \frac{2E}{3(1-\mu^2)} \int_{-k(x)}^{k(x)} h^3(x,y)dy\,,\\
K_2(x) = \frac{2E}{3(1-\mu^2)} \int_{-k(x)}^{k(x)} y^2 h^3(x,y)dy\,, \qquad (2.5)\\
\sigma_0(x) = \frac{G}{2}\int_{-k(x)}^{k(x)} h(x,y)dy\,, \quad \sigma_2(x) = \frac{G}{2}\int_{-k(x)}^{k(x)} y^2 h(x,y)dy\,,\\
\tau_0(x) = \frac{G}{6}\int_{-k(x)}^{k(x)} h^3(x,y)dy\,, \quad \tau_2(x) = \frac{G}{6}\int_{-k(x)}^{k(x)} y^2 h^3(x,y)dy\,,
\end{gathered}$$

$$\sigma_4(x) = \frac{G}{2}\int_{-k(x)}^{k(x)} y^4 h(x,y)dy\,.$$

We make the positivity assumption

(P) There is a positive number ν_0 such that the functions

$$K_0(x),\quad K_2(x),\quad \sigma_0(x),\quad \sigma_2(x),\quad \tau_0(x),\quad \tau_2(x),\quad \sigma_4(x)$$

are bounded below by a positive number ν_0.

The potential energy is now expressed by

$$\begin{aligned}\mathcal{V}(t) \;=\; & \frac{1}{2}\int_0^L \big\{K_0[(\phi_{0x}+\mu\psi_1)^2+(1-\mu^2)\psi_1^2]+K_2\phi_{1x}^2\\ &+\sigma_0[(\psi_0+w_1)^2+(\phi_0+w_{0x})^2]\\ &+\sigma_2[(\psi_1+2w_2)^2+(\phi_1+w_{1x})^2+2(\phi_0+w_{0x})w_{2x}]\\ &+\tau_0(\phi_1+\psi_{0x})^2+\tau_2\psi_{1x}^2+\sigma_4 w_{2x}^2\big\}\,dx\,.\end{aligned}\tag{2.6}$$

Remark 2.1 *It will be shown that under the positivity assumption (P), the potential energy functional given in (2.6) is positive definite.*

We next consider the kinetic energy of the system. Assuming a constant density function ρ, the kinetic energy is expressed as an integral by

$$\begin{aligned}\mathcal{T}(t) \;=\; & \frac{1}{2}\int_0^L\int_{-k(x)}^{k(x)}\int_{-h(x,y)}^{h(x,y)}\rho\{U_t^2(x,y,z,t)+V_t^2(x,y,z,t)\\ &+W_t^2(x,y,z,t)\}\,dz\,dy\,dx\,.\end{aligned}$$

Define the functions

$$\begin{gathered}I_{\rho,0}(x)=\frac{2\rho}{3}\int_{-k(x)}^{k(x)}h^3(x,y)dy\,,\quad I_{\rho,2}(x)=\frac{2\rho}{3}\int_{-k(x)}^{k(x)}y^2h^3(x,y)dy\,,\\ \rho_0(x)=2\rho\int_{-k(x)}^{k(x)}h(x,y)dy\,,\quad \rho_2(x)=2\rho\int_{-k(x)}^{k(x)}y^2h(x,y)dy\,,\\ \rho_4(x)=2\rho\int_{-k(x)}^{k(x)}y^4h(x,y)dy\,.\end{gathered}\tag{2.7}$$

From these assignments and from (2.3), we find, after performing integrations with respect to z and y, that

$$\begin{aligned}\mathcal{T}(t) \;=\; & \frac{1}{2}\int_0^L\frac{1}{6}\{I_{\rho,0}\phi_{0t}^2+I_{\rho,2}\phi_{1t}^2+I_{\rho,0}\psi_{0t}^2+I_{\rho,2}\psi_{1t}^2\\ &+\rho_0w_{0t}^2+\rho_2w_{1t}^2+2\rho_2w_{0t}w_{2t}+\rho_4w_{2t}^2\}\,dx\end{aligned}\tag{2.8}$$

Let us suppose that a body force, F, is exerted normal to the x-y plane. The work due to this force is

$$\mathcal{W}(t) = \int_0^L \int_{-k(x)}^{k(x)} \int_{-h(x,y)}^{h(x,y)} F(x,y,z,t) W(x,y,z,t)\, dz\, dy\, dx\,.$$

Define the functions

$$\begin{aligned} F_0(x,t) &= \int_{-k(x)}^{k(x)} \int_{-h(x,y)}^{h(x,y)} F(x,y,z,t)\, dz\, dy\,, \\ F_1(x,t) &= \int_{-k(x)}^{k(x)} \int_{-h(x,y)}^{h(x,y)} yF(x,y,z,t)\, dz\, dy\,, \\ F_2(x,t) &= \int_{-k(x)}^{k(x)} \int_{-h(x,y)}^{h(x,y)} y^2 F(x,y,z,t)\, dz\, dy\,. \end{aligned} \tag{2.9}$$

From (2.3) the work may be expressed upon integration as

$$\mathcal{W}(t) = \int_0^L \{F_0 w_0 + F_1 w_1 + F_2 w_2\} dx\,. \tag{2.10}$$

The Lagrangian is given by

$$\mathcal{L}(t) = \mathcal{T}(t) - \mathcal{V}(t) + \mathcal{W}(t).$$

Hamilton's principle indicates that the deformation experienced by the body is obtained as an extremal of the integral $\int_0^t \mathcal{L}(s) ds$, cf [4]. That is, the deformation assumed by the body has the property that the variation of the Lagrangian functional is zero:

$$\delta \int_0^t \mathcal{L}(s) ds = 0$$

with respect to functions $\delta\phi_0$, $\delta\phi_1$, $\delta\psi_0$, $\delta\psi$, δw_0, δw_1, and δw_2 satisfying the essential boundary conditions and equal to zero at times 0 and t. Upon integration by parts with respect to time and the spatial variable, we obtain the following equations of motion.

$$\begin{gathered} I_{\rho,0}\phi_{0tt} - (K_0(\phi_{0x} + \mu\psi_1))_x + \sigma_0(\phi_0 + w_{0x}) + \sigma_2 w_{2x} = 0 \\ I_{\rho,2}\psi_{1tt} - (\tau_2\psi_{1x})_x + \mu K_0(\phi_{0x} + \psi) + \sigma_2(\psi_1 + w_2) = 0 \\ \rho_0 w_{0tt} + \rho_2 w_{2tt} - (\sigma_0(\phi_0 + w_{0x}))_x - (\sigma_2 w_{2x})_x = F_0 \\ \rho_2 w_{0tt} + \rho_4 w_{2tt} - (\sigma_4 w_{2x})_x - 2(\sigma_2(\phi_0 + w_{0x}))_x = F_2 \\ I_{\rho,2}\phi_{1tt} - (K_2\phi_{1x})_x + \sigma_2(\phi_1 + w_{1x}) + \tau_0(\phi_1 + \psi_{0x}) = 0 \\ I_{\rho,0}\psi_{0tt} - (\tau_0(\phi_1 + \psi_{0x}))_x + \sigma_0(\psi_0 + w_1) = 0 \\ \rho_2 w_{1tt} - ((\sigma_2(\phi_1 + w_{1x}))_x + \sigma_0(\psi_0 + w_1) = F_1 \end{gathered} \tag{2.11}$$

where we have written the equations in an order which emphasizes the couplings between them.

The boundary conditions at 0 and L are associated with conditions

$$\begin{gathered}
(\phi_{0x} + \mu\psi_1)\delta\phi_0|_0^L = 0\,, \quad \psi_{1x}\delta\psi_1|_0^L = 0\,,\\
(\sigma_0(\phi_0 + w_{0x}) + \sigma_2 w_{2x})\delta w_0|_0^L = 0\,,\\
(2\sigma_2(\phi_0 + w_{0x}) + \nu_0 w_{2x})\delta w_2|_0^L = 0\,,\\
\phi_{1x}\delta\phi_1|_0^L = 0\,, \quad (\phi_1 + \psi_{0x})\delta\psi_0|_0^L = 0\,,\\
(\phi_1 + w_{1x})\delta w_1|_0^L = 0\,.
\end{gathered}$$

For example, if the narrow plate is clamped at $x = 0$ and free at $x = L$, we see that

$$\phi_0(0) = \phi_1(0) = \psi_0(0) = \psi_1(0) = w_0(0) = w_1(0) = w_2(0) = 0$$

and

$$\begin{gathered}
(\phi_{0x} + \mu\psi_1)(L) = 0\,, \quad \phi_{1x}(L) = 0\,,\\
(\phi_1 + \psi_{0x})(L) = 0\,, \quad \psi_{1x}(L) = 0\,,\\
(\sigma_0(\phi_0 + w_{0x}) + \frac{1}{2}\sigma_2 w_{2x})(L) = 0\\
(\phi_1 + w_{1x})(L) = 0\,,\\
(2\sigma_2(\phi_0 + w_{0x}) + \nu_0 w_{2x})(L) = 0\,.
\end{gathered}$$

It is convenient to rewrite the equations using vector notation. Towards this end, we introduce the column vector-valued function

$$v = \mathrm{col}(\phi_0, \psi_1, w_0, w_2, \phi_1, \psi_0, w_1)$$

and the matrices

$$\mathcal{E}_0 = \begin{bmatrix} 0 & \mu & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{E}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} K_0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \tau_2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \sigma_0 & \sigma_2 & 0 & 0 & 0\\ 0 & 0 & \sigma_2 & \sigma_4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & K_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \tau_0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2 \end{bmatrix}, \quad C = \begin{bmatrix} (1-\mu^2)K_0 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_0 \end{bmatrix}.$$

Under the above assumptions we see that the matrix functions $A(x)$ and $C(x)$ are positive definite for each x in $[0, L]$.

Lemma 2.2 *There exists a positive constant κ_0 such that for any vector $\mathbf{u} \in \mathbb{R}^7$,*

$$\mathbf{u}^* A(x)\mathbf{u} \geq \kappa_0 \mathbf{u}^*\mathbf{u}$$

and

$$\mathbf{u}^* C(x)\mathbf{u} \geq \kappa_0 \mathbf{u}^*\mathbf{u}\,.$$

Proof. Clearly, by choosing κ_0 sufficiently small, but positive, the condition holds for $C(x)$. From assumption (P), we have

$$\mathbf{u}^* A(x)\mathbf{u} \geq \nu_0\{u_1^2 + u_2^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2\} + [u_3, u_4]\begin{bmatrix}\sigma_0 & \sigma_2\\ \sigma_2 & \sigma_4\end{bmatrix}\begin{bmatrix}u_3\\ u_4\end{bmatrix}$$

But

$$[u_3, u_4]\begin{bmatrix}\sigma_0 & \sigma_2\\ \sigma_2 & \sigma_4\end{bmatrix}\begin{bmatrix}u_3\\ u_4\end{bmatrix} \geq \lambda_0(x)(u_3^2 + u_4^2)$$

where

$$\lambda_0(x) = \frac{1}{2}\{\sigma_0(x) + \sigma_4(x) - [(\sigma_0(x) + \sigma_4(x))^2 - 4(\sigma_0(x)\sigma_4(x) - \sigma_2^2(x))]^{\frac{1}{2}}\}.$$

From assumption (P) we find that the function

$$x \mapsto \lambda_0(x)$$

is piecewise continuous on $[0, L]$ and there is a positive number ν_1 such that $\lambda_0(x) \geq \nu_1$ for any $x \in [0, L]$ if $\sigma_0(x)\sigma_4(x) - \sigma_2^2(x) > 0$ for all $x \in [0, L]$. Using the definitions from the equations (2.5), we find that for each $x \in [0, L]$

$$\sigma_0(x)\sigma_4(x) - \sigma_2^2(x) = \int_{-k(x)}^{k(x)} h(x,y)dy \int_{-k(x)}^{k(x)} y^4 h(x,y)dy - (\int_{-k(x)}^{k(x)} y^2 h(x,y)dy)^2.$$

Thus, $\sigma_0(x)\sigma_4(x) - \sigma_2^2(x)$ is positive on $[0, L]$ by the Cauchy-Schwarz inequality. It follows that $\lambda_0(x)$ is real-valued and bounded away from zero. Selecting κ_0 to be less than ν_0 or ν_1 now yields the result.

With the above definitions, we may express the strain potential energy functional (2.6) as

$$\mathcal{V}(t) = \frac{1}{2}\int_0^L \{(v_x + \mathcal{E}_0 v)^* A(v_x + \mathcal{E}_0 v) + v^*(\mathcal{E}_1^* C \mathcal{E}_1)v\}dx\,. \tag{2.12}$$

The form of the kinetic energy is obtained by introducing the matrix

$$M = \begin{bmatrix} I_{\rho,0} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I_{\rho,2} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \rho_0 & \rho_2 & 0 & 0 & 0\\ 0 & 0 & \rho_2 & \rho_4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I_{\rho,2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & I_{\rho,0} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 \end{bmatrix}.$$

The kinetic energy functional is now expressed as

$$\mathcal{T}(t) = \frac{1}{2}\int_0^L {v_t}^* M v_t dx. \tag{2.13}$$

Finally, defining the column vector

$$F = \mathrm{col}(0, 0, F_0, F_2, 0, 0, F_1),$$

the linear functional expressing the work done by external forces is

$$\mathcal{W}(t) = \int_0^L F^* v dx.$$

The system of partial differential equations becomes

$$M v_{tt} - (A(v_x + \mathcal{E}_0 v))_x + \mathcal{E}_0^* A(v_x + \mathcal{E}_0 v) + \mathcal{E}_1^* C \mathcal{E}_1 v = F$$

with cantilevered boundary conditions

$$v(0) = 0 \quad \text{and} \quad (v_x + \mathcal{E}_0 v)(L) = 0\,.$$

Of course, initial conditions $v(0) = v_0$ and $v_t(0) = v_1$ must be specified as well.

We next discuss the weak formulation for our problems. Towards this end, we designate the Sobolev space

$$\mathbf{V} = H^1(0, L; \mathbb{R}^7) = \{(v_1, v_2, v_3, v_4, v_5, v_6, v_7) : v_i \in H^1(0, L)\ i = 1, ..., 7\}$$

with norm

$$\|v\|_{\mathbf{V}} \;=\; (\sum_{i=1}^{7} \|v_i\|^2_{H^1(0,L)})^{1/2}$$

and the Hilbert space $\mathbf{H} = L^2(0, L; \mathbb{R}^7)$.

We now define the bilinear form $a(\cdot, \cdot)$ on $\mathbf{V}$ by

$$a(u, v) \;=\; \int_0^L \{(u_x + \mathcal{E}_0 u)^* A(v_x + \mathcal{E}_0 v) + u^* \mathcal{E}_1^* C \mathcal{E}_1 v\} dx. \tag{2.14}$$

Note from Lemma 2.2 that for any $u \in \mathbf{V}$

$$a(u, u) \geq \kappa_0 \int_0^L \{|u_x + \mathcal{E}_0 u|^2 + |\mathcal{E}_1 u|^2\} dx\,.$$

Remark 2.3 *From the positive definiteness of $a(u, u)$ and from the Cauchy-Schwarz inequality, it follows that there are positive numbers γ_0 and γ_1 such that*

$$a(u, u) + \gamma_0 \|u\|^2_{\mathbf{H}} \geq \gamma_1 \|u\|^2_{\mathbf{V}}. \tag{2.15}$$

By the Cauchy Schwarz inequality it follows that there is a positive constant γ_2 such that

$$|a(u, v)| \leq \gamma_2 \|u\|_{\mathbf{V}} \|v\|_{\mathbf{V}}. \tag{2.16}$$

Proposition 2.4 *Suppose that* $\mathbf{V}_0$ *is a closed subspace of* $\mathbf{V}$ *with the property*

$$\text{if } u \in \mathbf{V}_0 \text{ is such that } a(u,u) = 0, \text{ then } u = 0, \tag{2.17}$$

then there exists a positive number γ *such that for any* $u \in \mathbf{V}_0$

$$a(u,u) \geq \gamma \|u\|^2_{\mathbf{V}}.$$

Proof. We show there is a positive constant α' such that

$$a(u,u) \geq \alpha' \|u\|^2_{\mathbf{H}}$$

for any $u \in \mathbf{V}_0$. If this were not the case, then for each n there exists u_n with $\|u_n\|_{\mathbf{H}} = 1$ and such that

$$0 \leq a(u_n, u_n) \leq 1/n.$$

It follows from (2.15) that

$$1/n + \gamma_0 \geq \gamma_1 \|u_n\|^2_{\mathbf{V}}.$$

Thus, there is a subsequence again denoted by $\{u_n\}_{n=1}^{\infty}$ such that

$$\begin{aligned} u_n &\to u \quad \text{weakly in } \mathbf{V}\,, \\ u_n &\to u \quad \text{strongly in } \mathbf{H} \end{aligned}$$

since $\mathbf{V}$ embeds compactly in $\mathbf{H}$. The limit satisfies $\|u\|_{\mathbf{H}} = 1$. The continuity condition (2.16) implies the weak lower semicontinuity of the bilinear form $a(\cdot,\cdot)$ on $\mathbf{V}$, i. e.,

$$\liminf a(u_n, u_n) \geq a(u,u).$$

Moreover, since $\mathbf{V}_0$ is a closed subspace of $\mathbf{V}$, it follows that $u \in \mathbf{V}_0$. Thus, we conclude that $a(u,u) = 0$ and therefore, $u = 0$, contradicting $\|u\|_{\mathbf{H}} = 1$. The results follow by setting $\gamma = 1/(1 + \mu\alpha')$,

Remark 2.5 *The previous proposition applies to those subspaces of* $\mathbf{V}$ *for which*

$$u_x + \mathcal{E}_0 u = 0 \tag{2.18)(i}$$

and

$$\mathcal{E}_1 u = 0 \tag{2.18)(ii}$$

imply that $u = 0$.

Note that the two conditions (2.18)(i) and (2.18)(ii) imply that

$$\psi_1 = w_2 = \phi_1 = \phi_{0x} = \psi_{0x} = w_{1x} = w_{0x} + \phi_0 = 0\,. \tag{2.19}$$

Suppose $\mathbf{V}_0$ is specified as above and that $\mathbf{V}_0$ is dense in $\mathbf{H}$. Designating $\mathbf{H}$ as the pivot space, let $\mathbf{V}_0'$ be the dual of $\mathbf{V}_0$. Let F be in $\mathbf{V}_0'$. The weak formulation of the static problem is then given as follows:

Find $u \in \mathbf{V}_0$ such that

$$a(u,v) = (F,v) \text{ for any } v \in \mathbf{V}_0 \tag{2.20}$$

where $(\cdot,\cdot)$ expresses the duality pairing between $\mathbf{V}_0$ and $\mathbf{V}_0'$. The existence of a unique solution of (2.20) is classical and follows from Proposition 2.4. Further, it satisfies the estimate

$$\gamma \|u\|_{\mathbf{V}_0} \leq \|F\|_{\mathbf{V}_0}.$$

Furthermore, results for the hyperbolic system and the associated eigenvalue problem are classical as well [3,8].

The weak formulation of the dynamic problem is obtained by introducing the bilinear form

$$m(u,v) = \int_0^L u^* M v dx$$

on $\mathbf{H}$. Observe, that under the assumptions above, there are positive constants β_0 and β_1 such that for any u and v in $\mathbf{H}$,

$$m(u,u) \geq \beta_0 \|u\|_{\mathbf{H}}^2$$

and

$$m(u,v) \leq \beta_1 \|u\|_{\mathbf{H}} \|v\|_{\mathbf{H}}.$$

The weak form of the dynamic problem is stated as follows:

Find $u \in L^2(0,T;\mathbf{V}_0)$ such that for any $v \in \mathbf{V}_0$

$$m(u_{tt}(t),v) + a(u(t),v) \;=\; (F(t),v)_{\mathbf{H}} \tag{2.21}$$

and

$$(u(0),v)_{\mathbf{H}} = (u_0,v)_{\mathbf{H}}$$
$$(u_t(0),v)_{\mathbf{H}} = (u_1,v)_{\mathbf{H}}$$

for any $v \in \mathbf{V}_0$.

The associated eigenvalue problem is posed as follows:

Find those numbers λ^2 such that there exist nontrivial solutions of the equation

$$a(u,v) = \lambda^2 m(u,v). \tag{2.22}$$

3 Numerical approximation and model validation

In this section, we give a finite element formulation of the above problems. The starting point is the weak formulation. Once we have obtained the system of approximating equations, we present results that constitute a validation of the model by comparing calculated natural frequencies for our model to compare with those observed experimentally in the laboratory.

To obtain the finite dimensional analogue, we specify a set of linearly independent real valued functions, $\{b_i\}_{i=1}^{M}$ defined on the interval $(0, L)$ contained in the Sobolev space, V. We define the M-row vector-valued function

$$b(x) = (b_1(x), \dots, b_M(x))$$

and the $7 \times 7M$ matrix-valued function

$$B(x) = \begin{bmatrix} b(x) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b(x) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(x) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b(x) \end{bmatrix} \tag{3.1}$$

where here, 0 represents a $M -$ row vector whose components are all zeros. Further, let

$$c = \mathrm{col}(c_1, \dots, c_{7M}).$$

For this approximation we look for vector-valued functions expressed as

$$u^M(x) = B(x)c.$$

In equation (2.20) setting $v(x) = B(x)d$, we obtain using (2.14)

$$c^*\{\int_0^L [(B_x + \mathcal{E}_0 B)^* A(B_x + \mathcal{E}_0 B) + B^* \mathcal{E}_1^* C \mathcal{E}_1 B] dx\} d = \{\int_0^L F^* B dx\} d.$$

It follows that we seek the solution of the linear system

$$\{\int_0^L [(B_x + \mathcal{E}_0 B)^* A(B_x + \mathcal{E}_0 B) + B^* \mathcal{E}_1^* C \mathcal{E}_1 B] dx\} c = \{\int_0^L F^* B dx\}. \tag{3.2}$$

In a similar manner for the dynamic system (2.21), we set $u(t) = Bc(t)$ to obtain the initial value problem

$$\begin{aligned} \{\int_0^L B^* M B dx\} c_{tt} + \{\int_0^L [(B_x + \mathcal{E}_0 B)^* A(B_x + \mathcal{E}_0 B) + B^* \mathcal{E}_1^* C \mathcal{E}_1 B] dx\} c \\ = \{\int_0^L F(t)^* B \, dx\} \end{aligned} \tag{3.3}$$

with initial conditions

$$\{\int_0^L B^* B dx\} c(0) = \int_0^L B^* u_0 dx$$

and

$$\{\int_0^L B^* B dx\} c_t(0) = \int_0^L B^* u_1 dx.$$

Finally, the generalized eigenvalue problem is given by

$$\{\int_0^L [(B_x+\mathcal{E}_0 B)^* A(B_x+\mathcal{E}_0 B)+B^*\mathcal{E}_1^* C\mathcal{E}_1 B]dx\}c = \lambda^2\{\int_0^L B^* MBdx\}c. \quad (3.4)$$

Remark 3.1 *The error analysis for the above approximations is standard and discussions may be found for example in [9].*

To test the model, we measured the natural frequencies for an aluminum structure in the shape of a paddle composed of 2 rectangles with the larger atop the smaller. The dimensions of the structure are as follows:

Total length	35 in
Length of the lower rectangle	14 in
Length of the upper rectangle	21 in
Thickness	0.125 in
Width of the lower rectangle	8 in
Width of the upper rectangle	22 in

Note that the width k is a piecewise constant function of x given by

$$k(x) = \begin{cases} 4\,, & x \in (0,14) \\ 11\,, & x \in [14,35]\,. \end{cases}$$

The observed frequencies are:

$$2.81 \quad 9.69 \quad 20.94 \quad 28.12 \quad 44.37 \quad 85.31 \quad 95.62 \quad 99.06 \quad 115.94\,.$$

Because the shearing and inplane motion is small compared to the motion normal to the plane, we set

$$\phi_0 = \phi_1 = \psi_0 = \psi_1 = 0\,.$$

We may also obtain affine motions with only linear cross-sectional deformations admissible by neglecting w_2 as well. We use a uniform mesh with 14 subdivisions on which the functions b_i are taken to be “hat” functions with boundary condition $b_i(0) = 0$ imposed to reflect the clamped boundary conditions at $x = 0$.

We calculate the following frequencies

$$3.16 \quad 9.42 \quad 19.7 \quad 26.0 \quad 45.1 \quad 84.7 \quad 95.6 \quad 99.1 \quad 118.4\,.$$

By calculating the frequencies in the affine case, we find that

$$3.16 \quad 9.42 \quad 19.7 \quad 26.0 \quad 95.6 \quad 99.1$$

d appear to be related to the affine motion of the structure while the frequencies

$$45.1 \quad 84.7 \quad 95.62 \quad 118.4$$

appear to be associated with bowed motions.

4 Coupling of Bowed Plates

In this section, we present models for structures that may be viewed as coupled narrow plates. Our approach is to designate the plates by assigning local coordinate systems. In a manner similar to that of the previous sections, we then determine the potential energy functional for each of the separate plates. The sum of these functionals forms the total potential energy of the deformations of the structure. Coupling and boundary constraints are imposed to determine the class of admissible deformations. The resulting constraints are then included in the functional by means of penalization. Inclusion of the constraints by penalization in the potential energy functional amounts to inclusion of certain potential energy functionals with large elastic constants.

We present our approach for several cases. Examples of mode shapes resulting from our formulation are given. For convenience, we assume that the plates are of the same length and constant thickness and width. Thus, in all cases the functions h and k are considered constants. We first explicitly give stiffness matrices corresponding to those in Section 3. The material functions given in the equations (2.5) are the constants

$$K_0 = \frac{4h^3kE}{3(1-\mu^2)}\,, \quad K_2 = \frac{4h^3k^3E}{9(1-\mu^2)}\,,$$
$$\sigma_0 = hkG\,, \quad \sigma_2 = \frac{hk^3G}{3}\,, \quad \sigma_4 = \frac{hk^5G}{5}\,,$$
$$\tau_0 = \frac{h^3kG}{3}\,, \quad \tau_2 = \frac{h^3k^3G}{9}\,.$$

Let the matrices $\mathcal{E}_0$, $\mathcal{E}_1$, A, and C be as defined in Section 2. The matrix valued function B is defined with a full basis without regard to boundary conditions so that essential boundary conditions are not imposed directly on the basis elements. Hence, dividing the interval $(0, L)$ into N subintervals and using piecewise linear elements with respect to the resulting mesh yields vector functions $b(x)$ with $M = N+1$ terms. The essential boundary conditions are imposed by penalization in the potential energy functional. With these assignments

$$B_x(x) + \mathcal{E}_0 B_(x) = \begin{bmatrix} b_x(x) & \mu b & 0 & 0 & 0 & 0 & 0 \\ 0 & b_x(x) & 0 & 0 & 0 & 0 & 0 \\ b(x) & 0 & b_x(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_x(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_x(x) & 0 & 0 \\ 0 & 0 & 0 & 0 & b(x) & b_x(x) & 0 \\ 0 & 0 & 0 & 0 & b(x) & 0 & b_x(x) \end{bmatrix}.$$

Define the $M \times M$ matrices

$$g_0 = \int_0^L b^*(x)b(x)\,dx\,, \quad g_1 = \int_0^L b_x^*(x)b(x)\,dx\,, \quad g_2 = \int_x^L b_x^*(x)b_x(x)\,dx\,.$$

We now obtain

$$\begin{aligned} G_2 &= \int_0^L [(B_x(x) + \mathcal{E}_0 B(x) * A(B_x(x) + \mathcal{E}_0 B(x)) + B(x)^* \mathcal{E}_1^* C \mathcal{E}_1 B(x)]dx \\ &= \begin{bmatrix} G_2(1,1) & 0 \\ 0 & G_2(2,2) \end{bmatrix} \end{aligned}$$

$$G_2(1,1) = \begin{bmatrix} K_0 g_2 + \sigma_0 g_0 & \mu K_0 g_1 & \sigma_0 g_1^* & \sigma_2 g_1^* \\ \mu K_0 g_1^* & \tau_2 g_2 + (\sigma_2 + \mu^2 K_0) g_0 & 0 & 2\sigma_2 g_0 \\ \sigma_0 g_1 & 0 & \sigma_0 g_2 & \sigma_2 g_2 \\ \sigma_2 g_1 & 2\sigma_2 g_0 & \sigma_2 g_2 & \sigma_4 g_2 + 4\sigma_2 g_0 \end{bmatrix}$$

$$G_2(2,2) = \begin{bmatrix} K_2 g_2 + (\tau_0 + \sigma_2) g_0 & \tau_0 g_1^* & \sigma_2 g_1^* \\ \tau_0 g_1 & \tau_0 g_2 + \sigma_0 g_0 & \sigma_0 g_0 \\ \sigma_2 g_1 & \sigma_0 g_0 & \sigma_2 g_2 + \sigma_0 g_0 \end{bmatrix}$$

To formulate equations describing the motion of two narrow plates 1 and 2 that are coupled along one side, we view the plates as being situated in such a way that they lie in the x-y plane. The junction between the two plates lies along the x-axis. Local coordinate systems for 1 and 2 are given by $x_1,\ y_1,$ and z_1 and $x_2,\ y_2,$ and z_2, respectively, where $0 < x_1 < L$ and $0 < x_2 < L$ with

$$-h_1 < z_1 < h_1 \quad \text{and} \quad -h_2 < z_2 < h_2$$

as well as

$$-k_1 < y_1 < k_1 \quad \text{and} \quad -k_2 < y_2 < k_2 .$$

For ease we suppose that $k_1 = k_2 = k$, $h_1 = h_2 = h$ and $L_1 = L_2 = L$. It follows that $x_1 = x_2 = x$, $y_1 = y + k$, $y_2 = y - k$, and $z_1 = z_2 = z$. For $i = 1, 2$, the displacement functions are given by

$$\begin{aligned} U_i(x, y_i, z) &= z(\phi_{0i} + y_i \phi_{1i}) , \\ V_i(x, y_i, z) &= z(\psi_{0i} + y_i \phi_{1i}) , \\ W_i(x, y_i, z) &= w_{0i}(x) + y_i w_{1i} + y_i^2 w_{2i} . \end{aligned} \tag{4.1}$$

For a coupling that gives rise to the two narrow plates behaving as a single plate, conditions are imposed at $y_1 = k$ and $y_2 = -k$ to assure continuity of the displacements and the first derivatives across the junction. These must be

$$\begin{aligned} U_1(x, k, z) &= U_2(x, -k, z) \\ V_1(x, k, z) &= V_2(x, -k, z) \\ W_1(x, k, z) &= W_2(x, -k, z) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} U_{1y_1}(x, k, z) &= U_{2y_2}(x, -k, z) \\ V_{1y_1}(x, k, z) &= V_{2y_2}(x, -k, z) \\ W_{1y_1}(x, k, z) &= W_{2y_2}(x, -k, z) \end{aligned} \tag{4.3}$$

From the narrow plate displacement relations for the i-th plate (4.1), we obtain the constraints for $x \in (0, L)$

$$\begin{gathered} \phi_{01}(x) + k\phi_{11}(x) - \phi_{02}(x) + k\phi_{12}(x) = 0 \\ \psi_{01}(x) + k\psi_{11}(x) - \psi_{02}(x) + k\psi_{12}(x) = 0 \\ w_{01}(x) + kw_{11}(x) + k^2 w_{21}(x) - w_{02}(x) + kw_{12}(x) - k^2 w_{22}(x) = 0 \end{gathered} \tag{4.4}$$

and

$$\begin{gathered} \phi_{11}(x) + \phi_{12}(x) = 0 \\ \psi_{11}(x) + \psi_{12}(x) = 0 \\ w_{11}(x) + 2kw_{21}(x) + w_{12}(x) - 2kw_{22}(x) = 0\,. \end{gathered} \tag{4.5}$$

Let $v_i = \mathrm{col}(\phi_{0i}, \psi_{1i}, w_{0i}, w_{2i}, \phi_{1i}, \psi_{0i}, w_{1i})$. Define the matrices

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & k & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k^2 & 0 & 0 & k \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2k & 0 & 0 & 0 \end{bmatrix} \tag{4.6}$$

$$C_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & k & 0 & 0 \\ 0 & k & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -k^2 & 0 & 0 & k \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2k & 0 & 0 & 1 \end{bmatrix}. \tag{4.7}$$

The system (4.2)-(4.3) now becomes

$$C_1 v_1 + C_2 v_2 = 0. \tag{4.8}$$

Remark 4.1. To model a hinged junction requires only that the deformation be continuous across the junction. Hence, it suffices to take only (4.5) as a constraint:

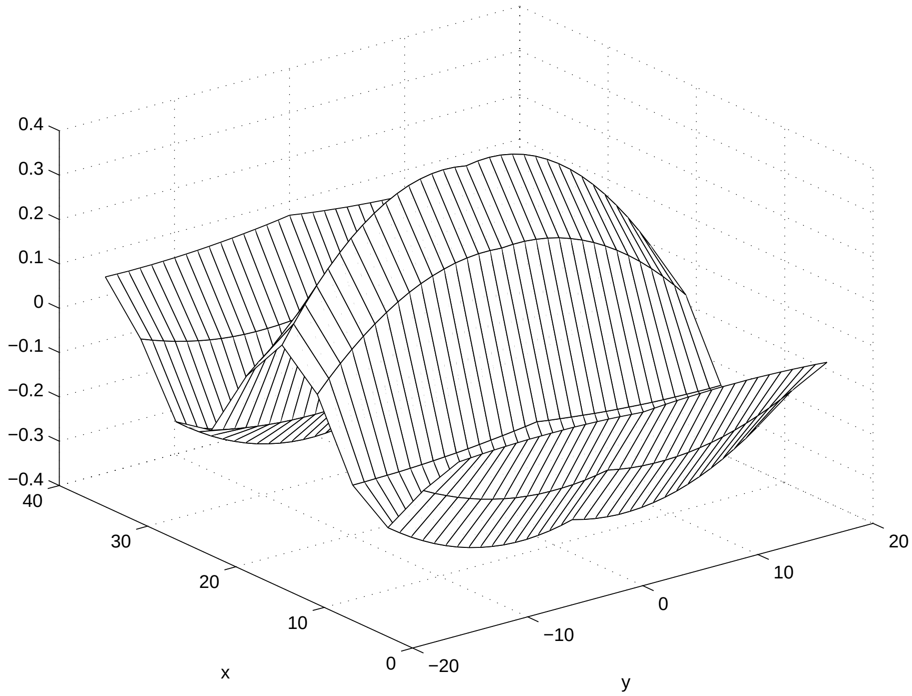
$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & k & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k^2 & 0 & 0 & k \end{bmatrix}$$

and

$$C_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & k & 0 & 0 \\ 0 & k & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -k^2 & 0 & 0 & k \end{bmatrix}$$

with the condition given in equation (4.8) as a constraint.

Figure 1: Sample mode shape with continuous junction



Remark 4.2. Plates may be coupled at points. For example, requiring the plates to be hinged at a point x_0 amounts to requiring

$$C_1 v_1(x_0) + C_2 v_2(x_0) = 0$$

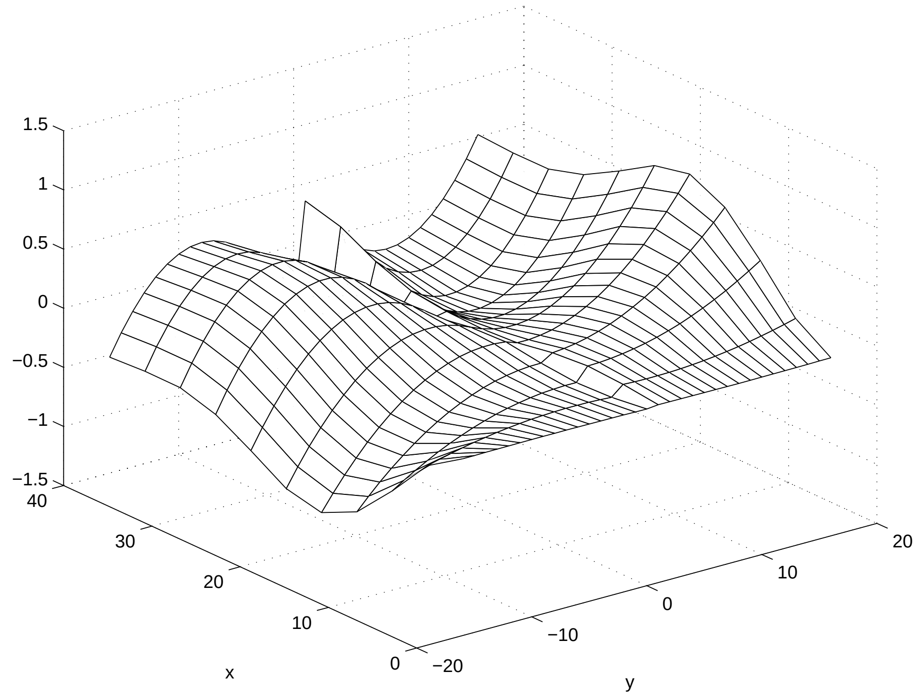
where C_1 and C_2 are given in Remark 4.1. Differentiable coupling at points, such as would occur by coupling the plates with dowels, may be modeled by using the matrices C_1 and C_2 in equations (4.6) and (4.7).

In Figures 1 and 2, we give two example modal shapes to illustrate the types of couplings possible. In Figure 1, the junction is hinged with a continuity condition between the plates so that a corner is allowed along the junction. In Figure 2, the differentiability across the point $x_0 = L/2$ may be thought as a mathematical idealization of a dowel connecting the two plates at that point.

Conclusions

The derivation of a narrow plate model that accommodates shearing, torsional, and bowing effects is presented. The model is validated against frequency data observed in laboratory experiments. The resulting system of boundary value problems has mathematical and computational advantages in the sense that it consists of a system of differential equations depending on only one spatial variable. Thus, it is easy to give formulations involving point effects even when

Figure 2: Sample mode shape with differentiable junction at L/2



structural properties may have discontinuities. Moreover, boundary value problems may be formulated to model complicated elastic structures obtained by coupling across all or portions of junctions between adjacent narrow plates. Thus, it is possible to model structures that are hinged or differentiable along interfaces or at points on the interfaces. Mode shapes of various couplings are presented as examples of the deformations that are possible under this model.

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