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Blow-up for p-Laplacian parabolic equations *

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Abstract

In this article we give a complete picture of the blow-up criteria for weak solutions of the Dirichlet problem

$$u_t = \nabla(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{q-2}u, \quad \text{in } \Omega_T,$$

where $p > 1$. In particular, for $p > 2$, $q = p$ is the blow-up critical exponent and we show that the sharp blow-up condition involves the first eigenvalue of the problem

$$-\nabla(|\nabla \psi|^{p-2}\nabla \psi) = \lambda |\psi|^{p-2}\psi, \quad \text{in } \Omega; \quad \psi|_{\partial\Omega} = 0.$$

1 Introduction

In this paper we study the Dirichlet problem

$$\begin{gathered} u_t = \nabla(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{q-2}u, \quad \text{in } \Omega_T, \\ u = 0, \quad \text{on } S_T, \\ u(x,0) = u_0(x), \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

$u_0(x) \in C_0(\overline{\Omega})$, where $p > 1$, $q > 2$, $\lambda > 0$ and $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$.

When $p = 2$, the blow-up properties of the semilinear heat equation (1.1) hasve been investigated by many researchers; see the recent survey paper [11]. For $p \neq 2$, the main interest in the past twenty years lies in the regularities of weak solutions of the quasilinear parabolic equations; see the monograph [4] and the references therein. When $\Omega = \mathbb{R}^N$, the Fujita exponents have been calculated; see [7, 8, 9, 10] and also the survey papers [3, 12].

To the best of our knowledge, when Ω is a bounded domain, the blow-up conditions are not fully established, especially, in the case $q = p > 2$. In [23], the author showed that $q = p$ is the critical case, that is, if $q < p$, (1.1) has a unique nonnegative global weak solution for all nonnegative initial values, and if $q > p$, there are both nonnegative, nontrivial global weak solutions and solutions which blow up in finite time. The blow-up result for $q > p$ is also proved in [14].

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Furthermore, in [24] the author proved that in the critical case $q = p > 2$, if the Lebesgue measure of Ω is sufficiently small, (1.1) has a global solution and if Ω is a sufficiently large ball, it has no global solution.

In this paper we shall give a complete picture of the blow-up criteria for (1.1). In particular, in the critical case $q = p > 2$, we will prove that if $\lambda > \lambda_1$, there are no nontrivial global weak solutions, and if $\lambda \le \lambda_1$, all weak solutions are global, where λ_1 is the first eigenvalue of the nonlinear eigenvalue problem

$$-\nabla(|\nabla\psi|^{p-2}\nabla\psi) = \lambda|\psi|^{p-2}\psi, \quad \text{in } \Omega; \quad \psi|_{\partial\Omega} = 0. \tag{1.2}$$

The following lemma concerns the properties of the first eigenvalue λ_1 and the first eigenfunction $\psi(x)$.

Lemma 1.1 *There exists a positive constant $\lambda_1(\Omega)$ with the following properties:*

- *(a) For any $\lambda < \lambda_1(\Omega)$, the eigenvalue problem* (1.2) *has only the trivial solution $\psi \equiv 0$.*
- *(b) There exists a positive solution $\psi \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ of* (1.2) *if and only if $\lambda = \lambda_1(\Omega)$.*
- *(c) The collection consisting of all solutions of* (1.2) *with $\lambda = \lambda_1(\Omega)$ is 1-dimensional vector space.*
- *(d) If Ω_j, $j = 1, 2$ are bounded domain with smooth boundary satisfying $\Omega_1 \Subset \Omega_2$, then $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$.*
- *(e) Let $\{\Omega_n\}$ be a sequence of bounded domains with smooth boundaries such that $\Omega_n \Subset \Omega_{n+1}$ and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$, then $\lim_{n\to\infty} \lambda_1(\Omega_n) = \lambda_1(\Omega)$.*

Proof (a)-(d) follow from [5, Lemma 2.1, 2.2]. The continuity of $\psi(x)$ is asserted in [22, Corollary 4.2]. We now prove (e). It follows from (d) that $\lambda_1(\Omega_n)$ is strictly decreasing and so it tends to some nonnegative constant $\lambda_1^*(\Omega)$ as $n \to \infty$. Denote by $\psi_n(x)$ the positive solution of (1.2) on Ω_n with $\lambda = \lambda_1(\Omega_n)$ such that $\int_{\Omega_n} \psi_n dx = 1$. By (c), ψ_n is unique. By the similar method in the proof of [5, Theorem 2.1], one can obtain from $\{\psi_n\}$ a positive solution ψ^* of (1.2) with $\lambda = \lambda_1^*(\Omega)$. Then by (b), we have $\lambda_1^*(\Omega) = \lambda_1(\Omega)$. $\diamondsuit$

We note that the blow-up conditions for (1.1) are similar to that of the porous media equations; see [6, 15, 16, 18]. Also our results clearly illustrate the observation that larger domains are more unstable than smaller domains; see [12].

To prove that $q = p$ is the critical case, we shall use the method of comparison with suitable blowing-up self-similar sub-solutions introduced by Souplet and Weissler [21]. This method enables us to treat the singular case $1 < p < 2$, which is not considered in [23, 24], as well as the degenerate case $p > 2$. Recently, the self-similar sub-solution method is proven to be useful in proof of blow-up theorems in the semilinear and porous media equations with gradient terms and nonlocal problems; see also [1, 17, 20]. This paper shows that this method can apply to the quasilinear problems with gradient diffusion. In the discussion of the critical case, we use a technique of comparison combined with the so-called "concavity" method, which is a different treatment with respect to the eigenfunction method for the porous media equations.

This paper is organized as follows: In the next section we consider comparison principles of the weak solutions of (1.1). In section 3 we first discuss the critical case $q = p > 2$. The last section is devoted to the proof of the blow-up results for (1.1) with large initial values.

2 Weak solutions and comparison principles

Following the book [4], we give the definition of the weak solutions of (1.1).

Definition 2.1 A weak sub(super)-solution of the Dirichlet problem (1.1) is a measurable function $u(x,t)$ satisfying

$$u \in C(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(\Omega_T),\ u_t \in L^2(\Omega_T)$$

and for all $t \in (0,T]$

$$\begin{aligned}&\int_\Omega u\varphi(x,t)dx + \int_0^t \int_\Omega \{-u\varphi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\} dx\, d\tau \\ &\leq (\geq) \int_\Omega u_0\varphi(x,0)dx + \lambda \int_0^t \int_\Omega |u|^{q-2}u\varphi\, dx\, d\tau\end{aligned}$$

for all bounded test functions

$$\varphi \in W^{1,p}(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(\Omega_T), \quad \varphi \geq 0.$$

A function u that is both a sub-solution and a super-solution is a weak solution of the Dirichlet problem (1.1).

It would be technically convenient to have a formulation of weak solutions that involves u_t. The following notion of weak sub(super)-solutions in terms of Steklov averages involves the discrete time derivative of u and is equivalent to (2.1),

$$\int_{\Omega\times\{t\}} \{u_{h,t}\varphi + [|\nabla u|^{p-2}\nabla u]_h \cdot \nabla\varphi - \lambda[|u|^{q-2}u]_h\varphi\} dx \leq (\geq) 0, \tag{2.1}$$

for all $0 < t < T - h$ and for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$. Moreover the initial datum is taken in the sense of $L^2(\Omega)$, i. e.,

$$(u_h(\cdot,0) - u_0)_{+(-)} \to 0, \quad \text{in } L^2(\Omega).$$

The Steklov average $u_h(\cdot,t)$ is defined for all $0<t<T$ by

$$u_h \equiv \begin{cases} \frac{1}{h}\int_t^{t+h} u(\cdot,\tau)d\tau, & t\in(0,T-h],\\ 0, & t>T-h. \end{cases}$$

The equivalence of (2.1) and (2.1) can be proven by the simple properties of Steklov averages.

Lemma 2.2 ([4, Lemma I.3.2]) *Let $v\in L^{q,r}(\Omega_T)$. Then let $h\to 0$, v_h converges to v in $L^{q,r}(\Omega_{T-\varepsilon})$ for every $\varepsilon\in(0,T)$. If $v\in C(0,T;L^q(\Omega))$, then as $h\to 0$, $v_h(\cdot,t)$ converges to $v(\cdot,t)$ in $L^q(\Omega)$ for every $t\in(0,T-\varepsilon)$, $\forall\varepsilon\in(0,T)$.*

The Hölder continuity of the above weak solution has been studied by many researchers in the past twenty years; see [4]. The following lemma is a special case.

Lemma 2.3 *For $p>1$, let u be a bounded weak solution of the Dirichlet problem* (1.1). *If $u_0\in C_0(\overline{\Omega})$, then $u\in C(\overline{\Omega_T})$. Moreover, let $T^*<\infty$ be the maximal existence time of u, then $\limsup_{t\to T^*}\|u(\cdot,t)\|_\infty=\infty$.*

The existence of the local weak solutions of the Dirichlet problem (1.1) can be proven by Galerkin approximations using the a priori estimates presented in the book [4, Theorem III.1.2 and Theorem IV.1.2]. For details for $p>2$, we refer to [24, Theorem 2.1].

To establish the comparison principle, we begin with a simple lemma that provides the necessary algebraic inequalities.

Lemma 2.4 *For all $\eta,\eta'\in\mathbb{R}^N$, there holds*

$$(|\eta|^{p-2}\eta-|\eta'|^{p-2}\eta')\cdot(\eta-\eta')\ge\begin{cases} c_2(|\eta|+|\eta'|)^{p-2}|\eta-\eta'|^2, & \textit{if } p>1,\\ c_1|\eta-\eta'|^p, & \textit{if } p>2, \end{cases}$$

where c_1 and c_2 are positive constants depending only on p.

For the detailed proof of this lemma, we refer to [2, Lemma 2.1].

Theorem 2.5 *Let $u,v\in C(\overline{\Omega_T})$ be weak sub- and super-solutions of* (1.1) *respectively and $u(x,0)\le v(x,0)$, then $u\le v$ in $\overline{\Omega_T}$.*

Proof We write (2.1) for u,v against the testing function

$$[(u-v)_h]_+(x,t)=\Big[\frac{1}{h}\int_t^{t+h}(u-v)(x,\tau)d\tau\Big]_+,$$

with $h\in(0,T)$ and $t\in[0,T-h)$. Differencing the two inequalities for u, v and integrating over $(0,t)$ gives

$$\begin{aligned} &\int_\Omega[(u-v)_h]_+^2(x,t)dx+2\int_0^t\int_\Omega[|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v]_h\cdot\nabla[(u-v)_h]_+dxd\tau\\ &\le\int_\Omega[(u-v)_h]_+(x,0)dx+2\lambda\int_0^t\int_\Omega[|u|^{q-2}u-|v|^{q-2}v]_h[(u-v)_h]_+dxd\tau. \end{aligned}$$

As $h \to 0$ the first term on the right tends to zero since $(u-v)_+ \in C(\overline{\Omega_T})$. Applying Lemma 2.2 and Lemma 2.4, we arrive at

$$\int_\Omega (u-v)_+^2(x,t)dx \le c_3 \int_0^t \int_\Omega (u-v)_+^2 dx d\tau.$$

The Gronwall's Lemma gives the desired result. ♢

In the following we consider the positivity of the weak solutions of the problem

$$\begin{gathered} v_t = \nabla(|\nabla v|^{p-2}\nabla v), \quad \text{in } \Omega \times \mathbb{R}_+, \\ v = 0, \quad \text{on } \partial\Omega \times \mathbb{R}_+, \\ v(x,0) = v_0(x) \ge 0, \quad \text{in } \Omega, \end{gathered} \tag{2.2}$$

where $p > 2$. Let

$$\begin{aligned} u_S(x-x_0, t-t_0) &= A_{p,N}[\tau + (t-t_0)]^{-N/[(p-2)N+p]} \\ &\quad \times \Big\{ \Big[a^{p/p-1} - \big(\frac{|x-x_0|}{[\tau + (t-t_0)]^{1/[(p-2)N+p]}} \big)^{p/(p-1)} \Big]_+ \Big\}^{(p-1)/(p-2)}, \end{aligned}$$

where

$$A_{p,N} = \big(\frac{p-2}{p}\big)^{(p-1)/(p-2)} \big\{ \frac{1}{(p-2)N+p} \big\}^{1/(p-2)},$$

$\tau > 0$, $a > 0$ are arbitrary constants. According to [19, p. 84], $u_S(x-x_0, t-t_0)$ satisfies the first equation of (2.2). Without loss of generality, we assume that $v_0(x) > 0$ in a ball $B(x_0, \delta_1)$. Let $\overline{x} \in \Omega$ be another point. In the following we prove that there exists a finite time $\overline{t}$ and a neighborhood $V_{\overline{x}}$ such that $v(x, \overline{t}) > 0$ in $V_{\overline{x}}$. Since Ω is connected, there exists a continuous curve $\Gamma : \gamma(s) \subset \Omega$, $0 \le s \le 1$, such that $\gamma(0) = x_0$ and $\gamma(1) = \overline{x}$. Denote $\delta_2 = \operatorname{dist}(\Gamma, \partial\Omega)$ and $\delta = \min\{\delta_1, \delta_2\}$. Let $x_1 = \Gamma \cap \partial B(x_0, \delta/2), \cdots, x_k = \Gamma \cap \partial B(x_{k-1}, \delta/2), \cdots$, such that $x_k \ne x_{k-2}$. It is clear that $\overline{x} \in B(x_n, \delta/2)$ for some n. Since $\overline{B(x_1, \delta/4)} \subset B(x_0, \delta)$, then $v_0(x) > 0$ in $\overline{B(x_1, \delta/4)}$. Choose suitable τ and a such that $\operatorname{supp} u_S \subset B(x_1, \delta/4)$ and $\|u_S\|_\infty \le \min_{x \in B(x_1, \delta/4)} v_0(x)$, then $u_S(x - x_1, t)$ is a weak sub-solution of (2.2) in $B(x_1, \delta)$. The comparison principle implies that there exists $\tau_1 > 0$ such that $v(x, \tau_1) > 0$ in $B(x_1, \delta)$. Thus $v(x, \tau_1) > 0$ in $B(x_2, \delta/2)$ since $B(x_2, \delta/2) \subset B(x_1, \delta)$. Repeating the above procedure, by finite steps, there exists a finite time $\overline{t}$ such that $v(x, \overline{t}) > 0$ in $B(x_n, \delta/2)$. The proof is completed. Thus we have the following lemma.

Lemma 2.6 *Assume that $v_0 \in C_0(\overline{\Omega})$ is nontrivial. Denote $\Omega_\rho = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \rho\}$. Let v be the weak solution of* (2.2). *Then there exists a finite time $t_\rho > 0$ such that $v(x, t_\rho) > 0$ in Ω_ρ.*

Proof It follows from the above proof that for any $x \in \Omega$, there exist $t_x > 0$ and a neighborhood $V_x \subset \Omega$ such that $v(x, t_x) > 0$ in V_x. Since $\bigcup_{x \in \Omega} V_x \supset \overline{\Omega_\rho}$, by the finite covering theorem, $\overline{\Omega_\rho} \subset \bigcup_{i=1}^n V_{x_i}$. Put $t_\rho = \max\{t_{x_1}, \cdots, t_{x_n}\}$. This lemma is proved. ♢

3 The critical case $q = p > 2$

Since in [23, 24], the authors have been established that $q = p > 2$ is the critical case of (1.1), we first consider what happens if $q = p$. Zhao showed in [24] that if the Lebesgue measure of Ω is sufficiently small, (1.1) has a global solution and if Ω is a sufficiently large ball, it has no global solution. In this section we shall prove that if $q = p > 2$, the crucial role is played by the first eigenvalue λ_1 of the eigenvalue problem (1.2), as in the porous media equations.

First we consider the global existence case $\lambda \le \lambda_1$.

Theorem 3.1 *Assume that $u_0 \in C_0(\overline{\Omega})$ and $q = p > 2$. If*

$$\lambda < \lambda_1, \tag{3.1}$$

then the unique weak solution of (1.1) *is globally bounded.*

Proof Since $\lambda < \lambda_1$, by Lemma 1.1, there exists $\Omega_\varepsilon \Supset \Omega$ such that $\lambda < \lambda_{1,\varepsilon} < \lambda_1$. Let $\psi_\varepsilon(x)$ be the first eigenfunction with $\sup_{x\in\Omega} \psi_\varepsilon(x) = 1$ of the eigenvalue problem (1.2) with $\Omega = \Omega_\varepsilon$. Choose K to be so large that $u_0(x) \le K\psi_\varepsilon(x) \equiv v(x)$. For all $0 < t < T - h$ and for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\varphi \ge 0$,

$$\begin{aligned}
&\int_{\Omega\times\{t\}} \{v_{h,t}\varphi + [|\nabla v|^{p-2}\nabla v]_h \cdot \nabla\varphi - \lambda[|v|^{p-2}v]_h\varphi\}dx \\
&= \int_\Omega \{|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi - \lambda|v|^{p-2}v\varphi\}dx \\
&= (\lambda_{1,\varepsilon} - \lambda)\int_\Omega |v|^{p-2}v\varphi dx \ge 0.
\end{aligned}$$

Hence $v(x) = K\psi(x)$ is a weak super-solution of (1.1) in terms of Steklov averages. The comparison principle implies this theorem. $\diamondsuit$

Remark 3.2 The global existence is still true for $\lambda = \lambda_1$ if u_0 satisfies the stronger assumption that $u_0 \le K\psi(x)$ for $K > 0$ large.

Remark 3.3 Theorem 3.1 and Remark 3.2 hold for mixed sign solutions as well. To see this, just use $-K\psi_\varepsilon$ in Theorem 3.1 and $-K\psi$ in Remark 3.2 as weak subsolutions of (1.1).

Now we consider the blow-up case $\lambda > \lambda_1$. In [24, Theorem 4.1], using the so-called "concavity" method, the author showed that if $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\mathcal{E}(u_0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p dx - \frac{\lambda}{p}\int_\Omega |u_0|^p dx < 0, \tag{3.2}$$

then there exists $T^* < \infty$ such that

$$\lim_{t\to T^*} \| u(\cdot,t) \|_{L^\infty(\Omega)} = \infty. \tag{3.3}$$

See also [13]. The result is crucial in the proof of the blow-up case $\lambda > \lambda_1$. The following lemma reproves the result using another version of the "concavity" argument.

Lemma 3.4 *Assume that* $u_0 \in W_0^{1,p}(\Omega) \cap C_0(\overline{\Omega})$ *satisfies* (3.2)*, then* (3.3) *holds.*

Proof Unlike in the usual "concavity" argument, we put

$$\mathcal{H}(t) = \frac{1}{2}\int_\Omega u^2 dx.$$

Taking u and u_t as testing functions in the weak formulation of (1.1), modulo a Steklov average, gives

$$\begin{aligned} \frac{d}{dt}\mathcal{H}(t) &= -p\mathcal{E}(u), \quad \text{in } \mathcal{D}'(\mathbb{R}_+), \\ -\frac{d}{dt}\mathcal{E}(u) &= \int_\Omega (u_t)^2 dx, \quad \text{in } \mathcal{D}'(\mathbb{R}_+). \end{aligned} \tag{3.4}$$

Differentiating (3.4), we have

$$\frac{d^2}{dt^2}\mathcal{H}(t) = -p\frac{d}{dt}\mathcal{E}(u), \quad \text{in } \mathcal{D}'(\mathbb{R}_+).$$

Note that

$$\frac{d}{dt}\mathcal{H}(t) = \int_\Omega u u_t dx, \quad \text{in } \mathcal{D}'(\mathbb{R}_+).$$

Then using the Hölder inequality, we have

$$\frac{p}{2}\Big[\frac{d}{dt}\mathcal{H}(t)\Big]^2 = \frac{p}{2}\Big[\int_\Omega u u_t dx\Big]^2 \le \frac{p}{2}\int_\Omega u^2 dx \int_\Omega (u_t)^2 dx = \mathcal{H}(t)\frac{d^2}{dt^2}\mathcal{H}(t),$$

in $\mathcal{D}'(\mathbb{R}_+)$, which implies

$$\frac{d^2}{dt^2}\mathcal{H}^{1-\frac{p}{2}}(t) \le 0, \quad \text{in } \mathcal{D}'(\mathbb{R}_+).$$

It follows that $T^* < \infty$. Indeed, otherwise, taking into account (3.2) and the continuity of $\mathcal{H}(t)$, there exists $T < \infty$ such that $\lim_{t\to T}\mathcal{H}(t) = \infty$: a contradiction. The proof is completed. $\diamondsuit$

The following theorem follows from the above lemma.

Theorem 3.5 *For* $q = p > 2$*, the unique weak solution of the Dirichlet problem* (1.1) *with nontrivial, nonnegative* $u_0 \in C_0(\overline{\Omega})$ *blows up in finite time provided that*

$$\lambda > \lambda_1. \tag{3.5}$$

Proof Let $\psi(x) > 0$ be the first eigenfunction of the eigenvalue problem (1.2) with $\max_{x\in\Omega}\psi(x) = 1$. Then we have, for any $k > 0$,

$$\mathcal{E}(k\psi) = \frac{1}{p}\int_\Omega |\nabla(k\psi)|^p dx - \frac{\lambda}{p}\int_\Omega (k\psi)^p dx = k^p\frac{\lambda_1 - \lambda}{p}\int_\Omega \psi^p dx < 0.$$

Therefore, by Lemma 3.4, the solution of (1.1) with the initial datum $k\psi(x)$ blows up in finite time. Given any nontrivial initial datum $u_0(x) \geq 0$, denote by T^* the maximal existence time of the weak solution of (1.1). Suppose by contradiction that $T^* = \infty$. Combining (3.5) with Lemma 1.1, there exists $\Omega_\rho \Subset \Omega$ such that $\lambda > \lambda_{1,\rho} > \lambda_1$. By Lemma 2.6 and the comparison principle, there exists $t_\rho > 0$ such that

$$u(x, t_\rho) > 0, \quad x \in \overline{\Omega_\rho}. \tag{3.6}$$

Consider the problem (1.1) in Ω_ρ with the initial datum $k\psi_\rho$, where ψ_ρ is the first eigenfunction of (1.2) in Ω_ρ with $\max \psi_\rho = 1$. We know that the weak solution $u_\rho(x, t)$ blows up in finite time for any $k > 0$. Choose k so small that $u(x, t_\rho) \geq k\psi_\rho$ in Ω_ρ, then a contradiction follows from the comparison principle. The theorem is proved. $\diamondsuit$

4 Global nonexistence for large initial values

In [24], the author used the so-called "concavity" method to prove that if $q > p > 2$, the unique weak solution of (1.1) blows up in finite time if $\mathcal{E}(u_0) < 0$. In this section we use the method of comparison with suitable blowing-up self-similar sub-solution to give a uniform treatment for all $p > 1$. In the following theorem we construct a suitable blowing-up self-similar subsolution.

Theorem 4.1 *Assume that $q > p > 1$ and $q > 2$. Given a nonnegative, nontrivial initial datum $u_0 \in C_0(\overline{\Omega})$, there exists $\mu_0 > 0$ (depending only upon u_0) such that for all $\mu > \mu_0$, the weak solution $u(x, t)$ of the Dirichlet problem* (1.1) *with initial data μu_0 blows up in a finite time T^*. Moreover, there is some $C(u_0) > 0$ such that*

$$T^*(\mu u_0) \leq \frac{C(u_0)}{\mu^{p-1}}, \quad \mu \to \infty. \tag{4.1}$$

Proof We seek an unbounded self-similar sub-solution of (1.1) on $[t_0, 1/\varepsilon) \times \mathbb{R}^N$, $0 < t_0 < 1/\varepsilon$, of the form

$$v(x, t) = \frac{1}{(1 - \varepsilon t)^k} V\Big(\frac{|x|}{(1 - \varepsilon t)^m}\Big), \tag{4.2}$$

where $V(y)$ is defined by

$$V(y) = \Big(1 + \frac{A}{\sigma} - \frac{y^\sigma}{\sigma A^{\sigma-1}}\Big)_+, \quad \sigma = \frac{p}{p-1},\ y \geq 0, \tag{4.3}$$

with $A, k, m, \varepsilon > 0$ and t_0 to be determined. First note that $\forall t \in [t_0, 1/\varepsilon)$,

$$\operatorname{supp}(v(\cdot, t)) \subset \overline{B}(0, R(1 - \varepsilon t_0)^m), \tag{4.4}$$

with $R = (A^{\sigma-1}(\sigma + A))^{1/\sigma}$. We compute (by setting $y = |x|/(1 - \varepsilon t)^m$ for convenience),

$$\begin{aligned}
Pv &= v_t - \nabla(|\nabla v|^{p-2}\nabla v) - \lambda |v|^{q-2} v \\
&= \frac{\varepsilon(kV(y) + myV'(y))}{(1-\varepsilon t)^{k+1}} - \frac{(|V'(y)|^{p-2}V'(y))' + (N-1)|V'(y)|^{p-2}V'(y)/y}{(1-\varepsilon t)^{(k+m)(p-1)+m}} \\
&\quad - \frac{\lambda}{(1-\varepsilon t)^{k(q-1)}} V^{q-1}(y).
\end{aligned}$$

It is easy to verify that

$$\begin{gathered}
1 \le V(y) \le 1 + \frac{A}{\sigma}, \quad -1 \le V'(y) \le 0, \quad \text{for } 0 \le y \le A, \\
0 \le V(y) \le 1, \quad -\frac{R^{\sigma-1}}{A^{\sigma-1}} \le V'(y) \le -1, \quad \text{for } A \le y \le R, \\
(|V'(y)|^{p-2}V'(y))' + (N-1)|V'(y)|^{p-2}V'(y)/y = -\frac{N}{A}\chi_{\{y<R\}} + \frac{R}{A}\delta_{\{y=R\}},
\end{gathered} \tag{4.5}$$

where χ is the indicator function. We choose

$$\begin{gathered}
k = \frac{1}{q-2}, \quad 0 < m < \frac{q-p}{p(q-2)}, \\
A > \frac{k}{m}, \quad 0 < \varepsilon < \frac{\lambda}{k(1 + A/\sigma)}.
\end{gathered}$$

For $t_0 \le t < 1/\varepsilon$ with t_0 sufficiently close to $1/\varepsilon$, we have, in the case $0 \le y \le A$,

$$Pv(x,t) \le \frac{\varepsilon k(1 + A/\sigma) - \lambda}{(1-\varepsilon t)^{k+1}} + \frac{N/A}{(1-\varepsilon t)^{(k+m)(p-1)+m}} \le 0.$$

In the case $A \le y < R$, we get

$$Pv(x,t) \le \frac{\varepsilon(k - mA)}{(1-\varepsilon t)^{k+1}} + \frac{N/A}{(1-\varepsilon t)^{(k+m)(p-1)+m}} \le 0.$$

Obviously, we also have $Pv \equiv 0$ for $y > R$. Since $v(x,t)$ is continuous and piecewise C^2 and due to the sign of the singular measure in (4.5) , then $v(x,t)$ is a local weak sub-solution of the Dirichlet problem (1.1).

Now by translation, one can assume without loss of generality that $0 \in \Omega$ and $u_0(0) = \max_{x\in\Omega} u_0(x)$. It follows from the continuity of u_0 that

$$u_0(x) \ge C, \quad \text{for all } x \in B(0,\rho),$$

for some ball $B(0,\rho) \Subset \Omega$ and some constant $C > 0$. Taking t_0 still closer to $1/\varepsilon$ if necessary, one can assume that $B(0, R(1-\varepsilon t_0)^m) \subset B(0,\rho)$. Therefore,

$$\mu u_0(x) \ge \mu C \ge \frac{V(0)}{(1-\varepsilon t_0)^k} \ge v(x,t_0), \quad x \in \Omega, \tag{4.6}$$

for all $\mu > \mu_0 = V(0)/C(1-\varepsilon t_0)^k$. By the Theorem 2.5, it follows that

$$u(x,t) \geq v(x,t+t_0), \quad x \in \Omega,\ 0 < t < \min\{T^*, \frac{1}{\varepsilon} - t_0\}.$$

Hence $T^* \leq 1/\varepsilon - t_0$.

To prove (4.1), given $\mu > V(0)/C(1-\varepsilon t_0)^k$, by the previous calculation, whenever $t_0 \leq T < 1/\varepsilon$ such that $\mu \geq V(0)/C(1-\varepsilon T)^k$, we have $T^*(\mu u_0) \leq 1/\varepsilon - T$. Then

$$T^*(\mu u_0) \leq \frac{1}{\varepsilon}\left(\frac{1+A/\sigma}{\mu C}\right)^{q-2}, \quad \text{for all } \mu \geq \frac{V(0)}{C(1-\varepsilon t_0)^{1/(q-2)}}.$$

The proof is completed. ♢

Under the conditions of the above theorem, the solutions of (1.1) exist globally for small initial data.

Theorem 4.2 *Assume that $q > p > 1$ and $q > 2$. There exists $\eta > 0$ such that the solution of* (1.1) *exists globally if $\|u_0\|_\infty < \eta$.*

Proof Let $\Omega_\varepsilon \Supset \Omega$ be a bounded domain and ψ_ε be the first eigenfunction of (1.2) on Ω_ε with $\sup_{x\in\Omega}\psi_\varepsilon(x) = 1$. Denote $\delta = \inf_{x\in\Omega}\psi_\varepsilon(x)$. Choose $k^{q-p} = \lambda_1/\lambda$ and $\eta = k\delta$. A direct computation yields that $k\psi_\varepsilon(x)$ and $-k\psi_\varepsilon(x)$ is a weak super- and sub-solution of (1.1) respectively. This theorem follows the comparison principle. ♢

Theorem 4.3 *Assume that* $2 < q < p$. *Then the solution of* (1.1) *exists globally for any initial datum.*

Proof The proof is very similar to the above. Let $\Omega_\varepsilon \Supset \Omega$ be a bounded domain and ψ_ε be the first eigenfunction on Ω_ε with $\inf_{x\in\Omega}\psi_\varepsilon(x) = 1$. We choose the super- and sub-solution to be $K\psi_\varepsilon(x)$ and $-K\psi_\varepsilon(x)$ for K so large that $\|u_0\| \leq K$ in Ω. ♢

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