AN INVESTIGATION OF THE RELATIONSHIPS BETWEEN CONCEPTUALIZATION OF LIMITS AND PROOF COMPREHENSION

by

Christine A. Herrera, M.S., B.S.

A dissertation submitted to the Graduate Council of
Texas State University in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
with a Major in Mathematics Education
May 2016

Committee Members:

M. Alejandra Sorto, Chair

Alexander White

Sharon Strickland

Beth Cory

FAIR USE AND AUTHOR'S PERMISSION STATEMENT

Fair Use

Duplication Permission

ACKNOWLEDGEMENTS

I first would like to thank the Gates Millennium Scholars Program for supporting my journey in higher education these past ten years. Their investment in my education demolished financial barriers and allowed me to chase my dreams. Also, this research was partially supported by the National Science Foundation CAREER grant DRL-1055067.

Secondly, I would like to thank my Master's advisor, Dr. Paul Kennedy. Without his guidance and support I would have never had the courage to leave Colorado and pursue a doctorate degree. Dr. Kennedy's ability to believe me, and see my potential changed my life forever. A mere thank you cannot express how grateful I am.

Next I would like to thank my dissertation chair and committee members. Your encouragement and insight these past four years has been invaluable. You all are the best! Thank you all for listening to every incoherent idea had, providing me with advise, and sharing your knowledge and expertise with me. Thank you!

Lastly, thank you to my family and friends. To my eldest brother, Brandan, I cannot thank you enough for the help you gave me during the darkest times of my life. To my dad, who's face lights up every time he sees me, thank you for loving me and being my biggest cheerleader. To my best friends in Colorado, who are more like family than friends, I would have not survived it Texas without all the love and support you sent my way! To my amazing friends here in Texas, thank you for being the most remarkable group of ladies I have ever met. Especially my awesome officemate, Shawnda Smith and

writing partner, Christina Starkey, you two are the best.

TABLE OF CONTENTS

LIST OF TABLES

Table **Page**

LIST OF FIGURES

Figure **Page**

ABSTRACT

This study investigates eighteen Real Analysis students' informal and formal understanding of the mathematical concept of limits and the relationship on their comprehension of limit proofs. This study utilizes Tall and Vinner's (1981) notion of concept image and concept definition. The framework examined students' conceptualization of limit by eliciting their mental images, associated properties, processes, and example space (Watson & Mason, 2005). The study analyzed surveys on the conceptualizations of limits of sequences and limits of functions, class observations, and task-based interviews to explore the different varieties of conceptual understanding held by Real Analysis students. From the data emerged two cognitive categorization of participants' thinking. Those whose concept image held serious potential conflict factors with the formal definitions of limits and those who had resolved their serious potential conflict factors by the end of the semester, called the *cognitive conflict* and *cognitive resolution* groups, respectively.

The students were given an end-of-semester proof comprehension assessment that was designed based on Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff's (2012) model for assessments for advanced mathematics' proof comprehension. The analysis showed that Real Analysis students with cognitive resolution had a better local understanding of limit proofs. However, both cognitive groups had difficulty generating examples that illustrated the main ideas of the limit proofs.

CHAPTER 1

INTRODUCTION

Background

Prior to advanced mathematics courses, such as Real Analysis, students encounter different levels of proof from the beginning of pre-kindergarten up to high school courses like geometry (CCSS, 2010; NCTM, 2000). The exposure to proof throughout students' academic careers gradually moves them from elementary to advanced mathematical thinking. During this transition, students' thinking progresses from "describing to defining, from convincing to proving in a logical manner based on definition" (Tall, 1991, p. 20). As a result of this progression, students are expected to comprehend formal definitions, axiomatic structures, and theorems about abstract mathematical entities. Ultimately, students at the undergraduate level must meet higher the standards of thinking, mathematical language, and proof to be successful in a Real Analysis course.

Despite the scaffolding and presence of argumentation and validation throughout grade levels, many students begin their upper-level mathematics courses "having no general perspective of proof or methods of proof" (Moore, 1994, p. 249). Undergraduate students labor with proof and "typically cannot comprehend proofs" (Weber, 2002, p. 14). Overall, their performance in proofs is weak (Harel & Sowder, 2007). Current literature proposes a variety of areas that can cause potential difficulties with proofs for undergraduates in advanced mathematics courses.

When learning proof, undergraduate students often encounter difficulties in the following areas:

- Logical reasoning and methods of proof are typically taught in transition-to-proof courses to aid students in their studies in advanced mathematics courses. Students do not always fully develop accurate reasoning and therefore struggle with using deductive reasoning to construct or validate proofs (e.g. Alcock, Bailey, Inglis, & Docherty, 2014; Stylianides & Stylianides, 2009).
- Students' perceptions of what makes a mathematical argument considered a valid proof can be a road block for students when learning proofs. (E.g. Harel & Sowder, 1998; Raman, 2003; Weber, 2010).
- Understanding the purpose of a proof potentially prohibit students' learning because they may not understand the benefits of learning mathematical arguments that they perceive to be trivial. Similarly, students encounter obstacles to learning proofs when they do not understand what knowledge they are to retain (e.g. Hanna, 1990; Weber, 2002).
- Mathematical language, which can include symbols and unfamiliar vocabulary, can present students with a variety of challenges such as translating the mathematical symbols into colloquial terms to decipher the mathematical meaning from everyday connotations of words (Laborde, 1990; McGee, 1997; Tall, 1992).

An additional difficulty some undergraduate students struggle with understanding a mathematical concept beyond an intuitive basis built on experience and computational understanding. Students must gain a formal understanding of concepts based on formal mathematical definitions rather than informal definitions. Applying their formal understanding of concepts accordingly in learning proofs is a difficult task for undergraduate students (Moore, 1994; Tall and Vinner, 1981; Weber, 2001). Current literature focuses on learning proofs is a multifaceted task dependent on “a complex constellation of beliefs, knowledge, and cognitive skills” (Moore, 1994, p. 250).

It is not clear which of these factors proves most relevant to understanding proofs in Real Analysis course but it is expected for students to learn the formal definitions and utilize them when writing and reading proofs. Limits, for example, are an important topic that students are expected to learn in order to read and write proofs in a Real Analysis course. Some evidence suggests that students conceptually struggle not only with the formal definition in a Real Analysis course but also informally in the prerequisite calculus courses (Davis and Vinner, 1986; Monaghan, 1991; Patel, McCombs, & Zollman, 2014; Roh, 2008; Szydlik, 2000; Williams, 1991). Thus, in Real Analysis courses, undergraduates must overcome understanding both the concept of a limit and mathematical proofs.

This study focuses on conceptualizing limits in relation to proof comprehension in a Real Analysis course, rather than on all the components of learning proofs (e.g. proof writing, proof methods, etc.). Proof comprehension is vital in advanced mathematics courses because undergraduates spend a significant amount of time reading them. The task of reading proofs provides students with the opportunity to develop their understanding of mathematical proofs (Mejia-Ramos & Weber, 2014). Yet, this component of learning proof has not been extensively studied in undergraduate mathematics courses, such as Real Analysis (Mejia-Ramos et al., 2012). Determining how students' conceptualization of limits relates to their understanding of reading proofs not only addresses the existing gap in the literature, but better informs the mathematical community about how the different conceptual levels of understanding can potentially impact proof comprehension.

Purpose of the Study

The purpose of this study will be to describe students' conceptual understanding of limits in terms of concept image and concept definition, as well as explore the relationship between students' concept image and concept definition of limits and students' proof comprehension. The analysis of the role students' concept image and concept definition has on their limit-proof comprehension will be explored using grounded theory. A person builds their cognitive structure of a concept over time through a variety of experiences that occur in and out of academic settings (Tall & Vinner, 1981). In order to gain the best description possible of Real Analysis students' concept image and concept definition, a collection of evoked concept images will be used to describe students' concept image and concept definition. The collection of evoked concept images will be analyzed to describe the students' concept image and concept definition at the end of the limits of sequences and limits of functions units, and determine how aligned the students' concept image and concept definition are to the formal definitions. To gather this rich description of the students' concept image and concept definition, six Real Analysis students will be interviewed to ensure measurement reliability. The data collection for concept image and concept definition will include surveys, task-based interviews, and observations of instruction of limit-proofs.

Research Questions

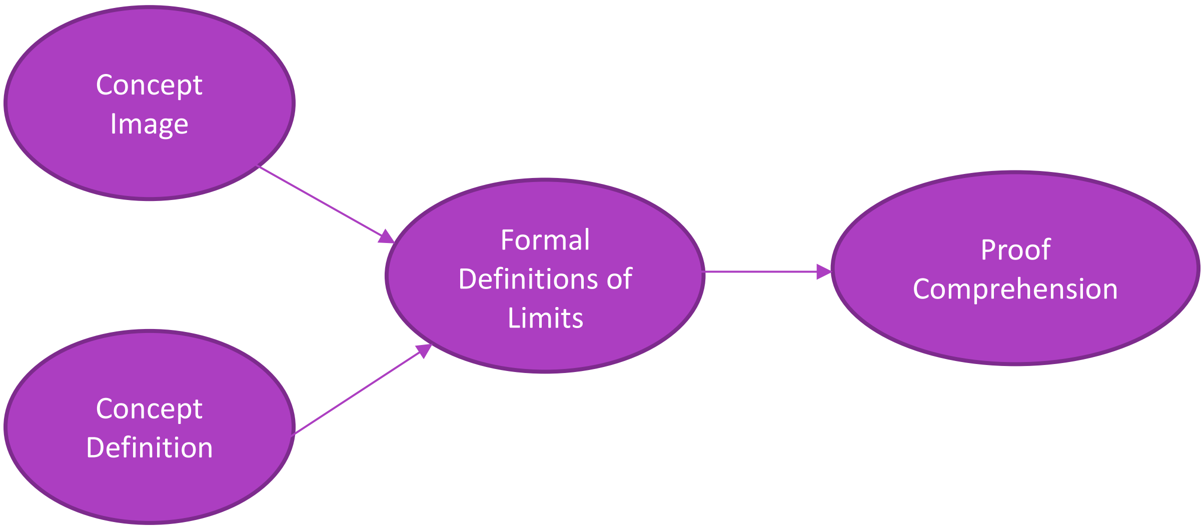


Figure 1. Model for Real Analysis students' comprehension of limit proofs.

The research questions for this study are as follows:

1. What are the concept images and concept definitions of limits held by Real Analysis students?
2. How does students' concept images and concept definitions of limits relate to their understanding of the formal definitions of limit?
3. How do the students' concept images relate to their comprehension of limit proofs?

Significance of the Study

An important concept in calculus are limits (Dawkins, 2012; Keene, Hall & Duca, 2014; Patel, McCombs, & Zollman, 2014; Tall, 1992). Typically, calculus students learn an informal conceptualization of limits that focuses on the process, properties and calculating limits. Therefore, students use different approaches, such as metaphors, to better understand the abstraction of limits of functions and limits of sequences (Keene, Hall & Duca, 2014; Patel, McCombs, & Zollman, 2014). Students' informal understanding of limits can cause conflicts when faced with the task of understanding the formal definitions of limits (Monaghan, 1991; Oehrtman, 2008; Roh, 2008; Tall and Vinner, 1981). The concept of limits is formally taught in advanced mathematics courses such as Real Analysis, and is potentially where conflicts between students' informal definitions of limits and formal definitions of limits occur. In a Real Analysis course students utilize these set definitions in the task of reading and writing proofs.

The task of proof in advanced mathematics courses is in and of itself a challenge for students (Harel & Sowder, 2007; Mejia-Ramos & Inglis, 2009; Stylianides & Stylianides, 2009; Weber, 2007, 2001). Students in Real Analysis courses face the challenge of using their formal understanding of limits while learning proofs. However, while there is literature that focuses solely on Real Analysis students' conceptual understanding of limits (Dawkins, 2012, Alcock & Simpson, 2002) and on Real Analysis students' proof ability (Alcock & Weber, 2005), current research does not focus on the relationship between the two. Therefore, there is the need to further investigate these two challenges Real Analysis students simultaneously encounter. Understanding how students face these challenges will better inform the role students' conceptualization of an advanced mathematical concept plays in proof comprehension.

This study will capture students' informal and formal understanding of limits in terms of their concept image and concept definition and how it relates to their proof comprehension in the proof-based Real Analysis course. Specifically, this study will attempt to understand the dynamics of students' concept image and concept definition of limits and the proof comprehension. Focusing on proof comprehension will contribute to the "limited research on what it means to understand mathematical proof at [the undergraduate] level” (Mejia-Ramos, Fuller, Weber, Rhoads, Samkoff, 2011; Mejia-Ramos and Inglis, 2009). Thus, this research will ultimately help inform mathematics educators about the role of undergraduates’ concept image and concept definition on limit proof comprehension in Real Analysis courses.

Definitions of Terms

Some of the terms used may have several interpretations and require clarification. The following definitions of those terms are provided to give clarity to the reader.

Formal definition of a limit of a function: The study will use the following $\varepsilon - \delta$ definition: The statement $\lim_{x \to c} f(x) = L$, means if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for all x, $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Formal definition of a limit of a sequence: The study will use the following $\varepsilon - N$ definition: The statement $\lim_{n\to\infty} a_n = A$, means if for any positive number ε, there is a natural number N such that $|a_n - A| < \varepsilon$ for all $n \geq N$. The limit of a sequence is also referred to as convergence.

Evoked Concept Image: This study will use Tall and Vinner’s (1981) definition of evoked concept images. They describe an evoked concept image to be the activated part of “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” at a certain time.

Concept Image: Tall and Vinner (1981) define concept image to be “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures” (p. 2). This study will not be able to focus on the entire total cognitive structure associated with the concept; thus concept image will refer to the collection of evoked concept images.

Concept Definition: In this study, concept definition will refer to the formal definition that a participant states and/or writes. The concept definition should explain the concept in a noncircular way.

Example: This study uses Watson and Mason's (2005) definition of example as any mathematical object from which it is expected to generalize. A mathematical object (Alcock, 2010) satisfies the definition of some concept.

Example Space: This study will use Watson and Mason's (2005) definition of example space. An individual's potential example space consists of a collection of examples and methods for generating examples that are drawn from a person's past experiences (both explicitly remembered and not). A subset of a potential example space that is triggered by a task, cues, environment, and recent experience, are the examples that compose an individual's example space.

Proof: This study will primarily refer to Weber's (2014) cluster concept definition of proof. Weber defines proof on the five categories that a proof is an argument: (a) that is deductive and non-ampliative, (b) that would convince a contemporary mathematician who knew the subject, (c) in a natural language and symbolic representation system where there are socially sanctioned rules of inference, (d) that convinces a particular community at a particular time, and (e) that is a blue print that knowledgeable mathematicians can use, in principle, to write a complete proof with no logical gaps.

Concept Usage: Based on Moore's (1994) idea, concept usage will be defined by how a student operates definitions and theorems in the process of writing and/or reading a proof.

Proof Comprehension: This study will refer to proof comprehension as the action or capability of understanding components of a proof and/or the entirety of the proof.

CHAPTER 2

LITERATURE REVIEW

Introduction

The purpose of the study is to investigate students' concept image of limits and how the students' concept image of limits relates to the students' comprehension of limit proofs in advanced mathematics courses. The study will answer the following research questions: (a) What are the student's concept images and concept definitions of limits? (b) How does the student's concept images and concept definitions of limits relate to their understanding of the formal definitions of limit? (c) How does the students' concept image relate their comprehension of limit proofs?

This chapter presents an overview of the research related to this topic to provide the foundation and justification for the study. The first section will present theories on the cognitive growth in advanced mathematical thinking and will discuss how students construct the mathematical concept of limits. This section highlights the research on the difficulties students encounter when they learn limits in calculus and when they transition from an informal understanding of limits to a formal understanding. The transition requires students to shift their thinking of mathematical concepts based on their informal experience with limits to formal definitions and properties. The next section will review the different types of difficulties students face with limits in the advanced mathematics proof-based course Real Analysis (Domingos, 2010). The last section will provide a brief introduction into what constitutes a proof in advanced mathematics (Weber, 2014), and research on proof comprehension in Real Analysis courses (Alcock & Weber, 2005).

Cognitive Theories

As students transition from Calculus to Real Analysis, their mathematical thinking of concepts transitions from being less formal, computationally focused, to one formulated around formal definitions and deductive reasoning. This change is a cognitive growth from elementary to advanced mathematical thinking. There is debate about whether the advanced in the phrase advanced mathematical thinking refers to the type of mathematics or the type of thinking (Harel, Selden, & Selden, 2006). Harel and Sowder (2005) distinguish the difference between "advanced-mathematical thinking" and "advanced mathematical-thinking." "Advanced-mathematical thinking" is thinking that occurs in advanced mathematics. "Advanced mathematical-thinking" is mathematical thinking of an advanced nature. This study will take on the latter definition, and thus conjecture that a developmental process occurs. This is an individualistic process that occurs mainly on a cognitive level. To be advanced in this process is not absolute, but is relative to the individual's way of thinking. Harel and Sowder (2005) define the way of thinking as "what governs one's way of understanding, and thus expresses reasoning that is not specific to one particular situation but to a multitude of situations" (p. 31).

The three worlds of mathematics. The notion of advanced mathematical thinking has given rise to different cognitive theories. Tall (1995, 2004, 2013) describes the development of mathematical thinking as an evolution of three worlds of mathematics. In the first world development begins from our perceptions of the world, both in the physical and mental. These objects are first experienced as visuo-spatial structures and upon the reflection of the object's properties are then classified based on their observed properties.

The second world is composed of the symbols that are used in mathematical actions, such as calculations and manipulations. Within these actions emerges the duality of intellectualizing the symbol as a concept (object) and using the symbols to do mathematics (the process). Students typically first learn a concept as a process, then the product of that process is symbolized in the same fashion, thus the symbol takes on the dual meaning of both process and product (object). Gray and Tall (1991) coined the term procept for the blend of process and concept (object).

The journey from the second world to the third world requires significant reconstruction in one's thinking. "The third world is based on properties, expressed in terms of formal definitions that are used as axioms to specify mathematics structures (e.g. groups and topological spaces)" (Tall, 2004, p. 28). This world is also known as the formal world of mathematics. Within this world, more properties can be deduced from proofs that are constructed from theorems. These axiom structures formulate new concepts built upon logically deduced theories. Harel, Selden, and Selden (2006) note that the essential difference of elementary and advanced mathematical thinking is the introduction of formal definition and proof, which according to this theory happens in the third world.

Process-object theory. Similar to the process-object theory described in the second world of symbolism is the operational-structural theory (Sfard, 1987, 1991, 1992). The distinction is that the structural perspective is more abstract and is shaped by the formal definition. The conceptualization of the structural concept is developed in three hierarchal stages: interiorization, condensation, and reification. The interiorization phase is where the students develop computational and procedural skills, which eventually give origin to a new concept. In the second stage, condensation, the student compiles a sequence of operations and develops an understanding of the process as a whole. The final segment is where the student is able to conceive a finished object of these manipulations. The reification stage is a sudden understanding of the finished object as a mathematical entity with meaning.

APOS theory. A third theory of concept acquisition constructed from the Piagetian theory of reflective abstraction is APOS, which is the four kinds of mental conceptions: action, process, object, and schema (Dubinsky, 1991; Dubinsky, Hawks, & Nichols, 1989). Action is composed of four kinds of mental constructions: acts of interiorization, coordination, encapsulation, and generalization. Dubinsky et al. define interiorization is the translation of a succession of mental actions into a repeatable whole. Coordination is constructing a new process from existing processes. Similar to reification is the translation of a dynamic process into a static object called encapsulation. The application of an existing schema to a wider collection of phenomena is generalization (Dubinsky, 1991). The mental conception of process and object are similar to procept. A schema is an individual mental construction connecting related processes.

Theory of conceptual change. The first three theories presented have the underlying theme of process and object. However, there are other kinds of theories of knowledge acquisition relevant to advanced mathematical thinking. One is the theory of conceptual change, which examines the process of obtaining knowledge and focuses on instances where incompatibilities occur between prior and new knowledge (Biza, Souyoul, & Zachariades, 2005). The process of obtaining knowledge begins with children creating an informal framework theory to understand the world around them. This explanatory framework is constructed as a coherence of ontological and epistemological presuppositions that are influenced from everyday experiences. These experiences, combined with preconceptions, transform their framework theory into specific theories. These specific theories are what influence the acquisition of new knowledge and cause cognitive problems.

Conceptual change theory concentrates on the revisions made on students' specific theories and interprets the difficulties and misconceptions that arise from the incompatibilities between the new and prior knowledge. During these revisions, students create synthetic models to assimilate the new information. Synthetic models are conceptual models, which are mental representations that they generate when there is a problematic situation present during a cognitive operation. These models are a mixture of individual beliefs and the scientific knowledge concerning the same notion. Conceptual change provides student-centered explanations about knowledge acquisition concerning counterintuitive mathematical concepts.

Concept image and concept definition. A different perspective on how students construct understanding of concepts is the distinction between how the mathematical concepts are defined (concept definition) and an individual's mental structure of the concept (concept image). The distinction between the two was first made by Vinner and Hershowitz (1980), and then was further elaborated on by Tall and Vinner in 1981.

Tall and Vinner (1981) recognized that the total cognitive structure that constructs the meaning of a concept is complex. It is more than imagery, diagrams, graphs, examples, symbols, and words. It is also formed by interactions one has with the concept.

Tall and Vinner defined concept image to “describe the total cognitive structure that is associated with the concept, which includes all the mental pictures, associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures” (Tall & Vinner, 1981, p. 2).

Tall and Vinner (1981) explained that the concept image is formed when the sensory input excites certain neuronal pathways and inhibits others. Different stimuli will activate different parts of the concept image. Thus, a person’s concept image is complex and not necessarily coherent at all times. The concept image that is activated at a specific time is called an evoked concept image; “it is desirable to have and evoke rich concept images” when completing mathematical tasks (Harel, Selden, & Selden, 2006). In contrast, a concept definition “refers to a formal verbal definition that accurately explains the concept in a noncircular way” (Moore, 1994, p. 252). It is to be noted that there is a concept definition can differ from a formal definition. A personal concept definition is one that is created by the student based on functionality and can be is expressly different than the formal definition that is used in the mathematical community.

Similar to the incompatibilities that occur in the conceptual change theory, there can be discrepancies between the formal definition and one’s concept image. Any part of the concept image or concept definition that may conflict with another part is referred to as a potential conflict factor. These clashes are possible because different aspects of a concept image can be evoked simultaneously and can cause cognitive conflict and confusion. The evoked factors that are activated and cause confusion are called cognitive conflict factors (Tall & Vinner, 1981). This theory revolves around the link of the concept itself and the determination of when the concept is correctly formed in somebody’s mind (Domingos, 2010).

Another subset of one’s concept image such as mental images, processes, and properties is example space (Fukawa-Connelly & Newton, 2014). Example space (Mason & Watson, 2008; Watson & Mason, 2005) is derived from an individual’s potential example space, a collection of models and methods for generating examples that are drawn from a person’s past experiences. The potential example space that is triggered by a task, cues, environment, and recent experience, are the examples that compose an individual’s example space. A student’s example space is influenced by conventional example spaces (Watson & Mason, 2005), which are the collections of examples that are generally understood by mathematicians and are displayed in textbooks. A student’s example space can consist of examples that are associated with a conventional space that help form one’s concept definition and formal understanding of a concept, along with examples that do not coincide with a conventional space and are seen as incorrect.

The Concept of Limits

This review of the cognitive theories on the construction of mathematical concepts has been applied to many different concepts in advanced mathematics courses such as calculus, linear algebra, and differential equations (Harel, Selden & Selden, 2006). However, this review of literature will focus solely on the concept of limits. It is important to note that this review will be refined to present the difficulties undergraduate students encounter when conceptualizing limits in calculus and in proof-based, Real Analysis courses.

Limits are a major concept taught in advanced mathematics courses (Dawkins, 2012; Patel, McCombs, & Zollman, 2014; Tall, 1992). Limits are one of the first concepts students grapple with that are not deduced from a direct mathematical computation. Students are challenged to “transition from a position where concepts have an intuitive basis founded on experience, to one where they are specified by logical deductions” (Tall, 1992, p. 495). Dawkins (2012) notes that learning about limits usually has two distinct stages. First in calculus, students are conceptually and computationally introduced to limits in the different contexts that include the limit of a sequence, a series, a function, and in the notion of continuity, differentiability, and integration. The latter stage occurs after the calculus courses in the form of the rigorous proof-based setting of r Real Analysis. This shift from learning about limits in calculus to understanding them in Real Analysis requires undergraduates to transition from elementary to advanced mathematical thinking (Tall, 1995) and requires considerable cognitive reconstruction. During such cognitive reconstructions, students face potential difficulties. The following is a review of how students learn limits and the potential difficulties.

Limits in calculus. Calculus typically marks the introduction to limits for most students. This mathematical concept elevates students from the traditional arithmetic and algebra to an infinite process; “it is the concept of a limit that signifies a move to a higher plane of mathematical thinking” (Tall, 1992, p.501). Despite the limit concept being a good mathematical foundation in advanced mathematics, the formal introduction of the $\varepsilon - \delta$ definition of a limit of a function is not always an appropriate cognitive root and therefore some instructors prefer not to heavily emphasize the formal definition (Tall, 1992, 1993). The choice to under-emphasize the formal definition is made partially because students do not typically use the formal definition when solving calculus problems (Dawkins, 2012). Thus, students’ introduction to limits is usually less formal, more computational, and conceptually based.

Regardless of the teaching approach, there are potential cognitive difficulties students encounter when learning limits. For instance, the everyday use of the word "limit" usually implies a boundary or maximum that is not to be passed like a speed limit or credit limit (Keene, Hall & Duca, 2014), whereas the mathematical interpretation is different. Another conflict that can occur is when someone describes the limiting process with such phrases as "tends to," "approaches," and "gets close to." Students have colloquial meanings attached to those phrases that differ from the mathematical interpretation (Monaghan, 1991). Thus, many students become confused when determining whether or not a limit is achieved, or may perceive a limit as only being a bound of a sequence or function (Davis & Vinner, 1986; Tall, 1992, 1993). These different phrases that students hear bring a level of abstraction to the concept of limits. To grasp the abstraction of limits students may use metaphors to understand limits. Research has shown that students do not necessarily use metaphors about limits correctly (Cappetta & Zollman, 2009; Oehrtman, 2009).

Understanding the relation of the infinite concept with the limit process is another potential difficulty for calculus students (Tall, 1980, 1992, & 1993). The process of computing a limit involves the concept of infinity. Students deal with this concept implicitly through commonly used phrases in mathematics such as "as N gets arbitrarily large," or "what happens at infinity." It also uses the infinitesimal concept when the limit process requires students to interpret phrases like "a variable getting arbitrarily small." Cognitive difficulties regarding the concept of infinity can potentially impact student's understanding of limits.

Two studies conducted by Williams (1991, 2001) investigated students' informal reasoning and use of intuitive understanding that employs metaphors. In Williams' 1991 study, 341 students from a second-semester calculus class at a large Midwestern university were initially surveyed. The questionnaire assessed how students thought about limits, specifically "what it means to say that the limit of a function f as $x \to s$ is some number L" (p. 221). From the information gathered from the survey, the students were classified in terms of their view of limits, and the researchers solicited volunteers from the students surveyed for interviews. Out of the 50 volunteers, 10 were selected, and met with the investigator for five sessions over a period of seven weeks. The sessions served to help students understand their own model of limits. In session one their viewpoints of limits were explored. During session two through four they were presented opposing perspectives in an effort to better understand limits.

Williams (1991) found that students held a personal procedural, dynamic viewpoint of limit (the function at points gets arbitrarily closer to the point of interest). The students used intuitive models based on dynamic imagery, limits being unreachable and a "generic metaphor" (p. 233). Over the five sessions, the students failed to adopt the formal view of limit. Williams also found that students viewed counterexamples as exceptions to their intuitive limits models. However, these counterexamples did not influence the student enough to abandon their incomplete understanding for the formal understanding. The research found that students did not appreciate or find use of the formal definition of a limit. Therefore, improving students' understanding of the formal definition of limits requires instructors to foster an appreciation of limits and show the value of learning the formal definition.

As an expansion of his earlier study, Williams (2001) explored the intuitive limit models of two students. Williams' 2001 study investigated students' understanding of limit models by having them categorize the models against one another, assigning properties based on the categories, and drawing inferences through deduction or metaphorical transfer. One way that both the students examined the validity of their intuitive models was by using the idea of getting "closer and closer" to a number. Both students encountered difficulty when understanding the formal definition of the limit was the image of infinity in relation to the limit process. Williams (2001) was able to the create categories for the two students, but recognized that such categories failed to capture the students' spontaneous reasoning.

Expanding on William's studies to systematically characterize students' metaphors for limits, Oehrtman did an in depth investigation with a larger sample of students. Oehrtman recruited 120 subjects from a yearlong Calculus sequence. There were nine students who participated in a sequence of two hour-long clinical interviews. Oehrtman also interviewed eleven students. Oehrtman created eight potential metaphor clusters, which are defined as a characterization of the application of a single domain in reference to a variety of limit applications.

There were three weak metaphors. The first used was motion imagery that used words such as "approaching" or "tends to." The second metaphor was zooming imagery and interpretations of local linearity and the third was the interpretations of arbitrarily and sufficiently. The five strong metaphors Oehrtman presented were collapse, approximation, proximity, infinity as a number, and physical limitation.

Oehrtman's (2009) research of students' metaphorical reasoning of calculus concepts was extended by Patel et al. (2014) to investigate the eleven instructors' metaphorical reasoning involving limit concepts. Patel et al. found instructors utilized both algebraic and graphical perspectives as well as Oehrtman's metaphor clusters. Most instructors used a variety of metaphors and shifted their usage in different contexts. The instructors were unaware of their use of metaphors, and the inconsistencies in their choices thereof. It was concluded that the instructors' inconsistencies could cause frustration for novice students and allow for them to create their own metaphors, which as prior research has shown can lead to misconceptions about limits.

Another approach to conceptually understanding limits in different contexts is visualization. Visual intuition in mathematics has some benefits and as well as disadvantages (Aspinwall, Shaw & Presmeg, 1997; Tall, 1991). A downside of visualization is that an "individual has inadequate experience of the concepts to provide appropriate intuitions" and visual representations (Tall, 1991, p. 3). This notion was found in Aspinwall, Shaw and Presmeg's (1997) case study. In this study the researchers sought to understand the role one calculus student's imagery plays in his learning process. The study had the student graphically interpret a function and its derivative. A derivative of a function $f(x)$ with respect to x is the function $f'(x)$ defined as $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. The study showed that the student had an uncontrollable image of the function $f(x) = x^2$ and its derivative $f'(x) = 2x$. The student was unsure how to compare the two functions when both were presented graphically together. The study concluded that imagery did not always increase students' conceptual understanding.

In a similar case study by Aspinwall and Shaw (2002), they investigate two calculus students' representational schemes for the derivative of functions. The study found that the students used different representational schemes and their student-generated representations gave insight into how students thought about the concept. They concluded that the development of students' understanding of a concept was restricted by the types of visual representations utilized.

Studies of students' understanding of formal definitions. Similar to how both of the earlier studies have focused on students' understanding about the informal aspect of limits, there have been studies about calculus students' formal conception of limits. Roh (2008) explored how calculus students' visualization of the limit of a sequence based on their informal understandings of limits, influenced their understanding of the formal definition of the limits of a sequence. In the study, eleven calculus students completed a series of task-based interviews. Roh found that students' understanding of definitions of limits were linked to how much their previously constructed images of limits were incompatible with the formal concept of limit. Roh saw that there were differing ideas about a sequence "approaching" a value, and their interpretation was related to their visualization of the limit of a sequence. For example, one student's definition of limit was that it approached a number but did not reach it. This student's interpretation of a limit approaching was attributed to asymptote for the limit of a sequence. Thus, exposing students to a wide variety of examples of sequences, not just examples of monotone sequences, is important to help them conceptualize and imagine the limit of a sequence.

Swinyard (2011) conducted a case study about how students construct their own definition of a limit into the conventional $\varepsilon - \delta$ definition. Swinyard surveyed twelve undergraduate students who had taken two or more courses of the calculus sequence about how they rationalized limits informally. Swinyard selected four students who demonstrated a strong informal understanding of limits, had no prior experience with the $\varepsilon - \delta$ definition and who he personally knew to have the ability to work together effectively. The participants took part in instructional activities that were intended to enrich the visual aspects of the students' respective concept images. During each teaching experiment, the participants refined their characterization of limit by encountering examples and counterexamples that caused cognitive conflict. Although the study mentioned 4 participants, the data and analysis presented was on only two, Amy and Mike.

Amy and Mike's reinvention of their definition of limit happened in six phases:

1. Presenting their informal ideas.
2. Initial x-first characterizations of limit.
3. Employment of a zooming metaphor.
4. Dissatisfaction with the infinite limiting process.
5. Characterizing limit at infinity.
6. Using limit at infinity as a template to define "limit at a point."

Swinyard found that the combination of purposefully designed tasks, guidance, and students taking ownership for learning mathematics, can prompt students to construct an accurate definition of limit.

Swinyard and Larsen (2012) used Swinyard's (2011) descriptive account of students reinventing a formal definition of limits to elaborate on Cottrill et al.'s (1996) genetic decomposition model of students' process of understanding the limit concept. Cottrill et al.'s (1996) genetic decomposition of the limit concept is a sequence of mental constructions students could use while developing informal and formal understandings of limits. Cottrill et al. conducted a study in a first-semester calculus course. The students in the course received an instructional treatment of limits where they partook in five types of computer activities, reflected on the computer tasks, and had exercises as reinforcement of the topic. The first computer activity was investigating approximation by having students write computer code to compute the average rate of change of a falling body over a small interval of time. The second activity was a graphical investigation, where the students estimated the slope of the tangent of a curve. The third activity had the students construct computer code that would evaluate a given function at each of a finite sequence of points. Students investigated a value approaching a limit point. The fourth computer activity had students study and modify a program that approximated values of limits, for both limits from the left and limits from the right. The final computer activity had students investigate epsilon-delta windows. This computer task was designed to help students formulate an understanding of the formal definition by having students draw the epsilon-delta "box."

Cottrill, et al. (1996) analyzed their data using the APOS (action, process, object, and schema) theoretical framework and provided insight into students' difficulties with the dynamic conception of the values of a function approaching a limiting value as the values in the domain approaching some quantity. This was a major difficulty for students, and therefore, presented impoverished covariational reasoning abilities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Cottrill, et al (1996) concluded the genetic decomposition of the limit concept to be the following sequence of seven mental constructions (p. 8):

1. The action of evaluating f as a single point x that is considered to be close to, or even equal to, a.

2. The action of evaluating the function *f* as a few points, each successive point closer to *a* than was the previous point.
3. Construction of a coordinated schema as follows.
    a. Interiorization of the action of Step 2 to construct a domain process in which *x* approaches *a*.
    b. Construction of a range process in which *y* approaches L.
    c. Coordination of (a), (b) via *f*. That is, and the function *f* is applied to the process of *x* approaching *a* to obtain the process of *f(x)* approaching L.
4. Perform actions on the limit concept by talking about limits of combinations of functions. In this way, the schema of (3) is encapsulated to become an object.
5. Reconstruct the process of 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of a limit.
7. A completed $\varepsilon - \delta$ conception applied to specific situations.

The Cottrill et. al (1996) study provided empirical evidence for the first four steps of the genetic decomposition but did not provide any for steps five through seven. Thus, Swinyard and Larson (2012) aimed to build upon this model by exploring how four Calculus III students' concept images of limits might be built into a formal definition. The study had two phases: the survey phase and the teaching experiment phase. There were two teaching experiments, and each began with the students generating examples of limits based on their concept image. The four students used these examples to reinvent a formal definition. From the analysis, Swinyard and Larson saw that students struggled to employ y-first perspective (looking at the y-values close to L) and relied on an x-first perspective (looking first at x-values close to a). The students also struggled with what it meant to be infinitely close to a point.

Despite these difficulties, two of the students, Amy, and Mike were able to create a formal definition (Swinyard, 2011; Swinyard & Larson, 2012). From this analysis, Cottrill et al.'s genetic decomposition framework was modified as such:

1. The action of evaluating f as a single point x that is considered to be close to, or even equal to, a.
2. The action of evaluating the function f as a few points, each successive point closer to a than was the previous point.
3. Construction of a coordinated schema as follows.
    a. Interiorization of the action of Step 2 to construct a domain process in which x approaches a.
    b. Construction of a range process in which y approaches L.
    c. Coordination of (a), (b) via f. That is, and the function f is applied to the process of x approaching a to obtain the process of $f(x)$ approaching L.
4. Constructing a mental process in which one tests whether a given candidate is a limit by:
    a. Choosing a measure of closeness to the limit values L along the y-axis;
    b. Determining whether there is an interval around the value a at which one is taking the limit for which every function value aside from the one at the point is close enough to L; and

c. Repeating this for smaller and smaller measures of closeness.

5. Associating the existence of a limit with the ability to continue (theoretically) this process forever without failing to produce the desired interval about *a*, or equivalently with the observation that there is not a point at which it will be impossible to find such an interval.
6. Encapsulating this process via the concept of arbitrary closeness. This involves realizing that one can establish that the process in Step 4 will work for every possible measure of closeness by proving that it will work for an arbitrary measure of closeness.

Multimedia learning of limits. A potential difficulty students encounter when learning the formal definitions of limits of sequences and limits of functions is understanding and manipulation the universal and existential quantifiers (Cory & Garofalo, 2011, Cottrill et al., 1996). An approached used to help students encapsulate the process of given a positive epsilon to find the corresponding variable, is the incorporation of multimedia learning.

Parks (1995) did a comparative study of two first semester calculus courses understanding of the formal definition of the limits of a function. One section was taught using graphing calculators and the other sections used Mathematica's dynamic capabilities. The Mathematica students performed significantly better on a quiz immediately after instruction, however, there was no significant difference between the two groups on their exam performances. It was found in follow-up interviews that the Mathematica students were less confused about the roles of epsilon and delta than the non-Mathematica students.

Cory and Garofalo (2011) extended Parks (1995) results by investigating three preservice secondary mathematics teachers' changing conceptions of limits of sequences and how their understanding developed from instruction that involved interactive, dynamic sketched of the formal definition. The three preservice teachers were able to reflect on their own concept image and compare their understandings to the dynamic sketches. Each student was allowed to spend time manipulating sketches to investigate the different components of the formal definition. The students were able to modify their conceptions and strengthen their understanding of the quantifiers and the roles of epsilon and N in formal definition of limits of sequences.

As seen above students' introduction to the concept of limits in a calculus course can range in its level of formality. Calculus students begin to conceptualize limits in an informal manner (Patel, McCombs, & Zollman, 2014), formally (Roh, 2008), or transition from thinking informally to formally about limits (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Swinyard, 2011; Swinyard & Larsen, 2012). Despite the varying levels of formality there are potential difficulties that can appear in calculus (Davis & Vinner, 1986; Monaghan, 1991; Tall, 1992, 1993). Unfortunately, these potential difficulties are not isolated to calculus but are also present in the proof-based setting of Real Analysis.

Limits in Real Analysis. Progressing from calculus to Real Analysis requires students to make "an even greater leap in advanced mathematical thinking to formal definitions and formal deductions" (Tall, 1995, p. 9). Mathematical definitions "are often used as a vehicle toward a more robust understanding of a given concept" (Edwards & Ward, 2008), which is essential in this evolution in mathematical thinking since there is a relationship between concept formation, definition construction, and proof (Harel, Selden, Selden, 2006). However, “in analysis, the availability of visual representations means that more students initially have access to a way of coming to understand the concepts. The understanding gained in this way means that they feel less need to engage seriously with” formal definitions (Alcock & Simpson, 2002, p. 34). Therefore, there is the potential difficulty of not having a robust understanding as well as not acquiring a conceptual understanding of the formal definition of limit that is essential in reading and writing Real Analysis proofs. Therefore, the research revolving around the concept limits in Real Analysis courses is geared to understanding and aiding students’ development of an applicable formal understanding that is conducive to reading and writing formal proofs.

A series of work done by Pinto and Tall (1999, 2002) was geared specifically towards developing a framework of how Real Analysis students build their formal theory of limits that is applicable to Real Analysis proofs. Building from a preliminary study, conducted by Pinto in 1998, Pinto and Tall (1999) documented students’ knowledge construction of three definitions, arguments, and images of a limit of a sequence in a twenty-week Real Analysis course to determine how students successfully and unsuccessfully construct formal theory (proofs). The students were interviewed on seven occasions and organized into a classification system. Based on the three themes (definitions, arguments, and imagery) students were classified as having an informal approach or a formal approach. Informal approaches had descriptive definitions, arguments based on concept images, and imagery that was not constructed from a definition. The formal approach had formal definitions (correct or distorted), arguments based on formal theory, and constructed their imagery from the definition. After further analysis, Pinto and Tall modified the informal approach to be “giving meaning” since the students give meaning to the concept definition from concept imagery. The formal approach became the “extracting meaning” approach since students extracted meaning from the concept definition by making formal deductions.

Pinto and Tall (1999) found that the transition from elementary mathematics to formal proof was difficult for students if their concept was unable to be morphed into formalism. Others, who were able to modify partially their old images, formulated a personal definition that is not formally operable. There were two ways some of the students were able to transition to formal proof. The “giving meaning” students who were constantly reconstructing their ideas based off of various images were able to build up formal theory. There were those “giving meaning” students, who used their concept image to construct generic proofs, which were intuitive but not necessarily formal. The “extracting meaning” students built up ideas primarily from formal deductions with few connections to their concept images. This approach avoided possible conflicts within the imagery, but also did not build any connection between the formal theory and the informal imagery.

Pinto and Tall (2002) continued their investigation into the transition to formal mathematical thinking by further analyzing the data of one the students in Pinto’s 1998 study. Their analysis revealed that the student Chris refined and reconstructed his existing imagery into a new imagery that was applicable to formal theory. The reconstruction process began with Chris using his prior knowledge of convergence to interpret the formal definition. He then explored convergence through thought experiments and reconstructed his concept definition.

Pinto and Tall (2002) also did a second analysis using the Cottrill et al.'s (1996) genetic decomposition theory for definitions and Dubinsky et al. 's (1988) theory that students approach quantified statement by working from the inner single-level quantification to successive, higher-level quantifications. Using this framework Pinto and Tall found that Chris could write down the definition and compute limits, but had difficulty dealing with the mixture of specific (e.g. $L = 1$) and general values (a_n). Chris approached the definition by first fixing $\epsilon > 0$ and then focus on a value N for which $|a_n - L| < \varepsilon$ whenever $n \geq N$. Once the student was successful for the fixed epsilon, he then allowed it to vary. Chris's approach differed from Dubinsky et al.'s theory that he would have approached the definition from just reading it left-to-right or decompressing the definition from the "inside-to-outside."

Pinto and Tall conjectured that an appropriate framework would be the APOS (action, process, object, and schema) theory. The A-P-O portion of the APOS theory aligns with Chris's development of constructing a concept image from defined objects, and through the process of abstracting "actions on objects." The A-P-O portion has the educator pose the construct on a student as an external object and through action the student internalizes the action as a process and encapsulates this into the mental object (limit). This framework is considered a natural way students learn formal theory.

It was determined that there are two categorizations of the development (Tall, 2013). The first, which was shown by Chris, was the natural approach based on theoretical mathematics involving embodiment, symbolism, or a blending of the two. The second was demonstrated by the student Ross, and was the formal approach. This approach used formal mathematics of set-theoretic definitions and deductions. In the formal approach, the student did not demonstration intuition but rather logically justified and reason through definitions and theorems.

In 2010, Domingos used the notion of concept definition and concept image and the theory of reification to investigate the difficulties that Real Analysis students endured when developing an understanding of the concept of limit. The qualitative study used observation of lessons and semi-structured task-based interviews to investigate students' level of concept image of limits. Based on the data, three levels of concept images were identified: incipient concept image, instrumental concept image, and relational image. The levels were determined by the basis of objects, processes, translation between representation, properties, and proceptual thinking that surfaced during cognitive tasks.

The cognitive task introduced the formal definition of a limit and the limit: $\lim_{x \to 1} \frac{x^2-1}{x-1}$ and the graphical representation of the function $\frac{x^2-1}{x-1}$. One student, Mariana displayed an incipient concept image by explaining the meaning of $\lim_{x \to 1} \frac{x^2-1}{x-1}$ by using the left and right neighbors of 1 and a neighborhood around 2. Her description solely used the language of the neighborhood and provided no symbolic translation to any part of the definition. Her conception was based essentially on a relation of proximity in geometric terms. When the interviewer supplied information and guidance for extending her understanding to the formal definition, Mariana showed difficulty following the suggestions.

Another student Jose displayed an instrumental concept image when discussing the meaning of the $\lim_{x \to 1} \frac{x^2-1}{x-1}$. Jose based his interpretation on the graphical representations (even in tasks where the graphical representation was not present). His interpretation began with discussing what happened as the function came closer to 2 and relates this back to the *y*-axis. Jose was able to demonstrate the dynamic relationship for the processes that relate the objects and the images. When Jose was asked to establish the symbolic representation of the limit he claimed that he could not, and described the process instead. He described the process of *x* approaching 1 to be represented as “1 minus *x* less than anything” and when he considered approaching 1 from both the left and the right, he was able to write |1-*x*|. Jose was also able to establish the neighborhood of the limit to be |2 –*f(x)*|. He knew that both neighborhoods were very small and not necessarily the same values. With the help of the interviewer he used the symbols of epsilon and alpha. Jose was able to write eventually the symbolic definition, but had difficulty drawing the quantifiers and was not able to describe their role in the definition.

Sofia demonstrated the third level, relational concept image, when she interpreted the meaning of $\lim_{x\to 1}\frac{x^2-1}{x-1}$. Sofia began with explaining her understanding of limit using the *y*- and *x*-axis, without representing the function graphically. She described how as *x* tends to 1 the image tends to the limit value, 2. She concluded that the point, at $x = 1$, does not have to belong to the domain. Sofia was able to translate successfully the limit symbolically (write the formal definition). She reasoned through writing the definition, demonstrating that she did not have it memorized. She established the roles of the parameters and was able to explain the influence of the quantifiers. Sofia did demonstrate some difficulty fully understanding the role of the quantifiers.

Based on their verbal performance of the concept, Domingos determined three levels of concept images. Thus, this study showed the cognitive demand Real Analysis students experience articulating their understanding of limits in accordance to a formal definition. While some of the difficulties students have can be accounted by the cognitive demand of limits, students also lack the logical maturity and understanding for formal theory needed in proof (Harel & Sowder, 2007).

To offer an explanation for why applying formal theory in Real Analysis courses is difficult for undergraduates, Alcock and Simpson (2002) explored three approaches to mathematical reasoning: generalizing, property abstraction, and working from definitions. The generalizing approach begins with students inspecting a prototype, (a representation that an individual considers prototypical of a mathematical category e.g. strictly increasing sequences), to generate a conjecture that they generalize to the entire category. The property abstraction approach has students abstract a significant property from their prototype and make deductions for the entire category. The last approach, working from definitions, has students use defined properties to make deductions for the entire category.

Alcock and Simpson found that certain reasoning strategies are inadequate for mathematics courses such as Real Analysis. Strategies like generalizing and property abstraction can be successful in non-technical contexts, but do not always allow students to deductively show an object as a member of a category in a formal proof, like the use of a mathematical definition does. This is a challenge for students since “definitions in analysis are logically complicated; they often involve multiple mixed quantifiers” (p. 33). Alcock and Simpson “do not argue that mathematicians think solely in terms of definitions” (p.32), they acknowledge that they do use the other approaches of mathematical thinking, but utilize definitions for valid arguments.

A theory of how teachers can help develop student’s understanding of the concept of limits was presented by Mamona-Downs (2001). Mamona-Downs discusses the challenge of learning the limit concept as both a cognitive demand as well as a metacognitive challenge that requires students to grapple with the content of the definition. Mamona-Downs thus presented a three-step didactical sequence that is intended to help alleviate the dual demand of understanding the limit concept by building students' images based on the formal statement.

Mamona-Downs' first step is to initiate and develop intuition through raising issues in a classroom discussion environment. Mamona-Downs suggests the tasks described below based on the literature that some students struggle with the concept of limit because of various notions such as: infinitesimals and infinite numbers (Tall, 1992), absolute values and inequalities (Cottrill et al, 1996), and beliefs and behaviors of limits (Davis & Vinner, 1986). These types of tasks expose students to the essential ideas that motivate the definition of limit and supply a strong cognitive base for the definition. Two examples follow:

Task 1: A Ping-Pong ball is dropped from a height *h* onto a level, hard floor. Each time it bounces, the highest height the ball attains is half that it attained for the previous bounce. (Height of the Ping-Pong ball is always measured relative to the 'lowest' point of the ball from the floor.)

> Question 1: How many times does the ball bounce? If the word 'infinity' occurs in the answer, ask the supplemental question what do you mean by infinity here?
>
> Question 2: How far will the ball travel in total?

Task 2: Imagine a stairway with just two steps, rise and tread both one-meter. From the original stairway we construct another with twice the number of steps, by halving the rise and tread. Following the same process inductively, we may construct a whole family of staircases.

Question 1: What can we say about the perimeter of the staircases?

Question 2: What is the final result of the inductive process?

The second step is to introduce the formal definition and to analyze it in tandem with the issues presented in the first step and to introduce particular representations. Mamona-Downs states the introduction of the concept of a limit should not be informal and not depend heavily on procedural methods. Rather, the students should be exposed to the definition and an illustration of a limit of a real sequence so that students can have a clear and consistent idea about the concept, rather than dealing with a foggy perception about the concept. The formal definition does not replace the original intuition, but it is meant to enhance it. This calls for an understanding of the definition of a sequence of a limit.

Mamona-Downs suggests that understanding the definition should first begin with making sense of the symbolism and have them relate their preconceived informal images with the symbols. A few difficulties students may encounter revolves around the focal component of the definition: $|a_n - L| < \varepsilon$. The combination of the absolute value and inequality may cause issues therefore it may be more preferable to introduce $|a_n - L| < \varepsilon$ as $a_n \in (L - \varepsilon, L + \varepsilon)$. Another issue for students is determining whether a_n, L, and ε are constants, varying, or parameters. After determining what kind of variables they are, the student must then understand their role and relation between one another, which is potentially problematic as well.

The third step endorses or revokes opinions made in step one by comparison with the formal definition, especially via the representation made in step two. Mamona-Downs suggests presenting images consonant to the notion of Cauchy sequences or to let the images developed from the definition alter their original intuition. The students should compare and contrast their intuitions and beliefs on the behaviors of limits of sequences with the formal image they constructed from the definition. Theoretically, these three steps should develop students understanding of a limit of a sequence; however, future research needs to be done to test this theory.

In Real Analysis courses, students are challenged with the cognitive demand of transitioning from elementary thinking to advanced mathematical thinking about the concept of limits. However, students also face the challenge of learning proofs. Following is a discussion of the literature on the task of reading proofs pertinent to the advanced mathematics course, Real Analysis.

Proof Comprehension in Real Analysis

Proof is irrefutably important in the field of mathematics and mathematics education (Common Core, 2015; Harel & Sowder, 2007; NCTM, 2000; Pinto, 1998; Weber, 2014). Despite the significance of proof, there is no universal definition of proof. The types of definitions of proof have changed over time. In the beginning of the 20th century mathematicians primarily saw proofs as a formal object (Pinto, 1998; Weber, 2014; Yoo, 2008). “Proof is a formal way of expressing particular kinds of reasoning and justification” (Cai & Cirillo, 2014, p. 133). Hence, proof can be characterized by an objective method that is well formed by deductive reasoning that verifies the truth of a mathematical statement.

Yet, there are mathematicians who have begun to diverge from the formal object definition to one that incorporates human dependency, construction, and interpretation. Proof has been seen as a deductive argument that is an age-appropriate representation system (Stylianides & Stylianides, 2008). Others define proof subjectively; “a proof is what establishes truth for a person or a community” (Harel & Sowder, 2007, p. 806). Whereas, other mathematicians characterize proof as based on human constructions, such as differentiating between proof through logic and proof through reasoning (Simpson, 1995). Even the incorporation of technology in mathematics has expanded some definitions to include computer-assisted proofs (Hanna, 1995), as well as “visual proofs” if they satisfy the necessary but not always sufficient requirements of being reliable, consistent, and repeatable (Borwein & Jörgenson, 1997).

The changing views of mathematics and the type of properties a proof should encompass fails to bring a consensus in the field on the definition of proof. The lack of agreement of a definition has caused some mathematicians to not focus on it and conclude that “however [proof] is defined, is secondary in importance to understanding” (Hanna, 2000, pp. 6-7) and emphasize the purpose of proof. Whereas, Weber (2014) has attempted to characterize proof as a cluster model in order to satisfy nearly all mathematicians’ constitution of what is and is not a proof. Weber’s cluster concept has the five following categories: A proof is an argument

- that is deductive and non-ampliative argument,
- that would convince a contemporary mathematician who knew the subject,
- in a natural language and symbolic representation system where there are socially sanctioned rules of inference,
- that convinces a particular community at a particular time,

- that is a blue print that knowledgeable mathematicians can use,

However, troubling the inconsistency is in defining what a proof is, the ambiguity allows for proof to be prevalent throughout mathematics, especially in advanced mathematics courses such as Real Analysis.

In Real Analysis courses students apply their conceptual understanding of limits to comprehending proofs. Students engage in this task while reading their textbook, their lecture notes and when they listen to proofs being presented to them by their professors. Unfortunately, mathematics majors do not always hold productive beliefs about proof reading and believe that reading a proof is a passive process that does not require them to construct things to help them understand, such as sub-proofs or diagrams. This may be attributed to how quickly proofs are presented in advanced mathematics courses combined with how rarely students are tested on their comprehension of a proof (Weber & Mejia-Ramos, 2014). Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff (2011) note that "students' comprehension of a given proof is often measured by asking them to reproduce it or modify it slightly to prove an analogous theorem, even though these types of assessments offer only a superficial view of students' comprehension" (p. 4).

Proof validation. Most of the literature on proof comprehension is focused on proof evaluation and the task of validating. Validating is the reading and mental processes of determining if a text is deemed a correct proof (Selden & Selden, 2003). The nature of the validation process makes the reader responsible for constructing the meaning of the text rather than placing such responsibility on the author. Thus, based on the personal background of the validator (e.g. the reader), different depths of understanding can occur. Regardless of the validator's background, validating proof is an important means for constructing sophisticated mathematical knowledge (NCTM, 2000). If undergraduates cannot validate proofs reliably, they reduce their ability to gain conviction or understanding in the advanced mathematical courses (Selden & Selden, 1995). Most work about proof validation is limited to instructors' and students' ability to accept or reject arguments based on their form (Hoyles, 1997; Martin & Harel, 1989; Segal, 1999; Selden & Selden, 2003).

Alcock and Weber (2005) investigated the skills needed to validate proofs in Real Analysis. They used Toulmin's (1969) model of argumentation that has the three essential parts consisting of the conclusion, the data, and the warrant. Eighteen volunteers from two introductory Real Analysis courses were presented the following mathematical argument and were asked to determine if the argument was a valid proof.

Theorem: $\{\sqrt{n}\} \to \infty \; as \; n \to \infty$.

Proof: We know that $a < b \implies a^m < b^m$. So $a < b \Rightarrow \sqrt{a} < \sqrt{b}. n < n + 1$ so $\sqrt{n} < \sqrt{n+1}$ for all n. So $(\sqrt{n}) \to \infty \; as \; n \to \infty$ as required.

The beginning of the proof contained minor errors in not defining *a* and *b*, restriction on *m*, and the scope of the variable of *n*. The fundamental flaw in the argument was the last line since not all increasing sequences diverge. The students had the opportunity to modify the argument and discuss the validity.

Alcock and Weber focused on the students' responses to the validity of the last line of the proof. There students' responses were sorted into three categories. The first category consisted of students who rejected the proof because of an invalid warrant. Three students did not believe that the last line of the proof followed from the previous statement and determined the claim was invalid. The three students were each able to produce counterexamples to show the warrants were not valid.

The second category contained students who rejected the proof as valid because the definitions were not employed. The students believed that only the definition of divergence to infinity could be used to support the claim. This interpretation is consistent with the emphasis to employ definitions in their Real Analysis course to produce valid proofs. Their understanding and discipline to use concept definitions demonstrates a sophisticated understanding of advanced mathematics. However, it is important to note that it is not always vital to use definitions, once theorems have been established.

The third category contained five students who accepted the proof as valid. In analysis of the data, Alcock and Weber saw that students accepted the proof due to their inadequate knowledge of the properties of sequences. But once they were guided to investigate the fourth line of the proof, they eventually were able to conclude the argument was invalid. There were two students who did not fit into any of the three categories. Dean was uncomfortable with the vagueness of the proof and could not articulate what aspect bothered him, despite the interviewer drawing his attention to the fourth line of the argument. The other student, Wendy, found the proof valid despite discussing the fourth line and being presented a counterexample by the interviewer.

Alcock and Weber’s (2005) study showed that when students were unable to reject the argument they focused on determining if assertions were true rather than determining if they were substantiated. Students who successfully attended to the proof’s validity did so either by determining the use of definitions or determining the warrant was invalid. This skill was also demonstrated by mathematicians in Weber and Alcock’s (2005) study where they presented the same task to mathematicians.

Weber (2010) investigated mathematics majors' ability to validate an argument and determine if they were convinced by the argument. The participants evaluated deductive arguments by their content and incorrectly found arguments valid because they did not recognize the logical flaw. Weber also found that most participants did not find empirical arguments to be convincing or a proof, but did find the diagrammatic argument (an argument that also incorporated an image or table) to be convincing and a proof.

Further investigations about whether or not students are convinced by mathematical arguments have been made by Inglis and Mejia-Ramos (2008, 2009). Inglis and Mejia-Ramos (2008) developed that there are at least five different ways students can evaluate their level of conviction and persuasion of argument when evaluating the text. The first two levels have students evaluate just a particular part of an argument, ignoring the rest. The next two levels have students focus on the core part of the argument and assessing the appropriateness of the qualifier. In the last level students focus on a specific context and evaluate whether or not the argument is admissible in that context. Inglis and Mejia-Ramos (2009) also found that undergraduates were less persuaded by visual proofs than mathematicians, and visual arguments were more convincing if text was also included.

The previous work investigating proof comprehension largely involves students evaluating if an argument is valid. However, these measures do not grasp the full depth of students' understanding. Therefore, building from the work of Yang and Lin (2008) and that of Conradie and Frith (2000), Mejia-Ramos, Weber, Fuller, Samkoff, Search and Rhoad's (2011, 2012) created a multi-dimension assessment model for proof comprehension to gain a better scope of students' proof comprehension.

Laying the foundation for Mejia-Ramos et al.'s (2012) model was Yang and Lin's (2008) *model of reading comprehension of geometry proof* (RCGP). This model consisted of four levels:

1. *Surface*: Students obtain basic knowledge regarding the meaning of statements and symbols within the proof.
2. *Recognizing the Elements*: Students recognize the logical placement of statements that are either implicitly or explicitly used in the proof.
3. *Chaining the Elements*: Students understand the manner in which the statements are connected and identify the logical sequence/relations between them.
4. *Encapsulation*: Students understand the proof as a whole and are able to apply the proof to other contexts.

Mejia-Ramos et al. (2011, 2012) adapted this model using Conradie and Frith's (2000) work on comprehension tests for advanced mathematics. It is important to note that the seven dimensions of Mejia-Ramos et al.'s (2012) model that are described below are not hierarchal. The model is separated into two groups: local and holistic. The local aspects of the proof address students' understanding about a specific statement in the proof or the relation of a statement to another statement. The holistic understanding of proof refers to students' understanding of the ideas or methods that motivate a major part of the proof or the entire proof.

The first fundamental way to understand text is to understand the meaning of terms, statements, symbols, terms, and definitions. If students are unable to understand the meaning of key terms, it could impact their understanding of aspects of the proof or the entirety of the proof (Conradie & Frith, 2000). Some readers can comprehend terms and definitions without reading the proof while other times, the student would need to read the proof to make sense of the new terminology. Mejia-Ramos et al.'s example questions for assessing whether students understand the meaning of a specific term may involve asking the individual to:

1. State the definition of a given term in the proof.
2. Identify examples that illustrate a given term in the proof.

Example questions for assessing an individual's comprehension of statements in a proof are of the type:

1. State a given statement in a different but equivalent manner.
2. Identify trivial implications of a given statement.
3. Identify examples that illustrate a given statement.

The next dimension of understanding a proof is to comprehend the logical status of statements and proof framework. The logical statements have different statuses that include: axiom or postulate, fact or theorem, hypothesis of the theorem to be proven, and a statement deduced from prior statements. Along with identifying the logical status of statements in proofs the reader must also recognize the logical relationship between the statement being proven, the assumptions, and the conclusions, which is what Selden and Selden (1995) call proof framework. Example items to assess this understanding are of the form:

1. Identify the purpose a sentence within a proof framework.
2. Identify the type of proof framework.

Proofs do not always include all of the logical details, and some of which left to the reader. Therefore, the reader has to infer what prior statements deduced an assertion in the proof. Thus, this comprehension dimension assesses the ability to justify claims. Questions that would allude to whether or not readers comprehend the justification of claims asks them to:

1. Make explicit an implicit warrant in the proof.
2. Identify the specific data supporting a given claim.
3. Identify the specific claims that are supported by a given statement.

The second group of dimensions assesses students' holistic understanding of the proof. The holistic comprehension of the proof is composed of the main ideas, methods, and application of other context. Educators view that to improve students' proof comprehension; it's important to have a top-level overview where they comprehend the main idea of the proof. To determine if students understand the main idea, one could ask questions of the form:

1. Identify or provide a good summary of the proof.
2. Identify a good summary of a key sub-proof in the proof.

Leron (1983) discussed partitioning a proof in modules into manageable parts. For example, a proof that contains a lemma can be thought of a separate unit from the proof that can be applicable for such a proof or elsewhere. The following questions are the different ways that the authors assess undergraduates' conception of the relationship of the modules in a proof:

1. Ask students to partition a proof into modules.
2. Identify the purpose of a module of a proof.
3. Identify the logical relation between modules of a proof.

Another aspect of the holistic level involves identifying the procedures implemented in a proof and how these procedures could be used in other proof writing tasks. The following questions are the sorts that evaluate students' ability to transfer the general ideas or methods to another proof context:

1. Transfer the method.
2. Identify the method.
3. Appreciate the scope of the method. (Example: Why can't the method used to prove this theorem not applicable to proving this other statement?)

The final dimension assesses whether students can infer the proof in terms of a specific example. This dimension allows the reader to gain a deeper understanding of the proof. At times, the example can guide the readers' interpretation, and make sense of how the proof works. Within this dimension, the question types include:

1. Illustrate a sequence of inferences with a specific example.
2. Interpret a statement or its proof in terms of a diagram.

This multi-dimension model was constructed to examine how much undergraduate students comprehend a proof. The model could also be used as a methodological tool for evaluating the effectiveness of instructional interventions that are geared toward increasing proof comprehension. Mejia-Ramos et al. also produced this model to redirect and stimulate more research on proof comprehension, such as the following Hodds, Alcock and Inglis's (2014) study.

Hodds, Alcock and Inglis (2014) investigated whether self-explanation training changes the process by which students read mathematical proofs. The authors conducted three different experiments. The first experiment had 76 mathematics undergraduate participants, 38 in the control group and 38 in self-explanation training group. Hodds et al. constructed a 14-item proof comprehension test according to the rubric specified by Mejia-Ramos et al. (2012), and used their proof for the comprehension task. The first experiment found that 45% of the variance in comprehension test scores involved knowledge/skills other than simple proof comprehension, self-explanation training increases the higher quality explanations during proof comprehension attempts, the control group scored on average four points lower on proof comprehension scores with a large effect size $d = 0.950$, and those further along in the program scored higher.

The second experiment investigated the impact of self-explanation training on the reading process when undergraduates were not required to explain verbally. The researchers used eye tracking to determine if the training changes the level of cognitive engagement with mathematical proofs. The 38 participants were randomly assigned to the self-explanation group and control group. There was no significant difference between the mean times spent on reading proofs. Similarly, to the findings in experiment one, the self-explanation training improved comprehension performance. The researchers also found that the training lead to deeper engagement with mathematical proofs and encouraged students to search for logical connections while reading proofs.

The third experiment consisted of 107 participants who were first-year undergraduate mathematics calculus student. The 107 students were randomly assigned to two groups. The self-explanation training materials were provided in a paper booklet, as well as the control materials. Participants were given a posttest and a delayed posttest, results showed that the self-explanation training improved proof comprehension significantly in the short term and had lasting effects. The three experiments demonstrate that students' deficits in proof comprehension "are not due to some inherent intellectual incapacity . . . a light-touch intervention can lead to better mobilization of these skills and thus to considerably better proof comprehension" (Hodds et al., 2014).

The current literature of proof comprehension has been largely dominated by validating proofs. The direction of this research field has been changed by Mejia-Ramos et al.'s (2012) multidimensional model of proof assessment, as seen by Hodds et al.'s (2014). At this time, more investigations into proof comprehension, not just validating proofs, is warranted. This study aimed to add to this gap in the literature by describing students' conceptual understanding of limits in terms of concept image and concept definition and exploring the relationship between students' concept image and concept definition of limits and students' proof comprehension.

Supporting Theories

Concept image and concept definition. To investigate Real Analysis students' conceptual understanding of limits, this study primarily used the theory of concept image and concept definition. As discussed earlier in the chapter there is a distinction between how the mathematical concepts are defined and an individual's mental structure of the concept (Tall & Vinner, 1981, Vinner & Hershowitz, 1980). Tall and Vinner (1981) define concept image to be the total cognitive structure that is associated with the concept, which includes all imagery, diagrams, graphs, examples, symbols, words, properties and processes. Therefore, this study will use the four domains of concept image to guide exploration of Real Analysis students' conceptual understanding of limits. The four domains are mental images, processes, properties, and example space (Fukawa-Connelly & Newton, 2014; Mason & Watson, 2008; Watson & Mason, 2002, 2005) as a subset of concept image. Examples may include a wide range of mathematical genres such as examples illustrating concepts, worked examples demonstrating techniques, examples of objects that satisfy certain conditions, and examples of constructing proofs.

This study incorporated Williams' Williams' (1991) six categories of how students think about limits to choose student with the most diverse mental images upon entering the Real Analysis course. The six categories also are used to categorize the students' mental images. (1) Formally, (2) limit as a bound, (3) limit as approximation, (4) limit as unreachable, (5) dynamic theoretical, meaning a limit describes a how a function moves as x moves toward a certain point and (6) dynamic practical, meaning a limit is determined by plugging in numbers arbitrarily close to a given number until the limit is reached.

This study activated the students' concept images and concept definitions at the end of the semester. Which a concept image and concept definition activated at a specific time is called an evoked concept image and evoked concept definition, respectively. Evoked concept images and concept definition is what guided this study to answering the first research question of determining the various cognitive structures associated with the concept of limit for a Real Analysis student.

Within a person's evoked concept image there may be factors that conflict with other factors within a particular domain on the concept image, with different factors across the domains, or with factors within their concept definition. Tall and Vinner (1981) defined these factors to be potential conflict factors. This theory was used to determine how students' conceptual understanding of limits relate to their understanding of the formal definitions of limits.

Proof Comprehension. The last framework that supports this study is the Mejia-

Ramos, Fuller, Weber, Rhoads, and Samkoff's (2012) proof comprehension assessment model. This model was described in much detail earlier in this chapter and is summarized in a table 1.

Table 1.
Proof comprehension assessment model.

Dimension	Definition	Type of questions
Meaning of terms and statements	Understanding the meaning of symbols, terms and definitions.	• State the definition of a given term in the proof. • Identify examples that illustrate a given term in the proof. • State a given statement in a different but equivalent manner. • Identify trivial implications of a given statement. • Identify examples that illustrate a given statement.
Justification of claims	Understanding how new assertions in the proof follow from previous ones.	• Make explicit an implicit warrant in the proof. • Identify the specific data supporting a given claim. • Identify the specific claims that are supported by a given statement.
Logical Structure	Understanding the logical relationship between lines or components of a proof.	• Identify the purpose of a sentence with a proof framework. • Identify the type of proof framework.
Higher level ideas	Identifying a good summary of the overarching approach of the proof.	• Identify or provide a good summary of the proof. • Identify a good summary of a key sub-proof in the proof.
General Method	Applying the methods within the proof to a different context.	• Transfer the method. • Identify the method. • Appreciate the scope of the method. •

Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff's (2012)

Table 1 continued.

Proof comprehension assessment model.

Dimension	Definition	Type of questions
Identifying modular structure	Understanding the main components and modules within a proof and the logical relationship between them.	• Ask students to partition a proof into modules. • Identify the purpose of a module of a proof. • Identify the logical relation between modules of a proof.
Application to examples	Using the ideas in the proof in terms of a specific example.	• Illustrate a sequence of inferences with a specific example. • Interpret a statement or its proof in terms of a diagram.

Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff's (2012)

CHAPTER 3

METHODOLOGY

Introduction

Real Analysis courses present students with the challenge of establishing a formal understanding of limits and appropriately using the concept to understand limit proofs. Research has shown that both tasks are difficult for students (Dreyfus, 1999; Patel, McCombs, & Zollman, 2014; Tall, 1992; Weber, 2001). Furthermore, there is limited research on Real Analysis students' comprehension of limit-proofs (Alcock & Weber, 2005), and none that investigates the relationship of the two cognitive tasks. This study will attempt to address this gap in the literature by investigating the following research questions:

1. What are the concept images and concept definitions of limits held by Real Analysis students?
2. How does the students' concept images and concept definitions of limits relate to their understanding of the formal definition of limit?
3. How do the students' concept image relate to their comprehension of limit proofs?

To examine how undergraduate Real Analysis students' conceptualization of limits relates to proof comprehension a grounded theory approach was utilized. This design was chosen to explore if a relationship was present and was grounded in the data collection of surveys, observations, and interviews from two Real Analysis sections. Purposeful sampling was used to select interview participants with different ideas about the behaviors of limits with aim to investigate a diverse set of concept images. This study used Williams' (1991) one-page questionnaire about limits of functions. After the initial survey and analysis six students were selected to participate in follow-up interviews throughout the semester. Classroom observations over the instruction were conducted during the limits portions of the Real Analysis course. At the end of both the limits of sequences and limits of functions units the students completed surveys about each topic, respectively. After each survey follow-up interviews occurred. During the final class the students took an in-class proof comprehension assessment. The collection of data provided diverse concept images of Real Analysis and insight into proof comprehension.

Pilot Study

In the spring of 2015, a pilot study was conducted to aid in the development of the concept image and concept definition instruments and research design. During the spring of 2015, students in a Real Analysis course were asked to volunteer to partake in an hour-long task-based interview with the incentive of a fifteen-dollar gift card. Prior to the interview, the students were not informed of the topic to prevent them from preparing in advance. Four students volunteered, two were undergraduate mathematics majors, Joey and Michelle, and two were post-bachelor (STEM) students, Jack and Samuel, who returned to pursue a degree in mathematics.

The interviews were intended to capture the students' concept image of limits of a function at a specific instance, which is referred to as their evoked concept image (Appendix D). The interview asked their mathematical background, how they conceive limits, how they visualize limits, and to state the formal definition for the limit of a function. The task-based portion of the interview asked the student to determine $\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$ algebraically. Following the computation, the students were asked to graph $\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$ plotting the function, and labeling the values of $L, c, \varepsilon,$ and δ, given $\varepsilon = 1$ with the corresponding $\delta = 1$. For this task the student was provided the formal definition of a limit of a function.

The next task asked the student to consider the graph (Figure 3.2.) where $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = L$, and determine if the range of possible $\delta's$ for the *f(x)* function the same range of possible δ's for the *g(x)* function given the indicated epsilon? The students were asked to explain their decision.

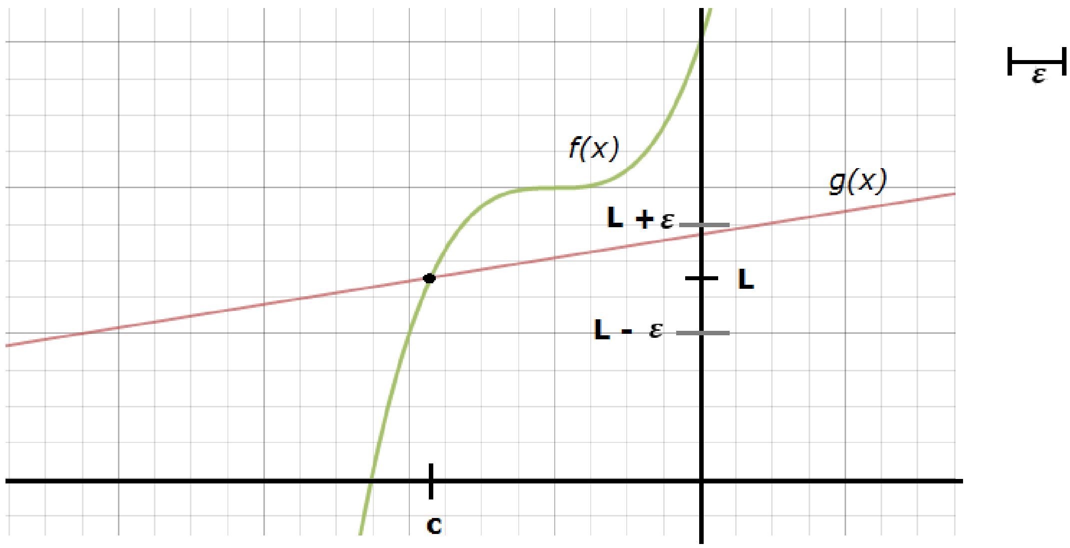


Figure 2. Comparing functions' corresponding deltas graphically.

The proceeding task asked the student to find a δ for the $\lim_{x \to 0} \sqrt{x+1} = 1$ and given $\varepsilon = 0.1$. The follow-up question asked the students if there is another delta possible, and if so, how the two deltas relate. The final task provided the students with the following theorem:

> If $\lim_{x \to c} f(x) = f(c)$, then there is a number $M > 0$ and $\delta > 0$ so that for each $x \in (c - \delta, c + \delta), |f(x)| < M$.

The students were asked to explain the theorem and determine if the statement was true.

The students were allowed to draw or write anything to help their explanation.

The tasks in the pilot study provided insight into what types of additional questions, and tasks should be added. One component of concept image that was not addressed in the tasks above was the students' example spaces. These questions did not provide the students with enough opportunity to generate examples. Therefore, the additional task of having the students generate as many example of limits of function that converge to an arbitrary real number was included

The set of four interviews demonstrated how the activated part of a concept image, called the evoked concept image, is influenced by different events. There was distinction between the students' evoked concept images based on when the interviews occurred. This difference showed why it is important to administer the surveys at the end of each unit in order to fully capture how the course and instruction influenced their concept image of limits of sequences and limits of functions.

Two of the interviews occurred prior to either of the units of limits in their Real Analysis course. Therefore, their evoked concepts were solely based off of the prior experiences and courses, Calculus I and II. These two students' concept images did not demonstrate a strong connection to the formal definition. For example, Samuel had neither a formal or informal definition of limits of functions. Samuel also claimed to have never seen the formal definition before the interview.

The other two interviews occurred during the unit of limits of sequences, and that unit strongly influenced their responses about limit of functions. For instance, Jack who took calculus a few years before stated, "I've haven't done this is [in] so long. I don't know if I've even done it, really." Jack was hesitant to attempt the task to label a graph of a rational function with respect to the formal $\varepsilon - \delta$ limit definition, given $\varepsilon = 1$ with the corresponding $\delta = 1$. Jack was unable to complete that task, and switched to the "more recent limit" to explain his formal understanding of limits. Jack generated the sequence $\frac{1}{n}$ to aid in his explanation and as he tried to connect limits of sequences with limits of functions Jack drew the sequence $\frac{1}{n}$ to be a continuous function (Figure 3.3.). This error of graphing a sequence as continuous has also appeared in Pinto and Tall's (2002) study. Pinto and Tall attribute this error to a person's inability to capture all possible cases of a concept with one generic example. However, with this instance, the discussion of limits of a function had caused the student cognitive conflict and may be the main contributing to factor to the error.

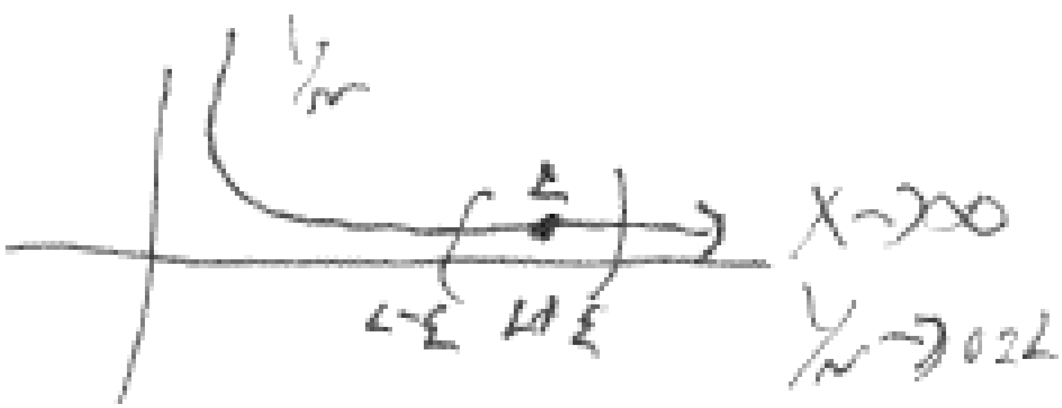


Figure 3. Jack's graph of a rational function and formal definition labeling.

Each participant had a distinct way of thinking about limits. The students chose to only describe how they perceive limits rather than draw a graphical representation. To attempt to have students generate graphical representations, the survey was altered to ask the students to provide a picture.

Tasks one and two of the interview provided insight into the participants' covariational reasoning of ε and δ. The first task had the students *consider*

$lim_{x \to -5} \frac{x^2+3x-10}{x+5}$. *The precise definition of a limit of a function states that* $lim_{x \to c} f(x) = L$ *if for every* $\varepsilon > 0$, *there exists a corresponding* $\delta > 0$ *such that for all* $x, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$. *For the function above and for* $\varepsilon = 1$, *it can be shown that the corresponding* $\delta = 1$. *Locate and label on your graph the values of* $L, c, \varepsilon,$ *and* δ. The second task asked the students to *consider the* $lim_{x \to 0} \sqrt{x+1} = 1$. *Let* $\varepsilon = 0.1$, *and find a* δ. *Is there another delta possible? If so how do the two deltas relate?*

These tasks reinforced that surveys should be given after the limits were taught so that the students could have prior exposure the formal definition first. For instance, Samuel who was unfamiliar with the formal definition of the limit of a function was reluctant to complete the task and struggled to understand how to perform the task. Samuel eventually, approached task one by first graphing and labeling $\delta = 1$, as a point. Samuel then graphed the rational function, the limit value of -7, and the c value of -5. Samuel graphed ε lastly, after simplifying the rational function to *x – 2* then substituted $f(x) = x - 2$ and $L = -7$ and simplified $|f(x) - L|$ to *x + 5*. Samuel decided to graph the line *x + 5* and labeled the line ε. Samuel had recognized that ε and δ somehow related but was uncertain of how. In an attempt to have them graphically relate Samuel changed δ as a point to a line (Figure 3.5.). Samuel again recognized that ε and δ were related in the second task by stating “I guess δ could be as big as you wanted to. Oh, it should correspond. They (ε and δ) should be somehow related.” Samuel did not describe the relationship between ε and δ but merely indicated there was one.

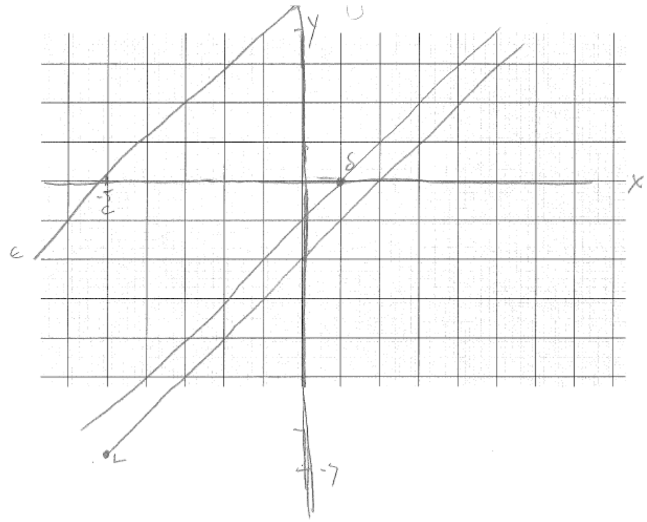


Figure 4. Samuel’s formal definition drawing of a limit of a function.

This pilot study showed that their understanding of absolute values impacted students’ covariational reasoning. For example, Joey, who previously demonstrated good covariational reasoning with graphical representations had difficulty determining delta algebraically. Joey understood that the delta represented a distance to the left and right of the *c* –value but incorrectly solved the problem by setting up an equation rather than an inequality with an absolute value. Therefore, these interviews demonstrated that inquiring about students’ understanding of absolute values graphically and algebraically is needed to determine if an error stems from a misconception of covariation or from an algebraic error.

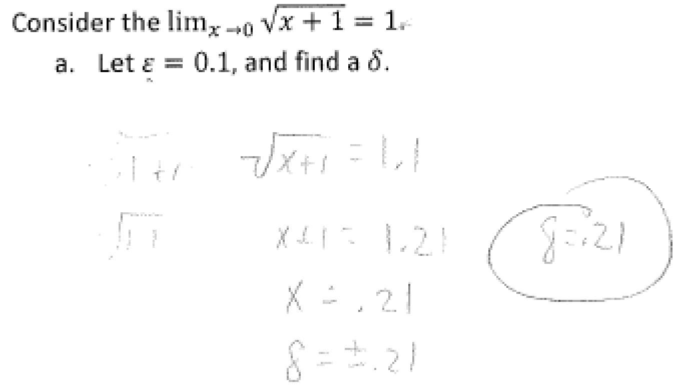


Figure 5. Joey’s calculation of delta.

The pilot study demonstrated that it would be appropriate to use Williams’ (1991) survey to select the group of interview participants. The students’ descriptions aligned with Williams’ (1991) six types of ways students thought about limits. Joey’s description of limits approaching from the left and right is “just a matter of plugging it in and see what it's going to answer,” aligned with the dynamical-practical characterization. Samuel’s description that, “the limit doesn't pass a certain point,” aligned with the limit acting as a boundary characterization. Jack’s explanation that limits are “a process of approaching but never really reaches it,” corresponds to the unreachable characterization. Since, there was a strong connection between the six characterizations and the participants’ concept image it was decided to use Williams’ (1991) survey to select a group of interview participants with a wide range of ways to think about limits.

Overall, this pilot study showed that there are tasks and questions that can capture students’ evoked concept images of limits. This study also provided insight to the researcher to create and modify the evoked concept image interview tasks and the evoked concept image surveys. The pilot study also informed the research design and when the instruments and interviews should occur.

Design

The study was designed to determine the depth of students' concept image and concept definition of limits in a Real Analysis course and if their concept image and concept definition relates to how they comprehend limit-proofs. To best address the study's first two research questions all the participants in the Real Analysis course were surveyed about their evoked concept image and evoked concept definition of limits of sequences and limits of functions after each unit, respectively. A grounded theory approach was used to investigate the students' conceptual understanding.

To better inform the student's concept image and concept definition of limits, data will include interviews. The decision to include interviews was based on the methodology of similar research investigating concept image and concept definition in advanced mathematic courses (Domingos, 2010; Pinto & Tall, 2002). This study as well as those, sought to provide a thorough description of Real Analysis students' concept image and concept definition, which vary depending on a person's experiences. The interpretive framework of social constructivism was used to allow participants to share with the researcher their subjective understanding of the concept. Additionally, observations on the instruction of limits were done throughout the semester. The information that will be gathered will be about how the concept is explained, examples, properties, processes, drawings, and other relevant information and discussions about the concept. This study used a constant comparative analysis, where the researcher analyzed the data as it was collected. This process allowed for the follow-up interviews to contain questions that addressed emergent themes among individuals' survey responses and addressed unique aspects.

To address the third research question about the relationship of Real Analysis students' concept image and concept definition about limits on proof comprehension incorporated Mejia-Ramos et al.'s (2012) proof comprehension assessment model. The proof comprehension model was used to construct the proof comprehension assessment. The proof comprehension assessment was administered on the last day of class. Follow-up interviews were done after the assessment as well.

Sample

The students to be in Real Analysis had successfully completed the prerequisite courses Calculus II and the Introduction to Advanced Mathematics course, with a grade of C or higher prior to the fall semester. According to the university's course catalog, the Introduction to Advanced Mathematics course presents the theory of sets, relations, functions, finite and infinite sets, and other selected topics.

There were two sections of Real Analysis during the fall semester of 2015 taught by different instructors. The sample was composed of volunteers enrolled in both sections of Real Analysis in the fall of 2015, who had given their informed consent to participate. There were three volunteers from one section, Amy, Kayla, and Vicky. Kayla was the only participant who was repeating this course. There were fifteen volunteers from the other section: Adam, Alan, Alicia, Arnold, Brandan, Carlton, Edith, Jessie, Maddie, Melody, Nick, Tim, Travis, Vincent, and Yolanda. Four of the students were applied mathematics majors, one mathematics major seeking teacher certification, eleven mathematics majors, and two mathematics majors double majoring in physics and finance, respectively.

The interview participants were selected with purposeful sampling and was based on the analysis of the students' responses on Williams' (1991) survey on how they conceptualize limits. The sampling method was chosen to maximize "a difference at the beginning of the study, [and] it increases the likelihood that the findings will reflect differences or different perspectives" (Creswell, 2013, p. 157). The ultimate deciding factor for interviewees was their willingness to partake in interviews.

Instruments

Limits of functions questionnaire. The first instrument administered served the purpose to determine how the participants thought about limits at the beginning of the semester. This was done to select the maximum number of participants with different perspectives for interviews. The study used (with permission) William's (1991) questionnaire about limits that consisted of the three following items:

A. Please mark the following six statements about limits as being true or false:

1. T F A limit describes how a function moves as *x* moves toward a certain point.
2. T F A limit is a number or a point past which a function cannot go.
3. T F A limit is a number that the *y*-values of a function can be made arbitrarily close to by restricting *x*-values.
4. T F A limit is a number or point the function gets close to but never reaches.
5. T F A limit is an approximation that can be made as accurate as you wish.
6. T F A limit determined by plugging in numbers closer and closer to a given number until the limit is reached.

B. Which of the above statements best describes a limit as you understand it? (Circle One)

1 2 3 4 5 6 None

C. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function f as $x \to c$ is number L.

Concept image instruments for limits. The two evoked concept image instruments were designed to capture a student's evoked concept image of a mathematical notion and determine how well it aligns with the concept definition. One of the evoked concept images focused on the limit of functions, and the other evoked concept image focused on the limit of sequences. Both instruments were constructed based off the pilot study and tasks found throughout the literature about concept image and concept definitions (e.g. Roh, 2008; Mamona-Downs, 2001; Pinto & Tall, 1999, 2002). The limits of sequences survey and limits of functions survey were each given at the end of their respective units. The instructors encouraged the students to do each survey as review for their respective exams. The surveys were administered at the end of the units to capture the student's concept image that was developed in the Real Analysis course.

These instruments were used to write the semi-structured follow-up interview protocols for the interview participants. Content validity for the evoked concept image instruments was done by referencing the literature on concept image and concept definition of limits. Subject-experts were mathematics professors with prior experience teaching Calculus and Real Analysis. The expert evaluated the themes and appropriateness of the items and provided feedback. The feedback was analyzed, and the instrument will be revised.

Proof comprehension assessment. The proof comprehension test was designed based on Mejia et al.'s proof comprehension model. The proof comprehension was composed of seven dimensions. Four limit-proofs were selected by the instructors and the researcher. Based on the selection of the proofs, appropriate question types from the seven dimensions were selected. Examples of types of questions from the literature of the seven dimensions are provided in Table 2. Content validity for the proof comprehension assessment was done by Real Analysis professors who served as subject-experts. The Real Analysis professors provided the appropriate proofs for the assessment. The proof comprehension assessment served as a test review for the students. The concurrent validity was conducted with interviews to ensure that the assessment score accurately estimates their proof comprehension.

Table 2.
Example questions for the seven dimensions of proof comprehension.

Dimension	Definition	Example Question*
Meaning of terms and statement	Understanding the meaning of symbols, terms, and definitions.	What does the symbol $\exists$mean?
Justification of claims	Understanding how new assertions in the proof follow from previous ones.	In the proof, which justification best explains why…?
Logical structure	Understanding the logical relationship between lines or components of a proof.	What is the logical relationship between the following two lines?
Higher level ideas	Identifying a good summary of the overarching approach of the proof.	Which of the following is the best summary of…?
General method	Applying the methods with the proof to a different context.	Could the method of the proof in line X be used to prove…?
Identifying the modular structure	Understanding the main components and modules within a proof and the logical relationship between them.	Which of the following explains why … was included in the proof?
Application to examples	Using the ideas in the proof in terms of a specific example.	Using the logic of the proof which best exemplifies why $x = 5$ is not a solution to $f(x)$?

*Hodds, Alcock, & Inglis (2014)

Procedures

Before the beginning of the semester the researcher met with both instructors to discuss the study and procedures. During the second class, the researcher introduced herself to the two classes and explained the research goals and rationale, as well as distributed and collected signed consent forms from the students. The participants also were given the first survey to complete in a week. An analysis of the first surveys was done during the second week of classes. Upon completion, the case study participants were selected and emailed about partaking in interviews. The follow-up interviews took place during the third week of the semester.

The unit of limits of sequences was taught during the month of October. The students were given the first evoked concept image as a reflection during the last week of October before their first exam. The unit of limits of functions was taught in November. The second evoked concept image survey about limits of functions, was similarly given as a reflection during the second to last week of the semester.

The follow-up interviews were scheduled via email after they submitted each round of the evoked concept image surveys. The interviews were conducted in a graduate office and were recorded with the signed permission of the students. The interviews were semi-structured and designed based on the students' responses on the surveys. Each interviewees' questions were tailored to their responses on the surveys. The interviews allowed the interviewees the freedom to add additional information or attempt to express the concepts in a different format and draw any images or graphs to help articulate their ideas. The interviews were transcribed using a transcription company. The transcriptions were coded and compared to their evoked concept image surveys.

Non-participant observations were done throughout both units and the observer sat in the back of the classroom. The observations audio recorded the lectures with consent of the instructor and the observer documented the board work. Pictures of the board work were taken with an IPad to crosscheck the written notes of the observation. The observations documented the instructors' word choice and written explanations, as well as students' comments and questions.

The observations that will be done with a Livescribe pen to recording the instructors' boards will be cross-referenced with pictures of the board work after each observation. The Livescribe recordings will be transcribed and coded. The semi-structured interviews will be transcribed and coded as well. Creswell (2013) warns against the use of preexisting codes since they may limit the analysis. Thus, open coding will also be used to allow for unexpected events that might provide surprising or interesting information (Creswell, 2013). Creswell's data analysis spiral (p. 183) will guide the overall structure of the qualitative analysis. The overall analysis of the students evoked concept image surveys, task-based interviews, and the themes from the observations triangulated will provide an in depth investigation into their concept image and concept definition of limits.

Around fall break, the instructors provided proofs for the proof comprehension assessment. The questions for the proof comprehension were generated based off on Mejia-Ramos et al.'s assessment model. With the permission of the instructors, the students took the assessment in the class. The timeline of data collection for the proposed study is presented in Table 3.

Table 3.
Semester timeline.

Week(s)	Data Collection	Data Analysis	Interviews
1	Williams' (1991) limits questionnaire		
2		Analyze initial data and select interview participants.	Contact potential interview participants.
3			Initial interviews
4-10	Observations		
10	Limits of sequences evoked concept image survey	Initial analysis of survey.	
11-14	Observations		Follow-up interviews
14	Limits of functions evoked concept image survey	Initial analysis of survey.	
15	Proof comprehension assessment		Follow-up interviews
16			Follow-up interviews

Data Analysis

Limits of functions. The first data collection was Williams' (1991) questionnaire about limits of functions. The questionnaire determined how the students thought about limits of functions. The data provided from the questionnaire provided the information needed to conduct purposeful sampling.

The first criterion for selecting interview participants was based on their responses of the true and false statements. The students were categorized based on their selections of the true and false statements. There were three groups: those who selected the statements to all be true, those who chose half the statements to be true and the other half to be false, and lastly those who selected only one or two of the statements as true. Two students who chose all the statements to be true were selected. Then from each of the other two groups one was chosen based on selecting the formal statement as true, and the other was chosen based on them selecting the formal statement to be false. The students who selected the formal statement as true either thought the statement was an accurate description for limits of functions and demonstrated that they incorporated the formal definition into their concept image. Whereas those who selected it be false thought that the statement was not an accurate description and did not demonstrate it to be a part of their concept image.

Next all of the students' selection of which statement best described a limit, as they understood it, were analyzed. From their selections, three groups emerged: formal, dynamical theoretical, and unreachable. The next criterion was that the initial six students chosen were representative of the formal, dynamical theoretical, and unreachable, and that the two people from the same true and false statement groups had selected different best statement choices. The six participants were then contacted via email and agreed to partake in interviews. A summary of the six participant's responses are in Table 4.

Table 4.
Summary of interview participant's responses to the initial survey.

	Alan	Kayla	Maddie	Vicky	Vincent	Edith
Dynamical-theoretical	True	True*	True	True*	True	True*
Acting as a boundary	True	True	True	True	False	False
Formal	True*	True	True	False	True*	False
Unreachable	True	True	True*	True	False	False
Acing as an approximation	True	True	False	False	False	False
Dynamic-practical	True	True	False	False	False	False

*The statement that best represents how they think about limits of functions.

Concept image and concept definition. The data from the surveys, assessment, observations, and interviewers were analyzed in three levels. The first was coding the data for the five themes: concept definition, and the four domains of the concept image, mental images, example space, processes, and properties. Next, open coding was done for each of the five themes to capture the diverse aspects held by Real Analysis students. The analysis for the five themes are described below.

Mental images. Mental images are one of the four components of a person's concept image. A mental image is a representation in a person's mind of the mathematical objects and concepts. Mental images can be a picture, metaphor, or description that are derived from a collection of explanations, drawings, and graphics built on a person's past experiences (both explicitly remembered and not). The students' sequence of limits survey, limits of functions survey, interviews, and observations were coded in three levels. The first level was open coding for pictures, metaphors, and/or descriptions of limits. Next, from axial coding emerged the following themes: correctness, mathematically usable, characterization of limits, and consistency. The first theme to appear was incorrect and correct descriptions and images. The data was coded as incorrect if there where errors in drawings and if the descriptions were mathematically flawed and coded as correct if no errors were present.

The second theme that emerged was whether the descriptions and/or mathematical drawing were mathematically usable. A description or drawing was considered usable if it was something that would be useful in the mathematics course. The students' descriptions and drawings were the following types: a memory relating to when they first learned limits, a real world application of limits, how the concept of limit was associated to another mathematical concept, the limiting process, the limit as an object, and/or the limit as a procept. At the selective level of coding those descriptions and drawings were categorized as unusable if it was associated to everyday content such as speed limit, or a word association to a vague and general mathematical context such as calculus. The descriptions and drawings were coded as usable if it describes the concept as a mathematical object, process, how the concept relates to another mathematical concept, or how the concept is applied.

The third theme that emerged was Williams' characterizations of limits. Therefore, Williams' (1991) six characterizations of functions were used to code the limits of functions' descriptions and mental images. Williams' six characterizations were adapted for limits of sequences (Table 5) and coded similarly. Lastly, each student's evoked mental image was coded for consistency. The student's descriptions' characterizations were analyzed to determine if the characterizations were consistent with their mathematical drawings' characterization.

Table 5.
Six characterizations of limits of sequences.

Characterization	Statement
Dynamical-Theoretical	A limit describes how a sequence moves as *n* moves towards positive infinity.
Acting as a Boundary	A limit is a number or a point past which a sequence cannot go.
Formal	A limit is a number that the a_n of a sequence can be made arbitrarily close to for all sufficiently large n.
Unreachable	A limit is a number or point the sequence gets close to but never reaches.
Acting as an Approximation	A limit is an approximation that can be made as accurate as you wish.
Dynamic-Practical	A limit is determined by plugging in natural numbers closer and closer towards infinity until the limit is reached.

Example space. The students were given prompts to generate as many examples as possible of sequences with a specific limit and functions with another specific limit. The provided prompt was designed to have students generate as many examples with the same limit without posing any restrictions. The students' example spaces were coded using Zazkis & Leikin's (2007) framework of generality, correctness, richness, and accessibility.

Zazkis & Leikin's (2007) dichotomous generality code categorized an example as either specific or general, but notes that "perception of generality is individual." An example could either be considered a specific variation of a classical example, or serve as a representative of a class of examples, even if the class is not described explicitly. Zazkis and Leikin defined an example as correct, if it satisfies the conditions of the task. Zazkis and Leikin's third code, richness evaluates all the examples generated by one student and determines if there was fluency in the variety, if the examples were routine or non-routine, what the different types were and how their examples related to the collective evoked example space. The last category of Zazkis & Leikin's (2007) framework was accessibility. Accessibility determines whether the student struggled to generate an example or not. This code was not applicable to coding since the students were not observed as they generated their examples.

Processes. The third domain of a concept image is the associated processes. The students' sequence of limits survey, limits of functions survey, interviews, and observations were coded in three levels for process. They were first open coded for calculations of any kind of method which included traditional arithmetic, algebra, approximating, plugging in numbers, making a table, graphically evaluating, and descriptions of any process. The processes were sorted based on whether they directly used the formal definition and those that did not. Then subcategorized as algebraic processes, graphical process, and descriptions of processes.

The algebraic processes were compared among each other to look for common methods. One method was to see if a graphical representation was used as a tool to assist in setting up the algebraic equations or calculations. Lastly, one method was to see if there were mathematical errors such algebra mistakes, misuse of notation, and incorrect conclusions. The explanations and descriptions were analyzed for correctness, and compared to the formal definition for accuracy and detail.

Properties. The last domain of a person's concept image is properties. Like the other three domains, a person's understanding of a mathematical concept's properties a drawn from a person's past experiences (both explicitly remembered and not). Open coding began noticing different groupings of properties were present for limits that exist, whereas some of those properties were missing for those limits that did not exist. Therefore, students' examples, were coded based on what properties were present.

The calculations of limits were analyzed to determine what properties of limits were evoked. The properties of limits included the constant, the multiplicative constant, the sum, the difference, the product, the quotient, and the power properties. The algebraic computations of the limits were analyzed to determine which properties were correctly or incorrectly used. The algebraic work was coded on whether the property was explicitly or implicitly applied. For instance, an explicit use of the multiplicative constant property would be$\lim_{n\to\infty}\frac{7\sin(n)}{n} = 7\lim_{n\to\infty}\frac{\sin(n)}{n} = 7(0) = 0$ and the implicit use of the property would be $\lim_{n\to\infty}\frac{7\sin(n)}{n} = 0$.

Concept definition. The students' concept definitions were coded as informal or formal. The students' evoked concept definitions were coded as formal if they incorporated any portion of the formal definition. For example, "the sequence of limit exists if there is a small number such that $|a_n - A| < \varepsilon$," incorporates the $|a_n - A| < \varepsilon$ portion of the formal definition and therefore would be coded as formal. The evoked concept definitions that were coded as formal were then analyzed for completeness. For instance, the above evoked concept definition example would be coded as incomplete because it did not incorporate all of the portions of the formal definition. The evoked concept definitions were also coded as having correct ordering or incorrect ordering. Lastly, the students' evoked concept definitions were coded for being a mixture of symbols and words, or purely symbolic.

Alignment with formal definition. The data was analyzed for potential conflict factors that misaligned with the formal definitions. Tall and Vinner (1981) defined factors that conflict with other factors within a particular domain on the concept image, with different factors across the domains, or with factors within their concept definition as potential conflict factors. Tall and Vinner (1981) defined a potential conflict factor to be serious if it misaligns with the formal definitions.

To determine if a student held a serious potential conflict factor, the student's responses, interviews, and comments were checked for accuracy and errors against the formal definition. If error was a minor algebraic, graphical, or notational error they were not classified as serious potential conflict factors. A potential conflict factor was considered serious if it added unnecessary conditions to the formal definition or contradicted the formal definition. If a person had demonstrated a serious potential conflict factor they were categorized as a part of the cognitive conflict group. If no serious potential conflict factors were present for that student, they were categorized as a part of the conflict resolution group.

Proof comprehension assessment. The proof comprehension assessment questions were first organized by the seven dimension of the proof comprehension model (Mejia Ramos et al., 2012). Only four dimensions of the model were deemed appropriate to use because of the proofs provided by the instructors. The proofs were relatively short and straightforward and therefore asking the students to identify the modular structures was deemed inappropriate. Lastly, a majority of the limit proofs presented throughout the course, including these three proofs were direct proofs. Therefore, asking the students to identify that the method was a direct proof, or direct proof with cases would not provide much insight. Therefore, both the dimensions of transferring the general method and logical proof framework were excluded from the limit proof comprehension assessment.

The four dimensions used in the proof comprehension assessment were: meaning of terms and statements, justification of claims, summarizing via high-level ideas, and illustrating with examples. For each dimension, all of the students' responses were first analyzed for correctness. If there were errors that occur in their responses, their evoked concept images and evoked concept definitions was compared to see if there were similar errors. If there were similarities found between a students' response and a specific domain of their evoked concept images and definitions, all of the students' responses were then compared to their respective domains. The students' responses were then compared and contrasted among the others for similarities and differences. Lastly, the students' responses were open coded to see if a relationship existed between to the four domains of concept image and concept definition.

CHAPTER 4

RESULTS

Introduction

The purpose of this study will be to describe students' conceptual understanding of limits in terms of concept image and concept definition. As well as explore the relationship between students' concept image and concept definition of limits and students' proof comprehension. In this chapter, the research questions will be answered based on data collected from surveys, interviews, and observations. The research questions for this study are as follows:

1. What are the concept images and concept definitions of limits held by Real Analysis students?
2. How does students' concept images and concept definitions of limits relate to their understanding of the formal definitions of limit?
3. How do the students' concept images relate to their comprehension of limit proofs?

To address the first research question, the findings begin with discussing the Real Analysis students' evoked concept images within each of the four domains: mental images, example space, properties, and processes. Followed by the Real Analysis students' evoked concept definitions for both limits of sequences and limits of functions.

There were two emerging types of concept images held by Real Analysis students, those that demonstrated cognitive conflict, and those that had cognitive resolution. To address the second research question, both groups' concept images' alignment with the formal definition will be presented, highlighting students' serious potential conflicts with the formal definitions. Lastly, the results will show how different domains of concept image and concept definition relate to the different dimensions of a proof comprehension assessment.

Concept Image of Limits

Concept image is the total cognitive structure that is associated with a concept, which includes mental pictures, example space (Mason & Watson, 2005), properties and processes. An individual's concept image is complex and built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures (Fukawa-Connelly & Newton, 2014; Tall & Vinner, 1981). To gain an understanding of the diverse concept images held by Real Analysis students, their evoked concept images were captured with surveys about limits of sequences and limits of functions, semi-structured interviews, and observations. A student's evoked concept, is a person's concept image that is activated at a specific time.

Eighteen students from two different Real Analysis courses participated in the study (Table 6). Twelve of the eighteen students submitted both the limits of sequences and the limits of functions surveys. Two students only submitted the limits of sequences surveys, and four students only submitted the limits of functions surveys. Alan, Edith, Kayla, Maddie, Vicky, Vincent were the six of the eighteen students participated in follow-up interviews. One interview participant, Kayla had stopped attending class and did not complete all the surveys and interviews. Therefore, one additional student Arnold was interview interviewed at the end of the study. The selection of including Arnold was based on his willingness to participate in an interview. Observations for both sections were both done throughout the semester. The next sections describe the eighteen students' evoked concept images within each of the four domains: mental images, example space, processes, and properties.

Table 6.
Participants' completion of surveys of limits.

	Limits of Sequences	Limits of Functions
Adam	-	x
Alan*	x	x
Alicia	x	-
Amy	x	x
Arnold*	x	x
Brandan	-	x
Carlton	x	x
Edith*	x	x
Jessie	x	x
Kayla*	x	-
Maddie*	x	x
Melody	x	x
Nick	x	x
Tim	x	x
Travis	-	x
Vicky*	x	x
Vincent*	x	x
Yolanda	-	x

*Interview participants.

The analysis of limits distinguishes between limits of sequences and limits of functions, because Real Analysis courses teach them distinctively. The Real Analysis courses first investigate the specific type of limits of functions, limits of sequences and then transition to study the general limits of functions. Despite that a sequence is a function with a modified domain, students do not necessarily recognize the connection. Therefore, this study collected data at the end of each unit and was analyzed to capture the differences and connections between limits of sequences and limits of functions within their concept images.

Evoked mental images. Mental images are one of the four components of an individual's concept image. A mental image is a representation in a person's mind of the mathematical objects and concepts. Mental images include pictures, metaphors, or descriptions that are derived from a collection of explanations, drawings, and graphics built by a person's past experiences (both explicitly remembered and not). Mental images are triggered by a task, cues, environment, and recent experience. Evoked mental images are the ones generated by the student at a given time.

Limits of sequences. The students' descriptions and mathematical drawings of how they think and visualize limits of sequences were analyzed using the following emerging codes: mathematically usable, correctness, characterization of limits of sequences, and consistency. Mathematically usable coded the descriptions and mathematical drawings as usable if it related to one of the following categories: a real world application of the concept of limit of sequences, how the concept of limit of sequences is associated to another mathematical concept such as series, the limiting process, the limit as an object, and/or the limit as a procept. Descriptions and mathematical drawings were coded as unusable if it was associated to non-mathematical content such as speed limit, or a word association to a vague and general mathematical context such as calculus. Each usable description and mathematical drawing was the analyzed on for correctness, which was based on whether there were any mathematical errors.

Each description and mathematical drawing was categorized based off the six characterizations of limits of sequences. The six characterizations of limits of sequences were an adaption from Williams' (1991) six characterizations of functions statements (Table 9). Lastly, each student's evoked mental image was coded for consistency.

Consistency analyzed whether the student's descriptions and mathematical drawings were of the same type of characterization.

Table 7.
Six characterizations of limits of sequences.

Characterization	Statement
Dynamical-Theoretical	A limit describes how a sequence moves as *n* increases towards positive infinity.
Acting as a Boundary	A limit is a number or a point past which a sequence cannot go.
Formal	A limit is a number that the a_n of a sequence can be made arbitrarily close to for all sufficiently large n.
Unreachable	A limit is a number or point the sequence gets close to but never reaches.
Acting as an Approximation	A limit is an approximation that can be made as accurate as you wish.
Dynamic-Practical	A limit is determined by plugging in natural numbers closer and closer towards infinity until the limit is reached.

Characterizations. Williams's (1991) six statements of characterization of limits of functions were altered to be characterization of limits. Each student's descriptions and graphical representations were coded as either dynamical-theoretical, acting as a boundary, formal, unreachable, acting as an approximation, or dynamical practical and then compared for consistency. There was only one student, Amy who generated unusable evoked mental images of limits of sequences (Table 8). Amy was very vague and stated she visualized limits of sequences graphically but did not generate a graph. Therefore, Amy's response was could not be categorized. Below a table summarizes the characterizations of the participants' usable mental images of limits of sequences (Table 8).

Table 8.
Characterization of participants' usable mental images of limits of sequences.

	Acting as an Approximation	Unreachable	Acting as a Boundary	Dynamical-Practical	Dynamical-Theoretical	Formal
Alan	-	-	x	-	x	-
Alicia	-	-	x	-	-	-
Amy*	-	-	-	-	-	-
Arnold	-	-	-	-	x	-
Carlton	-	x	x	-	-	-
Edith	-	-	-	-	x	-
Jessie	-	-	x	-	x	-
Kayla	-	x	x	-	-	-
Maddie	-	-	-	-	x	-
Melody	-	-	-	-	-	x
Nick	-	-	-	-	-	x
Tim	-	-	-	-	-	x
Vicky	-	-	-	-	x	-
Vincent	-	-	x	-	x	-

*Generated only unusable mental images.

There were six students, Alan, Alicia, Carlton, Jessie, Kayla, and Vincent who produced mental images that included the characterization of as acting as a boundary. Alicia generated graphical representations of a sequence approaching a limit from one direction but included the condition of being bounded. Alicia's first number line had a sequence bounded above and the second graph had a sequence bounded below, labeled the lower and upper bounds as *B*. (Figure 6). Alicia described that she visualized a sequence as "either converging (or) diverging and whether or not it (was) bounded." Alicia's mental images was solely characterized as acting as a boundary since Alicia did not refer to either the limit process, the formal definition or indicate in the limit was achieved or not.

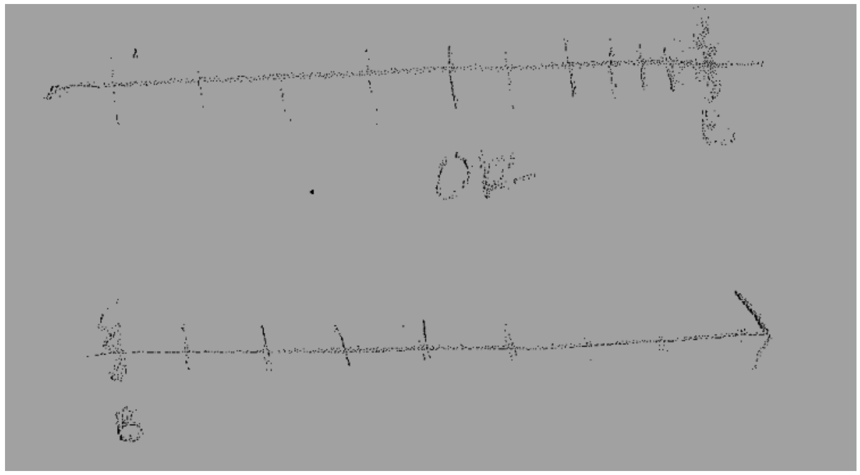


Figure 6. Alicia’s evoked mathematical drawing of limits of sequences.

The other student’s mental images that had the acting as a boundary characterization, also had another characterization. Carlton’s simile “like a wall where a function goes up to not until it gets close to hitting it or hits it” was dually categorized as unreachable and acting as a boundary. Carlton did not provide any graphical representations with his description. Similarly, Kayla’s graphical representation of an increasing and bounded above function (Figure 7) was categorized as unreachable and acting as a boundary. The graph was of a function rather than a sequence (continuous domain rather than a discrete domain) which demonstrated that Kayla’s mental images of the two types of limits were not distinct. In the interview, Kayla did recognize the limit process was “tending towards” a mathematical object, *L*.

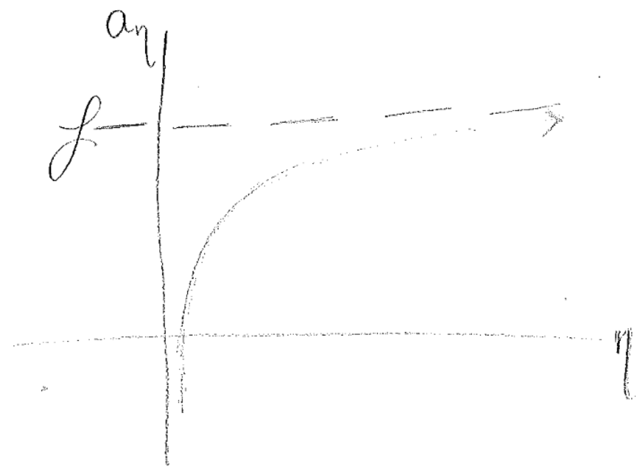


Figure 7. Kayla's evoked mathematical drawing of limits of sequences.

The third student was Jessie who generated a graphical representation of an increasing, bounded above sequence with a limit of zero (Figure 8). In Jessie's drawing the limit was acting as a boundary, and the arrow above the number line indicated the direction of the sequence above the number line showing process that was characterized as dynamic theoretical. Jessie's description did not encompass one of the six characteristics since Jessie generated the telescoping series as an example of how sequences relate to summations and series.

Figure 8. Jessie's evoked mathematical drawing of limits of sequences.

Alan and Vincent each generated three graphical representations of limits of sequences to show the process of limit and the limit as an object. Both produced an increasing, bounded above sequence and a decreasing, bounded below sequence were on the Cartesian plane. Since boundaries appeared in two of their images their mental images included the acting as a boundary characterization. The third graphical representation Alan generate was the specific sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$, drawn on a number. Alan generated this exampled to demonstrate that “the limit of a sequence is a number that a sequence approaches” such as zero (Figure 9). Alan also noted that “not all sequences have limits.” Vincent’s third graphical representation was another general example of an oscillating sequence that eventually approaches to a value on the Cartesian plane. Vincent’s three graphical representations are how he “imagin(es) an arbitrarily large amount of points converging toward some specific value, in both directions. In one direction, increasing, decreasing, and alternating” (Figure 10). Thus, their mental images were dually characterized as dynamical-theoretical since Alan and Vincent described and illustrated how a sequence moves as *n* moves towards positive infinity.

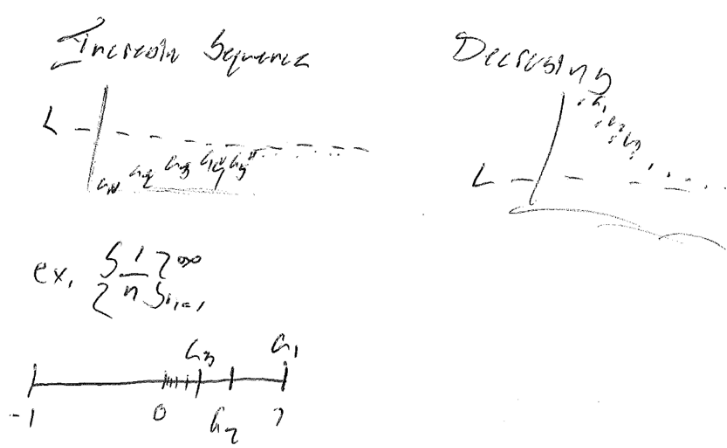


Figure 9. Alan’s evoked mathematical drawing of limits of sequences.

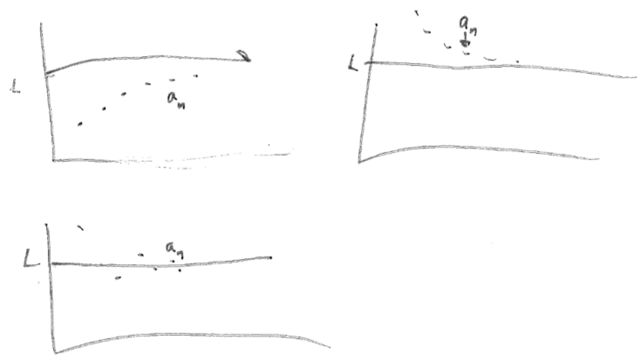


Figure 10. Vincent's evoked mathematical drawing of limits of sequences.

There were four other students, Arnold, Edith, Maddie, and Vicky whose mental images were solely characterized as dynamic-theoretical. Edith and Arnold generated a corresponding graphical representation. Arnold "visualize(d) limits of sequences on a number line and where the sequence is going along that number line … heading towards a fixed point, or possibly no point if there is no limit." Arnold's graphical representation (Figure 11) was different than Edith's (Figure 12) because it showed a sequence within an interval approaching a limit from one direction. Whereas, Edith explicitly discussed the sequence being able to approach from both sides. Edith, stated "as the variable gets larger and larger it (the sequence) approaches a certain number." Both Arnold and Edith's mental images informally incorporated the process of a limit.

Figure 11. Arnold's evoked mathematical drawing of limits of sequences.

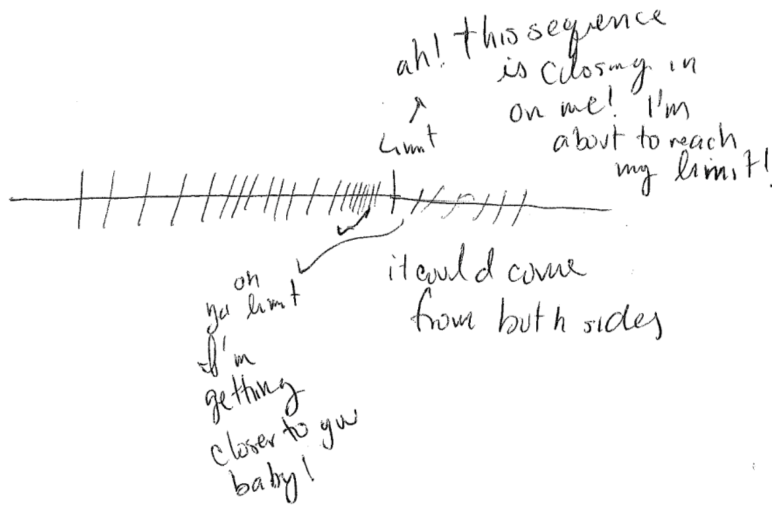


Figure 12. Edith's evoked mathematical drawing of limits of sequences.

Maddie provided the description that "the limit shows where a specific sequence of numbers approaches with respect to a given direction." In the follow-up interview Maddie clarified she thought that sequences' domains were increasing or decreasing, like limits of functions. Vicky's mental images held a similar incorrect understanding, Vicky described the process of limits of sequences "as n gets higher or lower, the solution of the sequence getting higher or lower." Both were incorrect mental images that described the process and were categorized as dynamical theoretical.

The last three students Melody, Tim, and Nick generated formal mental images. Nick only generated a description while Tim and Melody generated a description with a corresponding formal graphical representation. Nick wrote he visualized "that after some member in the range, say, a_N the members become arbitrarily close or equal to some number." Tim's description of his graphical representation (Figure 13) was similar to

Nick's. Tim wrote "I think of limits of sequences as a number which after a particular element in the sequence, the preceding elements' values settle at." Both Nick and Tim's mental images incorporate the role of the formal definition's *N* in their limit process.

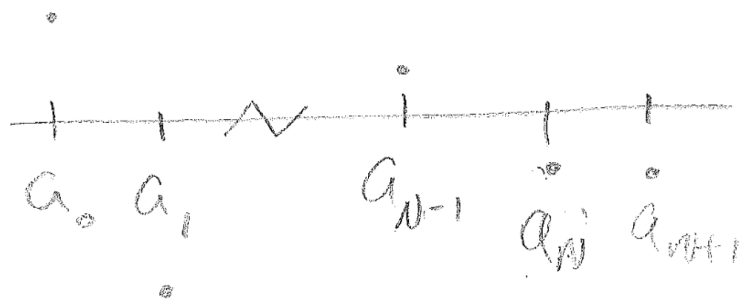


Figure 13. Tim's evoked mathematical drawing of limits of sequences.

Melody's mental images not only included the role of *N* but also the tolerance around the limit (Figure 14). Melody described the process as "when you get past a certain number in a sequence the rest of the sequence will be in a constrained bracket $|a_n - L| < c$." Melody's mental image was the most formal out of all of the students.

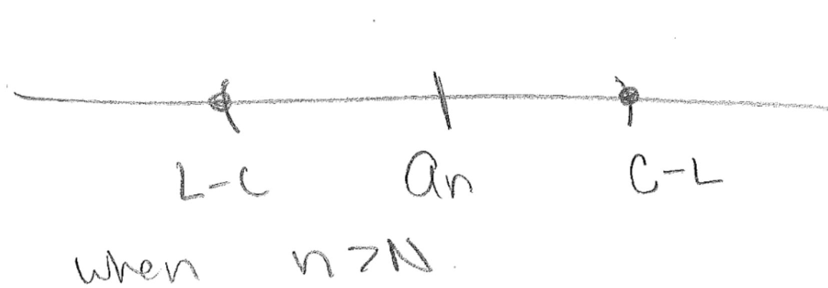


Figure 14. Melody's evoked mathematical drawing of limits of sequences.

Correctness. Thirteen of the students did not generate any mathematical errors or demonstrate a missing understanding of limits of sequences in their evoked mental images. Of the five students who had mathematical errors, two of the students, Vicky and

Maddie provided descriptions that highlighted their misunderstanding that the *n*-values of the limit of sequences increases towards infinity. Vicky's description incorrectly discussed the relationship between input and output increasing and decreasing together, respectively. Whereas, Maddie discussed the sequence approaching a value "with respect to a given direction" and not distinguishing that the *n*-value only increase towards infinity, which is one of the difference between the limits of sequences and the limits of functions.

Kayla's graphical representation (Figure 7) and Carlton's metaphor used functions in their mental images rather than a sequence, and did not allude that a sequence was a specific type of function. Therefore, it is not clear if this was an error or that they understood the connection between sequences and functions, but it is important to note. There was also one algebra error that appeared in all the evoked mental images. Melody had described that limit as "when you get past a certain number in a sequence the rest of the sequence will be in a constrained bracket $|a_n - L| < c$" with a graphical representation. To show the "constrained bracket" $|a_n - L| < c$, Melody expanded it incorrectly as $(L - c, c - L)$ (Figure 14). Melody did not provide the scratch work.

Limits of functions. The students' descriptions and mathematical drawing of how they think and visualize limits of functions were analyzed for mathematically usability, correctness, characterization of limits of functions, and consistency. Mathematically usable coded the descriptions and mathematical drawing as usable if it described one of the following: a real world application of the concept of limit of functions, how the concept of limit of functions is associated to another mathematical concept such as continuity, or derivatives, the limiting process, the limit as an object, and/or the limit as a procept. A description or mathematical drawing was coded unusable if it was associated to non-mathematical content such as speed limit, or a word association to a vague and general mathematical content such as calculus. Each usable description and mathematical drawing was the analyzed on for correctness, which was based on whether there were any mathematical errors.

The third categorized each description and mathematical drawing based off of Williams' (1991) six characterizations of limits of functions (Table 9). Lastly, each student's evoked mental image was coded for consistency. Consistency analyzed whether the descriptions' characterization was different than the mathematical drawings' characterization.

Table 9.
Six characterizations of limits of functions.

Characterization	Statement
Dynamical-Theoretical	A limit describes how a function moves as *x* moves toward a certain point.
Acting as a Boundary	A limit is a number or a point past which a function cannot go.
Formal	A limit is a number that the *y*-values of a function can be made arbitrarily close to by restricting *x*-values.
Unreachable	A limit is a number or point the function gets close to but never reaches.
Acting as an Approximation	A limit is an approximation that can be made as accurate as you wish.
Dynamic-Practical	A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

*Williams (1991)

Characterizations. Williams's (1991) six statements of characterization of limits of functions were used to characterize the students limits of functions' mental images. Jessie and Amy provided unusable evoked mental images. Jessie responded that he thought and visualized limits of function "same as last time" referring to his response to how he thought about limits of sequences. Amy initially had provided no response and when asked in a follow-up conversation how she thought about limits she responded that she thought about calculus and visualized limits of functions "by drawing graphs." Therefore, both responses were too vague to be characterized. Below a table summarizes the characterizations of the participants' usable mental images of limits of functions (Table 10).

Table 10.
Characterizations of participants' usable mental images of limits of functions.

	Acting as an Approximation	Unreachable	Acting as a Boundary	Dynamical-Practical	Dynamical-Theoretical	Formal
Adam	-	x	x	-	-	-
Alan	-	-	-	x	-	x
Amy*	-	-	-	-	-	-
Arnold	-	-	-	-	-	x
Brandan	-	-	-	-	x	-
Carlton	-	-	-	-	-	x
Edith	-	-	-	-	-	x
Jessie*	-	-	-	-	-	-
Maddie	-	-	x	-	x	-
Melody	-	x	x	-	x	-
Nick	-	-	-	-	-	x
Tim	-	-	-	-	-	x
Travis	-	-	-	-	x	-
Vicky	-	-	-	x	-	-
Vincent	-	-	-	-	-	x
Yolanda	-	-	-	-	-	x

*Generated only unusable mental images.

There were other students who provided both a description and a graphical representation, but either the description or graphical representation was unusable. Nick and Yolanda provide usable graphical representations but vague statements. Nick stated limits of functions are “beautifully analogous to limits of sequences, which are quite interesting”, and Yolanda stated “very similar to limits of sequences.” Carlton wrote, “They are like limits of sequences except with an added part to cover the plane.” The three students did also generate other mental images that were usable and were appropriately characterized.

There were three students, Adam, Melody, and Maddie who produced mental images that included the characterization of as acting as a boundary. Adam provided a graphical representation of a function (Figure 15) and labeled both the vertical and horizontal asymptotes with a description to plot the “graph values, then try to locate where the points approach to look for an asymptote.” Adam’s mental images consistently were oriented around the limiting process of asymptotic functions and therefore had the dual characterization of acting as a boundary and unreachable. Adam generated descriptions and graphical representations of functions that got arbitrarily close to but never reached the value and was bounded by the asymptote.

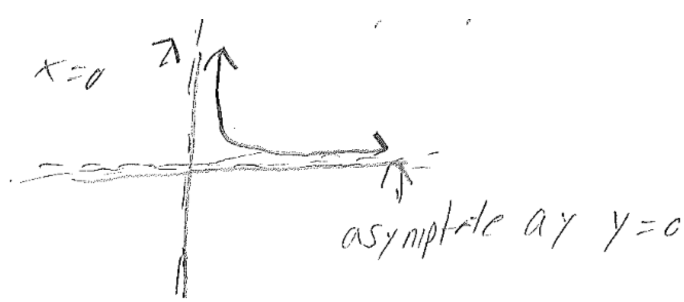


Figure 15. Adam’s evoked mathematical drawing of limits of functions.

Similarly, Melody generated a graphical representation that was of an asymptotic function and therefore was dually characterized as acting as a boundary and unreachable.

In her graphical representation Melody did not include the tolerance window (Figure 16). However, Melody's statement "I think of limits as a number goes to infinity or another number it approaches the number L" was characterized as dynamical theoretical since it described how the *x*-values and function move.

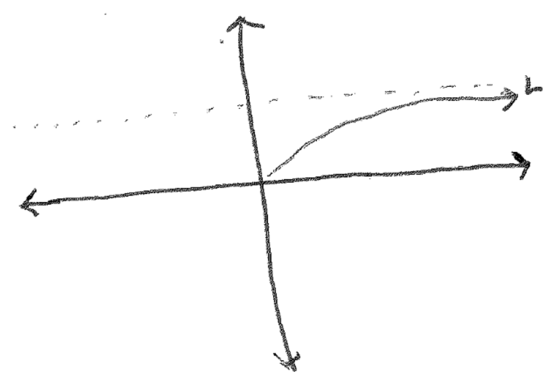


Figure 16. Melody's evoked mathematical drawing of limits of functions.

Maddie's mental images also had a dual characterization of dynamical theoretical and acting as a boundary. Maddie wrote that "for a limit of $f(x)$ to exist it must satisfy the three conditions of being continuous where the limit approach one value, bounded, and not oscillating." Maddie discussed bounded functions as well as the process of the *x*-values approaching a value and the function simultaneously approaching the limit. Maddie's included two-sided limits but also additional conditions that a limit must satisfy, which Maddie overgeneralized to include continuity.

Travis and Brandan, whose evoked mental images were consistently characterized as dynamical theoretical since both discussed the simultaneous movement of the *x*-values and function. Travis's proceptual description discussed limits of functions being a mathematical object that are "special points" that exist when a "function approaches (the) point from (the) left and right." Travis's evoked mental images were rooted in the concept that limits are two-sided limits, where the left- and right-handed limits must satisfy the condition of being equal. Therefore, jump discontinuities are an instance when the limit does not exist. Travis provided a graphical representation of this case, showing at $x = 4$, the function has a jump discontinuity thus the limit does not exist at that "special point" (Figure 17).

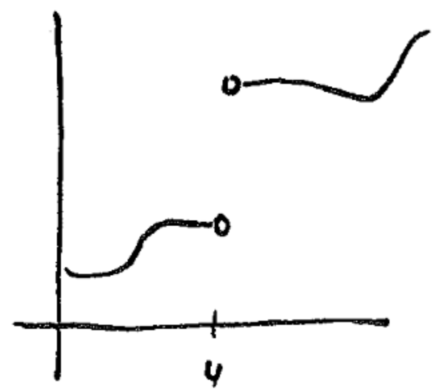


Figure 17. Travis's evoked mathematical drawing of limits of functions.

Brandan discussed "x approaching to c from both sides then the corresponding value of $f(x)$ also gets closer to A" and provided a graphical representation (Figure 18). Within the graphical representation he indicated the process of approaching c from both sides and indicated this was a continuous function so that $f(c) = A$. Brandan's evoked mental images demonstrated that he had a proceptual understanding and was characterized as dynamical theoretical.

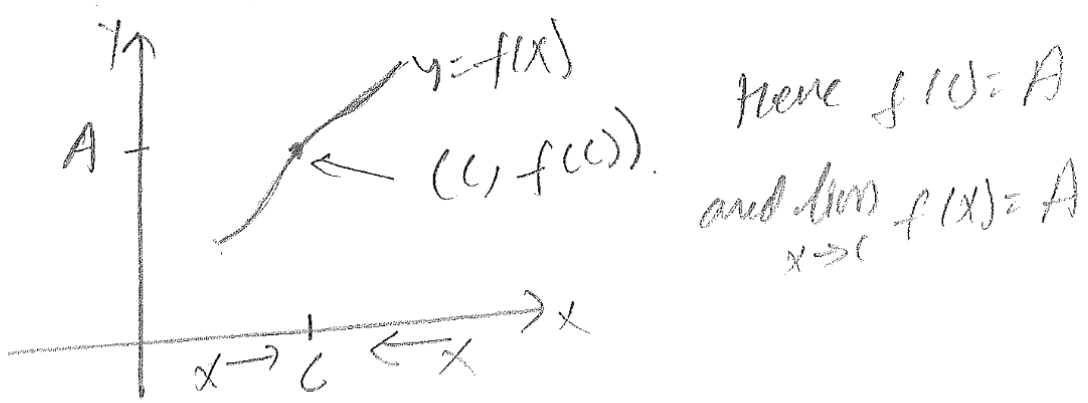


Figure 18. Brandan’s evoked mathematical drawing of limits of functions.

Vicky had a dynamical-practical evoked mental image were she discussed plugging in numbers closer and closer to a given number until the limit is reached. When Vicky was asked to describe or draw how she visualizes limits of functions she generated a graphical representation of a function with asymptotic behavior (Figure 19). Vicky did not offer any clear indication about what the limit but vaguely discussed “plugging in numbers.”

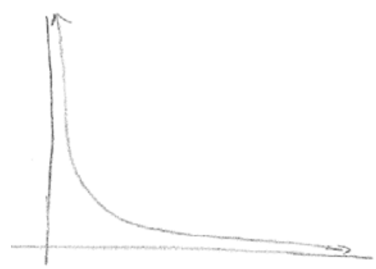


Figure 19. Vicky’s evoked mathematical drawing of limits of functions.

Adam, Melody, Maddie, Travis, Brandan, and Vicky were the students who did not discussed limits as a number that the *y*-values of a function can be made arbitrarily close to by restricting *x*-values nor did they include the formal definition’s window of tolerance in their graphical representations. The remaining students incorporated the formal definition in their mental images.

Nick’s evoked mental image was characterized as formal, however his description discussed how the function looks within the tolerance window but gave no description of how the function approached the limit by arbitrarily restricting the *x*-values. Nick described the process of constructing the window of tolerance. “I imagine that I am constructing a box for which the function must enter/exist through the left/right edges so as to contain the function with the top/bottom edges.” Nick’s description connected to the formal definition’s window of tolerance but only to the function without any reference to the limit as a mathematical object or the limit as a process.

Carlton and Yolanda who provided some unusable and usable mental images generated some incorrect formal mental images by including the window of tolerance around a point with no function (Figures 20 and 21). Both plotted the limit as a point, which showed isolated mathematical object within the formal definition’s window of tolerance. Yolanda did incorporate the process of the *x*-values approaching the *p*-value.

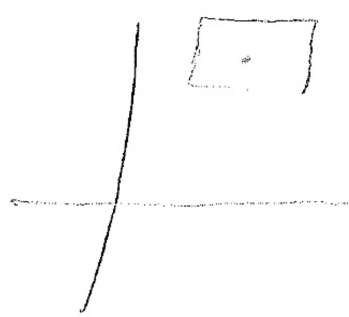


Figure 20. Carlton’s evoked mathematical drawing of limits of functions.

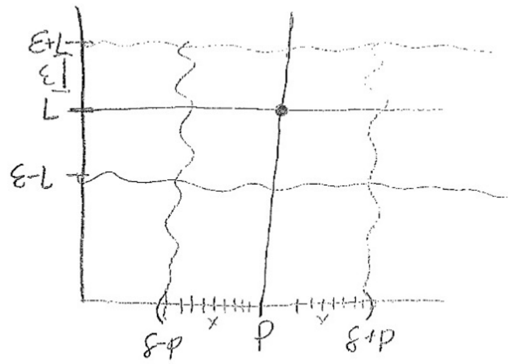


Figure 21. Yolanda’s evoked mathematical drawing of limits of functions.

Alan’s mental images were characterized as formal for including tolerances that were inaccurately labeled (Figure 22), but was also dually characterized as dynamical practical. Alan stated, “limits of functions are a sequence plugged into a function and it's not the terminating point of the function, but the value of the function approaches and then stabilizes that.” While Alan’s description incorrectly linked sequences and functions it did discuss the dynamical practical process of plugging in numbers closer and closer to a given number until the limit is reached.

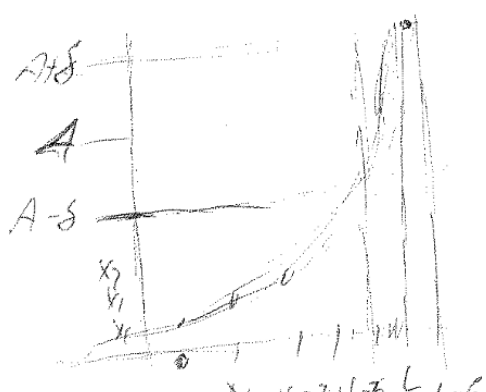


Figure 22. Alan’s evoked mathematical drawing of limits of functions.

Arnold, Edith, Tim, and Vincent all consistently and correctly generated mental images that were characterized as formal. Arnold's description was of the process of a two-sided limit, where "the y (or $f(x)$) values are going from the left and right as x goes to a certain value." Arnold's proceptual graphical representation (Figure 23) reiterated this by showing that within the window tolerance the function approached the limit from both left and right sides, and the *x*-values approach the value *p* from the left and right as well.

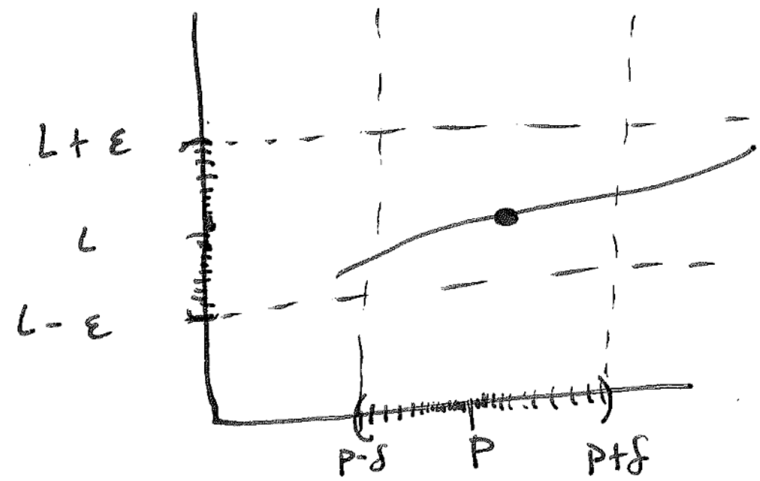


Figure 23. Arnold's evoked mathematical drawing of limits of functions.

The other students discussed the process of limits of functions were more general and did not specify approaching from both the left and right side. For example, Edith provided a graphical representation (Figure 24) and described the limiting process "as the constraint around a point on the x axis gets smaller, the constraint on the y axis gets closer and closer to the function value at the axis." Likewise, Tim generated a similar graphical representation (Figure 25) and explanation, "as values for which as a functions input '*x*-value' approaches a particular value with in a tolerance the function approaches that value." Both student's evoked mental images incorporated the formal definition's window of tolerance.

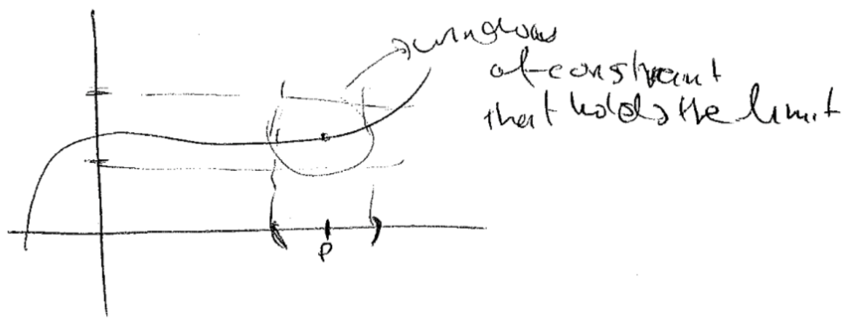


Figure 24. Edith's evoked mathematical drawing of limits of functions.

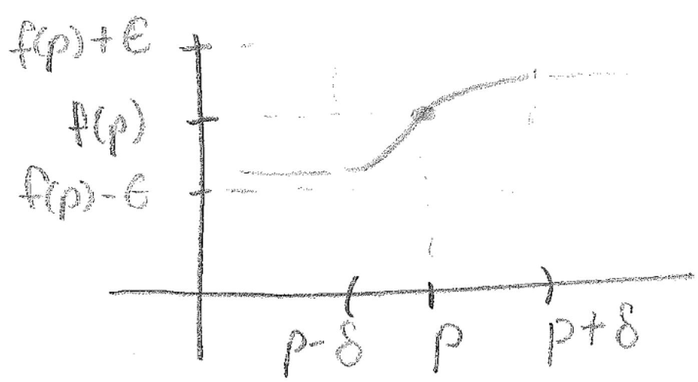


Figure 25. Tim's evoked mathematical drawing of limits of functions.

Lastly, Vincent's evoked mental images were oriented around the formal definition and explicitly showed that limits do exist at removable discontinuities (Figure 26). Vincent's evoked mental images were proceptual and strongly aligned with the formal definition of limits of functions. Vincent described the limit process "as (the) act of picking the "right" delta so that as x approaches c, f(x) can be made arbitrarily close to

L.” Vincent described his mental images “as an intuitive, dynamic picture. That is, a ‘moving picture’. From the intuitive mental construction, I typically derive the formal definition."

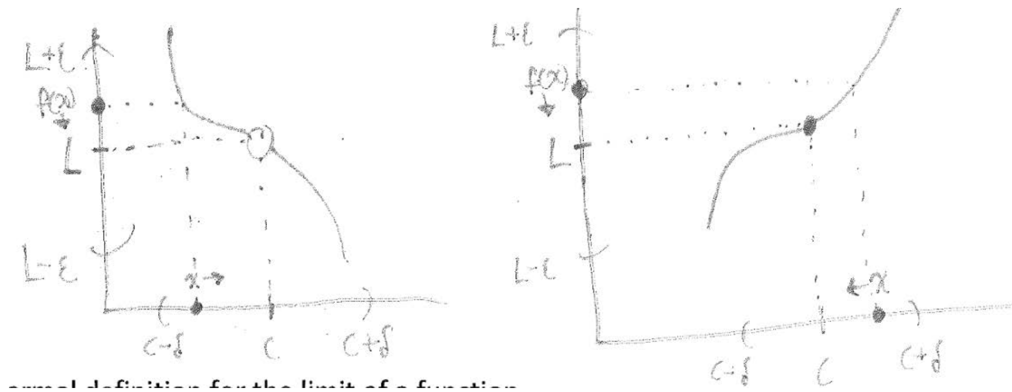


Figure 26. Vincent’s evoked mathematical drawing of limits of functions.

Correctness. A majority of the students’ evoked mental images consisted of generalized descriptions and graphical representations. Some students in their attempts to generalize, incorrectly overgeneralized. For instance, Adam’s evoked mental images consist of a graphical representation of a function with the vertical asymptote at $x = 0$ and the horizontal asymptote at $y = 0$, with the description of his process of determining a function’s limit is by locating where the points approach to by looking for an asymptote. His generalization of an asymptote did not distinguish between when a limit exists and does not exist. In his example, both cases of a vertical and horizontal asymptote arise. If he were to follow his process with the *x*–values approaching zero, the function would increase to infinity and the limit does not exist. While, if the process had the *x*–values approached infinity the function would approach zero, and the limit would exist.

Another incorrect overgeneralization was done by Maddie wrote that "for a limit of $f(x)$ to exist it must satisfy the three conditions of being continuous where the limit approach one value, bounded, and not oscillating." The first overgeneralization Maddie made was that a function had to be continuous. This is not true, since a function's limit does exist at removable discontinuities. Maddie also overgeneralized saying a function cannot oscillate. Maddie was only considering the cases when the *x*-values tend towards positive and negative infinity, and was not considering the instance where e *x*-values tend towards a real number. Also this generalization was only considering function oscillates with a fixed amplitude. If the amplitude of the function decreases, there is a chance for the function's limit to exist, for instance the function $f(x) = \frac{\sin(x)}{x}$.

Another error appeared in Alan's evoked mental images. Alan stated "limits of functions are a sequence plugged into a function and it's not the terminating point of the function, but the value of the function approaches and then stabilizes that." Alan had cognitive conflict between the relationship between a function and a sequence and trying to make a connection between the limits of functions with the previous unit's topic of limits of sequences. This was repeated in his graphical representation (Figure 22) where he plotted both a sequence and function and incorrectly oriented the epsilon tolerance on the *x*-axis and the delta tolerance on the *y*-axis. Alan was not the only student who attempted to generate mental images related to the formal definition with errors. Carlton and Yolanda generated graphical representations that included the formal definition's window of tolerance but only include the limit point. There was no inclusion of a function (Figures 20 and 21).

Evoked mental images summary. Overall, the evoked mental images of Real Analysis students are diverse amongst the students and could even be diverse among a single students' evoked mental images (Table 11). Students mental images that are evoked for sequences may have different characterizations than the evoked mental images for functions' characterizations.

Table 11.
Characterizations of participants' mental images of limits.

	Acting as an Approximation	Unreachable	Acting as a Boundary	Dynamical-Practical	Dynamical-Theoretical	Formal
Alan	-	-	Sequences	Functions	Sequences	Functions
Amy*	-	-	-	-	-	-
Arnold	-	-	-	-	Sequences	Functions
Carlton	-	Sequences	Sequences	-	-	Functions
Edith	-	-	-	-	Sequences	Functions
Jessie	-	-	Sequences	-	Sequences	-
Maddie	-	-	Functions	-	Both	-
Melody	-	Functions	Functions	-	Functions	Sequences
Nick	-	-	-	-	-	Both
Tim	-	-	-	-	-	Both
Vicky	-	-	-	Functions	Sequences	-
Vincent	-	-	Sequences	-	Sequences	Functions

*Generated only unusable mental images.

There were only three students who had consistent characterizations between their two evoked metal images. Tim and Nick both generated formal evoked mental image for limits of sequences and limits of functions. Maddie consistently had both of her evoked mental images characterized as dynamical-theoretical, but her mental images for limits of functions also include acting as a boundary. The remaining students did not have consistent characterizations between their mental images of limits of sequences and limits of functions.

Carlton did however, attempt to generate a mental image that related limits of functions to limits sequences by saying “they are like limits of sequences except with an added part to cover the plane.” However, this mental image was too vague and unusable. Instead Carlton did generate the usable but incorrect formal mental image of a window of tolerance around a point with no function, for the limits of functions. For limits of sequences he generated the metaphor that was dually categorized as unreachable and acting as a boundary. Carlton’s combined mental images were not connected or similar, and where of different characterizations.

Alan, Arnold, Edith, and Vincent’s evoked mental images varied between dynamical-theoretical mental images for limits of sequences and formal mental images for limits of functions. Alan and Vincent’s evoked mental images of limits of sequences additionally were characterized as acting as a boundary. Lastly, Melody generated the most formal evoked mental image for limits of sequences, and generated evoked mental images for limits of functions with the three characterizations of acting as a boundary, unreachable, and dynamical theoretical. These students did not have consistency of characterizations between their two evoked mental images.

It is worth noting that none of the students generated the least applicable characterization for the proof-based course, acting as an approximation. Similarly, dynamical-practical characterization is not the strongest semantic characterization for a Real Analysis course since it requires the limit to be determined by plugging in numbers until the limit is reached. This characterization is more appropriate for determining specific limits in a calculus. There were only two students, Alan and Vicky who held a dynamical-practical evoked mental image of limits of functions. In general, the majority of the students generated mental images that were appropriate characterizations for a Real Analysis course.

Evoked example spaces. Another domain of a person's concept image is their example space (Fukawa-Connelly & Newton, 2014; Tall & Vinner, 1981). A person's example space is derived from a collection of examples and methods for generating examples that are drawn from a person's past experiences (both explicitly remembered and not) which are triggered by a task, cues, environment, and recent experience (Mason & Watson, 2008; Watson & Mason, 2005). An evoked example space is the collection of examples that were generated by the student at a given time.

Limits of sequences. The students were given the prompt: *Please provide as many examples as possible of sequences with the limit of 5.* The provided prompt was designed to have students generate as many examples of sequences with the same limit without posing any restrictions. The students evoked example space of limits of sequences were coded using Zazkis & Leikin's (2007) framework of generality, correctness, richness, and accessibility.

Zazkis & Leikin's (2007) framework was used to analyze the 14 student's evoked example space of limits of sequences based on the four categories of generality, correctness, and richness. Zazkis and Leikin's dichotomous generality code categorizes an example as either specific or general but note that "perception of generality is individual." Therefore, this study defines a sequence example as specific if the example was symbolically named and not necessarily indicated to be representative of a class of examples. For instance, $s_n = an + b$ where *a* and *b* are real numbers, is not a specific example since it represents all examples of that form but $s_n = 5n$ is a specific example. An example was defined as general if it was a representation of a class of sequences with certain properties or representative of a symbolic type of sequence such as $s_n = an + b$ where *a* and *b* are real numbers.

Zazkis and Leikin defined an example as correct, if it satisfies the conditions of the task. Thus, study defined the example as correct if it satisfied the definition of a sequence and the direction of the limits approaches positive infinity. A sequence is a function, which is defined to be a relation from the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ or either the non-negative integers, $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$ to the set real numbers, such that every element of the domain is uniquely associated with an element of the range. The examples of limits of sequences were also considered correct in the *n*-values only increased to positive infinity. The examples were then sorted on whether it explicitly identified the direction the *n*-values were approaching or not. For instance, the example $\lim_{n\to\infty} \frac{1}{n} + 5$ specified that the *n*-values were approaching positive infinity and the example $s_n = \frac{5n}{n}$ was not explicit. For those that explicitly stated the direction the *n*-values were approaching, it was coded correct if it was positive infinity and incorrect if another value was indicated. The explicit examples that were correct and those that implicitly implied positive infinity were then calculated. If the sequence's limit was different than five, the example was coded incorrect. Each example was analyzed to determine whether the example was written mathematically correct.

Zazkis and Leikin' third category of richness evaluates all the examples generated by one student and determines if the examples vary in type and how their evoked example space relates to the collective evoked example space. Therefore, this study first identified the common sequences among the collective evoked example spaces. Next, the type of sequences for each example was classified, followed by identifying the different types of sequences generated in each student's evoked example space. The last category of Zazkis & Leikin's (2007) framework was accessibility. Accessibility determines whether the student struggled to generate an example or not. This code was not applicable to coding since the students were not observed as they generated their examples.

Generality. The evoked example space of Alan, Kayla, and Vincent consisted of graphical representations of general examples of limits of sequences. Kayla provided a single graphical representation of an increasing function that was bounded above by the horizontal asymptote $y = 5$ (Figure 27).

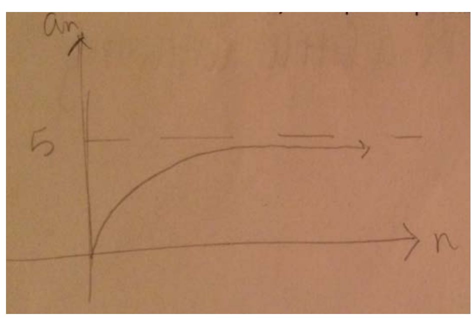


Figure 27. Kayla's evoked example space for sequences.

Alan generated two graphical representations (Figure 28). The first example was a function with a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 0$. The second was of an oscillating sequences that tended toward $y = 5$ as the n-values increased to infinity.

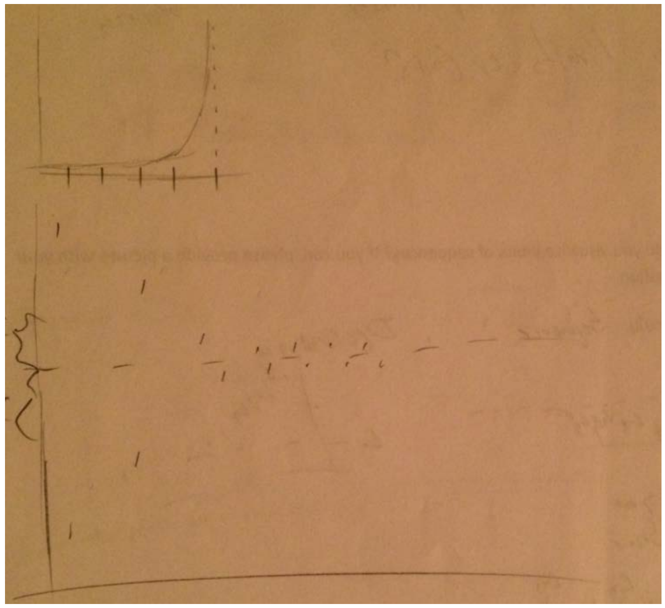


Figure 28. Alan's evoked example space for sequences.

Vincent provided four different graphical representations (Figure 29). The first three graphical representations Vincent described as "general ideas … an increasing sequence, or monotonically going up, and so forth. Then, I would also have the decreasing. Then I would have an alternating one." The first example Vincent was referring to was a monotonically increasing sequence that was bounded above by the horizontal asymptote $y = 5$. The second example was a monotonically decreasing sequence that was bounded below by the horizontal asymptote $y = 5$. Vincent's third base example was an oscillating sequence that tended towards 5. Vincent explained "these are the three, to me in my mind, the three base components to all sequences that you could come up with. After that you are just doing some kind of linear transformation on them to make them into a different picture."

Vincent extended his evoked example space to include the fourth example because the first three were “my base examples, but the I recognize that I’m missing certain types of examples.” Vincent included an example of a sequence that initially increased towards a maximum value that was greater than the limit and then eventually decreased passed the limit that would eventually converge towards the limit. Vincent described that he included this exampled because before taking this course he had thought that “once the sequence falls in that (epsilon) tolerance it has to stay there. But (my instructor) was like no it doesn't… What if it's up and down? What if it goes in, goes out? It totally can do that. I was like oh, yeah. Well, crap okay.”

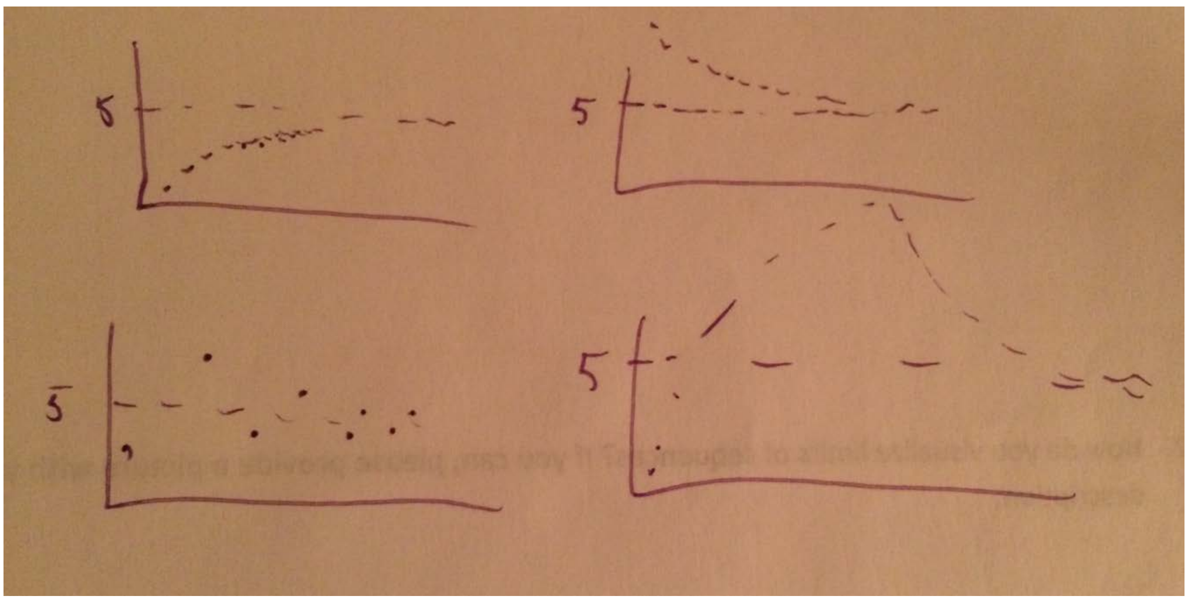


Figure 29. Vincent’s evoked example space for sequences.

There were eleven students whose evoked example space consisted of specific algebraically-named examples. Four students, Travis, Vicky, Nick, and Arnold explicitly identified their sequences’ domains by writing their examples in the form $\{a_n\}_{n=1}^{\infty}$, such as $\{\frac{1+5n^4}{n^4+8n^3}\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty} = \{5 + \frac{1}{n}\}$. The other six students did not specify the examples’ domains. Amy wrote her sequences as $s_n = \frac{5n^2+n}{n^2}$, which was consistent with the sequence notation introduced by her instructor that implicitly indicated the domain to be the natural numbers. Tim and Maddie wrote their examples in the form $\lim_{n \to\infty} \frac{5n+1}{n}$, which also implicitly identified the domain as the natural numbers. Lastly, Alicia, Melody, Carlton did not explicitly or implicitly identify the domains with the notational form $5 + \frac{1}{n}$.

Correctness. All of the students generated correct examples except for four students. There were three students, Kayla, Alan and Maddie whose evoked example space contained examples that did not satisfy the definition of a sequence. Both generated graphical representation of a function rather than a sequence. When Kayla was asked to describe her example of a limit of a sequence she even identified her example as a function but renamed both the axes to correspond the domain and range of a sequence. "I would just say that my limit would be 5 and then my function would be doing this, not really approaching it, I mean approaching it, but not really touching it … This, obviously, being n, (writing n on the *x*-axis) and this being a_n (writing a_n on the *y*-axis)."

Alan's first example was a function with a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 0$. This example was also incorrect because the limit as n increases to infinity was not 5. Alan indicated the limiting value of 5 on the incorrect axis. Maddie whose evoked examples space consisted of algebraically written examples included functions that had the limit of 5. Maddie who wrote her examples in the form $\lim_{n\to 4} n + 1$, included seventeen examples whose *n*-values approach integers and negative infinity.

Out of the fourteen students, only six made minor notational errors. One error that appeared was incorrectly indexing examples whose sequences' domains were explicitly identified. Travis, who only provided one example $\{\frac{1+5n^4}{n^4+8n^3}\}_{n=0}^{\infty}$, began indexing at $n = 0$ when the sequence is undefined. The third error that appeared was providing sequences whose limits were not 5. Two students, Carlton and Alicia generated the incorrect examples $5n$ and Alicia also included the incorrect example $\frac{5}{2}n^2$. Additionally, Carlton, Melody and Alicia did not use the standard mathematical notation of a sequence besides using *n* as the input variable.

Richness. The number of correct examples an individual student generated ranged from two to seven. Each example generated was coded by type of sequence (Tables 12 and 13). Since the examples that were graphical representations were not explicitly named, the sequence type was coded based off the graph's properties, such as constant, increasing/decreasing, bounded above or below, and alternating. The type of sequence that the students generated the most examples for was rational sequences. The sequences that were generated by two or more students were the constant sequence $\{5\}_{n=1}^{\infty}$, the identity sequence $\{5\}_{n=1}^{\infty}$, and five plus or minus the harmonic sequence $\{5+\frac{1}{n}\}_{n=1}^{\infty}$ and $\{5-\frac{1}{n}\}_{n=1}^{\infty}$. The common sequence that appeared in the graphical representations was an oscillating sequence that converged to 5.

Table 12.
Types of symbolically-named limits of sequences examples.

	Constant	$\{5 \pm \frac{1}{n}\}$	Polynomials	Rational	Trigonometric
Alicia	-	-	x*	x	-
Amy	x	-	-	x	-
Arnold	x	x	-	x	-
Carlton	x	-	x*	x	-
Edith	x	x		x	-
Jessie	-	-	-	x	-
Maddie	x	x	-	x	x
Melody	x	x	x	x	-
Nick	x	-	-	x	-
Tim	-	-	-	x	-
Vicky	x	-	-	x	-

*Incorrect example.

Table 13.
Types of graphical representations of limits of sequences examples.

	Increasing/ Bounded Above	Decreasing/ Bounded Below	Alternating	Other
Alan	x*	-	x	-
Kayla	x*	-	-	-
Vincent	x	x	x	x

*Incorrect example.

The number of correct different types of sequences that a student generated ranged from zero to four. The student who did not produce a correct example of a sequence was Kayla. Kayla's example was an incorrect graphical representation of an increasing bounded above function. Jessie generated the rational sequence $\{\frac{1+5n^4}{n^4+8n^3}\}_{n=0}^{\infty}$, but this example did contain an indexing error.

There were three students who generated only one correct type of sequence. Alicia who generated three examples, only generates a correct example of the rational sequence $\frac{10n+3}{2n+1}$. The incorrect examples Alicia generated was the linear sequence $5n$ and the quadratic sequence $\frac{5}{2}n^2$. Similarly, Alan generated two graphical examples, however one was an incorrect increasing unbounded function. Therefore, Alan only generated one correct type of sequence, which was an alternating sequence that converged to 5. Tim generated only one correct example, the rational sequence $\lim_{n \to \infty} \frac{5n+1}{n}$.

There were four students who generated two correct types of sequences. Two of the students, Vicky and Nick generated exactly two examples that was the constant sequence and the rational sequences $\{\frac{1+5n^4}{n^4+8n^3}\}_{n=1}^{\infty}$ and $\{\frac{5n}{n+1}\}_{n=1}^{\infty}$, respectively. It is important to note that the examples Jessie and Vicky both generated (just with different indexing) was very similar to the sequence $\{\frac{1-5n^4}{n^4+8n^3}\}_{n=1}^{\infty}$ provided in a following task in the survey. The other two students who generated two correct types of sequences were Carlton and Amy. Carlton generated a total of four sequences, one which was the incorrect linear sequence $5n$. Carlton's three correct examples and Amy's seven correct examples were the constant sequence and rational sequences. Carton's two rational sequences were $\frac{15n-8}{7n-6}$ and $\frac{15n}{3n}$. Amy's six rational sequences were $s_n = \frac{4n+10n^2}{2n^2}$, $s_n = \frac{5n^2+n}{n^2}$, $s_n = \frac{5n}{n}$, $s_n = \frac{15n+20n^3}{4n^3}$, $s_n = \frac{10n^2}{2n^2}$, and $s_n = \frac{(n^3+5n^4)n^4}{(n^4+7n^3)n^4}$.

There were two students, Edith and Arnold who generated three correct types of sequences. Both generated the constant sequence, five plus or minus the harmonic sequence and rational sequences. Both generated five plus the harmonic sequence, $\{5 + \frac{1}{n}\}_{n=1}^{\infty}$ but only Edith generated five minus the harmonic sequence, $\{5 - \frac{1}{n}\}_{n=1}^{\infty}$ as well. The rational sequences Edith and Arnold generated were $\{\frac{5n^2}{n^2}\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty} = \{\frac{5(n-1)}{n}\}$, respectively.

There were three students who generated four different types of sequences. Vincent generated the general graphical representations of a monotonically increasing sequence that was bounded above by the horizontal asymptote $y = 5$ and a monotonically decreasing sequence that was bounded below by the horizontal asymptote $y = 5$. Vincent's third general types was an alternating sequence that tended towards 5. Vincent's fourth example was of a sequence whose range initially entered and exited the arbitrary epsilon tolerance but then eventually stayed within the epsilon tolerance and converged to the limit of five.

The other two students who generated four different types of sequences, provided algebraic representations. Melody generated exactly four examples of the following types: the constant sequence, the five plus the harmonic sequence, the rational sequence $\frac{5}{n} * n$, and the polynomial sequence, $5n^0$. Maddie who generated a total of 24 examples, however only seven were correct. The correct seven examples were of the following types: the constant sequence, the five plus the harmonic sequence, the rational sequence $\lim_{n\to\infty} \frac{5n^2+n+1}{n^2}$, and the trigonometric sequences, $\lim_{n\to\infty} sin\frac{1}{n} + 5$, $\lim_{n\to\infty} \cos\frac{1}{n} + 4$, $\lim_{n\to\infty} \frac{sin\frac{1}{n}}{\cos\frac{1}{n}} + 5$, and $\lim_{n\to\infty} tan\frac{1}{n} + 5$. Maddie made a note on here survey that "if you add 5 to most sequences, where the answer is 0 without the 5, the limit will equal 5. I can't think of any situations that it does not work."

Limits of functions. The students were given the prompt: *Please provide as many* *examples as possible of functions with the limit of 2.* The provided prompt was designed to have students generate as many examples of functions with the same limit without posing any restrictions. The students' evoked examples space of limits of functions were coded using Zazkis & Leikin's (2007) framework of generality, correctness, richness, and accessibility.

Zazkis & Leikin's (2007) framework was used to analyze the 16 student's evoked example space of limits of functions based on the four categories of generality, correctness, and richness. Zazkis and Leikin's dichotomous generality code categorizes an example as either specific or general but notes that "perception of generality is individual." Therefore, this study defines a function example as specific if the example was symbolically named and non-representative of a class of examples. For instance, $f(x) = ax^2 + b + c$, where *a, b,* and *c* are real numbers, is not a specific example since it represents all functions of that form, but $f(x) = 3x^2$ is a specific function. An example was defined as general if it was a representation of a class of functions with certain properties. Similarly, the limiting value was considered specific if the example's *x*-value approached a specified real number, such as 0 or ∞ and was considered general if the *x*-values approached an arbitrary value, such as p or c_1.

Zazkis and Leikin defined an example as correct, if it satisfies the conditions of the task. Thus, this study defined the example as correct if it satisfied the definition of a function. A function, *f* is defined to be a relation from the set *A* to the set *B* such that every $a \in A$ is uniquely associated with an object $f(a) \in B$. The examples were also coded as correct or incorrect based on whether the function's limit existed and was the value of 2. Each example was analyzed to determine whether the example was written mathematically correct.

Zazkis and Leikin' third category of richness evaluates all the examples generated by one student and determines if they vary in type and how their evoked example space relates to the collective evoked example space. Next, the type of function for each example was classified followed by identifying the different types of functions generated in each student's evoked example space. Lastly each student's evoked example space was analyzed to classify the type of numbers, the example's x-values approached. The last category of Zazkis & Leikin's (2007) framework was accessibility. Accessibility determines whether the student struggled to generate an example or not. This code was not applicable to coding since the researcher was not able to observe the students as they generated their examples.

Generality. The evoked example space of Vincent, Carlton, and Alan consisted of graphical representations of general examples of limits of functions. For instance, Vincent generates a graph of a function with a discontinuity at the point (c, 2). All of Vincent's examples were graphical representations that indicated the x-values were approaching an arbitrary real number c, which was explicitly not positive or negative infinity (Figure 30).

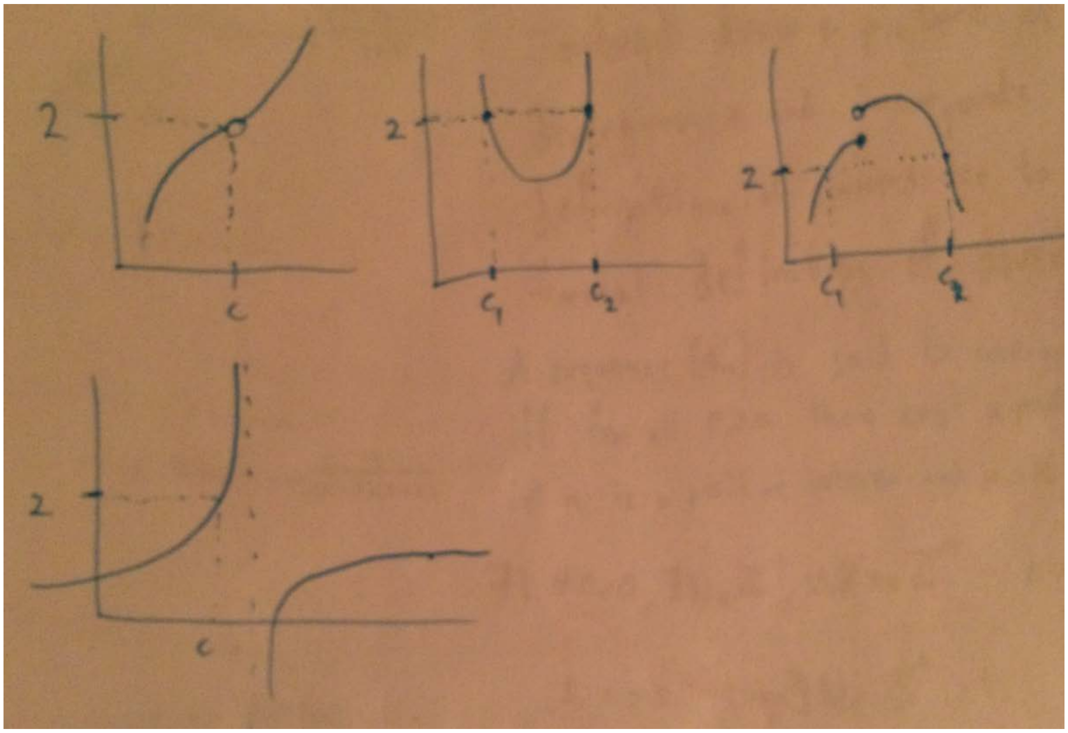


Figure 30. Vincent's evoked example space for functions.

Carlton's entire evoked example space consisted of general graphic representations (Figure 31), with no explicit indication of what *c*-value the *x*-values were approaching. It was left for the reader to interpret that as the *x*-values approached infinity, the *y*-values approached the value 2.

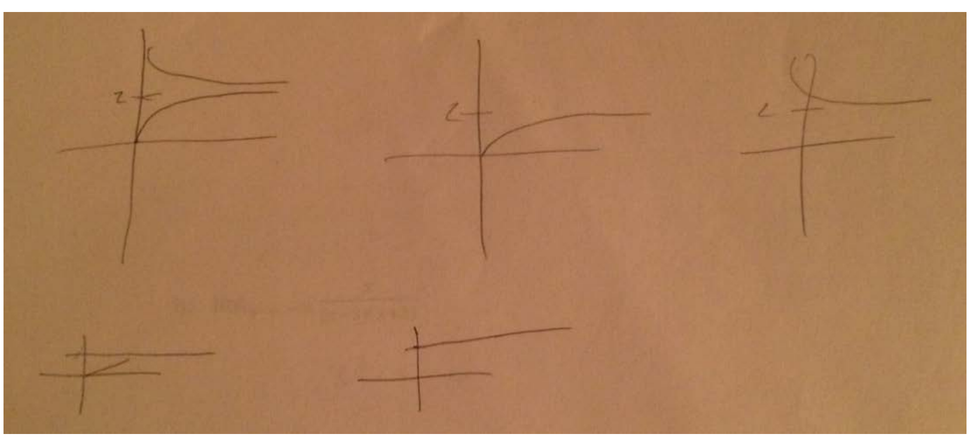


Figure 31. Carlton's evoked examples space for functions.

Alan generated a graphical representation and indicated the *c*-value to be two, as well as two graphical representations with no explicit indication of what *c*-value but implied the *c*-value to be infinity (Figure 32).

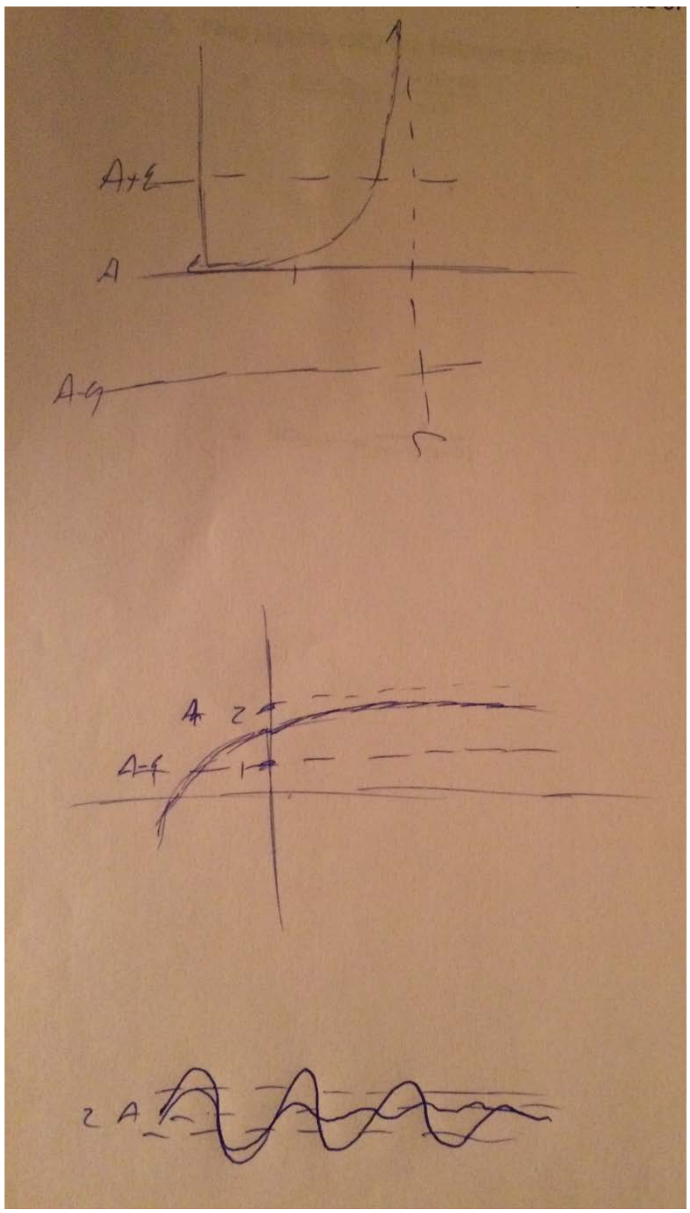


Figure 32. Alan’s evoked example space for functions.

The thirteen students' evoked example space consisted of specific algebraically-named examples. Out of the thirteen students, only one student, Arnold generalized that the *x*-values could approach an arbitrary value, *p* for the constant function $f(x) = 2$. The other eight students, who included the algebraic representation of the constant function, chose the *x*-values to approach a positive integer, or positive and/or negative infinity. Maddie generate the limit of the constant function twice, once with the *x*-values approaching negative infinity and the other approaching positive infinity. There were two students, Melody and Jessie who did not indicate the direction of where the *x*-values were approaching.

Correctness. Of the sixteen students who generated an evoked example space for limits of functions six students there were six students who had mathematical errors. There were two students whose evoked example space contained examples that did not satisfy the definition of a function. Carlton, who generated graphical representations, had provided an example that failed the vertical line test. Another example generated by Carlton technically failed the vertical line test as well. However, it is important to note that it was undeterminable whether both lines were meant to represent one "function" or if the horizontal line was meant to be an asymptote. The other student who held an error with the concept of function was Jessie. Jessie's evoked example space consisted of the two sequences, $\{2 + (\frac{1}{10})^n\}_{n=0}^{\infty}$ and $\{2 + \frac{1}{n}\}_{n=1}^{\infty}$, and Jessie incorrectly identified that functions were special instances of sequences. Jessie had misunderstood the relationship between functions and sequences, and thus his examples were considered incorrect.

Other errors that appeared in three students' evoked example spaces, was examples of limits of functions whose limits do not exist at the indicate *c*-value or the limits value were different than 2. Maddie included two incorrect examples in her evoked example space. Maddie included the example, $\lim_{x\to 0}\frac{5x^2+2}{x^2}$, whose limit does not exist at the indicated *c*-value, zero. Another example Maddie generated was $\lim_{x\to 4}\frac{x^2+3x-10}{x-5}$, whose limit was not 2. Edith also included the example, $\lim_{x\to -5} x+3$ whose limit was -2, rather than 2. Arnold also included the example $\lim_{x\to\infty}\frac{2x^2}{x+1}$ whose limit does not exist.

Alan included a graphical representation of a function with a vertical asymptote at $x = 2$, (which he also labeled as *c*) and as the *x*-values approached 2, the *y*-values tended towards infinity. Therefore, Alan included a function whose limit does not exist at 2. When Alan described the example he identified that the function was "an undefined value" at *x* = 2. Alan continued to explain that he was "probably looking at it more from a sequential point of view" and was unable to recognize that the example was incorrect.

The mathematical notational errors among all the evoked example spaces, ranged from switching the input variable from *x* to *n*, as seen in the example $\lim_{x\to\infty}\frac{2n+1}{n+1}$. Another notational error was including the name of the function *f(x)* in the expression, for instance $\lim_{x\to 2} f(x) = \frac{2^n}{n}$, and $\lim_{x\to 2} f(x) = x$. Lastly, there were examples that did not indicate which *c*-value the *x*-values were tending towards, such as $f(x) = 2$ and $f(x) = 2(\frac{x}{x+1})$.

Richness. The number of correct examples an individual student generated ranged from two to thirteen. Each example generated was coded by type of function (Tables 14 and 15), and what indicated *c*-value the *x*-values were approaching. Since the examples that were graphical representations were not explicitly named, the function type was coded based off the graph's properties, such as constant, increasing/decreasing, bounded above or below, continuity, and oscillating. The type of function that the students generated the most examples for was rational functions. All of the rational functions were different. The algebraically named functions that were generated by two or more students were the constant function $f(x) = 2$, the identity function $f(x) = x$, the quadratic function $f(x) = x^2$, and the trigonometric function $f(x) = \frac{2\sin(x)}{x}$. The graphically represented examples that were generated by two or more students were the increasing and bounded above by the horizontal asymptote $y = 2$ function.

Table 14.
Types of symbolically-named limits of functions examples.

	Constant	Identity	Linear	Quadratic	Rational	Trigonometric	Radical	Log
Adam	x	x	x	x	x			
Amy	x		x	x	x			
Arnold	x	x			x		x	
Brandan			x	x	x			
Edith	x	x	x	x	x			
Jessie					x*			
Maddie	x	x	x	x	x			
Melody	x				x			
Nick	x		x		x	x		
Tim		x		x		x		x
Travis	x				x			
Vicky					x			
Yolanda		x			x			

*Incorrect example.

Table 15.
Types of graphical representations of limits of functions examples.

	Other	Increasing/ Bounded Above	Decreasing/ Bounded Below	Oscillating	Constant	Continuity
Alan	x*	x		x		
Carlton	x*	x	x		x	
Vincent						x

*Incorrect example.

There were five students, (Melody, Travis, Vicky, Jessie, and Yolanda) who provided exactly two examples with either both of the same type or two different types. Jessie's evoked example space was the least rich, instead of generating functions Jessie generated two rational sequences, $\{2 + (\frac{1}{10})^n\}_{n=0}^{\infty}$ and $\{2 + \frac{1}{n}\}_{n=1}^{\infty}$. Vicky produced two functions of the same type, both were rational functions, $\lim\limits_{x\to 1} \frac{2}{x}$ and $\lim_{x\to 0} \frac{x^2+5x+6}{x^2+3x+3}$. Melody and Travis generated the constant function and the respective rational functions, $f(x) = 2(\frac{x}{x+1})$ and $\lim_{x\to\infty} \frac{1+2x}{x}$. Yolanda generated the identity function and the rational function $\lim_{x\to 2} f(x) = \frac{2^n}{n}$.

There were three students, Brandan, Alan, and Carlton who produced three different types of functions. Brandan generated a total of nine examples. Four of Brandan's examples were linear functions, $\lim_{x\to 1}(2x)$, $\lim_{x\to -1}(-x+1)$, $\lim_{x\to 5} 2x - 8$, and $\lim_{x\to 4}(10-2x)$. Brandan included one quadratic function, $\lim_{x\to 2} x^2 - 2$ and four rational functions, $\lim_{x\to 1} \frac{x^2-1}{x-1}$, $\lim_{x\to 0} \frac{x^2+2x}{x}$, $\lim_{x\to -3} \frac{x^2+8x+15}{x+3}$, and $\lim_{x\to 6} \frac{x^2-10x+24}{x-6}$.

Alan, who provided three graphical representations, provided two functions with asymptotic behavior. The correct example was of an increasing function with the horizontal asymptote was bounded above by the limit A. The other function that was an incorrect example of a function with a limit of 2, had a two asymptotes. The first was a vertical asymptote at *x* = 2 and was unbounded as the *x*-values approached 2. The second asymptote was the *x*-axis, Alan had also labeled the *x*-axis as *A*, indicating to be the limit of the function. Alan's third example was a function that oscillated and became arbitrarily close to y = 2 as the *x*-values increased towards infinity.

The third student who generated three different types of functions was Carlton. Carlton, generated five different graphical representations, but similarly to Jessie, Carlton produced two of the graphical representations failed to satisfy the definition of a function. Of three examples that were functions, two were functions with horizontal asymptotes. One function was increasing and bounded above by the horizontal asymptote, *y* = 2 and the other was a decreasing function that was bounded below by the horizontal asymptote, *y* = 2. The last example Carlton generated was of the constant function *y* = 2. None of Carlton's graphical representations indicated a value that the *x*-values were approaching.

There were six students, who each produced four different types of functions. Nick and Vincent both generated exactly four examples, all of which were different types of functions. Nick generated the constant function, a linear function, $\lim_{x\to 1} 2x,$ a rational function, $\lim_{x\to\infty} \frac{2n+1}{n+1} = 2$ and a trigonometric function, $\lim_{x\to 0} \frac{2sinx}{x}$. Nick's examples had the *x*-values either approaching an integer or positive infinity.

Vincent generated four different general graphical representations, each had different properties. Vincent's selection of examples was central to the concept of continuity. Vincent explained, "Limits are a necessary condition for continuity," but it is “not sufficient to be continuous.” The first was an increasing function with a removable discontinuity at the point (*c*, 2). Vincent described the example as “a general example of the point not being defined because it doesn't have to actually get there, so to speak, in order for the limit to exist.” The second example was a continuous function that achieves the limit of 2 at multiple *x*-values. Vincent included this example to show the limit “one exists whenever it approaches C1 (and) C2. I'm basically trying to create a concept that I could have multiple points that give me that limit.” The third example was a function with a jump discontinuity, but which still achieves the limit of 2 at multiple *x*-values that are different than the point of discontinuity. Vincent generated this example to show that “even though (a function is) discontinuous, it could still have a limit of 2, as long as we're not talking about 2 being (at the jump discontinuity).” For the final example Vincent included a function with a vertical asymptote at a value different than c. Therefore, Vincent’s evoked example space consisted of four different types of functions based on the concept of continuity.

There were three students, Tim, Arnold, and Amy who generated four different types of functions. Tim and Arnold generate five examples each and included the identity function. Arnold also included the constant function and the rational function, $\lim_{x\to 4}\frac{x}{2}$ .What was unique about Arnold’s evoked example space was the he was the only student to generate a radical function. Arnold included the radical function, $\lim_{x\to 4}\sqrt{x}$. Tim’s evoked example space was also different than the other students’ evoked example space because he generated the only logarithmic function, $\lim_{x\to e^2}\ln(\,x)$ and two trigonometric functions, $\lim_{x\to 0}\frac{2sinx}{x}$ and $\lim_{x\to 0}\cos(x)$. The last type of function Tim generated was the quadratic function, $\lim_{x\to\sqrt{2}}x^2$. Amy’s evoked example space consisted of a total of ten functions. Amy generated the following rational functions: $\lim_{x\to 0}\frac{x+2}{x^2+1}$, $\lim_{x\to 2}\frac{x^2}{x}$, $\lim_{x\to 1}\frac{x^2+2x-5}{x^2+3x-5}$, $\lim_{x\to 2}\frac{4}{x}$, $\lim_{x\to 3}\frac{x^2-1}{x+1}$, and $\lim_{x\to 4}\frac{(x-2)(x+5)}{(x+5)}$. Amy generated the constant function, two linear functions, $\lim_{x\to 1} 2x$ and $\lim_{x\to 1} 3x-1$, and the quadratic function $\lim_{x\to 1} 2x^2-3x+3$.

Lastly there were three students Adam, Edith, and Maddie who generated the same five types of functions: a constant function, the identity function, a linear function, a quadratic function, and a rational function. Edith generated a total of six examples, which included the incorrect linear function example$\lim_{x\to -5} x+3$. Edith's quadratic function was $\lim_{x\to\sqrt{2}} x^2$, and her two rational functions were $\lim_{x\to\frac{1}{2}}\frac{1}{x}$ and $\lim_{x\to 10}\frac{(x-10)(x-8)}{(x-10)}$. Adam generated a total of six examples that included the linear function $\lim_{x\to 1} 2x$, the quadratic function $\lim_{x\to 1} x^2+x$, and the two rational functions $\lim_{x\to 2}\frac{x^2+2x}{x+2}=2$ and $\lim_{x\to 2}\frac{x^3+2x}{x^2+2}=2$. Maddie generated the most examples, with the total of fifteen. Maddie's evoked example space was unique because she included two different examples using the constant function, $\lim_{x\to\infty} 2=2$ and $\lim_{x\to-\infty} 2=2$. She provided two linear functions, $\lim_{x\to 0} 5x+2$ and $\lim_{x\to 1} 5x-3$, and three quadratic functions, $\lim_{x\to 1} x^2+1$, $\lim_{x\to 2} x^2+3x-8$, and $\lim_{x\to 0} x^2+4x+2$. Maddie generated a total of seven rational functions, in which two were incorrect examples. The seven rational functions were:$\lim_{x\to -3}\frac{x^2+3x-10}{x-2}$, $\lim_{x\to 4}\frac{x^2+3x-10}{x-5}$, $\lim_{x\to\infty}\frac{2x^2+5}{x^2+4}$,$\lim_{x\to-\infty}\frac{2x^2+5}{x^2+4}$, $\lim_{x\to 0}\frac{5x^2+2}{x^2+1}$, $\lim_{x\to 0}\frac{5x^2+2}{x^2}$, and $\lim_{x\to 0}\frac{x+2}{x^2-x+1}$.

Each student's evoked example space should have had a collection of all the types of numbers that each example's *x*-values were approaching. However, there were two students, Melody and Jessie who generated algebraically-named examples who did not include which *c*-values the example's *x*-values were tending towards. Similarly, Carlton, who generated graphical representations, did not indicate which *c*-values the example's *x*-values were tending towards. Alan, who also generated graphical representations did indicate *c*-values on his correct examples, but did on his incorrect example, whose limit did not exist at the *c*-value.

There were twelve students who indicated a specific or general *c*-value for each of their examples. The collections of *c*-values differed in which types of numbers were included. Travis's collection only consisted of positive infinity. Amy, Brandan, Vicky, and Yolanda's collections of *c*-values consisted only of integers. Adam, Maddie and Nick's collections of *c*-values consisted of integers, and positive and negative infinities. Travis and Edith had generated two similar collections, both generated integers and irrationals. However, Edith additionally included a rational number.

Vincent's collection was unique because it not only consisted of arbitrary real numbers, Vincent also identified that there are functions who achieve the same limit at multiple *c*-values. There was only one other student, Maddie who explicitly demonstrated that this was possible by including the constant function twice, with one example's *x*-values approaching negative infinity and the other example's *x*-values approaching positive infinity. Arnold generalized Maddie's implication by indicating the constant function's *x*-values could approach any arbitrary *p*-value. Arnold did not define *p*. Arnold's collection was also very similar to Maddie's collection by consisting of integers and positive infinity.

Evoked example spaces summary. The majority of the students' evoked example space of limits consisted of specific algebraically named examples. Some of those students, like Amy and Maddie approached the prompt as trying to generate a long list of examples with no emphasis on generating examples to be representative of what this study refers to as types and what Watson and Mason (2005) refer to as a "flavour of possible examples." Other students like Tim and Arnold provided a shorter list of specific examples to represent what they thought to be different "flavour(s) of possible examples." For instance, Tim's evoked example space of functions, consisted of the four examples: $\lim_{x\to 2} x$, $\lim_{x\to\sqrt{2}} x^2$, $\lim_{x\to e^2} ln(x)$, $\lim_{x\to 0} 2cos(x)$, and $\lim_{x\to 0} \frac{2sin(x)}{x}$. These four examples were categorized as linear, quadratic and trigonometric functions. While, both $\lim_{x\to 0} 2cos(x)$ and $\lim_{x\to 0} \frac{2sin(x)}{x}$ are examples of limits of trigonometric functions, Tim saw the two examples are representatives of two different types. Arnold, who also approached generating examples as types of limits of functions uniquely included the example $\lim_{x\to p} 2$ to explicitly showed that there are infinitely many examples of a constant function whose limits is equal to 2.

Being able to generate more general examples in mathematics education, especially in proof-based courses, appears to be more valuable than specific. "Usually, we strive for seeing the general in the particular" (Mason & Primm, 1984; Zazkis & Leikin, 2007). Therefore, students who generated graphical representations generalized the different "flavours of possible examples" even further. Vincent recognized that he could not generate an exhaustive list of all the limits of sequences and functions that had specific sets of properties. Therefore, to encapsulate the most examples possible, Vincent generated a "base" example that could be transformed to generate a different example with the same “base” properties.

Vincent’s selections of “base” examples of functions were not created to represent parent functions, such as linear functions or logarithmic functions. Rather, his “base” examples were chosen to clarify his “concept map or mind map” between the relationship of limits of functions and continuity of functions. His four types of examples of function demonstrated three different types of discontinuity and one example of continuity when the limit could be achieved at multiple *x*-values (Figure 32).

Another important connection between Vincent’s evoked example space of sequences and evoked example space of functions was that he did not generated graphical representations of examples he thinks to be trivial cases. For instance, Vincent did not include a continuous function, who achieves the limit of 5 at exactly one *x*-value. Similarly, Vincent did not include the constant sequence in his evoked example space for sequences because “it’s the example that’s really uninteresting.” He explained that the trivial example gave you “nothing and everything”, meaning that it met the necessary condition of having a limit of 5 but it provides no interesting information of how it relates to an epsilon tolerance. Only that for any positive epsilon value, the entire range of the constant sequence is within the epsilon tolerance, which Vincent found to be “boring.”

Therefore, even though Vincent did not produce the longest list of examples, Vincent’s evoked examples spaces were also consistently rich with producing four types of limits of sequences and four types of limits of functions. Each of these types were different characterizations of a group of properties and contained no errors.

It is not necessarily true that every Real Analysis student who generates an evoked examples consisting of graphical representations has no misconceptions. While general graphical representations “may be seen as an indication of mathematical understanding, other general examples may point to deficiencies in understanding” (Zazkis & Leikin, 2007). This was seen in Alan’s evoked examples spaces. For both of Alan’s evoked example spaces he began first by generating the same incorrect type of graphical representation. The graphical representations were of a function with a horizontal asymptote at $y = 0$, and a vertical asymptote. The difference between the two examples was that, the example for a sequence with the limit of *5*, had the vertical asymptote at $x = 5$ (Figure 33) and the other example for a function with the limit of 2, had the vertical asymptote at $x = 2$ (Figure 34).

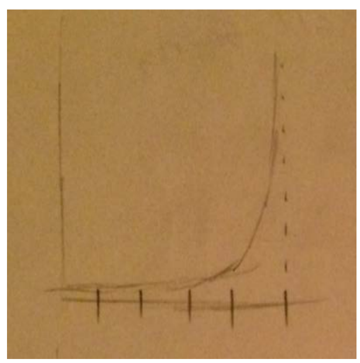


Figure 33. Alan’s dominant example for limits of sequences.

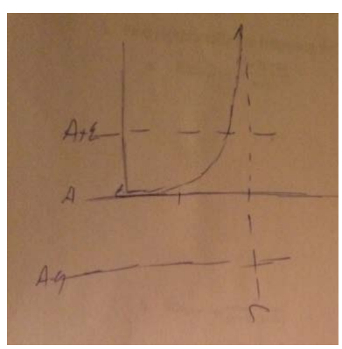


Figure 34. Alan’s dominant example for limits of functions.

These general examples alluded to the major misconceptions Alan had about limits of functions and the formal definition. Alan thought “limits of function are a sequence plugged into a function . . . a_1, $a_{2,}$ a_3 plugged into your f as for *x*-value.” Alan provided a graphical representation (Figure. 35) to help explain how he thought about limits of functions by labelling the *x*-axis with the sequence $\{x_n\}_{n=0}^{\infty}$. The graphical representation Alan generated was very similar to the two graphical representations Alan provided in his evoked example spaces. It was a function with the horizontal asymptote at *y* = 0, and a vertical asymptote at *x* = *L*. Not only did Alan label the limit *L* on the *x*-axis, he also placed the epsilon tolerance on the *x*-axis and the delta tolerance on the *y*-axis around some *A*-value. Alan was unable to state the formal definition or provide a coherent explanation. Thus, asking students to generate examples that are graphical representations could potential expose some misconceptions they have about limits.

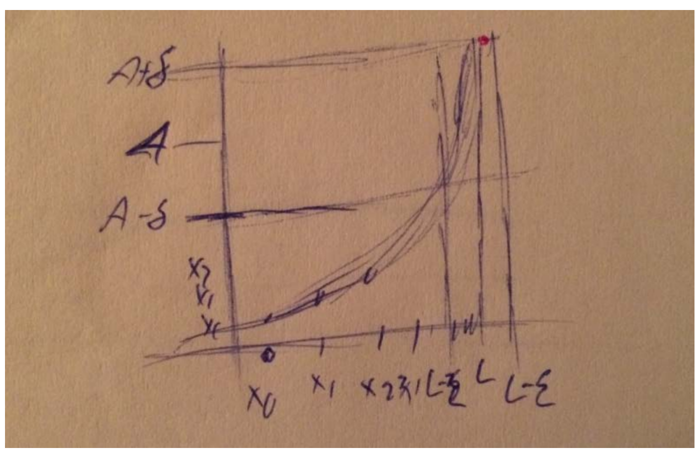


Figure 35. Alan’s mental image for limits of functions.

Overall, there was a balance between the richness of the students’ evoked example space of sequence and their evoked example space of functions. On average, the students produced two different types of sequences and three different types of functions.

The inclusion of an additional type of function in the students' evoked example space of functions could be contributed to the flexibility the students had in selecting the *x*–values to approach different types of *c*-values. Whereas, their evoked example space of sequences was restricted to generating examples whose *n*-values were to increase to positive infinity. Tim, who generate more types of functions than types sequences was able to include functions he was familiar with such as the quadratic function and natural log function, by creatively adjusting the *c*-values to $\sqrt{2}$ and e^2, respectively.

There were two students, Jessie and Maddie, whose evoked example space of limits, despite the different level of richness, were dominated by sequences or functions, respectively. Jessie had the least rich example space of limits out of the twelve students. Jessie only generated a total of three examples of sequence and did not provide any examples of functions throughout the surveys. Jessie's example space of limits was dominated by sequences and became apparent when Jessie wrote he thought of limits of functions as the same as limits of sequences.

The other students Maddie, had one of the richest example space of limits. Maddie provided the most examples (39 in total), and generated the most types of sequences (4) and the most types of functions (5). However, out of the twenty-four examples generated with the intention to be sequences with the limit of 5, only seven examples were correct. The other seventeen were actually examples of functions with the limit of 5. Maddie who could accurately described the "direction" of the *n*-values for a limit of a sequence as "approaching infinity … so a_n, and then a_{n+1}, a_{n+2}", was experiencing cognitive conflict when generating examples of limits of sequences. Maddie had not made the distinction in her example space of limits, that examples whose input values approached positive infinity could be examples for both limits of sequences and functions. Whereas, examples whose input values approached values other than positive infinity were limits of functions examples.

Evoked processes. The third domain of a concept image is the associated processes. Processes are calculations by mathematical methods and logical reasoning. There are different processes associated with limits. In calculus, students elevate the process of computing a limit from the traditional arithmetic and algebra process from approximating and plugging in numbers closer and closer until the number is reached into an infinite process (Tall, 1992). The process of determining a limit does not typically use the formal definition (Dawkins, 2012). This process is less formal, more computational, and conceptually-based.

Another process typically taught in calculus is how to determine an appropriate $\delta > 0$ that satisfies the formal definition of a limit of a function for a given $\varepsilon > 0$. This process is then generalized in Real Analysis to proving there exists a $\delta > 0$ that satisfies the formal definition for any $\varepsilon > 0$. Similarly, for limits of sequences, the process of determining an appropriate natural number N, that satisfies the formal definition of a limit of a sequence for a given $\varepsilon > 0$ is generalized to proving there exists a N for any $\varepsilon > 0$.

The above processes associated with limits utilize a wide range of methods that incorporate a variety techniques and skills students learn and develop through different experiences (both explicitly remembered and not). Processes evoke different methods that are sparked by a task, cues, environment, and recent experience and their evoked processes are the ones generated by the student at a given time.

Limits of sequences. *Computing limits of sequences algebraically.* The students were given the prompt: *Find algebraically the following limits:* $\left\{\frac{7sin(n)}{n} - 2\right\}_{n=1}^{\infty}$, $\{cos(n) + 3\}_{n=1}^{\infty}$, $\{-\frac{1}{2}\}_{n=1}^{\infty}$, *and* $\{\frac{1-5n^4}{n^4+8n^3}\}_{n=1}^{\infty}$. The prompt provided insight into the students' evoked process of computing limits of sequences. The student's responses first coded for the algebraic method used to determine the limit, and if a graphical representation was present. The student's responses were then coded for mathematical errors such algebra mistakes, misuse of limit notation, and correct limit value.

Of the fourteen students who submitted the limits sequence, there was only one student Alan who did not attempt to compute any of the limits of sequences, the other 13 students attempted to compute at least two limits of sequences (Table 16).

Table 16.
Participants who attempted to compute the limits of sequences.

	$\left\{\frac{7sin(n)}{n} - 2\right\}_{n=1}^{\infty}$	$\{cos(n) + 3\}_{n=1}^{\infty}$	$\{-\frac{1}{2}\}_{n=1}^{\infty}$	$\{\frac{1-5n^4}{n^4+8n^3}\}_{n=1}^{\infty}$
Alan	-	-	-	-
Alicia	-	-	x	x*
Amy	x	x	x	x
Arnold	x*	x	x	x
Carlton	-	-	x	x
Edith	x	x	x	x
Jessie	x	x	-	-
Kayla	-	-	x	x
Maddie	x	x*	x	x
Melody	-	-	x*	x*
Nick	x	-	x	x
Tim	x	x*	x	x
Vicky	x	x*	x	x
Vincent	x	x	x	x

*Made an error when computing the limit.

Of 13 students who responded to the prompt only nine students attempted to algebraically compute the first limit of $\left\{\frac{7sin(n)}{n} - 2\right\}_{n=1}^{\infty}$. All nine applied the difference property of limits of sequences $\lim_{n\to\infty} \frac{7sin(n)}{n} - \lim_{n\to\infty} 2$, and computed each term separately. Only one student, Arnold determined $\lim_{n\to\infty} \frac{sin(n)}{n}$ was equal to one, when asked how he determined the limit. Arnold stated "it's either equal to one or zero, it's one that I had memorized." The other students determined $\lim_{n\to\infty} \frac{sin(n)}{n}$ equal to zero, but only some provided justification. For instance, Amy wrote "since $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$ and $\lim_{n\to\infty} -\frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, then $\lim_{n\to\infty} \frac{sin(n)}{n} = 0$."

There were eight students who attempted to determine the limit of the sequence $\{cos(n) + 3\}_{n=1}^{\infty}$. Jessie was the only student who provided no work or explanation for how he determined the limit of the sequence to be 3. Maddie also incorrectly determined the limit to be three, but explicitly applied the sum of limits of sequences property and determined $\lim_{n\to\infty} \cos(n) = 0$.

The other six students also explicitly applied the sum of limits of sequences property and determined that the range of the sequence *cos(n)* was bounded between [-1,1]. Two of the students, Tim and Vicky had correctly determined the range of the sequence $\cos(n)$but did not correctly determine that the limit does not exist. Tim rewrote the $\lim_{n\to\infty} \cos(n)$ as $\lim\limits_{n\to\infty} (-1)^n$ but provided no further work or conclusion. Whereas, Vicky determined that the $\lim_{n\to\infty} \cos(n)$ was either -1 or 1, and therefore, $\lim_{n\to\infty} \cos(n) + 3$ was either 2 or 4. Vicky didn't recognize that the limit of a sequence cannot be two different values.

The other four students determined that the limit of $\lim_{n\to\infty} \cos(n) + 3$ doe not exist because the $\lim_{n\to\infty} \cos(n)$ does not exist. Arnold and Vincent both justified that "the limit of $\cos(n)$ doesn't exist because it oscillates." Edith provided an explanation that incorporated the formal definition of the limit of a sequence. "The limit does not exist, (because) the range of $\cos(n)$ is bounded by -1 to 1. The limit of 3 is 3. But it doesn't fit the definition of a limit because this sequence will bounce back between 2 and 4 so as the (epsilon) constraint gets smaller the sequence will be getting out of it." Amy was the only student who generated a graphical representation (Figure 36) of the cosine function on the domain $[0, 2\pi]$, to help her determined that the range of the function was bounded between [-1,1].

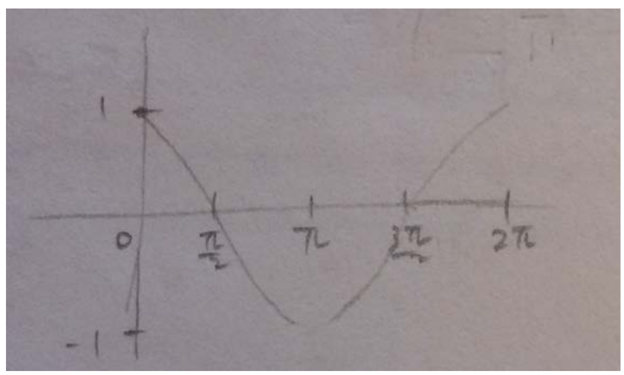


Figure 36. Amy's graphical representation of cosine.

Thirteen students attempted to determine the limit of the sequence $\{-\frac{1}{2}\}_{n=1}^{\infty}$. Melody was the only student who incorrectly determined the limit to be zero. The other twelve students correctly determined the limit to be $-\frac{1}{2}$. Kayla and Carlton only provided the solution with no explanation. Nick's explanation was an expansion of the sequence that showed the pattern: "$\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots\}n \to \infty = -\frac{1}{2}$." Alicia and Vicky justified the limit to be $-\frac{1}{2}$ because "there's no n." The remaining six students all explained that "the limit of a constant is a constant."

Thirteen students attempted to determine the limit of the sequence $\{\frac{1-5n^4}{n^4+8n^3}\}_{n=1}^{\infty}$. There were only two students who computed the limit to be positive five instead of negative five. There were four different algebraic methods used to determine the limit. Melody was the only student who rewrote the quotient as two different quotients and the computed the limit of the two. Vicky was the only student who did a mathematically incorrect method but determined the correct value.

Five of the students did what Alicia referred to as the “highest power comparison.” The students determined that since the highest powers of both the numerator and denominator’s polynomials were the same, the limit of the quotient is the quotient of coefficients of the the highest power terms in the numerator and denominator. Edith stated that the method “has to do with the coefficients of the leading degrees. And I know it fits the definition of limit.” Six students divided each term by the highest powered term x^4, and simplified each term. Then the students determined each term’s limit and reduced the quotient to be -5.

Calculating an N for a corresponding epsilon algebraically. The students were given the prompt: *Consider the* $lim_{n \to \infty} \frac{1-2n}{1+4n} = -\frac{1}{2}$ *with the given* $\varepsilon = 0.1$*, determine a corresponding* N.The prompt provided insight into the students’ evoked process of algebraically determining a *N* value for a given sequence and specific epsilon value. The student’s responses were first coded whether the student generated a graphical representation or not. If the student generated a graphical representation, it was determined if the graphical representation was usable and assisted the student in determining the corresponding *N*. The responses were coded for the different algebraic approach used. Each response was coded on whether there were algebraic errors and if they correctly determine a corresponding *N* value.

Of the twelve students who responded to the prompt, none provided a graphical representation. The methods used are summarized in Table 17.

Table 17.
Method for calculating an N for a corresponding epsilon.

	Dynamical practical	Absolute value inequality	Attempted a proof
Alan	-	x*	-
Alicia	-	x	-
Amy	-	-	x
Arnold	x	-	-
Carlton	-	x*	-
Edith	-	x*	-
Kayla	-	x	-
Maddie	-	-	x
Melody	-	x	-
Nick	-	x	-
Tim	-	x	-
Vicky	-	-	x

**Did not simplify.*

Arnold was the only student who used a dynamical practical method to determine the corresponding *N*. Arnold computed the first four terms of the sequence and stopped once he found a sequence value within the epsilon tolerance. Arnold had explained that he had “assumed that the sequences would be within the epsilon tolerance from *N*= 4 on.”

Eight of the students began with the absolute value inequality $|a_n - L| < \varepsilon$ from the formal definition and applied it to the specific prompt. Two of the students, Carlton and Kayla only set up the absolute value inequality. With no other scratch work, Kayla had written “$N = -1?$" In the interview Kayla could not explain the role of *N* or identify that *N* was supposed to be a natural number. One student, Alan additionally did an additional step of expanding the absolute value inequality into a compound inequality, and did no further simplifications.

The other five students simplified the absolute value completely. Nick made algebraic errors simplifying the absolute inequality to incorrectly determined *N* as 1. Melody and Alicia also made algebraic errors simplifying and determine *N* to be $\frac{1}{11}$. Neither student recognized that *N* is supposed to be a natural number. Edith and Tim were the only two students who simplified the absolute inequality correctly and correctly determined a corresponding *N* to be 4.

Amy, Maddie, and Vicky all attempted to generate a proof for the limit $\lim_{n \to \infty} \frac{1-2n}{1+4n} = -\frac{1}{2}$ for a given $\varepsilon = 0.1$. Maddie stated there existed a positive integer *N*, but did not define *N*. Maddie also stopped once she expanded the absolute value inequality into a compound inequality. Amy generated a proof and simplified the absolute value inequality to find that "$n > 3.5$ thus, $N = 3$ since $N \in \mathbb{N}$." Lastly, Vicky also generated a proof but with no algebraic justification chose $N > \frac{1}{\varepsilon}$, and substituted in the $\varepsilon = 0.1$ to determine the corresponding $N = 10$, which was not the smallest possible corresponding *N*, but an acceptable value. Vicky justification on why she chose $N > \frac{1}{\varepsilon}$, was because her instructor had done that in multiple proofs.

After the student were asked to algebraically find a corresponding *N*, the students were asked the follow-up question: *Is there another N possible? If so how do the two N's relate?* These responses were coded based whether they determined that another *N* value was possible or not. The responses were coded as either understanding the possible range of corresponding *N* or not correctly determining the range.

There were ten students who responded to the prompt. Carlton who was one of the students who had only set of the absolute value inequality wrote that he was "not sure how to do this." Kayla who had originally guessed that *N* = -1, wrote "*N = 1* because of the absolute value." Both students did not demonstrate an understanding of the algebraic process or the relationship of *N* with a given epsilon. Vicky and Melody were the only two students who explicitly wrote that there was not another possible *N*.

The remaining students, Amy, Arnold, Edith, Nick, and Tim determined that there was another possible *N.* Amy provided incorrect justification. She had stated there was another possible *N* "but the two *N*'s do not relate to each other because *N* is dependent on the values of epsilon." Although Amy understood that there was a covariational relationship between epsilon and *N*, she did not demonstrate that she understood that she had found the smallest possible *N*, and any *N* larger would satisfy the definition.

Arnold similarly responded to the prompt by writing that "if you change the epsilon another *N* values is possible." Arnold was taking the formal definition to be literal. When asked in the follow-up interview if the specific epsilon could have another *N*, Arnold stated "yeah, any one bigger." Edith and Tim had both chose *N* = 5 to be there other possible *N*, because "any value greater than *N*=4 is a possible value because after the fourth term the sequence converges." Melody made an algebraic error and chose her original *N* to be $\frac{1}{11}$, stated "*N* +1 could also work because all $n > N + 1$ will still be in the range. You pick the smallest *N*." Melody conceptually understood how the possible corresponding *N*'s related to a given epsilon. However, Melody did not recognize that *N* was supposed to be a natural number.

There was student, Nick who had specifically asked about this task, when he submitted his survey. Nick explain that he wrote "since we know that $\lim_{n\to\infty} a_n = -\frac{1}{2}$ there exists a *N* that makes it so", because the formal definition that there was only one *N*, but he also thought that there could possibly be another. Nick expressed that he was hesitant to write that there was another possible *N*, because he did not want to contradiction the definition. When Nick was asked to explain what he meant by contradict Nick drew a sequence and given epsilon and reasoned graphically that it was only necessary to have one *N* to satisfy the formal definition, but there could exist another *N* that was larger. Nick was having difficulty understanding the existential quantifier, but after a discussion of what it meant. By the end of the discussion Nick was confident in his original thought that there was another possible *N*, and no longer had confusion about the existential quantifier.

The process of determining the smallest N for a corresponding epsilon graphically. The students were given the prompt: *Consider the graph below (Figure 37), where both sequences, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to the real number L. Let ε be the indicated distance on the graph. Determine the smallest possible N for the $\{a_n\}_{n=1}^{\infty}$sequence and the smallest possible N for the $\{b_n\}_{n=1}^{\infty}$ sequence.*

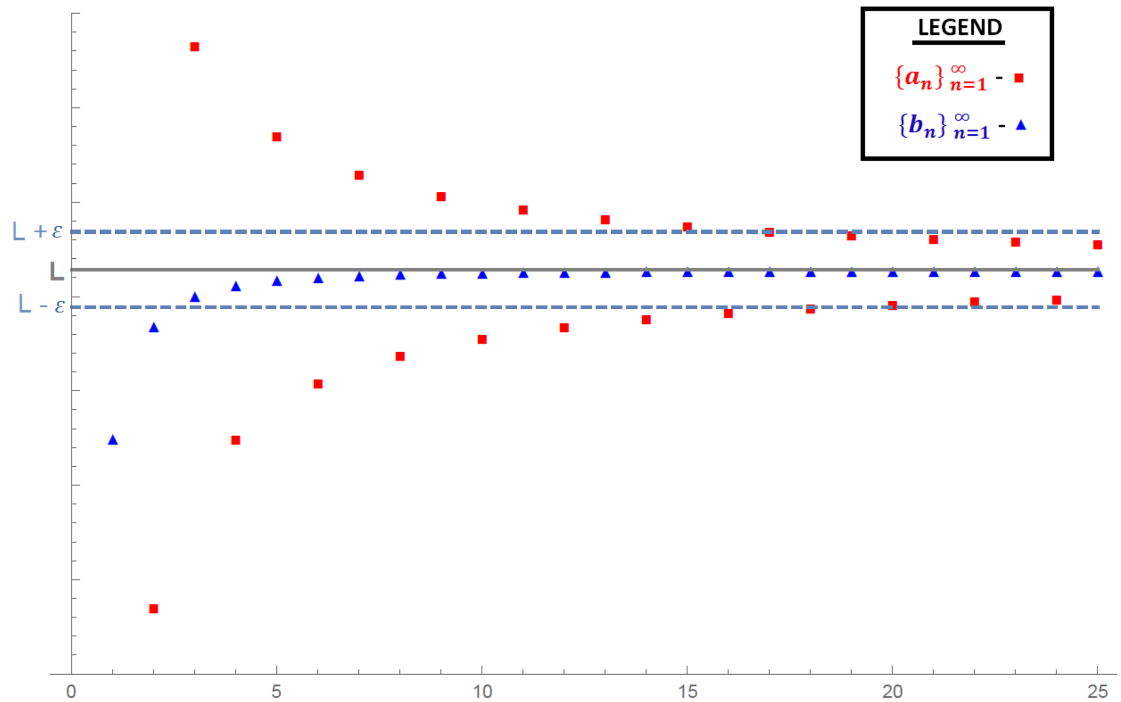


Figure 37. Graph of convergent sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$.

The prompt provided insight into the students' evoked process of graphically determining an *N* value for a given sequence and specific epsilon value. The student's responses were first coded whether the student determined if the *N* was a natural number (positive integer) to satisfy the formal definition. Next the corresponding *N* was coded if the student chose the smallest acceptable *N*.

The same 12 students responded to this prompt. Nine of the students correctly chose the smallest acceptable *N's* for both sequences. There were three students who had errors determining the smallest corresponding *N* values. One student, Melody chose an *N* that was not a natural number. Another student, Vicky treated the alternating sequence $\{a_n\}_{n=1}^{\infty}$ as two different sequences; the first sequence consisted of the terms of a_n when *n* is even, and the second sequence consisted of the terms of a_n when *n* is odd .

Therefore, Vicky determined two different smallest corresponding *N* values, N_{even} and N_{odd} for the alternating sequence $\{a_n\}_{n=1}^{\infty}$.

Maddie had written "aren't the *N*'s of a_n constant? They don't change throughout the sequence." When Maddie was asked to explain why she thought the *N*'s were constant, she explained she was confused because she originally thought that "from one point to the next (point in the sequence), between the following *x*-values, clearly there is like, infinitely many *x*-values. The distance (between) the *x's* is going to be your *n*." As Maddie described her understanding she indicated on the graph that the *n*'s were the distance between the *x*-values (Figure 38). Maddie then described that when she attempted this task, that the question "totally confused me because then it was like, smallest *N*", and she thought that they were a constant distance.

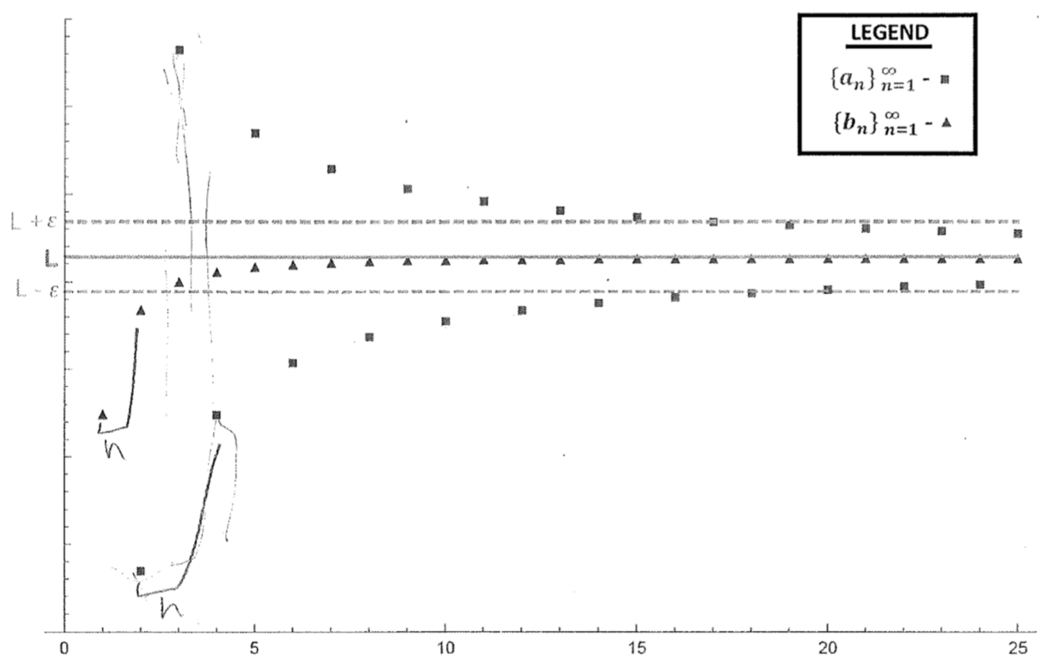


Figure 38. Maddie's graph of convergent sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$.

Understanding of the impact of the sequence on the process of determining the smallest possible N. One of the follow-up questions asked the students to: *Explain why the smallest possible N for the* $\{a_n\}_{n=1}^{\infty}$ *sequence is different for the smallest possible N for the* $\{b_n\}_{n=1}^{\infty}$ *sequence for the given* $\varepsilon > 0$. This prompt provided information about the students' understanding of how the relationship between epsilon and the smallest possible corresponding *N* depends on the sequence.

There were eight students who responded to the follow-up question. Seven of those students explained that the *N* values were two different sequences that approached the limit differently and therefore "entered" the epsilon tolerance at different rates. For instance, Edith explained "$\{b_n\}_{n=1}^{\infty}$ enters the restraints and stays there before $\{a_n\}_{n=1}^{\infty}$ does." Maddie was the only student of the eight who just stated the sequences were different and did not explain how the sequences interacted with the epsilon tolerance.

After determining the smallest *N*, the students were then asked to *explain how to determine the range of possible* $N's$ *for the sequence* $\{a_n\}_{n=1}^{\infty}$ *and the possible* $N's$ *for the sequence* $\{b_n\}_{n=1}^{\infty}$ *for the given* $\varepsilon > 0$. This prompt gave insight into the students' process of determining not only find the smallest possible corresponding *N* for a given epsilon, but all possible corresponding *N*'s. There were eight students who responded to the prompt.

Two students, Amy and Maddie incorrectly used the formal definition to attempt explain how to determine the possible *N*'s. Neither responses were coherent. Thus, Amy and Maddie tried to incorporate the formal definition but did not fully understand how the *N* values could vary for a given epsilon and how the relationship depends on the sequence. Nick also used the formal definition to explain how to determine the range of possible $\mathrm{N}'s$ for the sequence $\{a_n\}_{n=1}^{\infty}$. Nick stated that you find *N* such that " $|a_n - L| < \varepsilon$ for all $n \geq N$." Nick chose the possible range of *N's* to either be the *n* right when the sequence entered the epsilon tolerance or the *N* to be when the second *n* such that $|a_n - L| < \varepsilon$. Similarly, for the sequence $\{b_n\}_{n=1}^{\infty}$. Nick was determining his ranges of *N*'s to be dependent on the different versions of the formal definitions, which state that "there exists $N \in \mathbb{Z}^+$ such that if $\forall\, n \in \mathbb{Z}^+$ and $n > N$" or "there exists $N \in \mathbb{Z}^+$ such that if $\forall\, n \in \mathbb{Z}^+$ and $n \geq N$." Nick's reasoning for determining the ranges of *N* was rooted in the direct application of the formal definition.

Three students, Arnold, Carlton, and Alicia only described how to graphically determine the smallest corresponding *N.* Two students, Edith and Tim, both correctly described the possible range of corresponding *N's*. Edith explained that you determine the range of *N*'s to be greater than or equal to the smallest possible corresponding *N* value. Tim described the process of determining if an *n* is an appropriate *N,* if *n* was the index of a term for which every term afterwards falls within the epsilon tolerance.

Limits of functions. *Computing limits of functions algebraically.* The students were given the prompt: *Find algebraically the following limits:* $lim_{x\to -5}\frac{x^2+3x-10}{x+5}$ *and* $lim_{x\to -\infty}\frac{x}{(x-3)(x+2)}$. The prompt provided insight into the students' evoked process of computing limits. The student's responses first coded for the algebraic method used to determine the limit, and if a graphical representation was present. The student's responses were then coded for mathematical errors such algebra mistakes, misuse of limit notation, and correct limit value.

Table 18.
Participants who attempted to compute the limits of sequences.

	$\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$	$\lim_{x \to -\infty} \frac{x}{(x-3)(x+2)}$
Adam	x	x
Alan	-	-
Amy	x	x
Arnold	x	x
Brandan	x	x
Carlton	x	x*
Edith	x	x
Jessie	x	x
Maddie	x	x
Melody	x*	x
Nick	x	x
Tim	x	x
Travis	x	x
Vicky	x	x*
Vincent	x	x
Yolanda	x	x

* Made an error when computing the limit.

To solve $lim_{x \to -5} \frac{x^2+3x-10}{x+5}$ eleven of the students began by factoring the numerator then reduced the quotient to $(x-2)$. Once the students reduced the quotient to a polynomial they evaluated the limit at -5 to be -7. Yolanda computed the limit in this manner but additionally defined the function as $f(x) = \begin{cases} x-2, x \neq 5 \\ undefined, x = 5 \end{cases}$ and provided a graph (Figure 39) to show there was a removal discontinuity at the point $(-5, -7)$.

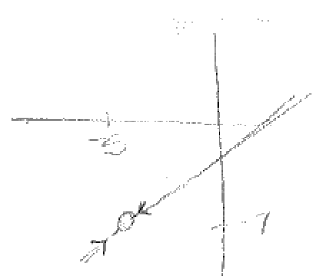


Figure 39. Yolanda's graphical representation of the function $\frac{x^2+3x-10}{x+5}$.

Two students Amy, and Nick began the problem by substituting the *x*-values with -5 and reduced the fraction to be of the indeterminant form $\frac{0}{0}$. Afterwards Amy solved the problem like the other ten students by factoring of the numerator, reducing the quotient and the evaluating the polynomial at $x = -5$ to be -7. Alternatively, Nick recognized the function was of indeterminant form and applied L'Hopital's rule. Nick evaluated the derivate of the numerator and the denominator to reduce to the polynomial $2x + 3$. Nick then evaluated the limit at -5 to be -7.

Melody was the only student who approached solving $lim_{x \to -5} \frac{x^2+3x-10}{x+5}$ by plugging in x-values closer to -5. Melody provided a graph (Figure 40), and plotted a few isolated points. Melody was the only student who did not correctly determine the limit of the function.

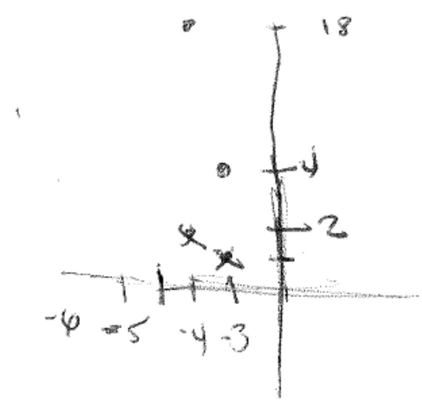


Figure 40. Melody's graphical representation of the function $\frac{x^2+3x-10}{x+5}$.

There were five students who used proper mathematical notation and wrote the limit symbol, $\lim\limits_{x \to -5}$ as they simplified the function. Another four students simplified the function without the symbol $\lim\limits_{x \to -5}$, but after they simplified the quotient to a polynomial the students returned the limit symbol. All nine students removed the limit symbol once they substituted -5 into the *x*-value, signifying the transition from the limit process to the limit as a mathematical object. Vincent explained that once the limit of the quotient is simplified to the "limit of $x - 2$, as *x* approaches -5, you evaluate (the limit and) drop limit symbol and (the limit) becomes -7." Vincent explained that the limit symbol is dropped because "we evaluated (the limit), just like whenever you add, you drop the plus sign. If I add 10 + 2, I don't write 12 +, I just write 12." There were three students who didn't include the limit symbol. These students' work did not distinguish the transition between the limit process and the limits as a mathematical object.

For the second $lim_{x\longrightarrow -\infty} \frac{x}{(x-3)(x+2)}$, none of the students provided graphs. There were five students who first simplified the limit by multiplying the binomials in the denominator, then they divided each term by x^2. Vincent explained that dividing each term by x^2 is the same as multiplying the quotient "by $\frac{1}{x^2}$ over $\frac{1}{x^2}$, which is just 1 ... which is equivalent to (the original) function, you would never guess. At least I wouldn't." Afterwards the students evaluated the limit of each term in the function. For determining the limit of the term $\frac{1}{x}$, as *x* goes to infinity, Vincent explained that as "*x* goes to negative infinity, 1 over a really big number eventually goes to 0. (The denominator) goes to infinity, and (the numerator) is just 1, that $\frac{1}{x}$ would go to 0." Once, each term's limit was calculated in a similar manner, the students reduced and concluded the limit to be zero.

Four other students multiplied the binomials in the denominator and then wrote the limit to be zero. Two of those students did not provide additional justification. Edith justified her answer by stating that there was a horizontal asymptote at *y* = 0. Nick stated the limit was zero "because the rational function has higher degree polynomial in the denominator."

Yolanda was the only student who after multiplying the binomials in the denominator out then rewrote the limit as $lim_{x\to -\infty}\frac{x}{x^2}$. Yolanda then reduced the quotient to $\frac{1}{x}$ and evaluated the limit to be zero. Two students, Jessie and Adam recognized that the limit was of indeterminant form and applied L'Hopital's rule. Jessie applied the rule by taking the derivative of both the numerator and the denominator and then evaluated the limit to be zero. Jessie justified his conclusion by stating "$\pm\frac{1}{\infty}=0.$" After Adam indicated L'Hopital's rule, he reduced the quotient to $\frac{1}{x}$ and evaluated the limit to be zero.

There was one student, Carlton who incorrectly rewrote the quotient $\frac{x}{(x-3)(x+2)}$ as $\frac{x}{x^2}-\frac{x}{x}-\frac{x}{6}$. Carlton then simplified and evaluated each of the three new quotients and then concluded the limit was $-\infty$. Vicky as well concluded the limit to be $-\infty$ but did not provide an algebra or explanation. Melody also did not provide any algebra but correctly concluded the limit to be zero.

There were six students who used proper mathematical notation and wrote the limit symbol, $\lim\limits_{x\to-\infty}$ as they simplified the function. One student, Arnold simplified the function without the symbol $\lim\limits_{x\to-\infty}$, but afterwards returned the limit symbol and then computed the limit. There were three students who didn't include the limit symbol. These students' work did not distinguish the transition between the limit process and the limits as a mathematical object. Carlton was the only student who had an algebraic error when simplifying the quotient.

The process of determining if a limit of a function exists graphically. The students were given the prompt: *Use the graph below* (Figure 41) *to answer the following questions: Does the limit of f(c) exist for each of the following c values: c = -4, c = -3, and c = 2? Explain why or why not.*

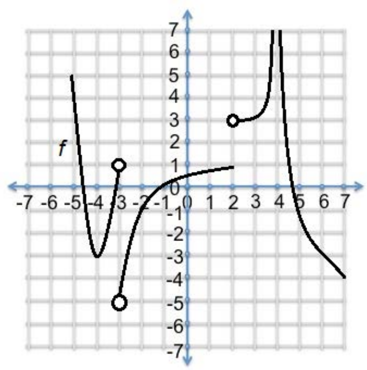


Figure 41. The graph of a discontinuous function.

The prompt provided insight into the students' evoked process of graphically determining if a limit exists at a specific *x*-value. The student's responses first coded whether they determined the limit exists, does not exist, or inconclusive. Next, the student's justifications were sorted based on the type of reasoning. Lastly, each response was coded for correct or incorrect justification and if it was a sufficient response. For instance, to determine whether a two-sided limit exists, the left- and right-sided limits must exist *and* must be equal to each other.

There was only one student, Amy that did not conclude if the limit existed for any of the *c*-values. Amy wrote the formal definition and did not provide any further explanation. The other fifteen students determined that the limit existed at the *c*-value *-4*.

Four students justified that the limit existed because the point (-4, -3) existed. Five students explained that limit existed at *c = -4* because the left- and right-sided limits were equal: $\lim_{x\to -4^-} f(x) = \lim_{x\to -4^+} f(x)$. Three different students used the concept of continuity to justify why the limit existed at that point. Two students presented justification that aligned with the formal definition. Edith discussed that the *y*-value -3 would be within the tolerance window. Vincent discussed how the function would be arbitrarily close to the limit value as *x*-values became arbitrarily close to -4. Lastly, Arnold reasoned that since "the function is defined at (the) point with no jump discontinuity."

There were seven students whose justifications were not sufficient or held an error. The students' who justified that the limit existed because the point (-4, -3) existed was not considered a sufficient mathematical argument. Carlton, who explained both sides of the function met at the same value, stated the limit to be -2. Lastly, Melody over generalized that the entire function was continuous, rather than stating it was continuous at the *c = -4*. The justifications provided were summarized in Table 19.

Table 19.
Justifications for why the limit existed at c = - 4

	Incorrect Justification	(-4, -3) existed	$\lim_{x\to-4^-} f(x) = \lim_{x\to-4^+} f(x)$	Continuous at *c* = - 4	Formal
Alan	-	x	-	-	-
Adam	-	x	-	-	-
Arnold	-	-	-	x	-
Brandan	-	x	-	-	-
Carlton	x	-	-	-	-
Edith	-	-	-	-	x
Jessie	-	x	-	-	-
Maddie	-	-	-	x	-
Melody	x	-	-	-	-
Nick	-	-	x	-	-
Tim	-	-	x	-	-
Travis	-	-	x	-	-
Vicky	-	x	-	-	-
Vincent	-	-	-	-	x
Yolanda	-	-	x	-	-

For determining if the limit exists at the *c*-value -3 there was one student, Melody incorrectly determined that the limit existed "because it has a left and a right limit." There was one student, Adam who did not definitively state if the limit existed or not. Adam provided the explanation that "as it approaches from the left the limit is 1, and approaching from the right the limit is -5." The other thirteen students concluded that the limit did not exist at -3.

Eight of the students explained that the limit does not exist at -3 because the left- and right-sided limits were different values: $\lim_{x\to-3^-} f(x) \neq \lim_{x\to-3^+} f(x)$. One student, Edith again used a formal argument that the "window (tolerance) does not close in on a value." Two students used the justification that it had a removable discontinuity/ "hole" and therefore the limit did not exist at -3. Alan and Carlton's explanation was a combination of the left- and right-sided argument with “holes at two points.” When Alan was asked the follow-up question to determine in the function’s limit exist at the hole (Figure 42), Alan stated that the limit did not exist “because it's non-continuous.”

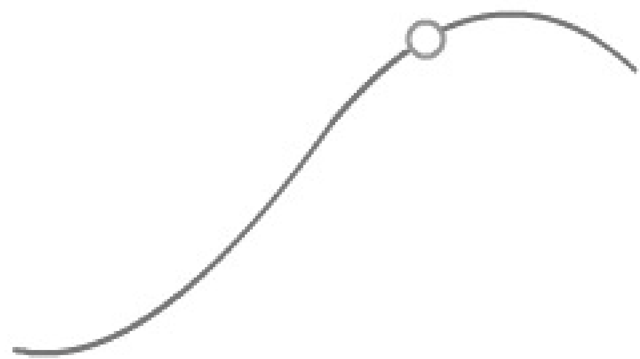


Figure 42. Function with a removable discontinuity.

Four of the students’ justifications were not sufficient or held an error. Melody determined that the limit existed because the left- and right-sided limits existed. However, it is not sufficient to determine that a two-sided limit exists. The left- and right-sided limits also need to be equal. The three students’ justification discussed removal continuities were incorrect mathematical arguments. The other ten mathematical arguments held no errors. The justifications of why the limit did not exist $c = -3$ is summarized in Table 20.

Table 20.
Justifications for why the limit does not existed at c = - 3.

	Insufficient justification	Removable Discontinuity	$\lim_{x\to-3^-} f(x) \neq \lim_{x\to-3^+} f(x)$	Formal
Alan		x	x	
Adam	x			
Arnold			x	
Brandan			x	
Carlton		x	x	
Edith				x
Jessie		x		
Maddie		x		
Melody	x*			
Nick			x	
Tim			x	
Travis			x	
Vicky			x	
Vincent			x	
Yolanda			x	

**Determined the limit existed.*

For determining if the limit exists at the c-value 2, Adam again did not conclude whether the limit at 2 existed, but did determine the left- and right-sided limits. Adam determined that the left-sided limit does not exist and the right-sided was 3. There were three students who determine the limit did exist at 2. Melody provided the same explanation that the limit exists "because it has a left and a right limit." Carlton concluded the limit existed because there was "no hole at $x = 2$, where $y = 1$." Yolanda argued that "$\lim_{c\to 2} f(c) = 3$ because $\lim_{c\to 2^-} f(c) = \lim_{c\to 2^+} f(c) = 3$."

The other eleven students concluded that the limit did not exist with varying justifications. Jessie concluded that the limit does not exist because of the removable discontinuity/ "hole." Edith again used the formal argument that the "window (tolerance) does not close in on a value." Eight of the students stated it did not exist because the left-and right-sided limits were different values: $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$. Lastly, Maddie argued that "although the *x*-value is continuous, $f(x)$ is not."

Six of the students' justifications were not sufficient or held an error. Adam incorrectly determined the left- and right-sided limits' values. Melody determined that the limit existed because the left- and right-sided limits existed. However, it is not sufficient to determine that a two-sided limit exists. The left- and right-sided limits also need to be equal. Jessie's justification that the limit does not exists because of the removal continuity/hole was an incorrect mathematical argument. Carlton who concluded the limit existed because $f(c)$ existed was incorrect in determining that if a one-sided limit did not have a removal discontinuity that the two-sided limit existed. Maddie incorrectly used the mathematical term continuity to describe the domain of the function. The other eight mathematical arguments held no errors. The justifications of why the limit did not exist $c = 2$ is summarized in Table 21.

Table 21.
Justifications for why the limit does not existed at c = 2.

	Incorrect justification	Removable Discontinuity	$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$	Formal
Alan	-	-	x	-
Adam	x	-		-
Arnold	-	-	x	-
Brandan	-	-	x	-
Carlton	x*	-	-	-
Edith	-	-	-	x
Jessie	-	x	-	-
Maddie	-	x	-	-
Melody	x*	-	-	-
Nick	-	-	x	-
Tim	-	-	x	-
Travis	-	-	x	-
Vicky	-	-	x	-
Vincent	-	-	x	-
Yolanda	x*	-	-	-

**Determined the limit existed.*

Overall, eight of the students used the same type of reasoning for each of the three different *c*-values. Edith used the formal definition's window of tolerance to determine if each of the limits existed. Tim, Nick, Brandan, Travis, and Yolanda determined if the two-sided limits existed whether the left- and right-sided limits exist and are equal. Jessie determined whether the limit existed if there was an open hole or closed hole for each *c*-values. Lastly, Maddie used the concept of continuity to determine if the limit existed or not.

There were four students who used different reasoning for the different *c*-values. The different justifications Vincent used was the formal definition and determined if the two-sided limits existed whether the left- and right-sided limits exist and are equal. The two types of justifications Melody, Carlton, and Arnold used was continuity and that the two-sided limits existed need the left- and right-sided limits to exist and are equal. Lastly, Vicky used to justifications that *f(c)* existed, and secondly that the left- and right-sided limits are not equal.

Calculating a delta for a corresponding epsilon algebraically. The students were given the prompt: *Consider the* $lim_{x \to 0} \sqrt{x+1} = 1$ *with the given* $\varepsilon = 0.1$*, determine a corresponding* δ.The prompt provided insight into the students' evoked process of algebraically determining a delta value for a given function and specific epsilon value. The student's responses were first coded whether the student generated a graphical representation or not. If the student generated a graphical representation, it was determined if the graphical representation was usable and assisted the student in determining the corresponding delta. Each response was coded on whether there were algebraic errors and if they correctly determine a corresponding delta value.

Of the thirteen students who responded to the prompt, five students generated a graphical representation. Two students, Vicky and Carlton generated unusable graphical representations (Figure 43 and 44). The only additional information Carlton provided was $\sqrt{1.1}$. Vicky did not provide an additional work. Both were unable to use the graphical representation, or algebraically conclude a delta value.

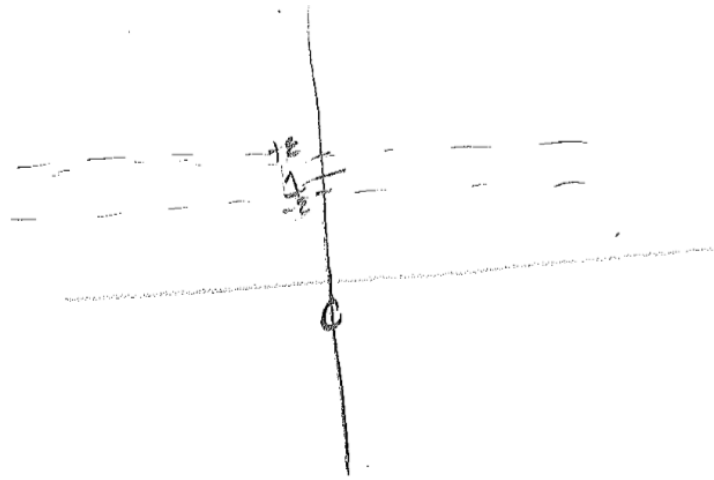


Figure 43. Vicky's graphical representation of $lim_{x \to 0} \sqrt{x+1}$.

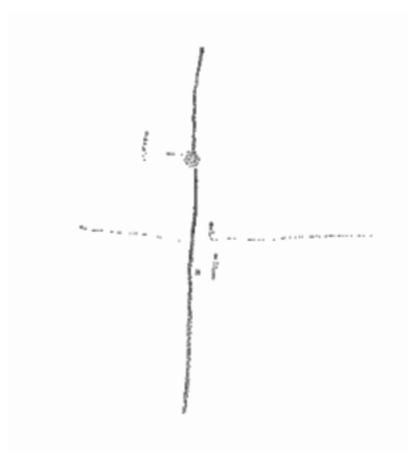


Figure 44. Carlton's graphical representation of $lim_{x \to 0} \sqrt{x+1}$.

Melody generated a graphical representation that plotted the function values (2, *f(2)*), (1, *f(1)*), and (0, *f(0)*). Melody indicated the limit and the epsilon tolerance (0.9, 1.1). Melody, provided no additional work, she solely used the graphical representation to incorrectly conclude that delta was 0.2. Therefore, the graphical representation was usable for Melody (Figure 45).

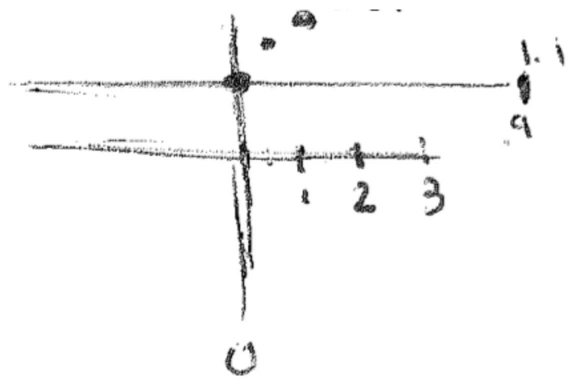


Figure 45. Melody's graphical representation of $lim_{x \to 0} \sqrt{x+1}$.

Maddie generated a usable graphical representation (Figure 46). Maddie graphed the function, the limit, the epsilon tolerance, and a delta tolerance. Maddie used the graph to set up the two different equations $\sqrt{x+1}$ = 0.9 and $\sqrt{x+1}$ = 1.1. Maddie correctly solved each equation and concluded that $c - \delta = -0.19$ and $c + \delta$ = 0.21. She incorrectly chose the corresponding delta to be "$c + \delta$ = 0.21."

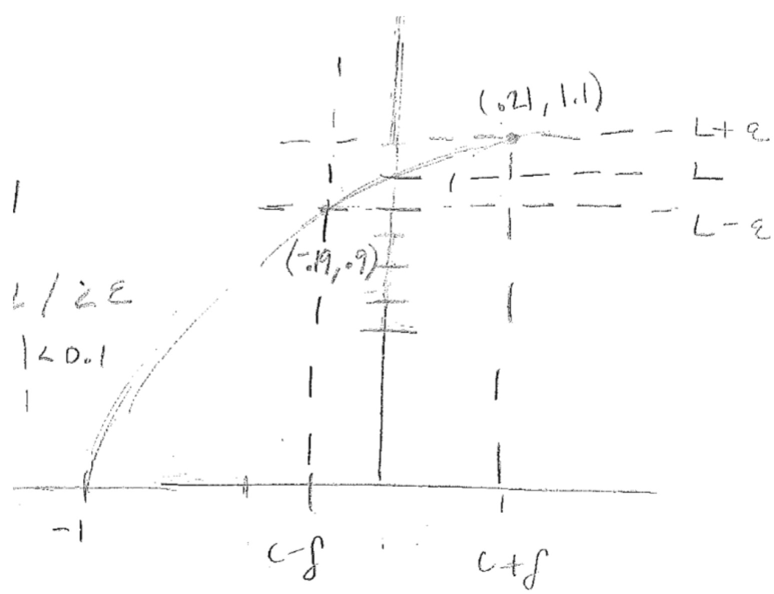


Figure 46. Maddie's graphical representation of $lim_{x \to 0} \sqrt{x+1}$.

The last person to generate a usable graphical representation was Brandan (Figure 47). Brandan graphical representation included the function, limit, epsilon tolerance, and a delta tolerance. Although Brandan provided the $|f(x) - L| < \varepsilon$ and $|x - p| < \delta$ parts of the formal definition and substituted the specifics from the given problem to have $|\sqrt{x+1} - 1| < 0.1$ and $|x - 0| < \delta$. Brandan did not simplify the absolute value inequalities. Rather, like Maddie, Brandan used the graphical representation to set up the two different equations $\sqrt{x+1}$ = 0.9 and $\sqrt{x+1}$ = 1.1. Brandan correctly solved each equation and concluded that $\delta_1 = -0.19$ and δ_2 = 0.21. Brandan correctly determined the delta value and explained that "δ_1 could not be δ because it will throw away from the epsilon range, therefore, δ should be 0.19."

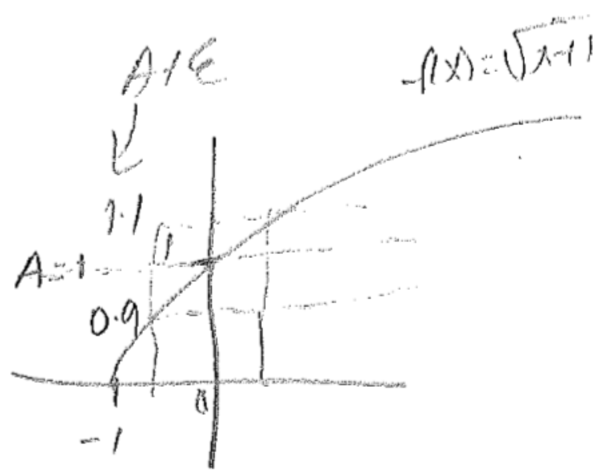


Figure 47. Brandan's graphical representation of $lim_{x \to 0} \sqrt{x+1}$.

There were eight students who did not provided graphical representations. One of the students, Adam only computed *f(0)*, and did not determine a delta value. The other seven students all began with setting up the absolute value inequality as $|\sqrt{x+1} - 1| <$

0.1. Three of those students made algebraic errors when simplifying the absolute value inequality $|\sqrt{x+1}-1|<0.1$. Travis, altered his original inequality to become $|x-1|<0.1$ and solved x to be $1.21<x<2.21$ and incorrectly chose $\delta=2$. Nick incorrectly expanded the absolute value inequality $|\sqrt{x+1}-1|<0.1$ to $-0.1<\sqrt{x+1}-1<0.1$. Nick also did not explicitly determine a delta value. Lastly, Edith separated $|\sqrt{x+1}-1|<0.1$ into the two inequalities $\sqrt{\delta+1}-1<0.1$ and $\sqrt{-\delta+1}-1<0.1$. Edith simplified the inequalities to be $\delta<0.21$ and $\delta>0.21$, respectively. Edith incorrectly concluded $\delta=0.22$.

Four of those students simplified the absolute values correctly, and only Arnold chose the correct delta. The remaining three, Vincent, Amy, and Tim all chose their delta value for different reasons. Amy also simplified the absolute value inequality correctly to $-0.19<x<0.21$. However, Amy did not recognize that delta represented a distance. Amy determined that the range of possible delta values were between the minimum value -0.19 and the maximum value 0.21. Amy chose the maximum value of the possible deltas. Vincent recognized that delta is a positive value that represents a distance and must be equal distance to the left and right of the *c*-value, 0. When Vincent was asked whether the distance 0.21 satisfied the epsilon tolerance, Vincent realized he did not check he computations, and later corrected his selection to be the smaller distance.

Like Vincent, Tim also knew that delta was distance equal distance to the left and right of the *c*-value, 0. Additionally, Tim recognized that the x-values' image within $|x|<\delta$ must be within the epsilon tolerance (0.9, 1.1). Therefore, Tim's error was not a conceptual misunderstanding related to delta, it was a misconception about comparing negative fractions. Tim had simplified the inequality in terms of fractions $-\frac{19}{100} < x < \frac{21}{100}$, and then used the formal definition to reason that $\frac{20}{100}$ would satisfy $|x| < \delta$. Tim mistakenly thought $-\frac{19}{100} < -\frac{20}{100}$, because the 19 < 20.

Arnold, who was the only student to correctly determine the delta value without a graphical representation reasoned similarly to Tim. Arnold also recognized that the x-values' image within $|x| < \delta$ must be within the epsilon tolerance (0.9, 1.1). Arnold thus determined that delta was the smaller distance, 0.19.

Table 22.
Methods for calculating delta for a corresponding epsilon algebraically.

	Generated a graphical representation	Absolute value inequality	Set up two equations	Incorrect method
Adam	-	-	-	x
Amy	-	x	-	-
Arnold	-	x	-	-
Brandan	x	x*	x	-
Carlton	x	-	-	-
Edith	-	x	-	-
Maddie	x	-	x	-
Melody	x	-	-	-
Nick	-	x	-	-
Tim	-	x	-	-
Travis	-	x	-	-
Vincent	-	x	-	-
Vicky	x	-	-	-

**Did not simplify.*

The students were asked the follow-up question: *Is there another delta possible? If so how do the two deltas relate?* These responses were coded based whether they determined that another delta value was possible or not. The responses were coded as either understanding the possible range of corresponding deltas or not correctly determining the range. There were ten students who responded to the follow-up prompt.

Brandan was the only student who determined that there was not another possible delta. Brandan had been one of the two students who correctly determined a corresponding delta for the given epsilon. Despite that, Brandan did not demonstrate an understanding that while it is sufficient to show that there exists a positive real number, delta, there may be other deltas that satisfy the formal definition for each epsilon.

The other nine students had determined that there was another possible delta. Three students, Carlton, Amy, and Maddie incorrectly stated that the other possible delta was the "opposite" delta value, $c - \delta = -0.19$. Amy's justification was that the possible range of delta values were between the minimum value -0.19 and the maximum value 0.21.

The other five students all provided correct explanations of for how the deltas would relate. These students recognized that the corresponding deltas are within zero and the largest possible delta, to keep the function within the given epsilon tolerance. Four of these five students had incorrectly found the largest possible delta, but demonstrated an understanding of the dependent relationship between epsilon and delta.

Determining the largest delta for a corresponding epsilon graphically. The students were given the prompt: *Consider the graph below* (Figure 48), *where* $lim_{x \to c} f(x) = L$ *and* $lim_{x \to c} g(x) = L$. *Let* ε *be the indicated distance on the graph. (a) Using the graph above of the function f(x) and the given* $\varepsilon > 0$ *on the graph, what is the largest possible* δ*? (b) Using the graph above of the function g(x) and the given* $\varepsilon > 0$ *on the graph, what is the largest possible* δ*?*

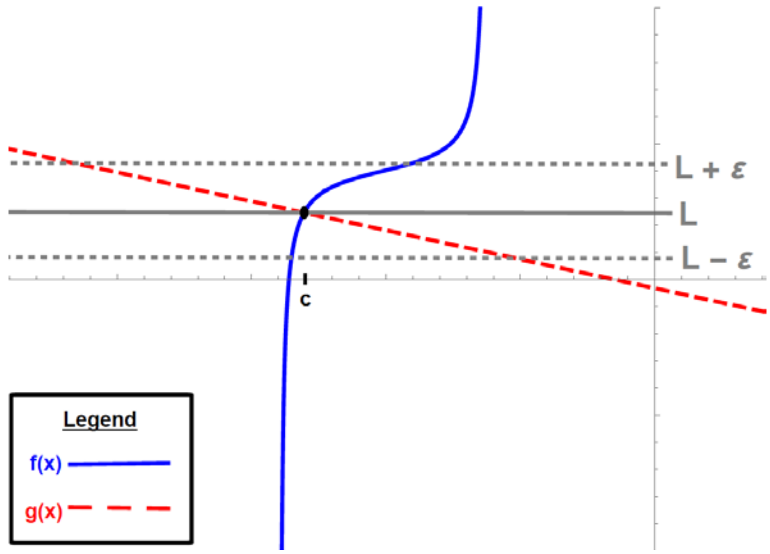


Figure 48: Graph of the limits: $lim_{x\to c}\, f(x) = L$ and $lim_{x\to c}\, g(x) = L$.

The prompt provided insight into the students' evoked process of graphically determining a delta value for a given function and specific epsilon value. The student's responses were first coded whether the student determined if the delta was greater than zero, to satisfy the formal definition. Next the corresponding delta was coded if the student chose the maximum of the two possible deltas or the minimum of the two possible deltas.

There were 15 students who responded to the prompt. There were seven students who correctly chose the deltas to be the minimum of the two possible deltas for each of the functions, and indicated that the deltas were positive values. Vicky explicitly explained " It's δ_1, because $\delta_1 < \delta_2$, therefore $\delta_1$is the largest appropriate delta."

There were three students who did not determine a numerical value for delta. Amy, who had responded that the possible range of delta values to be the minimum value $(c - \delta)$ and the maximum value $(c + \delta)$ in the algebraic prompt, responded similarly when determining the corresponding delta graphically. Amy wrote the formal definition and expanded the absolute value inequality $|x - c| < \delta$ to $c - \delta < x < c + \delta$, and indicated that the largest possible delta was the maximum value, $(c + \delta)$. Jessie had chose the corresponding delta for the function $f(x)$, δ_f to be the point $(\delta + c, L - \varepsilon)$ and the corresponding delta for the function $g(x)$, δ_g = $(\delta + c)$. Lastly, Adam have chosen the maximum values of the two possible deltas for both functions, however Adam reported the *y*-values rather than the *x*-values, $\delta_f = L + \varepsilon$ and $\delta_g = L - \varepsilon$.

There was one student Carlton, who did not recognize that there were two possible largest corresponding deltas for the given epsilon. Carlton chose delta for $g(x)$ and $f(x)$ to be the total distance between $(c - \delta_g, c + \delta_g)$ and $(c - \delta_f, c + \delta_f)$, respectively. The other three students selected the maximum of the two possible corresponding deltas for each function. One of those three students, Travis didn't recognize that both δ_g and δ_f must be greater than zero. Travis did not demonstrate an understanding that delta represents a distance and therefore must be a positive value.

The process of determining a corresponding delta for a given epsilon is dependent on the on the function. To obtain information about the students' understanding of how the relationship between epsilon and delta depend on the function, the students were asked the follow-up question: *Explain why the largest possible* δ *for the f(x) function is different than the largest possible* δ *for the g(x) function for the given* $\varepsilon > 0$. There were three types of responses the students generated. The first type of response only described the deltas being different because the two functions were different with no explicit reference to the epsilon. The second type of response demonstrated an understanding that there was a relationship between the function, epsilon and delta.

There were eleven students who responded to the follow-up question. Melody and Travis did not explicitly talk about the epsilon tolerance. Melody had stated that "the functions are different and have a different domain, so the area of the $|x - p| < \delta$ is of different length." Travis stated the deltas for the two functions were different because "the functions grow at different rates for some *x*-values." It is not clear whether Melody and Travis were referring to the functions' domains or their domains within the epsilon tolerances.

Nine of the students' explanations included how the function and epsilon tolerance impact the choice of the deltas. The students' description of how the function's relation to the epsilon tolerance impacts the selection of delta varied. For instance, Edith stated "the functions are different and the functions' lines gets out of the epsilon constraint at different times." Edith's description was an example of how some of the students thought about the function "entered" and "exited" the epsilon tolerance. Whereas, Nick stated that the functions "behave differently within the tolerance of *L* and so (the deltas) require different constraint conditions with respect to *c*." Nick's description was an example of how the students described the relationship within the epsilon tolerance rather than entering/exiting the tolerance.

Evoked processes summary. There were ten students who computed the limits of functions and the limits of sequences. Of those ten, six attempted to compute all of the limit problems. Vincent, Edith, and Amy were the three students who correctly computed all of the limits with no notational errors. There were four students, Maddie, Tim, and

Arnold, who correctly computed all the limits except one of the trigonometric sequences. Vicky also attempted each limit problem but incorrectly computed on trigonometric sequence and one rational function. Arnold and Maddie both had not correctly included the limit symbol in at least one limit problem.

There were three students who did not attempt to calculate all of the limit. Nick correctly computed all the limits with no notational errors except for the sequence $\{cos(n) + 3\}_{n=1}^{\infty}$. Nick did not attempt to calculate the limit of $\{cos(n) + 3\}_{n=1}^{\infty}$. Both Carlton and Melody did not attempt to compute the limits of the two trigonometric sequences. Melody had incorrectly computed the limit of the constant sequence. Both incorrectly calculated the limit of one of the rational functions. Both students did not correctly use the limit symbol.

A method for computing the corresponding delta and *N* for a given epsilon for both limits of functions and limits of sequences, respectively, does involve simplifying absolute value inequalities. There were nine students who attempted to calculate the corresponding variable for a given epsilon for both limits of functions and limits of sequences. Of those nine, only four attempted to simplify absolute value inequalities for both instance.

Two of those for students, Amy and Tim who correctly simplified both absolute values inequalities. However, Amy incorrectly choice the incorrect delta value and Tim had a misconception about comparing negative fractions, which impacted his selection of a corresponding delta. Edith correctly simplified the absolute inequality for limit of sequences and unsuccessfully attempted to do so for limits of functions. Nick in both instances made algebraic errors when simplifying absolute value inequalities.

There was one student, Arnold who found the corresponding variables for a given epsilon for both instance correctly. However, Arnold only set up an absolute inequality for the limit of functions. Arnold determined the corresponding *N* for the limit of a sequence using a dynamical practical method.

There were four students, Maddie, Carlton, Melody, and Vicky who generated graphical representations and were unsuccessful in determining a corresponding delta for a given epsilon of a limit of function. These four students all unsuccessfully attempted to determine the corresponding *N* for a given epsilon of a limit of sequence using an absolute value inequality. These four students were unsuccessful in computing the corresponding variables for both instances.

Evoked properties. The last domain of a person's concept image is properties. Like the other three domains, a person's understanding of a mathematical concept's properties a drawn from a person's past experiences (both explicitly remembered and not). Depending on the task, cues, environment, and recent experience different properties and combinations of properties may be prompted.

Limits of sequences. Determining if a limit exists or does not exist for a sequence can be determined by whether or not it satisfies either a single property or a combination of properties. For instance, it is sufficient for the sequence to have the property of being a constant to determine the sequence converges. Whereas, in contrast the single property of increasing is not sufficient to determine if a limit of a sequence exists or does not. Combine the property of increasing with the property of bounded above then one can determine that the limit of sequence does exist.

To gain insight into student's evoked properties, the students' evoked example spaces were analyzed. Fourteen students responded to the prompt: *Please provide as many examples as possible of sequences with the limit of 5.* Three of those students generated graphical representations and the other eleven students generated symbolically named sequences. To analyze each student's collection of examples' properties, each symbolically named sequence was graphed. All of the graphed examples were then categorized as either a sequence that converges to 5 or diverges. Next the sequences were sorted based on the properties that they portray. Convergent sequences' property types were either constant, increasing and bounded above, decreasing and bounded below, or alternating and eventually converging. Divergent sequences' property types were either increasing and not bounded above, decreasing and not bounded below, or alternating. Below is a chart that summarizes the percentage of the different property types of limits of sequences each student generated.

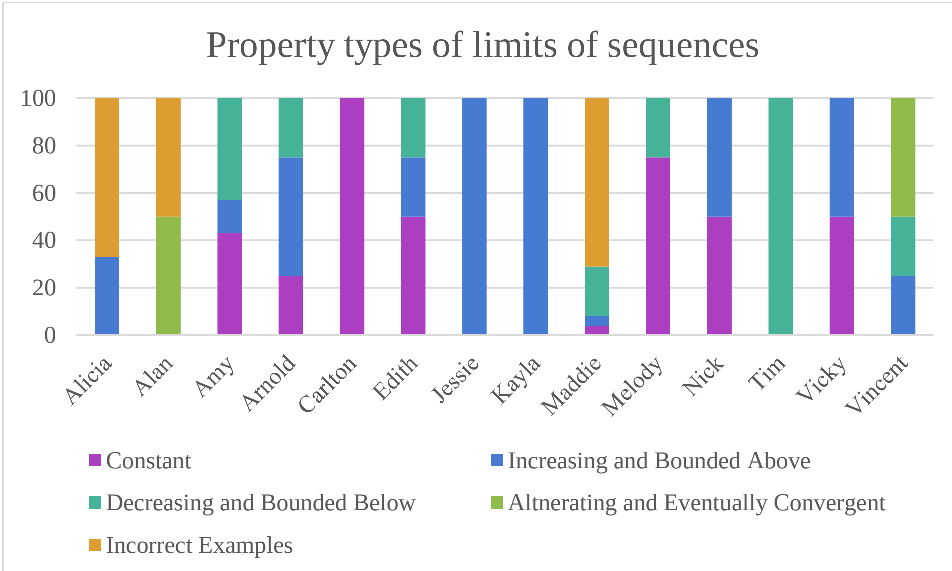


Figure 49. Property types of limits of sequences.

Out of the fourteen students only three students, Kayla, Alan and Maddie generated examples whose property types did not satisfy the definition of a sequence. Both, Kayla and Alan generated graphical representation of a function rather than a sequence (Figures 27 and 28). Maddie also generated examples with the domain as real numbers, rather than natural numbers. This was seen in her seventeen incorrect examples whose *n*-values approach integers and negative infinity. The other eleven students all generated examples that satisfied the properties of sequences.

Properties of convergent sequences. There were five students who generated only one of convergent property type. Tim generated a limit of a sequence with the two properties of being a decreasing sequence that is bounded below. Alan generated an alternating sequence that was eventually convergent to the value of 5. Carlton generate only constant convergent sequences. Alicia, Kayla, and Jessie both generated an example that was increasing and bounded above. Kayla and Alan's examples were graphical representations and Alicia, Carlton, and Tim's were symbolically names examples.

There were three students, Melody, Nick, and Vicky, who generated two types of convergent sequences' property types. All three generated symbolically named sequences that had the property of being constant. Vicky and Nick also generated an example with the properties of being an increasing, bounded above. Melody's second example type had the properties of being decreasing, bounded below.

There were five students who generated named sequence three types of convergent sequences' property types. Maddie, Amy, Arnold, and Edith all generated named symbolically sequences with the same three property types. The three property types were constant, increasing and bounded above, and decreasing and bounded below. Lastly, Vincent generated graphical representations of three sequences with the following properties increasing and bounded above, decreasing and bounded below, and an alternating sequence.

Properties of divergent sequences. There were four students who generated sequences that did not have all the necessary properties to be convergent. Alicia, Carlton, and Maddie generated symbolically named sequences that diverged and had the property type of being strictly increasing. Alan generated an increasing example with a vertical asymptote, Alan incorrectly bounded the domain rather than the range.

Limits of functions. Determining if a limit exists or does not exist for a function can be determined by whether or not it satisfies either a single property or a combination of properties. For instance, it is sufficient for the function to have the property of being a constant to determine the function converges. In contrast the single property of increasing is not sufficient to determine if a limit of a function exists or does not. However, if the property of increasing is combined with the property of being continuous at the *c*-value, and then one can determine that the limit exists.

To gain insight into student's evoked properties, the students' evoked example spaces were analyzed. Sixteen students responded to the prompt: *Please provide as many examples as possible of functions with the limit of 2.* Three of those students generated graphical representations and the other thirteen students generated symbolically named sequences. To analyze each student's collection of examples' property types, each symbolically named sequence was graphed. All of the graphed examples were then categorized as either a sequence that converges to 2 or diverges at the indicated *c*-value. Next the functions were sorted based on the properties that they portray around the indicated *c*-value. Functions whose limits were 2 at the indicated *c*-value had the following property types: constant, increasing and bounded above by the horizontal asymptote $y = 2$, decreasing and bounded below by the horizontal asymptote $y = 2$, oscillating and continuous at the indicated *c*-value, increasing/decreasing and continuous at the indicated *c*-value, or increasing/decreasing with a removable discontinuity at the indicated *c*-value. Figure 50 is a chart that summarizes the percentage of the different property types of limits of function each student generated.

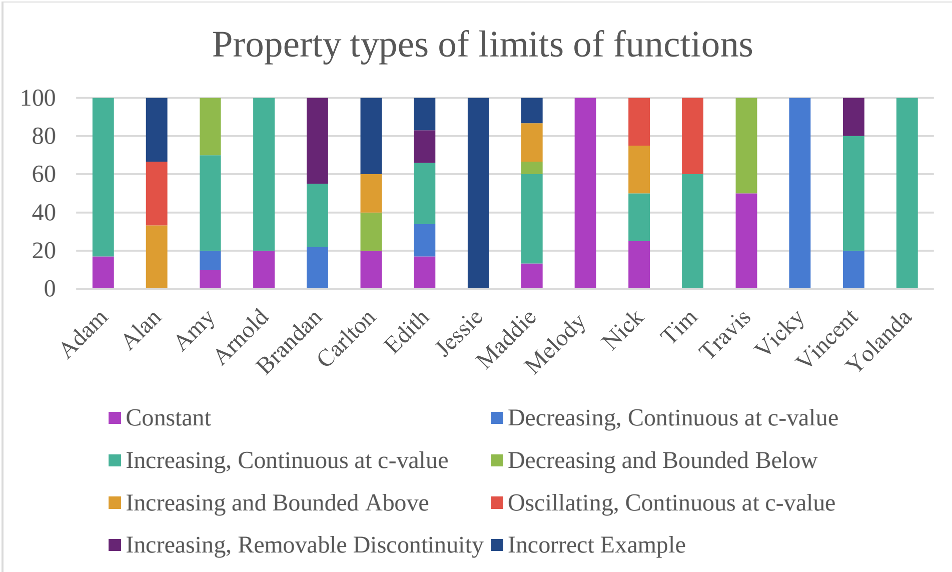


Figure 50. Property types of limits of functions.

Of the sixteen students who generated examples of limits of functions there were two students whose evoked ES contained examples that did not satisfy the definition of a function. Carlton, who generated graphical representations, had provided an example that failed the vertical line test (Figure 31). Another example generated by Carlton technically failed the vertical line test as well. However, it is important to note that it was undeterminable whether both lines were meant to represent one "function" or if the horizontal line was meant to be an asymptote. The other student, Jessie provided examples that were sequences, $\{2 + (\frac{1}{10})^n\}_{n=0}^{\infty}$ and $\{2 + \frac{1}{n}\}_{n=1}^{\infty}$ in which Jessie incorrectly thought functions were a special type of sequences. The other fourteen students generated examples that satisfied the properties of functions.

Limits that existed at the indicated c-value. There were three students, Melody, Vicky, and Yolanda who only generated examples who held one property type. Melody only generated examples that were constant functions. Vicky generated examples that were decreasing functions that were continuous at the indicated *c*-value. Lastly, Yolanda generated examples that were increasing functions that were continuous at the indicated *c*-value.

There were five students, Travis, Tim, Alan, Arnold, and Adam who generated examples that satisfied two different property types. Travis generated an example that was constant and the other example was a decreasing function that was bounded below by the horizontal asymptote at $y = 2$. Tim generated examples that had increasing behavior around the *c*-value and were continuous at the indicated *c*-value, and oscillating functions that were continuous at the indicated *c*-value. Alan produced an increasing function that was bounded above by the horizontal asymptote at $y = 2$, and an oscillating functions that were continuous at the indicated *c*-value. Lastly, Arnold and Adam both generated a constant function and the other examples had increasing behavior around the *c*-value and were continuous at the indicated *c*-value.

There were three students, Vincent, Carlton, and Brandan who generated examples that satisfied three different property types. The three property types that Vincent generated were decreasing and continuous at the indicated *c*-value, increasing and continuous at the indicated *c*-value, and an increasing function with a removable discontinuity at the indicated *c*-value. Carlton generated the following three property types: constant, decreasing function that was bounded below by the horizontal asymptote at $y = 2$, and increasing function that was bounded above by the horizontal asymptote at $y = 2$. Brandan generated examples that had increasing behavior around the *c*-value and were continuous at the indicated *c*-value, decreasing behavior around the *c*-value and were continuous at the indicated *c*-value, and an increasing function with a removable discontinuity at the indicated *c*-value.

There were four students, Nick, Edith, Amy, and Maddie who generated examples that satisfied four different property types. All four generated constant functions, and examples that had increasing behavior around the *c*-value and were continuous at the indicated *c*-value. Both Nick and Edith generated examples that were an increasing function with a removable discontinuity at the indicated *c*-value. Nick and Maddie generated an example that were increasing function that was bounded above by the horizontal asymptote at $y = 2$. Edith and Amy also generated a decreasing behavior around the *c*-value and were continuous at the indicated *c*-value. Amy and Maddie both generated a decreasing function that was bounded below by the horizontal asymptote at $y = 2$.

A function with discontinuities. There was one student, Vincent who generate a graphical representation with the purpose of explicitly indicating his understanding of the relationship of the property of a jump discontinuity and limits (Figure 51). In the follow-up interview Vincent explained that "limits are a necessary condition for continuity," but it is "not sufficient to be continuous." Vincent generated an example to show that "even though (a function is) discontinuous, it could still have a limit of 2", as long as the jump discontinuity does not occur at the indicated *c*-value. Vincent presented the idea that the properties around indicated property were of importance and the behavior not arbitrarily close to the indicated *c*-value were irrelevant. Other students such as Nick and Vicky generated examples that also demonstrated that they understood that discontinuities and other properties that are not arbitrarily close to the indicated *c*-value were irrelevant. These students generated functions that had vertical asymptotes not at the indicated *c*-

value.

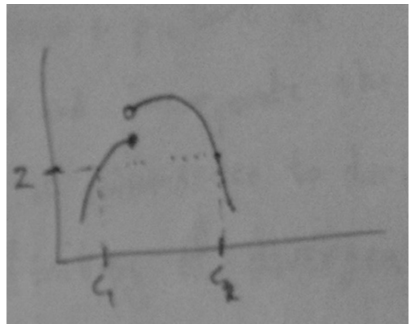


Figure 51. Vincent's example with a removable discontinuity.

There was one student, Alan who demonstrated a misunderstanding about the combination of the two properties of increasing and bounded above. Alan had generated an increasing function with the vertical asymptote $x = 2$, which was Alan's indicated *c*-value (Figure 52). Alan's interpretation of an increasing function that is bounded above by the limit value was oriented incorrectly. Alan interpreted that bounded above meant that the domain was bounded above, rather than the function's range. In the follow-up interview Alan was unable to recognize his misunderstanding of the properties, and did not recognize that he generated an incorrect example.

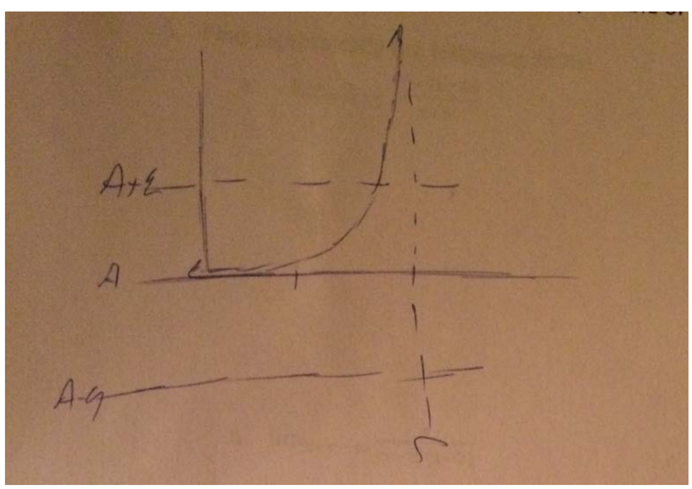


Figure 52: Alan's incorrect example of increasing and bounded above.

Evoked properties of limits during computations. The process of computing limits algebraically evokes students' knowledge of the properties of limits. The algebraic properties of limits of sequences are similar to the algebraic properties of limits of functions, since a sequence is a function. A sequence is defined to be a relation from the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ or either the non-negative integers, $\mathbb{Z}^+ = \{0, 1, 2, 3, ...\}$ to the set real numbers, such that every element of the domain is uniquely associated with an element of the range. Therefore, the five algebraic properties of limits are applicable to both limits of functions and limits of sequences. Assuming that the $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and $c \in \mathbb{R}$, is a constant, the algebraic properties for limits of functions are as follows:

- the constant property: $\lim_{x \to a} c = c$
- the multiplicative constant property: $\lim_{x \to a}[c * f(x)] = c \lim_{x \to a} f(x)$
- the sum and difference properties: $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- the product property: $\lim_{x \to a}[f(x) * g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
- the quotient property: $\lim_{x \to a}[\frac{f(x)}{g(x)}] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$
- the power property: $\lim_{x \to a}[f(x)]^p = [\lim_{x \to a} f(x)]^p$, where p is any real number

Similarly, for the algebraic properties for limits of sequences assuming that the $\lim_{n \to \infty} a_n$, and $\lim_{n \to \infty} b_n$ exist and $c \in \mathbb{R}$, is a constant. The algebraic properties for limits of sequences are as follows:

- the constant property:$\lim_{n \to \infty} c = c$

- the multiplicative constant property: $\lim_{n\to\infty}[c * a_n] = c \lim_{n\to\infty} a_n$
- the sum and difference properties: $\lim_{n\to\infty}[a_n \pm\ b_n] = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- the product property: $\lim_{n\to\infty}[a_n *\ b_n] = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$
- the quotient property: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided that $\lim_{n\to\infty} b_n\ \neq 0$
- the power property: $\lim_{n\to\infty}(a_n)^p = [\lim_{n\to\infty} a_n]^p$ where *p* is any real number

To gain insight into student's evoked properties of limits, the students' algebraic computations of the limits were analyzed to determine which properties were correctly or incorrectly evoked. The students' algebraic work was coded on whether the students explicitly or implicitly applied the property. For instance, an explicit use of the multiplicative constant property would be$\lim_{n\to\infty} \frac{7\sin(n)}{n} =\ 7 \lim_{n\to\infty} \frac{\sin(n)}{n} = 7(0) =$ 0 and the implicit use of the property would be $\lim_{n\to\infty} \frac{7\sin(n)}{n}$ = 0. The analysis of the evoked property of limits did not place an emphasis of whether the properties were applied to sequences or functions.

Algebraically evoked limit properties. There were ten students who algebraically computed limits of both sequences and functions. Since there are different algebraic methods to compute limits not all ten students explicitly or implicitly evoked all of the six algebraic properties of limits. Nine students correctly and explicitly applied the constant property. Melody incorrectly evoked the constant property by determining the $\lim_{n\to\infty} -\frac{1}{2}$ to be zero.

There were eight students who explicitly applied the multiplicative constant property when algebraic solving the limit $\left\{\frac{7sin(n)}{n} - 2\right\}_{n=1}^{\infty}$. Maddie was the only students who implicitly applied the multiplicative constant property. The other two students, Melody and Carlton did not attempt to solve the limit $\left\{\frac{7sin(n)}{n} - 2\right\}_{n=1}^{\infty}$, nor did they apply the multiplicative constant property to the other limits.

Nine of the students explicitly applied the sum property. For example, Amy applied the sum property to determine the limit does not exist for the sequence $\{cos(n) + 3\}_{n=1}^{\infty}$. Melody was the only student who attempted to solve the limit $\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3}$ using the sum property, and made the error of not including the negative sign in front of $\frac{5n^4}{n^4+8n^3}$. Carlton was the only student who did not apply explicitly or implicitly the sum property. Carlton however did inappropriately attempt to explicitly use the difference property to solve $\lim_{x\to-\infty} \frac{x}{(x-3)(x+2)}$. Carlton incorrectly rewrote the quotient $\frac{x}{(x-3)(x+2)}$ as $\frac{x}{x^2} - \frac{x}{x} - \frac{x}{6}$. None of the other nine students utilized the difference property.

Three of the students, Amy, Tim and Vincent explicitly applied the quotient property and implicitly applied the power property when solving both $\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3}$ and $\lim_{x\to-\infty} \frac{x}{(x-3)(x+2)}$. Arnold explicitly applied the quotient property and implicitly applied the power property when solving $\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3}$. The other six students did not explicitly or implicitly utilize the quotient and the power properties.

None of the limit computations explicitly required the students to use the product property. Therefore, to gain insight into the students understanding of the product property of limits they were asked to determine if the following statement was true or false: *Suppose that* $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences with* $lim_{n\to\infty} s_n = s$ *and*

$lim_{n\to\infty} t_n = t. \ Then \ lim_{n\to\infty} s_n t_n = st.$ The students were then asked to provide an example if the statement was true, or provide a counterexample if the statement was false.

The ten students all determined that the product property was true. Only one student, Vicky wrote "I'm not sure" instead of generating an example. The other nine students generated examples and showed that the product of the limits was equal to the limit of the products. For instance, Tim chose $s_n = \frac{1}{n-1}$ and $t_n = \frac{2n+1}{n+1}$. Tim showed that the $\lim_{n\to\infty} \frac{1}{n-1} = 0$ and the $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$, thus their product was equal to 0 as well as showing that the limit of the product $\lim_{n\to\infty} [\frac{1}{n-1} * \frac{2n+1}{n+1}]$ is equivalently zero.

Evoked properties summary. Overall, most students were able to generate limits of sequences and limits of functions satisfied the definitions of sequences and functions, and also different sets of properties that satisfies the limit to exist. Some of these students demonstrated misunderstandings of the properties to satisfy the definitions of functions and/or sequences, such as Carlton who generated a graphical representation that did not satisfy the definition of a function. There were a few students who generated examples that did not satisfy all properties necessary for a limit to exist, such as Alan who misunderstood that the property bounded above to imply the domain of a function was bounded above. Also, most of the students were able to correctly apply the properties of limits when computing limits.

Concept Definition

Tall and Vinner (1981) define "concept definition to be a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole"

(Tall & Vinner, 1981, p. 2). A student's concept definition may not necessarily align with the formal concept definition. A student's evoked concept definition may vary at different activated times. To gain an understanding of the diverse concept definitions students have in a Real Analysis course the students evoked concept definitions were captured with a survey about limits of sequences, a survey about limits of functions, semi-structured interviews, and observations.

Limits of sequences. The students were given the prompt: *Without using any resources please state the formal definition for the limit of a sequence.* The provided prompt was designed to activate the students' concept definition. The formal definitions presented in the two Real Analysis courses were:

1. The statement that the sequence $a_1, a_2, a_3, \ldots = \{a_n\}_{n=1}^{\infty}$ has a limit means there is a number L such that if $c > 0$ then there is an $N \in \mathbb{Z}^+$ such that if $k \in \mathbb{Z}^+$ and $k > N$ then $|a_k - L| < c$.
2. A sequence (s_n) is said to converge to s provided that $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N$ with $n \in \mathbb{N}$, it holds $|s_n - s| < \varepsilon$.

The students evoked concept definitions were coded as informal or formal. The students' evoked concept definition was coded as formal if it incorporated any portion of the formal definition. For example, "the sequence of limit exists if there is a small number such that $|a_n - A| < \varepsilon$," incorporates the $|a_n - A| < \varepsilon$ portion of the formal definition and therefore would be coded as formal. The evoked concept definitions that were coded as formal were then analyzed for completeness. For instance, the above evoked concept definition example would be coded as incomplete because it did not incorporate all of the portions of the formal definition. The evoked concept definitions were also coded as having correct ordering or incorrect ordering. Lastly, the students' evoked concept definitions were coded for being a mixture of symbols and words, or purely symbolic.

Formalization. Three students, Amy, Jessie, and Kayla generated informal concept definitions. Amy's evoked concept definition was "sequence s_n gets closer to a specific point s as $n \to \infty$." Jessie generated the evoked concept definition as the "values of a sequence adds up to or tends to." Lastly, Kayla's evoked concept definition of limits of sequences was "$\lim_{n\to\infty} s_n$ (is) the number s_n approaches as $n \to \infty$."

The other eleven students generated formal definitions. Five students' evoked concept definition were categorized as formal, but were incomplete and/or not in the correct order. Alicia's evoked concept definition was : "A sequence $\{a_n\}_{n=1}^{\infty}$ has a limit if $|a_n - L| < c$ for each $n \in \mathbb{Z}$ and $c > 0$." Alicia's evoked concept definition was both incomplete and had in correct ordering. Alicia did not include the quantifiers and the corresponding $N \in \mathbb{Z}^{+}$and $n > N$, and she placed $c > 0$ last instead of first.

Maddie's evoked concept definition was: "Let $\{a_n\}_{n=1}^{\infty}$ be a sequence s.t. $\exists\ N \in \mathbb{Z}$ if $n\ > N$ and $n \in \mathbb{Z}^{+}$then $c > 0$ and $|a_n - L| < c$." Maddie's evoked concept definition had incorrectly placed $c > 0$ after N, which is incorrect since N depends on c. Maddie incorrectly defined the domain of N as integers, rather than positive integers. Maddie also never defined L.

Carlton and Melody's evoked concept definitions were correctly ordered. However, both were missing the universal quantifier for $c > 0$. Carlton's evoked concept definition was: "Let a_n has a limit L for a $c > 0$ then there exists an $N \in \mathbb{Z}^{+}$ and n such that if $n \in \mathbb{Z}^+$then $n\ > N$ and $|a_n - L| < c$." Similarly, Melody's evoked concept definition was: "The limit of a sequence exists if there is an L and $c > 0$ then there exists an $N \in \mathbb{Z}^+$such that if $n \in \mathbb{Z}^+$and $n\ > N$ then $|a_n - L| < c$." The *if* $c > 0$ in their Real Analysis course's formal definition was discussed in class to be equivalent to *for every* $c > 0$, *for any* $c > 0$, or *for each* $c > 0$. Therefore, their evoked concept definition did not completely capture that the definition should hold true for every $c > 0$, rather than a single $c > 0$.

Lastly, Vicky's evoked concept definition was: "$s_n \to s$ iff $\varepsilon > 0\ \forall\ N \in \mathbb{N}$ s.t. $n \geq N$, $|s_n - s| < \varepsilon$." Vicky's evoked concept image had the correct ordering. However, Vicky's evoked concept definition was incomplete since Vicky did not include the universal and existential quantifiers.

The following six students' evoked concept definitions were formal, complete and in correct order:

- Alan: "$\exists$ some number L s.t. if $c > 0$ then there is an $N \in \mathbb{Z}^+$ such that if $n \in \mathbb{Z}^+$ and $n > N$ then $|a_n - L| < c$."
- Arnold: "There is a number L such that if $\varepsilon > 0$ there is an $N \in \mathbb{Z}^+$ such that if $n \in \mathbb{Z}^+$ and $n > N$ then $|a_n - L| < \varepsilon$."
- Edith: "The statement $\lim_{n\to\infty} a_n = L$ means there is a number L such that if $c > 0$ then there is an $N \in \mathbb{Z}^+$ where if $n \in \mathbb{Z}^+$ and $n > N$ then $|a_n - L| < c$."
- Nick: "L is the limit of a sequence, $\{a_n\}_{n=0}^{\infty}$ if for each $c > 0$, there exists $N \in \mathbb{Z}^+$ s.t. if $\forall\ n \in \mathbb{Z}^+$ and $n > N$ then $|a_n - L| < c$."

- Tim: “The limit of a sequence, $A(n) = \{a_n\}_{n=0}^{\infty}$ there exists a number l so that for some positive value ϵ there is a positive integer N, so that for $n > N$ $|a_n - l| < \epsilon$.”
- Vincent: “$\exists\ L, \forall\ c > 0$ s.t. $\exists\ N \in \mathbb{Z}^+$, $(\forall\ n \in \mathbb{Z}^+ \wedge\ n \geq N) \Rightarrow |a_n - L| < c$.”

None of the six evoked concept definition were identical. Five of the students' evoked concept definitions used a mixture of symbols and words. Vincent was the only student who completely generated a symbolic definition. When asked to verbally state the definition Vincent stated “I need to write you symbols first.” As Vincent generated a symbolic evoked concept definition he explained that “I use a mnemonic device that I picked up from (my professor). Which is pick, let, pick, let, show. Which means ... to pick something means that there exists something, let means for all, to pick exists, and then let means for all again. So, I equate in my mind the words pick with the idea. . . I am kind of breaking it up in pieces. That's how the definition of limit is. It has those 4 components. Whenever I say, pick, let, pick, let, show. What I am actually referencing is how I would actually prove that definition holds for some sequence.” Vincent was able to translate his symbolic definition into an equivalent complete, coherent, and correct ordered verbal definition.

Limits of functions. The students were given the prompt: *Without using any resources please state the formal definition for the limit of a function.* The provided prompt was designed to activate the students' concept definition. The formal definitions presented in the two Real Analysis courses were:

1. Suppose that f is a function and p is a number. The statement that "the limit of f exists as x has limit p" means there is a number A such that if $\epsilon > 0$, then there is a $\delta > 0$ such that if $0 < |x - p| < \delta$, then $|f(x) - A| < \epsilon$.
2. A limit of a function f exists provided that $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if $0 < |x - p| < \delta,$ then $|f(x) - L| < \varepsilon.$

The students evoked concept definitions were coded as informal or formal. The students' evoked concept definition was coded as formal if it incorporated any portion of the formal definition. For example, "the limit of a function exists if there is a small number such that $|f(x) - L| < \varepsilon$," incorporates the $|f(x) - L| < \varepsilon$portion of the formal definition and therefore would be coded as formal. The evoked concept definitions that were coded as formal were then analyzed for completeness. For instance, the above evoked concept definition example would be coded as incomplete because it did not incorporate all of the portions of the formal definition. The evoked concept definitions were also coded as having correct ordering or incorrect ordering. Lastly, the students' evoked concept definitions were coded for being a mixture of symbols and words, or purely symbolic.

Formalization. There were two students, Jessie and Alan, who did not generate evoked concept definitions for limits of functions. In the follow-up interview, Alan was asked to state the formal definition, Alan couldn't and stated "I can't recite the formal definition. I haven't memorized it yet." When asked to provide an informal definition, Alan stated "the function part is throwing me for a loop. I think it's more of an anxiety thing when I see how long everything gets. I always start second guessing myself and then I just lock up." Alan was unable to generate an evoked concept definition.

Of the fourteen students who generated an evoked concept definition, only one student Vicky produced an informal definition. Vicky's evoked concept definition was "a fundamental concept concerning the behavior of the function near a particular input." There was also only one student who generated a formal evoked concept definition that included an incorrect condition $f(x)$ exists, toward the end of the definition. Yolanda's evoked concept definition was "there exists a number *L* such that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - p| < \delta$ then $f(x)$ exists and is $|f(x) - L| < \varepsilon$."

The following 12 students' evoked concept definitions were formal, complete and in correct order. The evoked concept definitions were all very similar, and used a mixture of words and symbols. Amy and Maddie were the two students who generated symbolic evoked concept definitions.

- Adam: "there is a number *L* such that if $\varepsilon > 0$, then there is a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.."
- Amy: "$\forall\ \varepsilon > 0, \exists\ \delta > 0$ s.t if $0 < |x - c| < \delta,$ then $|f(x) - L| < \varepsilon$."
- Arnold: "there exists a number *L* such that if $\varepsilon > 0$, then there exists a $\delta > 0$ such that if $0 < |x - p| < \delta$, then $|f(x) - L| < \varepsilon$."
- Brandan: "means there is a number *A* such that if $\varepsilon > 0$, then there is a $\delta > 0$ such that if $0 < |x - p| < \delta$, then $|f(x) - A| < \varepsilon$."
- Carlton: "means there is a number *A* such that if $\varepsilon > 0$, then there is a $\delta > 0$ such that if $0 < |x - p| < \delta$, then $|f(x) - A| < \varepsilon$."
- Edith: "means there is a number *A* such that if $\varepsilon > 0$, then there is a $\delta > 0$ such that if $0 < |x - p| < \delta$, then $|f(x) - A| < \varepsilon$."

- Maddie: "∃ a # A if > 0 ∃ $\delta > 0$ s.t if $0 < |x - p| < \delta,$ then $|f(x) - L| < \varepsilon$."
- Melody: "means there is a number A such that if $\varepsilon > 0$, then there exists a $\delta > 0$ such that if $0 < |x - p| < \delta$, then $|f(x) - A| < \varepsilon$."
- Nick: "$\forall\ \varepsilon > 0, \exists\ \delta > 0$ s.t if $x \in \mathbb{R}$ and $0 < |x - c| < \delta,$ then $|f(x) - A| < \varepsilon$. $A = f(p)\ \forall\ x \in \mathbb{R}$ iff $f(x)$ is continuous."
- Tim: "if $\varepsilon > 0$, then there is a $\delta > 0$ such that if $0 < |x - p| < \delta$, then $|f(x) - A| < \varepsilon$."
- Travis: "means there is a number A such that if $\varepsilon > 0$, then there's a $\delta > 0$ such that if $0 < |x - p| < \delta$, then $|f(x) - A| < \varepsilon$."
- Vincent "there exists a real number L such that for all positive real numbers ε, there exists a positive real number δ so that if $0 < |x - p| < \delta,$ then $\left|f(x) - L\right| < \varepsilon$."

Evoked concept definitions summary. Overall, most of the students were able to generate evoked concept definitions, there were only two students who were not able to. Most of the evoked concept definitions incorporated the formal definitions. There were only three students who generated an informal evoked concept definition for limits of sequences, and one student who generated an informal evoked concept definition for limits of functions. There were more formal evoked concept definitions that were incomplete or had incorrect ordering for limits of sequences than limits of functions. The predominate flaw in the incomplete concept definitions for sequences was excluding the equivalent universal quantifier: *if* $c > 0$, *for every* $c > 0$, *for any* $c > 0$, *for each* $c > 0$, or $\forall\ c > 0$. However, the majority of evoked concept definitions were formal, complete, and in the correct order.

Potential Conflict Factors

Within a person's concept image there may be factors that conflict with other factors within a particular domain on the concept image, with different factors across the domains, or with factors within their concept definition. Tall and Vinner (1981) defined these factors to be potential conflict factors. These factors may be subtle and are not always apparent to the individual. These potential conflict factors also may never be activated or cause confusion. However, depending on the task, cues, environment, and recent experiences different potential conflict factors may simultaneously arise and when they do so, they become cognitive conflict factors. The cognitive conflicting factors can cause confusion and misunderstandings or even a sense of unease. Below is an example of cognitive conflict and how the student did not yield a correct resolution.

Maddie's evoked mental image of a limit of a function is one that "must satisfy the three conditions of being continuous where the limit approach one value, bounded, and not oscillating." Maddie's concept image contains the potential conflict factor that she incorrectly thinks only continuous functions have limits and the potential conflict factor that can cause cognitive conflict is functions with removable discontinuities. Maddie was given the prompt: *Consider* $lim_{x \to -5} \frac{x^2+3x-10}{x+5}$. *The precise definition of a limit of a function states that* $lim_{x \to c} f(x) = L$ *if for every* $\varepsilon > 0$, *there exists a corresponding* $\delta > 0$ *such that for all* $x < 0$, $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$. *For the function above and for* $\varepsilon = 1$, *it can be shown that the corresponding* $\delta = 1$. *Locate and label on your graph the values of* L, c, ε, *and* δ.

Maddie graphed the function $f(x) = \frac{x^2+3x-10}{x+5}$ as the linear function $f(x) = x - 2$ rather than a function with a removable discontinuity at $x = -5$ i.e. $f(x) = \begin{cases} x-2, x \neq -5 \\ undefined, x = -5 \end{cases}$ (Figure 53). Maddie's incorrect cohesion to the cognitive conflict was to graph the continuous linear function. Therefore, Maddie was not able to address the underlying misconception of limits of functions.

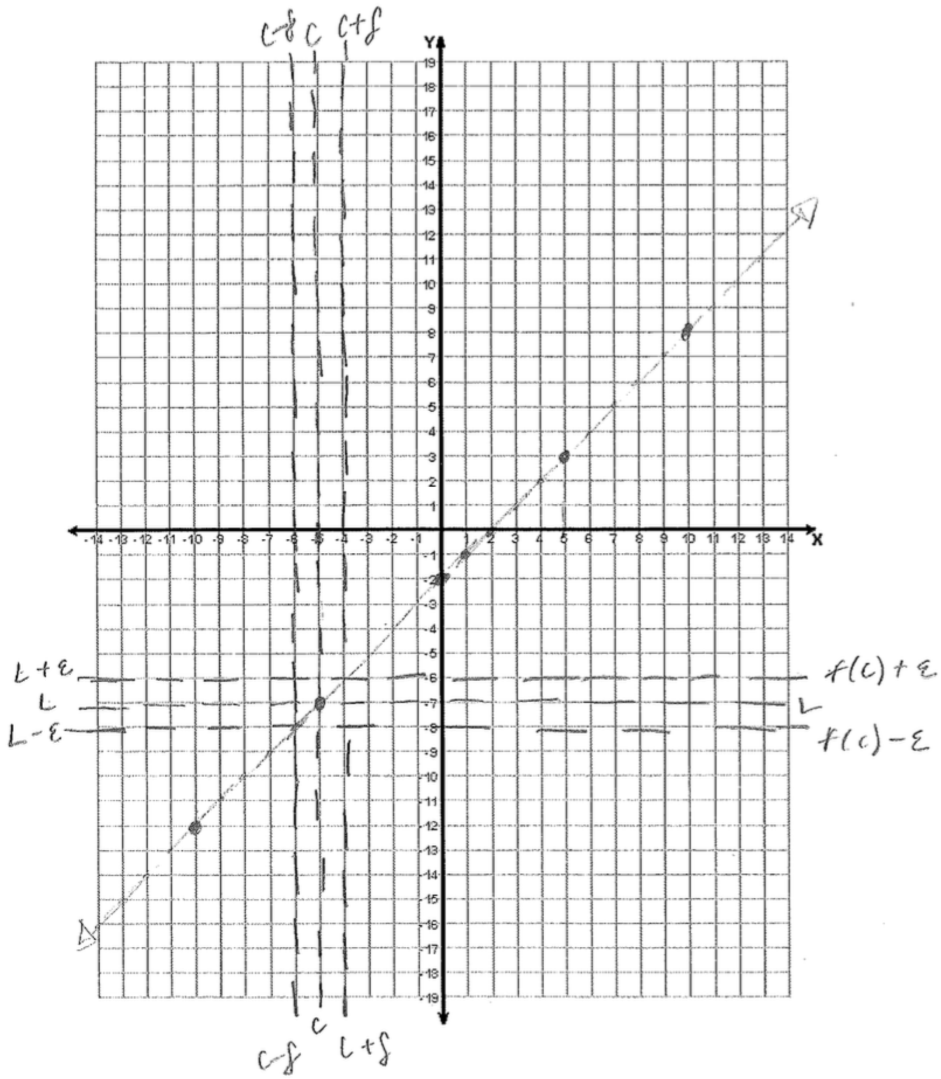


Figure 53: Maddie's potential cognitive conflict.

There are also other instances when students do recognize their cognitive conflict and make correct resolutions. The resolution occurred during the follow-up interview when Vicky was discussing her responses to the prompt: *Use the graph below* (Figure 54) *to answer the following questions: Does the limit of f(c) exist for each of the following c values: c = -4, c = -3, and c = 2? Explain why or why not.*

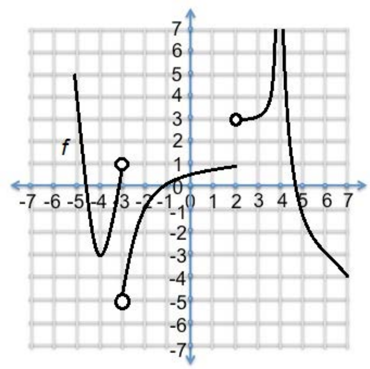


Figure 54. The graph of a discontinuous function.

Vicky had an understanding that jump discontinuities were instance when a limit does not exist. Vicky had correctly determined that the limits did not exist at *c = -3,* because “there’s two possibilities.” However, in the interview Vicky expressed she was uncertain about her reasoning because of the “open holes” at the jump discontinuity. Therefore, Vicky’s understanding of limits and jump discontinuities were having conflict with a jump discontinuity that was also had removable discontinuities. After Vicky was told the “open holes” represent a removable discontinuities Vicky reasserted her reasoning was correct. To better understand if Vicky had made a resolution she was asked about two different scenarios with removable discontinuities. The first was to determine if whether the limit existed at *c = -2,* where the jump discontinuity only had “one hole.” Vicky said “no, there’s two possibilities.” The second scenario was of a function with a removable discontinuity and no jump discontinuity. Vicky was provided an image of function with a removable discontinuity (Figure 55) and was asked to determine if a limit exists at the “open hole.” Vicky stated “I guess so. I guess it's, well no because it stops there. Oh, but it says the limit is what it's going towards, so then yeah it would be right there.” Vicky was able to reach a resolution, of the different discontinuities and limits with the use of her concept definition.

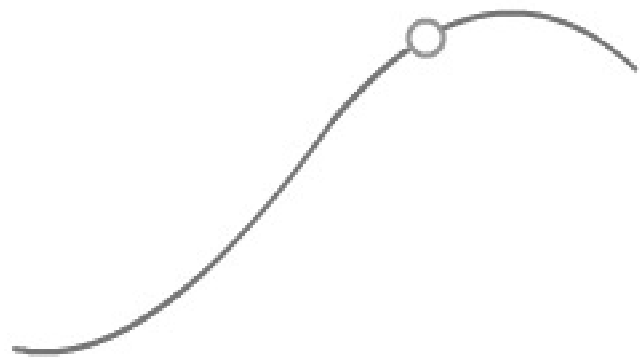


Figure 55. Function with a removable discontinuity.

Tall and Vinner (1981) determined that “a more serious type of potential conflict factor is one in the concept image which is at variance not with another part of the concept image but with the formal concept definition itself.” These potential conflict factors are serious because if someone experiences cognitive conflict with the formal definition it made impede on someone’s ability to adopt the formal definition. The individual may be reluctant to reconstruct their own concept image to coherently align with the formal definition, and potentially have the individual “regard the formal theory as inoperative and superfluous.”

From the data there emerged two groups of students. Those who’s evoked concept images and evoked concept definitions demonstrated potential conflict factors that misaligned with either one or both of the formal definitions were a part of the cognitive conflict group. It is however, not necessarily true that the cognitive conflict was apparent to the individual. The other group consists of individuals whose evoked concept images and evoked concept definition did not present potential conflict factors or cognitive conflict with the formal definition, called the cognitive resolution group. It is not to say that those in the cognitive resolution group do not have potential conflict factors that misalign with the formal definitions. It means that none of the potential conflict factors emerge at the activated times or were apparent in their responses.

To determine if a student held a serious potential conflict factor, the student's responses were checked for accuracy and errors against both of the formal definitions. If error was a minor algebraic, graphical, or notational error they were not classified as serious potential conflict factors. A potential conflict factor was considered serious if it added unnecessary conditions to the formal definition or contradicted the formal definition. To accurately determine that a student was a part of the cognitive resolution group both evoked concept images of limits of sequences and limits of functions were analyzed. Therefore, the students who only completed one of the limit surveys were not included in the analysis, because it could not conclusively be determined whether they did or did not have serious potential conflict factor about types of limits.

Cognitive conflict group. Since a person's concept image of limits is composed of both limits of sequences and limits of functions, it was appropriate to only classify the students who had responded to both surveys. Therefore, serious potential conflict factors were determined from the examination of both evoked concept images. Out of the twelve students who completed both the limits of sequences and limits of functions surveys, there were seven students' responses demonstrated serious potential conflict factors.

Alan generated three similar images that displayed the same cognitive conflict with the formal definition. Each of the three graphs (Figure 56) were of a continuous function with a horizontal asymptote at $y = 0$ and a vertical asymptote to the right of the function, graphed on the Cartesian plane. The behaviors of the three functions were the same; as the *x*-values decreased to negative infinity the *y*-values became arbitrarily close to zero and as the *x*-values became arbitrarily close to the vertical asymptote the *y*-values increase to infinity. The first graphical representation was generated as an example of a sequence with the limit of five, and vertical asymptote was $x = 5$. The second graphical representation was generated as an example of a function with the limit of two, and the vertical asymptote at $x = 2$, which Alan also labeled as *c*. The last graph was generated as a general graphical representation of function with the limit of *L*. Alan indicated the vertical asymptote to be $x = L$.

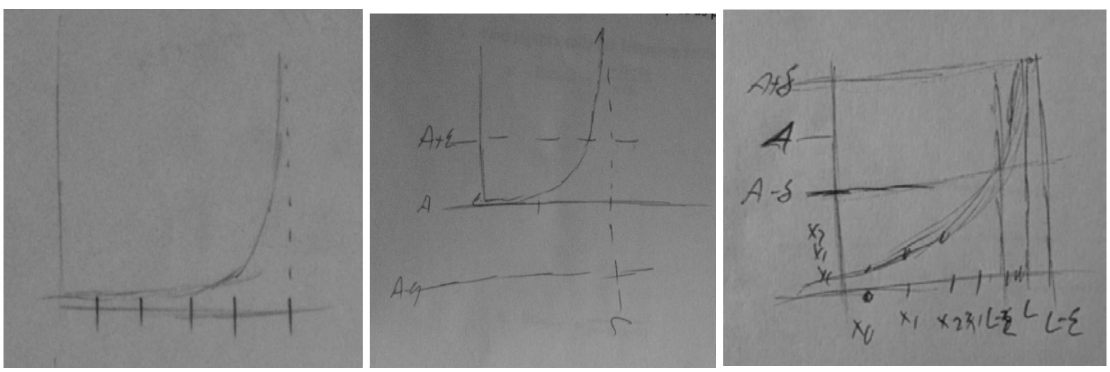


Figure 56. Alan's cognitive conflict factors.

Alan's three graphical represents were the result of Alan trying to blend conflicting mental images that were occurring simultaneously. The mental image of this particular function on a Cartesian plane was conflicting with images of limits of sequences on a number that Alan had seen in class. Some of Alan's classmates presented number line graphics to aid in explanations of their proofs of limits of sequences, as seen below (Figure 57).

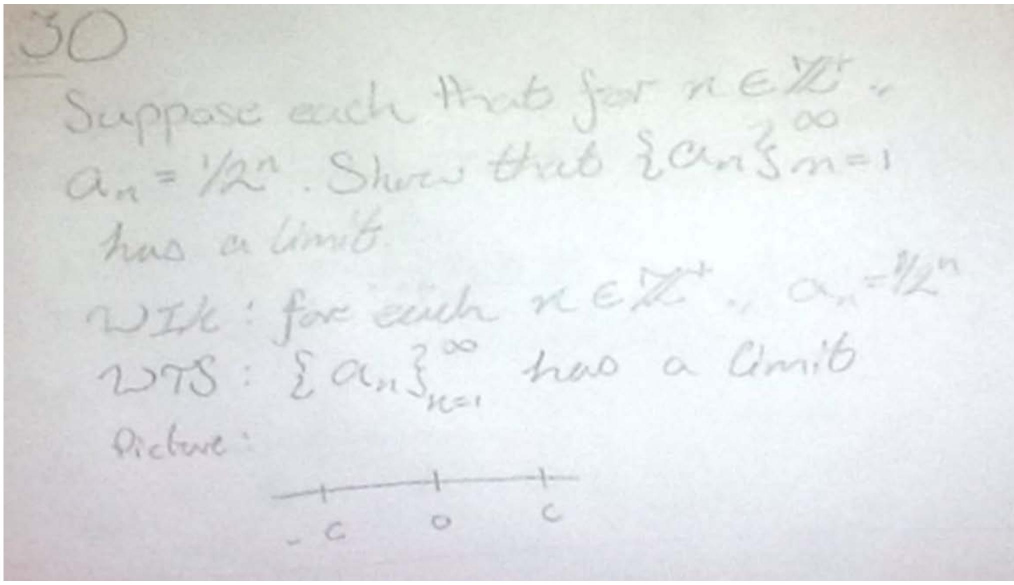


Figure 57. Limit of a sequence on a number line class observation example.

Alan was aware that he had confusion between limits of sequences and limits of functions and explained that he was "probably looking at it more from a sequential point of view" when he drew the limit as the horizontal asymptote. Alan was incorporating the limit vertically like the images he saw presented in class. To make cohesion between these two conflicting factors Alan incorrectly determined that "limits of functions are a sequence plugged into a function and it's not the terminating point of the function, but the value of the function approaches and then stabilizes that."

This cognitive conflict was serious since it influenced Alan to incorrectly adapt the formal definition to have cohesion with these mental images. Alan swapped the orientation of the epsilon tolerance and the delta tolerance. Alan did incorporate his instructors' notation of the limit value being *A* for a limit of a function rather than *L*, but did not recognize the different variables were meant to denote the same thing. Alan was also not able to generate a concept definition and stated "I can't recite the formal definition … the function part is throwing me for a loop." Overall, the cognitive conflict was a major prohibited Alan from adopting the formal definition correctly.

Maddie also demonstrated cognitive conflict between simultaneous conflicting images of a Cartesian plane graphic and a number line graphic. The cognitive conflict appeared in the following prompt: *Consider the graph, where both sequences,* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *converge to the real number L. Let ε be the indicated distance on the graph. Determine the smallest possible N for the* $\{a_n\}_{n=1}^{\infty}$*sequence and the smallest possible N for the* $\{b_n\}_{n=1}^{\infty}$ *sequence.*

Maddie wrote "aren't the *N*'s of a_n constant? They don't change throughout the sequence." When Maddie was asked to explain why she thought the *N*'s were constant, she explained she was confused because she originally thought that "from one point to the next (point in the sequence), between the following *x*-values, clearly there is like, infinitely many *x*-values. The distance (between) the *x's* is going to be your *n*." As Maddie described her understanding she indicated on the graph that the *n*'s were the distance between the *x*-values (Figure #). Maddie then described that when she attempted this task, that the question "totally confused me because then it was like, smallest *N*", and she thought that they were a constant distance.

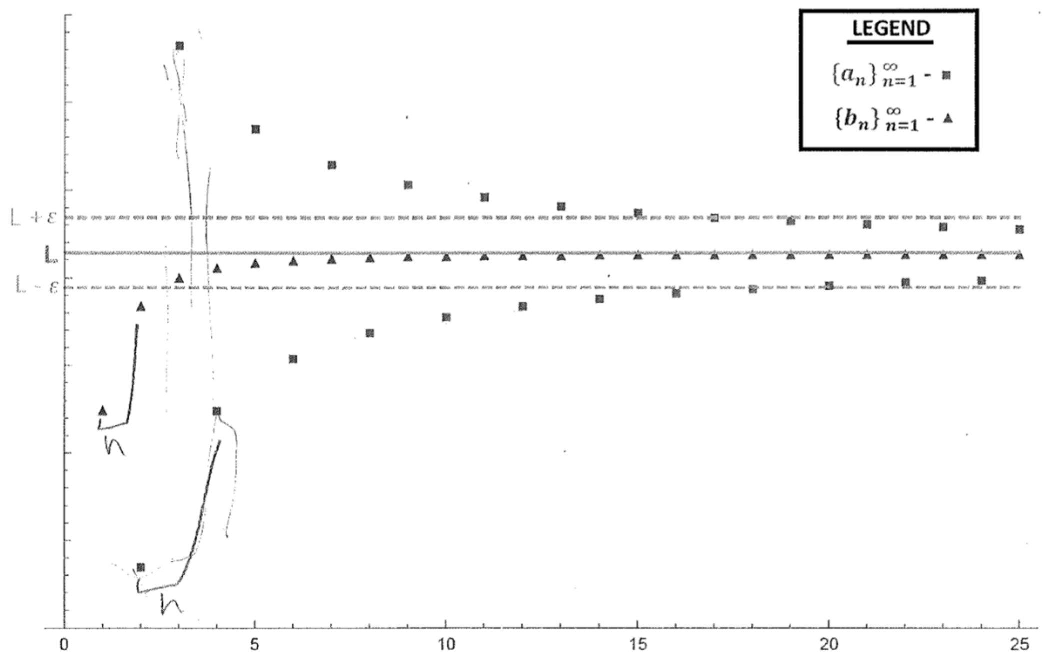


Figure 58. Maddie's graph of convergent sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$.

When Maddie was asked to generate her own sequence to further discuss her understanding of this distance, the underlying cognitive conflict factors appeared. Maddie was having cognitive conflict between the prompt's limit of a sequence on a Cartesian plane and her mental image of a limit of a sequence on a number line (Figure 59). The number line graph was a common image that was presented in class discussions about "how do we pick *N*?" The instructor would direct the students to use the picture and ask "what do they know about members of the sequence and their possible locations." The instructor would wait for the students to describe the sequences behavior and notice when the sequence would stay within $(L - c, L + c)$. Students were able to determine *N*, during these discussions. Once student determine *N*, the class automatically let *n* be a positive integer greater than *N*.

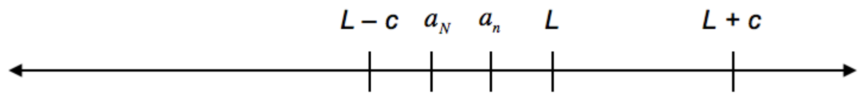


Figure 59. Number line graphic of a limit of a sequence.

Maddie addressed her cognitive conflict by asking about the two types of graphs, “it's like the same thing though right?” Once, Maddie made cohesion that the two different images represented the same thing just in different dimensions, she generate a graph (Figure 60) and began trying to recreate a class discussion. Maddie described the sequences’ behavior and the distances between the sequence terms and the limit in relation to the epsilon tolerance. It was when Maddie tried to explain the step of “picking *N*” that she described the horizontal distance between the terms on the Cartesian plane were constant, and called the distances *n’s*. Maddie concluded that since the distances were constant and all the terms of the sequence were within the epsilon tolerance she drew, that someone could pick any *N* and then “let *n* be a positive integer greater than *N*.” Overall, this was a serious cognitive conflict because Maddie had a lasting misconception of the *n*’s and their relationship with the *N* in the formal definition.

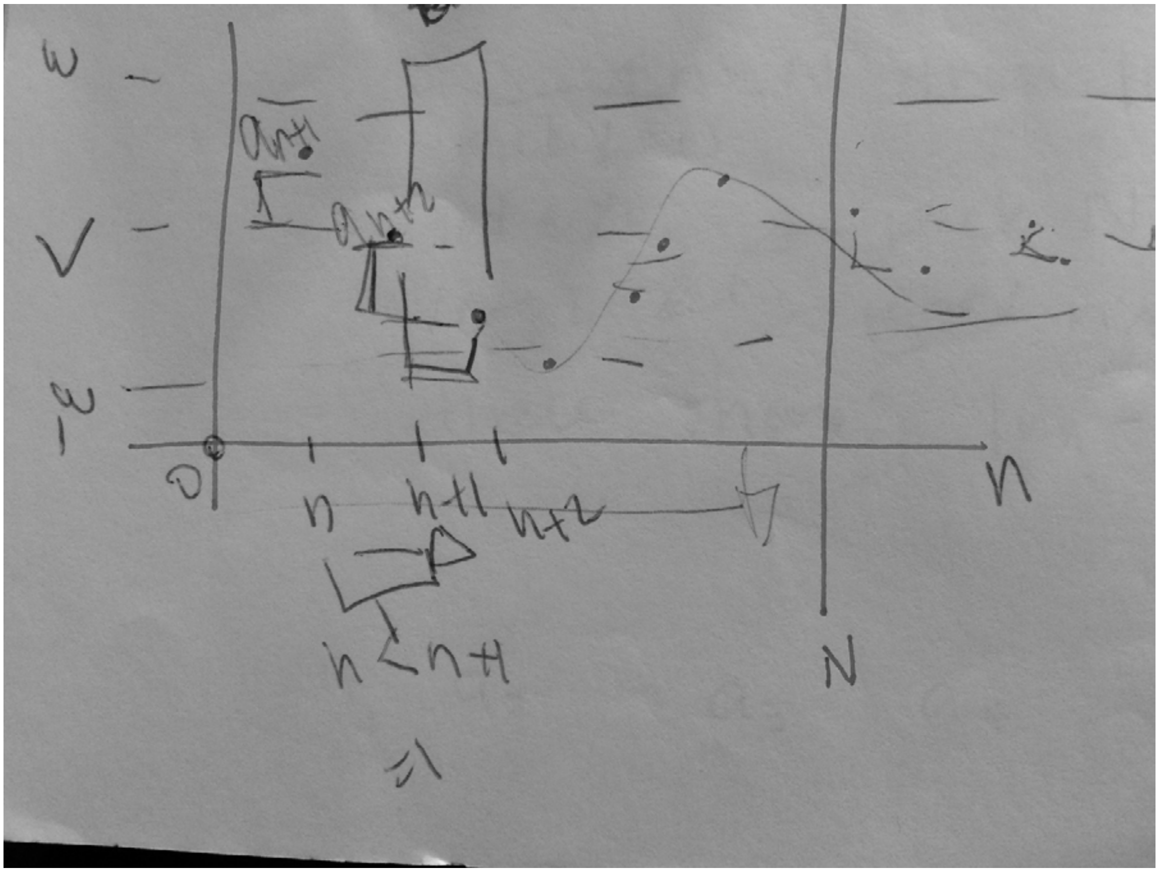


Figure 60. Maddie's graphical explanation of the distances between terms.

Vicky also had a serious potential conflict with the formal definition. Vicky had considered alternating sequences to special cases of convergent sequences. Vicky thought that alternating sequences could have two possible *N* values since the even terms "hits the $L + \varepsilon$ line at a different time than (the odd terms) hits the $L - \varepsilon$." Thus, Vicky determined there two different smallest *N* values N_{even} and N_{odd}, for the alternating sequence $\{a_n\}_{n=1}^{\infty}$. Vicky also treated alternating sequences as special convergent sequence when computing limits. Vicky had determined that the $\lim_{n\to\infty} \cos(n)$ was either -1 or 1, and therefore, $\lim_{n\to\infty} \cos(n) + 3$ was either 2 or 4. This was a serious potential conflict factor since Vicky chose to morph the formal definition for this special case.

Carlton demonstrated cognitive conflict responding to the prompt: *Consider the* $lim_{n\to\infty}\frac{1-2n}{1+4n} = -\frac{1}{2}$. *Let* $\varepsilon = 0.1$, *and find a* N. *Is there another N possible? If so how do the two N relate?* Carlton had previously demonstrated an understanding that to satisfy the formal definition one must show there exists an *N*. But was facing uncertainty when asked if there were another possible *N*, and wrote "not sure how to do this." This prompt conflicted with Carlton's interpretation of that it was a necessary condition that *N* was the smallest possible.

The same prompt generated a potential conflict factor with Melody. Melody made an algebraic error and chose *N* to be $\frac{1}{11}$. Melody's reasoning on how to choose was correct and stated "*N* +1 could also work because all $n > N + 1$ will still be in the range. You pick the smallest *N*." However, Melody did not recognize that *N* was supposed to be a natural number. Melody not recognizing that *N* was supposed to be a natural number conflicts with the formal definition.

The other two students who had presented serious potential conflicting factors were Amy and Jessie. Both had potential conflicts factors with determining delta for limits of functions. Amy's serious potential conflict factor appear in her response to the prompt: *Consider the* $lim_{x\to 0}\sqrt{x+1} = 1$ *with the given* $\varepsilon = 0.1$, *determine a corresponding* δ. Amy had correctly simplified the absolute value inequality to $-0.19 < x < 0.21$. However, Amy did not recognize that delta represented a distance and to satisfy the formal definition delta must be a positive value. Amy determined that the range of possible delta values were between the minimum value -0.19 and the maximum value 0.21.

Jessie's serious potential conflict factor was that he was unable to graphically determine a corresponding delta for a given epsilon in two different prompts. The first prompt was: *Consider the graph, where* $lim_{x \to c} f(x) = L$ *and* $lim_{x \to c} g(x) = L$*. Let* ε *be the indicated distance on the graph. (a) Using the graph above of the function f(x) and the given* $\varepsilon > 0$ *on the graph, what is the largest possible* δ*? (b) Using the graph above of the function g(x) and the given* $\varepsilon > 0$ *on the graph, what is the largest possible* δ*?*

The second prompt that demonstrated the serious potential conflict was: prompt: *Consider* $lim_{x \to -5} \frac{x^2+3x-10}{x+5}$ *. The precise definition of a limit of a function states that* $lim_{x \to c} f(x) = L$ *if for every* $\varepsilon > 0$*, there exists a corresponding* $\delta > 0$ *such that for all* $x < 0, |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$*. For the function above and for* $\varepsilon = 1$*, it can be shown that the corresponding* $\delta = 1$*. Locate and label on your graph the values of* $L, c, \varepsilon,$ *and* δ.

In response to the first prompt, Jessie had chosen the corresponding delta for the function $f(x)$, δ_f to be the point $(\delta + c, L - \varepsilon)$ and the corresponding delta for the function $g(x)$, δ_g = $(\delta + c)$, and incorrectly indicated $(\delta - c, \delta + c)$ on the graph (Figure 61). For the second prompt, the Jessie did not include the delta tolerance, even though the corresponding delta was provided (Figure 62). Both show a potential conflict factor with Jessie's understanding of the delta tolerance which could cause conflict with the formal definition.

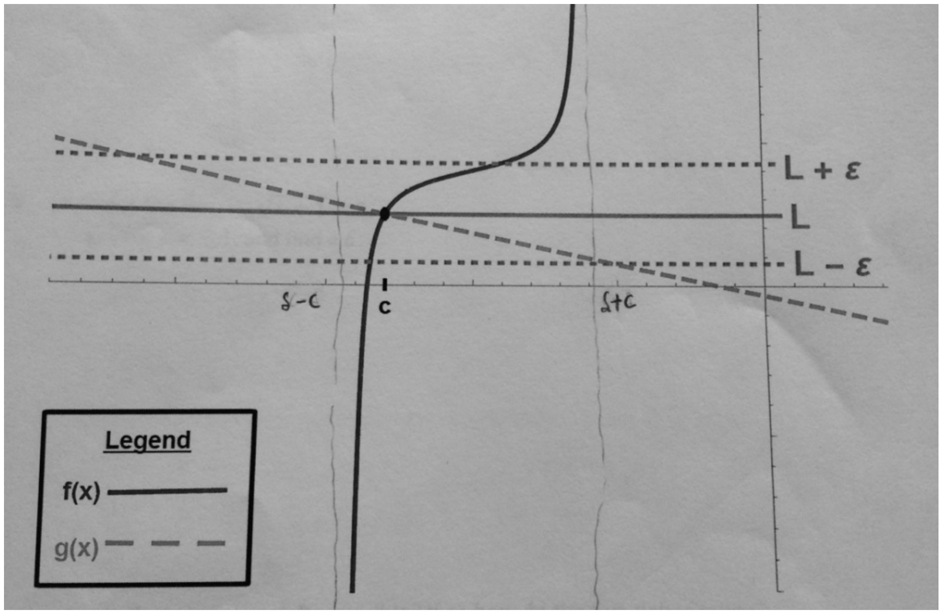


Figure 61. Jessie's image of the $(\delta - c, \delta + c)$ neighborhood.

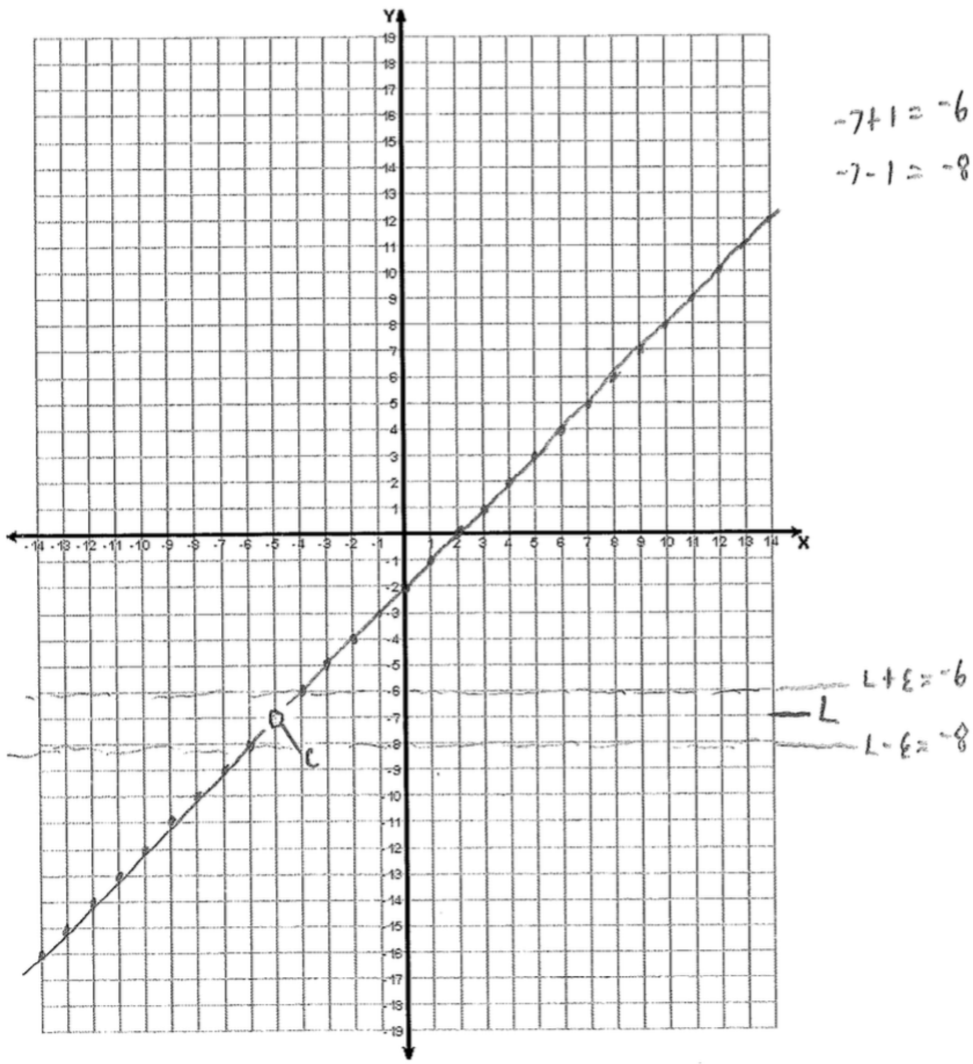


Figure 62. Jessie's image missing the $(\delta - c, \delta + c)$ neighborhood.

Cognitive resolution group. There were five students who were classified to have cognitive resolution with the formal definition: Arnold, Edith, Nick, Tim, and Vincent. These students' evoked concept image's four domains aligned with the formal definition and did not demonstrate any serious potential conflict factors. These students evoked concept images were not without errors, however the errors were analyzed and determined not serious.

The mental images that were generated by the conflict resolution group satisfied the formal definition. The characterizations that each of their mental images were predominately dynamical-theoretical and formal (Table 23).

Table 23.
Cognitive resolution groups' usable mental images characterizations.

	Acting as an Approximation	Unreachable	Acting as a Boundary	Dynamical-Practical	Dynamical-Theoretical	Formal
Arnold	-	-	-	-	Sequences	Functions
Edith	-	-	-	-	Sequences	Functions
Nick	-	-	-	-	-	Both
Tim	-	-	-	-	-	Both
Vincent	-	-	Sequences	-	Sequences	Functions

All of the cognitive resolution students' evoked examples and evoked properties of these examples satisfied the formal definitions. There were two incorrect examples generated by Edith and Arnold. Edith had included the example, $\lim_{x\to\,-5} x+3$ whose limit was -2, rather than 2. When Edith was asked about it she mentioned she forgot the negative sign around the entire function. Similarly, when Arnold had included the incorrect example $\lim_{x\to\infty}\frac{2x^2}{x+1}$ and when asked about it Arnold said he had for to square the *x* term in the denominator. Therefore, both algebraic errors were not considered potential conflict factors to the formal definition.

The five students' evoked processes did have a variety of errors; however, these different errors were not serious potential conflict factors that caused incompatibility with the formal definition. The five students were able to compute all of the limits except for Nick who did not attempt one problem. Arnold and Tim made errors algebraically calculating limits of trigonometric sequences. These errors were determined not to be serious potential conflict factors (Table 24 and 25). All five of the students were able to correctly determine and justify if a limit of function existed at a specified *c*-value. All of the students correctly used the properties of limits as well.

Table 24.
Cognitive resolution groups' performance on computing limits of sequences.

	$\left\{\frac{7sin(n)}{n} - 2\right\}_{n=1}^{\infty}$	$\{cos(n) + 3\}_{n=1}^{\infty}$	$\{-\frac{1}{2}\}_{n=1}^{\infty}$	$\{\frac{1 - 5n^4}{n^4 + 8n^3}\}_{n=1}^{\infty}$
Arnold	x*	x	x	x
Edith	x	x	x	x
Nick	x	-	x	x
Tim	x	x*	x	x
Vincent	x	x	x	x

*Made an error when computing the limit.

Table 25.
Cognitive resolution groups' performance on computing limits of functions.

	$\lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5}$	$\lim_{x \to -\infty} \frac{x}{(x - 3)(x + 2)}$
Arnold	x	x
Edith	x	x
Nick	x	x
Tim	x	x
Vincent	x	x

All five of the students were able to graphically determine a corresponding *N* for a given epsilon. Only four of the five students attempted to algebraically determining a corresponding *N* for a given epsilon. Vincent did not respond to the prompt. Of the four who did respond, Arnold was the only student who used a dynamical practical method to determine the corresponding *N*, and correctly determined an *N*. The other three students had algebraic errors when simplifying the absolute inequality. These algebraic errors were not considered serious potential conflict factors to the formal definition.

The four students' follow-up responses to whether or not there was another possible *N* varied. Edith and Tim both correctly explained that any value greater than the smallest *N* was a possible value since after the smallest *N* the sequence converges. Nick and Arnold had explained that it was only necessary to have one *N* to satisfy the formal definition. Both were applying the "there exists an *N*" from formal definition to directly to their response. Follow-up conversations with both Nick and Arnold, yield that they correctly understood that any *N* larger than the smallest *N* would satisfy the formal definition. It was also seen that it both Nick and Arnold were able to apply their understanding in their proofs for the following statement were analyzed: *Suppose that* $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences with* $lim_{n\to\infty} s_n = s$ *and* $lim_{n\to\infty} t_n = t$. *Then* $lim_{n\to\infty} s_n t_n = st$. Both determined that the *N* = max$\{N_s, N_t\}$, where N_s and N_t where the corresponding values for the $lim_{n\to\infty} s_n = s$ and $lim_{n\to\infty} t_n = t$, respectively.

Arnold was the only student to make no errors algebraically determining a delta value for a given function and specific epsilon value. Edith, Tim, and Nick made algebraic errors solving the absolute value inequality. Vincent had incorrect chose the wrong delta value, but when asked about his choice Vincent recognized his error and corrected his selection to be the smaller distance in order to satisfy the formal definition. All five were able to correctly determine a delta graphically and had correct reasoning about determining a corresponding delta and the range of possible deltas.

Overall, these students' evoked concept image's four domains aligned with the formal definition and did not demonstrate any serious potential conflict factors. These students evoked concept images were not without errors. However, the errors were algebraic and were determined not serious.

Proof Comprehension Assessment

Both instructors provided a collection of proofs that incorporated the concept of limits that they expected their students to understand for the final exam. For each collection of proofs Mejia-Ramos et al.'s (2012) proof comprehension model was used to generate questions for the assessment. During the last day of class, each section took their respective proof comprehension assessment, which served as their final exam review.

Out of the twelve students who had completed both of the limit surveys, only one student did not complete a proof comprehension assessment. Out of the eleven students who took the proof comprehension assessment two students, Amy and Vicky who were in a different Real Analysis section. The instructors provided very different proofs. Therefore, their assessments were not compared and were excluded from the overall analysis. The nine students' assessments that were analyzed were Alan, Arnold, Carlton, Edith, Jessie, Maddie, Melody, Nick, and Tim. The nine students were given the following three proofs.

If $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence that is bounded above, then $\lim_{n\to\infty} a_n$ exists.

Proof:

(Line 1) Suppose $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence that is bounded above by some real number M.
(Line 2) Let $\varepsilon > 0$.
(Line 3) Since, $\{a_n\}_{n=1}^{\infty}$ is bounded, there exists some least upper bound of $\{a_n\}_{n=1}^{\infty}$, say L.
(Line 4) Since $L - \varepsilon < L$ then $L - \varepsilon$ is not an upper bound of $\{a_n\}_{n=1}^{\infty}$.
(Line 5) There exists a natural number N such that $L - \varepsilon < a_N \leq L$.
(Line 6) Since $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence, for every $n \geq N$, $L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon$.
(Line 7) Thus, for every $n \geq N$, $-\varepsilon < a_n - L < \varepsilon$.
(Line 8) Hence, $\lim_{n\to\infty} a_n$ exists.

Figure 63. Proof one.

Suppose f is an increasing function and $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence such that $\lim_{n\to\infty} x_n = p$. Then $\{f(x_n)\}_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to\infty} f(x_n)$ exists.

<u>Proof:</u>

(Line 1)	Suppose f is an increasing function and $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence such that $\lim_{n\to\infty} x_n = p$.
(Line 2)	Since $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence and f is an increasing function then $\{f(x_n)\}_{n=1}^{\infty}$ is an increasing sequence.
(Line 3)	Let $\varepsilon > 0$.
(Line 4)	Since $\lim_{n\to\infty} x_n = p$, there exists a natural number *N* such that for all $n \geq N$ $x_n \in (p - \varepsilon, p + \varepsilon)$.
(Line 5)	Therefore, the increasing sequence $\{x_n\}_{n=1}^{\infty}$ is bounded above by $(p + \varepsilon)$.
(Line 6)	Since x_n is bounded above for all natural numbers *n*, then $f(x_n)$ is bounded above for all natural numbers *n*.
(Line 7)	$\{f(x_n)\}_{n=1}^{\infty}$ is an increasing and bounded above sequence hence, $\lim_{n\to\infty} f(x_n)$ exists.

Figure 64. Proof two.

Suppose $f(x) = mx + b$ and p is a real number, then $\lim_{x\to p} f(x) = f(p)$.

<u>Proof:</u>

(Line 1)	Suppose $f(x) = mx + b$ and p is a real number.
(Line 2)	Let $\varepsilon > 0$.
(Line 3)	Case i.) $m = 0$. Let $\delta = \varepsilon$.
(Line 4)	Assume $\lvert x - p\rvert < \delta$, then $\lvert f(x) - f(p)\rvert = \lvert(mx + b) - (mp + b)\rvert = \lvert m(x - p)\rvert = \lvert m\rvert\lvert x - p\rvert = 0 < \varepsilon$.
(Line 5)	Case ii.) $\lvert m\rvert > 0$. Let $\delta = \frac{\varepsilon}{\lvert m\rvert}$.
(Line 6)	Assume $\lvert x - p\rvert < \delta$, then $\lvert f(x) - f(p)\rvert = \lvert(mx + b) - (mp + b)\rvert = \lvert m(x - p)\rvert = \lvert m\rvert\lvert x - p\rvert < \lvert m\rvert\delta = \varepsilon$.
(Line 7)	Hence, $\lim_{x\to p} f(x) = f(p)$.

Figure 65. Proof three.

As Mejia Ramos et al.’s (2012) cautioned not all of the seven dimension of the proof comprehension model were suitable for these three limit proofs. These three proofs were relatively short and straightforward and therefore asking the students to identify the modular structures was deemed inappropriate. Lastly, a majority of the limit proofs presented throughout the course, including these three proofs were direct proofs.

Therefore, asking the students to identify that the method was a direct proof, or direct proof with cases and if it the method would be applicable to other limit proofs and would not gleam much insight. Therefore, both the dimensions of transferring the general method and logical proof framework were excluded from the limit proof comprehension assessment.

Meaning of terms and statements. A fundamental element to understanding a proof is understanding the meaning of individual words and sentences. This is considered a surface level understanding since it is possible that a reader could explain the meaning of the terms and sentences without reading the proof if the terms and statements are not new terminology. For this assessment selected terms and statements from the selected proofs, that had been discussed in class. Therefore, the terms and statements were not meant to be challenging for the students. The first was for the students to provide the definition of a limit of a sequence. Second, the students were to explain what $\lim_{x \to p} f(x) = f(p)$ means in the third proof.

Definition of a limit of a sequence. Six of the nine students correctly provided the formal definition of the limit of a sequence. There were four students, Alan, Carlton, Jessie, and Melody who did not generate correct definitions of a limit of a sequence. Alan generated the correct formal definition for the limit of a function. Both Carlton and Melody provided a definition that did not include all of the quantifiers. Carlton's definition was "the limit exists for $\{x_n\}_{n=0}^{\infty}$ if $c > 0$ for an $N \in \mathbb{Z}^+$ where $n > N$ so that $|x_n - L| < c$." Melody's definition was "the limit of a sequence exists and is L if $c > 0$ and $N \in \mathbb{Z}^+$ such that if $n \in \mathbb{Z}^+$ and $n > N$ so that $|x_n - L| < c$." Lastly, Jessie's definition was only the symbolic representation of the limit of a sequence, "$\lim_{n \to \infty} a_n =$

L."

The relationship between students being able to generate the formal definition on the proof comprehension was strongly related to their evoked concept definition they had generated at the end of the limit of sequences unit. The four students who provided an incorrect definition were a part of the cognitive conflict group. Carlton, Jessie, and Maddie were students whose evoked concept definitions had misalignment with the formal definition. For both the assessment and the limits of sequences survey, Jessie did not generate the formal definition. Jessie's evoked concept definition at the end of the limit of sequences unit was the informal description that limits are "values of a sequence adds up to or trends too." By the end of the semester, Jessie still had not adopted the formal definition. Carlton and Melody's evoked concept image both had missing quantifiers like the formal definitions they provided on the assessment. Alan who had provided the correct formal definition of a limit of a function, had demonstrated cognitive conflict between limits of sequences and limits of functions throughout the semester. When asked in the follow-up interview to provide the definition, Alan had explained he couldn't because he hadn't gone back to re-memorize it yet for the final exam. Alan's concept definition that he generated at the end of the unit was memorized as well, but was correct because he had memorized it for the upcoming exam.

The only student who was a part of the cognitive conflict group who correctly generated the formal definition. Maddie's concept definition of a limit of a sequence held errors, such as incorrect ordering. However, by the end of the semester Maddie had resolved the errors and was able to generate a correct formal definition. The four students in the cognitive resolution group had no errors in their concept definition or in the definition provided on the assessment.

Explaining $\boldsymbol{lim_{x \to p} f(x) = f(p)}$. The students' explanations of what $\lim_{x \to p} f(x) = f(p)$ means, varied. There were two students who did not correctly discuss the concepts of the symbols. Jessie provided an incorrect explanation that "the limit of $f(x)$ as *x* goes to p equals p." Maddie described the process of substitution rather than the concept. Maddie's response explained "when you replace the *x* with the *p* of $f(x)$ you get $f(p)$." Maddie, who held the potential conflict factor that only continuous functions have limits did not recognize that this was a specific case. In the interview Maddie considered $\lim_{x \to p} f(x) = f(p)$ trivial. Therefore, Maddie's prior potential conflict factors arose during this prompt.

The other nine students did not have conflict factors arise and presented different interpretations of $\lim_{x \to p} f(x) = f(p)$. Alan's response was the least descriptive of the nine, he wrote "the limit of f(x) exists and is f(p)." Nick elaborated more and wrote "$\forall$ p in the domain of $f(x)$, the function has a limit, and is $f(p)$." Others like Carlton and Tim also identified the limit existed and was $f(p)$, but additional provided an informal explanation of the process of a limit. Tim wrote, "that as *x*, a member of the domain, approaches the value *p* the values in the range approach what *p* maps to, $f(p)$." In contrast, Melody provided the formal definition of a limit of a function with $f(p)$ as the limit value.

There were two students who incorporated the concept of continuity in their explanation. Arnold did not explicitly discuss limits but wrote "it means the function is continuous at p, and so $f(x) = f(p)$." When continuity was discussed with Arnold, he explained that a limit at *p* must exist in order to a be continuous at *p*. Therefore, Arnold's response implicitly applied the limit exists at *p*. Lastly, Edith also stated the function was continuous at *p* but explicitly stated that the function exists at *p* as well.

Justification of claims. Statements within proofs are deduced from previous ones by the application of mathematical principles, and it is left for the reader to infer the logical relationship. This dimension of proof comprehension is considered the chaining level which requires students to provided justifications for the new assertions by either making implicit claims explicit or identifying which specific claims support other assertions. This proof assessment asked the students to make implicit claims explicit and to identify why a claim was used to prove the mathematical statement.

Proof one. For the first proof (Figure 63) the students were asked to refer to line 4, and explain what it means to not be an upper bound of $\{a_n\}_{n=1}^{\infty}$. This question intended the students to make the implicit connection between lines 4 and 5 explicit. Seven of the students explicitly identified that either "$L - \varepsilon$ is not greater than or equal to all the members of the range of the sequence" or that "there exists an element of the sequence greater $L - \varepsilon$." Of those seven students, Edith was the only one to make a notational error and wrote $\varepsilon - L$ rather than $L - \varepsilon$.

There were two students who did not correctly make the connection between lines 4 and 5 explicit. Melody incorrectly wrote, "$L - \varepsilon$ is a lower bound (since) $L - \varepsilon < a_n$ $\therefore L - \varepsilon$ is not an upper bound." Melody's interpretation of $L - \varepsilon$ not being an upper bound meant that it was a lower bound. Jessie wrote "because $L - \varepsilon$ is less than L and L is an upper bound of $\{a_n\}_{n=1}^{\infty}$." Jessie provided the justification of why $L - \varepsilon$ was not an upper bound, which connected the logical relationship between lines 3 and 4 of the proof. However, Jessie did not explicitly infer what it means to not be an upper bound of sequence.

Proof two. For the second proof (Figure 64) the students were asked to explain why the following assertion was included, *since* $\{x_n\}_{n=1}^{\infty}$ *is an increasing sequence and* f *is an increasing function then* $\{f(x_n)\}_{n=1}^{\infty}$ *is an increasing sequence.* The cognitive conflict group were the students who did not explain how the statement relates to other assertions or its role in proving that the limit exists. Alan, Carlton, Jessie, and Melody wrote that it was included to show that $\{f(x_n)\}_{n=1}^{\infty}$ was an increasing sequence but provided no insight into how that claim related to what was being proven, or to other assertions in the proof. Maddie was the only student who did not even include the key property that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ was increasing. Maddie wrote the statement was included "to show that $f(x_n)$ is a sequence." Maddie, also did not include the key assertion that $\{f(x_n)\}_{n=1}^{\infty}$ was an increasing sequence in her proof summary.

The cognitive resolution group were able to identified the claim was one of the two components of the monotone convergence theorem that were needed to show that the limit of $\{f(x_n)\}_{n=1}^{\infty}$ exists. The students' Real Analysis section did not name theorems therefore each of the three students, Edith, Nick, and Tim described the theorem. For instance, Nick wrote that the statement was included "in order to be able to execute the proof obligation, that is if $\{f(x_n)\}_{n=1}^{\infty}$ is increasing and bounded, then $\lim_{n\to\infty} f(x_n)$ exists, by a previous theorem." Arnold initially provided a general justification that it "helps to clarify why certain parts of the proof work." However, Arnold was able to clarify that it was a key element to showing that $\{f(x_n)\}_{n=1}^{\infty}$ was an increasing and bounded sequence, and therefore the limit of $\{f(x_n)\}_{n=1}^{\infty}$ existed.

The groups' performance on the local aspects of proof comprehension. In a Real Analysis course students' understanding of limits transition from informal to formal; where students learn the formal definitions, build connections between limits and other concepts, and explore limit theorems and proofs. The necessary surface level understanding in a Real Analysis course requires students to adopt formal definitions of limits and be familiar with the meaning of the terms and notation discussed in class. Therefore, the students should know the formal definition of a limit of a sequence and be able to explain $\lim_{x\to p} f(x) = f(p)$, without having to understand the proof in its entirety. The proof comprehension assessment showed that the cognitive conflict group was more likely not to have a well-developed surface level understanding. For instance, Jessie, wrote the definition of a limits of a sequence was "$\lim_{n\to\infty} a_n = L$." Jessie had still not adopted the formal definition of limits of sequences by the end of the Real Analysis course.

Similarly, the students in the cognitive conflict group had more difficulty deductively reasoning about assertions in the proof and justifying claims. Thus, this proof comprehension assessment showed that if a student has difficulty understanding the local aspects of a limit proof they are more likely to be a part of the cognitive conflict group and have a serious conflict factor. It does not necessarily mean the serious conflict factor is the reason why they have difficulty understanding the local aspects of the proofs, but their serious conflict factor could be a contributing reason. Future research would be needed to explore this relationship further.

Summarizing via high-level ideas. A holistic comprehension of a proof is being able to understand the proof as a whole and in terms of its main ideas. The students should demonstrate an understanding of the big idea, as well as the logic between the key components of the proof. Therefore, this assessment asked the students to either explain the logical ordering of the key ideas in the proof or provide a short summary for each of the three proofs.

Proof one. For the first proof, Alan and Jessie did not respond to the prompt. Of the seven who did respond, Maddie provided line by line explanations, identifying that the definition of a limit of a sequence was used, but did not show that provided the structure for the proof. Maddie wrote,

“1. Given

2. Let $\varepsilon > 0$

3. Finding an L that works with bounded-ness

4. Facts

5. More facts

6. Sequence definition

7. Connecting the facts and definition

8. Conclusion.”

In contrast, Edith and Carlton provided an outlines showed how the formal definition guided the proof. Carlton’s outline was a variation of the mnemonic “Pick, Let, Pick, Let, Show." This mnemonic, was introduced by the instructor to show how the structure of a limit of a function proof followed the formal definition of a limit of a sequence. Edith went line by line describing how you set up your premise and eventually get "together the pieces for the limit definition" to show that the limit exists.

Arnold and Tim also identified the key ideas were connected by the formal definition. Arnold additionally noted that the definition of upper bounds was a key idea to show that the limit existed. The last two students, Melody and Nick, provided more informal explanations of the big idea of the proof. For example, Melody wrote, "they talk about it being bounded above and then show it is increasing. And prove that it has a limit since the sequence can't go beyond the bound." Nick wrote "to show that $\lim_{n\to\infty} a_n$ exists of an increasing and bounded sequence, the proof demonstrates that the LUB of the sequence <u>is</u> the limit. Which makes incredible intuitive sense." Both Melody and Nick's summaries showed the big idea, but did not show that the logical structure of the proof used the formal definition.

The first proof activated either the student's concept definition during their interpretation of the big idea of the proof or a mental image. For the student's whose concept definition was activated their responses incorporated or acknowledged the definition. Other students' mental images were activated since this proof's main idea can be easily illustrated using a mathematical diagram. Therefore, those students' informal explanations were generated from their evoked mental image.

Alan and Jessie were two of the students who did not generate an informal or formal definition of a limit of a sequence during the assessment. Alan had provided the formal definition of limit of a function, and Jessie had written the symbolic representation of a limit of a sequence. Therefore, this proof had not activated a concept definition for them that they could use to interpret the main idea. As for mental images, Jessie did not produce an evidence of having or not having a mental image for the proof. As for Alan, this proof activated his conflict factor of increasing and bounded above. Alan held a strong incorrect mental image of a function whose limit was the vertical asymptote to be $x = L$. Alan was not able to resolve his cognitive conflict during the assessment and chose to not respond. During the interview after the assessment Alan explained how overwhelmed he felt, "I think it's more of an anxiety thing when I see how long everything gets," and it made feel as if he couldn't put "it into writing and a coherent thought."

Proof two. For the second proof the students were asked to provide a summary of the proof. Alan also did not respond to this prompt. Jessie provided the following incorrect summary, "state that everything is increasing, prove p is increasing, since p is increasing $f(x_n)$ exists." Jessie did interpret $\lim_{n\to\infty} x_n = p$ correctly and did not recognize that the limit of the sequence was not increasing. Jessie summary did not mention that the sequences were bounded above, which was a key component to the proof. Carlton's summary also only identified that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ was increasing but not bounded above.

Another student who did not include a main idea in their summary was Maddie. Maddie stated that "since we know $\lim_{n\to\infty} x_n = p$ we get definition parts and use them with $f(x)$ to show it is bounded to created that limit definition." Maddie did not include that the proof first showed that $\{f(x_n)\}_{n=1}^{\infty}$ was an increasing sequence. Therefore, Maddie's summary did not encompass that the proof was satisfying the monotonic convergence theorem.

Melody's proof summary incorrectly applied the property of bounded above to the function. Melody also used a pronoun, rather than precisely defining which mathematical object's limit existed, therefore it was inconclusive if Melody was inferring that the function's limit existed or that the new sequence's limit existed. Melody's summary was as follows, "since the sequence is bounded above and increasing and the function is bounded above and increasing then it has a limit." Edith provided a similar summary but correctly showed how the properties of the function and sequence were connected to the new sequences' properties to show that the limit of the new sequence exists, "since *f* is increasing and x_n is increasing and has a limit $\rightarrow f(x_n)$ is increasing, x_n has a limit and x_n is bounded $\rightarrow f(x_n)$ is bounded $\Longrightarrow \lim_{n\to\infty} f(x_n)$ exists."

Arnold provided a summary that addressed all the key ideas of the proof but included the unnecessary condition that the function *f* was continuous. It is not necessary that a function is continuous to hold the theorem to be true, therefore Arnold's inclusion of the condition was incorrect. There were two students who provided a complete and accurate summary that incorporate both the monotonic convergence theorem and the formal definition. Nick wrote, "let *f* be increasing and $\{(x_n)\}_{n=1}^{\infty}$ increasing sequence such that $\lim_{n\to\infty} x_n = p$. Then x_n is bounded by $(p + \varepsilon)$. Then $f(x_n)$ is bounded. Thus $f(x_n)$ is bound and increasing $\therefore \lim_{n\to\infty} f(x_n)$ exists." Tim wrote "suppose *f* is increasing and $\{(x_n)\}_{n=1}^{\infty}$ increasing sequence, $\lim_{n\to\infty} x_n = p$. Since $\{x_n\}$ and *f* are increasing then $\{f(x_n)\}_{n=1}^{\infty}$ is increasing. Since x_n falls within a tolerance for $n \geq N$ then $\{x_n\}$is bounded above. So, $\{f(x_n)\}_{n=1}^{\infty}$ is increasing and bounded above, $\lim_{n\to\infty} f(x_n)$."

Tim was able to identify the main ideas and be explicit that $\{f(x_n)\}_{n=1}^{\infty}$ was a sequence.

For the summary of the proof the students again evoked mental images and concept definitions to interpret the main idea of the proof. The students used their activated concept image to interpret the main idea with the complexity of the functions and sequences differently. For instance, Nick who had thought the first proof was “more intuitive” relied more on his mental image now saw this proof as “less intuitive” since it involved sequences and functions. Thus Nick relied more on his concept definition to capture the main idea of the proof. Other students like Melody simplified their mental image by looking at the sequence and the function separately. Melody had two non-conflicting images and was able to determine both had limits.

For those students who were not able to generate fully correct responses had added or omitted a property to their mental images. Arnold’s evoked mental image of the proof, was of a continuous function and therefore, his summary incorporated this component. Maddie and Carlton did not include key properties in their mental image and did not identify them in their summary. Jessie’s mental image also omitted bounded above and added a property by over generalizing that everything was increasing including the limit value. Alan again was reluctant to interpret the proof, and explained that he felt “overwhelmed” with the information.

Proof three. The students were asked to explain the logical ordering of the key ideas in the third proof. All of the students identified that there were two cases. All the students but Jessie determined that it was necessary to show both cases to prove the limit exists. As Melody wrote “they show two possible cases and prove them to cover all bases." The students were more dependent on their mental images of constant functions, and linear functions to capture the main idea of the proof.

The one student who did not use a mental image but solely evoked his concept definition was Jessie. Jessie wrote "the cases I and II, since $0 < \varepsilon$ when $m = 0$ brings a contradiction and since $|m|\delta = \varepsilon$ when $|m| > 0$ brings a contradiction." Jessie did not recognize that both cases satisfied the formal definition of limit of a functions. For the first case, Jessie did not infer that $0 < \varepsilon$ is equivalent to $\varepsilon > 0$. For the second case, Jessie did not consider the entire string of equalities. Jessie only read the last portion, and determined that since it ended with an equal sign instead of a less than sign, it was a contradiction. Jessie did not demonstrate an understanding of the big idea of the proof.

Tim was the only student who provided a detailed outline that not only identified there were two cases but showed that the proof aligned with the formal definition of a limit of a function. Tim wrote:

- "Breaking the proof into two cases, if m is zero or not.
- Letting δ be a value.
- Assuming the distance from x to p is less than δ.
- Showing the distance from $f(x)$ and $f(p)$ is less than ε."

The only student beside Tim who incorporate a component of the formal definition of a limit of a function in their response was Nick. Nick presented a flow chart (Figure 66) that showed that each case need a specific delta but reach the same conclusion. Both Tim and Nick used both their mental images and their concept definitions.

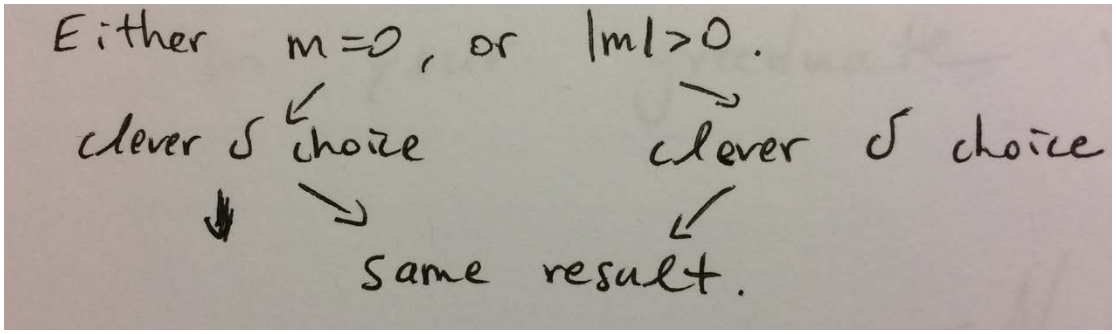


Figure 66. Nick's flow chart.

The groups' performance on the holistic aspects of proof comprehension. A holistic understanding of a proof cannot be achieved by only interpreting a few statements in a proof but rather by inferring the key ideas of proof or the entire proof. The cognitive resolution group demonstrated that they were able to interpret the different assertions within the proof and piece together the key ideas and explain the main idea of the proofs. Their summaries were consistently more accurate. Whereas, the students who were unable to identify all of the key ideas, or understand the importance of an assertion in relation to the main idea was in the cognitive conflict group.

Illustrating with examples. Another holistic understanding of a proof is being able to encapsulate the main ideas of a proof and illustrate them with specific examples. Generating an example is an essential tool that many mathematicians use to check their own understanding of a proof (Mejia Ramos et al., 2012). Generating an example can be used to make sense of different assertions within the proof, or a proof that captures the main idea. Since these three proofs were relatively short and straightforward, the students were asked to provide an example that represents the mathematical statement being proven, rather than specific assertions in the proof.

Proof one. The students were asked to provide an example of the following statement: if $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence that is bounded above, then $\lim_{n\to\infty} a_n$ exists. There were three students, Alan, Jessie, and Tim who did not generate examples. There were only two students who generated correct examples. These were the two students, Melody and Nick, who used their mental images rather than the formal definition to generate their summary of the key ideas of the proof. The students who relied more on their concept definition to generate their summary provided incorrect examples.

Melody was the only student who generated a correct specific example. Melody wrote "$a_n = 1 - \frac{1}{n}$, it's increasing and bounded above and the limit is 1." Nick was the other student who generated a correct general example. Nick provided a mathematical drawing (Figure 67). Both are acceptable examples since both encompassed the main ideas and the key components of being an increasing and bounded above sequence.

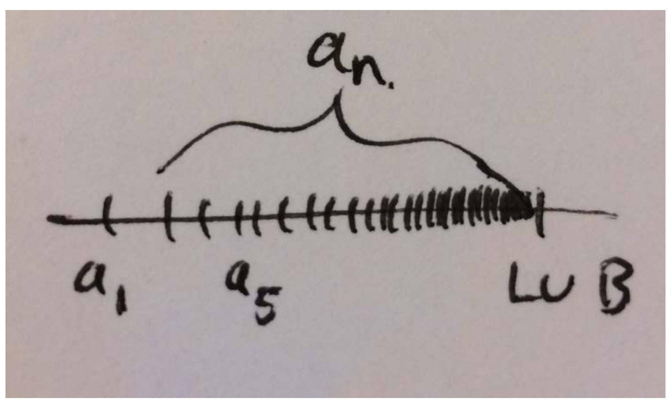


Figure 67. Nick's general example of the monotonic convergence theorem.

Two students generated increasing sequences that were not bounded above, but to satisfy that condition they restricted the sequence's domain. Arnold's example was "if $\{a_n\}_{n=1}^{\infty} = \{4n\}$ bounded on $[0, 20]$ then $\lim_{n\to\infty} 4n = 20$." Carlton's example was "$\{x_n\} = \{2, 4, 6\}, = 6, 6 - 5 < x_n < 6 + 5, 1 < x_n < 11$." Neither example respected that the original statement's sequence domain was the natural numbers. Both students were unable to evoke an example that satisfied both properties of increasing and bounded above.

Maddie and Edith both generated incorrect examples. Maddie wrote "let $a_n = n + 1$ hence as *n* increases, $(n + 1)$ increases, so $\lim_{n\to\infty} n + 1 = \infty$." Maddie's example was increasing, however was not bounded above. Maddie also identified that the limit of the increasing sequence was infinity. When Maddie was asked about the difference between a limit existing and not existing, Maddie did not identify positive and negative infinity as non-existing limits. Edith wrote "$a_n = \frac{5n+1}{n}$, increasing and bounded, $\lim\limits_{n\to\infty} a_n = 5$." Edith incorrectly generated a decreasing sequence.

Proof two. The students were asked to provide an example of the following statement: *Suppose f is an increasing function and $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence such that $lim_{n\to\infty} x_n = p$. Then $\{f(x_n)\}_{n=1}^{\infty}$ is a sequence such that $lim_{n\to\infty} f(x_n)$ exists.* There were three students, Alan, Jessie, and Arnold who did not generate examples. Edith was the only one to generate a correct example, Edith provided a general mathematical drawing (Figure 68), that incorporated all the key ideas. On the *x*-axis Edith demonstrated that x_n was an increasing sequence that was bounded above, whose limit existed. Edith chose an increasing step function *f*. Edith tried to illustrate $\{f(x_n)\}_{n=1}^{\infty}$ by showing that x_n was the pre-image, and $f(x_n)$ was the image. Edith also clearly defined the limit of $\{f(x_n)\}_{n=1}^{\infty}$. Edith's general mathematical drawing was very similar to her proof summary.

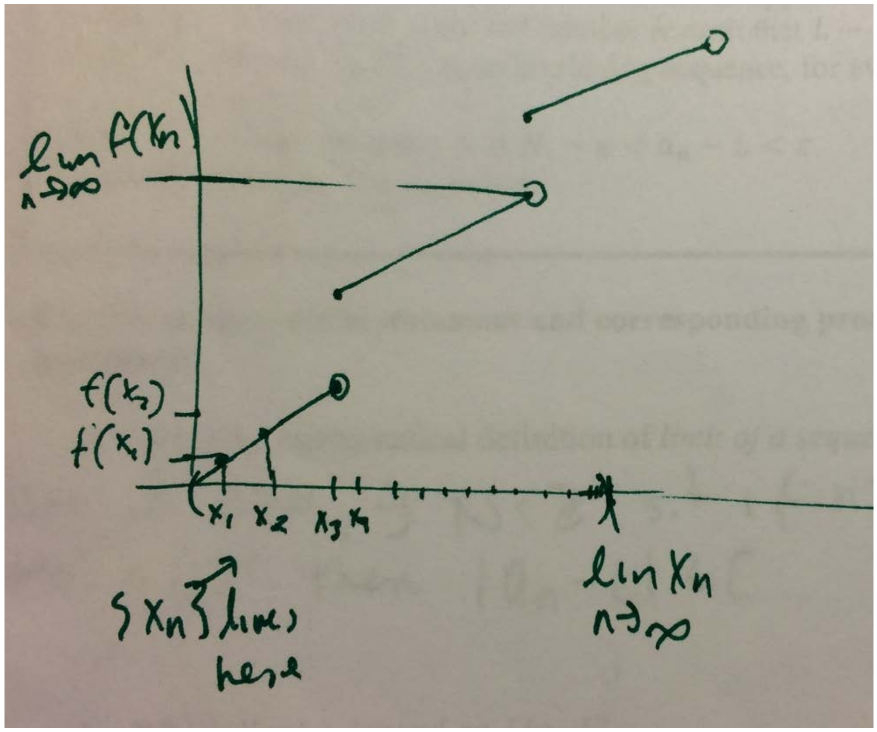


Figure 68. Edith's general example of proof two.

The other students who generated examples were all able to generate increasing functions, but were not able to generate increasing, bounded above, infinite sequences. Carlton again generated a finite sequence, like he did for the other example. Carlton's example was "$\{x_n\} = \{1, 2, 3\}$, $f = x$, $f(x_n) = \{1, 2, 3\}$, $L = 3$." Melody's example was "$f(x) = x - \frac{1}{x}$ and $\{x_n\} = \left\{n - \frac{1}{n}\right\}$." Melody's sequence was increasing but not bounded above. Maddie, Nick, and Tim all generated increasing functions, but decreasing bounded below sequences. Maddie's example was "$\{x_n\} = \left\{\frac{1}{n}\right\}$ and $f(x) = 5x + 2$." Nick's example was "$f(x) = x$ and $\{x_n\}_{n=1}^{\infty} = \{\frac{4n+1}{n}\}_{n=1}^{\infty}$." Tim's example was "$\{x_n\}_{n=1}^{\infty} = \{\frac{1}{n} + 1\}_{n=1}^{\infty}$ and $f(x) = x^2$."

Proof three. The students were asked to provide a specific example for both cases within the following statement's proof: *Suppose* $f(x) = mx + b$ *and* p *is a real number, then* $lim_{x \to p} f(x) = f(p)$. Alan and Jessie did not generate examples. All of the other students were able to correctly generate a constant function for the first case and a linear example for the second case except for Tim. Tim only provided a linear function for the second case. Similarly, Tim did not generate a limit of a constant in either of his evoked example spaces.

Summary. Overall the cognitive resolution group was more successful with assessing the local aspects of the proofs. The cognitive resolution group tended to be more precise with defining and explaining terms and statements within the proof. The cognitive resolution group was also able to correctly identify how assertions relate within the proof and to the conclusion of the proof. Some of the members of the cognitive conflict group were successful in some of the parts of assessing the local aspects of a proof. However, they were more likely to have errors, or not demonstrate a chaining level understanding of the assertions in the proofs.

The relationship between the students' conceptual understanding of limits with their holistic comprehension was seen in activation of their mental images and concept definition. If students were experiencing conflict generating a mental image or concept definition the students experienced difficulty producing summaries, such as Alan. The student's additions and omissions of properties to their mental images also appeared in their summaries. As seen in Arnold, Carlton, and Maddie's second proof summaries. If the students' summaries aligned more with their activated mental image, and did heavily rely on the formal definition the students were able to generate examples, as seen with Melody, Nick, and Edith. In general, the members of the cognitive resolution generated more accurate summaries of the proofs, with the exception of Arnold including the additional property in the second proof summary.

There was not a strong connection to the students' previous evoked example spaces and the examples the students generate for the first two proofs. This is unsurprising since the limits of sequences and functions surveys asked the students to generate examples that only needed to satisfy one condition, as specific limit value. The survey's prompt did not require the students to adhere their examples to specific properties, whereas the proof comprehension assessment's first two proofs did. The students were no longer restricted to a specific limit value, but had to satisfy specific properties. Generating examples for the first two proofs did activate two concept image domains, example space and properties. But what was seen from the proof comprehension was that overall the students' two domains were not strongly connected enough for them to generate correct examples.

There was a connection between the third proof's example and the students' evoked example spaces of functions. Six of the seven students who generated examples for the third proof also had generated a specifically-named linear function in their evoked example space of functions. The only exception was Carlton, who had generated general graphical representations. The students seemed to be more successful at generating an example for the third proof because the symbolic structure of the example was provided. Which was not the case for the other two proofs, the examples were to be generated based on the specified properties. The symbolic structure was something the students were more familiar with and was a tool they used to help generate their evoked example space. For instance, Maddie used the linear function symbolic structure and generated different examples by changing the slope, y-intercept and the respective *c*-value to satisfy the specified limit value.

Therefore, while a lot of the students' example spaces contained functions with combinations of properties that satisfied the function's limit existed, the examples were not generated because they had those properties. For instance, Nick and Maddie generated examples that were increasing function that was bounded above by the horizontal asymptote at $y = 2$, but didn't include them because of this property. Both had algebraically approached generating their symbolically named examples.

CHAPTER 5

DISCUSSION

Introduction

Undergraduate students in a Real Analysis courses are expected to develop a formal conceptualization of limits and appropriately apply their understanding to limit proofs. Research has shown that both tasks are difficult for students (Dreyfus, 1999; Patel, McCombs, & Zollman, 2014; Tall, 1992; Weber, 2001). There lacks research that investigates the relationship of the two cognitive tasks. Thus, this study examined how undergraduate Real Analysis students' conceptualization of limits relates to proof comprehension by utilizing a grounded theory approach.

Two Real Analysis sections from a single public university in central Texas were asked to participate. Eighteen Real Analysis students participated in completing surveys, observations, and a subgroup of the eighteen participated in follow-up interviews. Purposeful sampling was used to select the interview participants with different ideas about the behaviors of limits aiming to investigate a diverse set of concept images. This study used Williams' (1991) one-page questionnaire about limits of functions. Classroom observations over the instructions were conducted throughout the semester. At the end of both units for the limits of sequences and the limits of functions, the students completed surveys about each topic. After each survey, follow-up interviews occurred with the purpose of triangulating the students' conceptual understanding expressed in the surveys and observed in class. During the final class, the students took an in-class proof comprehension assessment that was created based on Mejia Ramos et al.'s (2012) proof comprehension model. The collection of data provided diverse concept images of Real

Analysis and insight into their proof comprehension. The data was coded in three levels: open coding, axial coding, and selective coding.

This chapter summarizes the results of the study and positions the findings within the body of research in the field. Implications of the findings are discussed and suggestions for future research are provided.

Summary of Findings

Concept image. Tall and Vinner (1981) have defined the total cognitive structure associated with a concept includes mental images, associated properties, and processes as a students' concept image. Within a concept image, students also possess an example space (Watson & Mason, 2002) which consists of a variety of examples related to a mathematical concept. This study investigated eighteen students' complex and diverse concept images and concept definitions of limits of sequences and limits of functions at the end of instruction for each respective unit.

Mental images. The students provided both graphical and verbal representations of how they envisioned limits. The students' evoked mental images were unique and varied from unusable intuitive word associations and informal metaphors (Keene, Hall, & Duca, 2014; Monaghan, 1991; Patel, McCombs, & Zollman, 2014) to images and descriptions that encapsulated the formal definition (Domingos, 2010).

Williams' (1991) six characterizations of limits were used to classify the students' graphical representations and descriptions. This study extended the six statements of limits of functions to also include six equivalent characterization statements for limits of sequences to adequately characterize both types of limits. Some of the students' evoked mental images held a single characterization. Whereas, other students' evoked mental images incorporated graphical representations that showed limits acting as boundaries and provided a dynamic theoretical description that discussed how the sequence moved as *n* increased to positive infinity (Davis & Vinner, 1986; Cornu, 1991; Sierpinska, 1987; Tall& Vinner, 1981; Williams, 1991).

There were only two students who consistently characterized both of their evoked metal images for limits of functions and limits of sequences as formal. The remaining students had variations in their characterizations between limits of sequences and limits of functions. None of the students generated the least applicable characterization for the proof-based course, acting as an approximation. Similarly, dynamic-practical characterization is not the strongest semantic characterization for a Real Analysis course since it requires the limit to be determined by plugging in numbers until the limit is reached. This characterization is more appropriate for determining specific limits in a calculus. There were only two students, who held a dynamic-practical evoked mental image of limits of functions. In general, the majority of the students generated mental images with appropriate characterizations for a Real Analysis course, such as dynamical, theoretical, and formal. The students typically generated more formal evoked mental images of limits of functions than limits of sequences. This is perhaps due to students generating one-dimensional mental images of limits sequences that did make the roles of *n*, and *N* explicit.

There were misconceptions that appeared in the students evoked mental images. The first misconception was that not all students recognized that the domain of the sequences was specified to natural numbers and the *n*-values increased to positive infinity. Some thought it was similar to limits of functions and that the *n*-values could decrease or increase. The second misconception was consistent to the literature and was about the relationship of the concept of continuity and limits of functions (Tall & Vinner, 1981). One student, Maddie thought only continuous functions' limits existed. Lastly, the evoked mental images exposed misconceptions students held about the formal definition, as seen in Alan's evoked mental images.

Example space. The students generated as many examples of sequences with the same limit and as many examples of functions with a different limit. There were no other restrictions. The students evoked example space of limits of sequences was coded using Zazkis & Leikin's (2007) framework of generality, correctness, and richness. Students' evoked example spaces were either composed of general graphical representation or symbolically-named examples.

Students who created general graphical representations included different examples based on the sequences' or functions' properties. Therefore, a rich general example space contained a variety of examples of limits with different combinations of properties, such as constant, increasing and bounded above, and decreasing and bounded below. An additional property that appeared in limits of functions general graphical representations was continuity. Examples were specifically included to show how limits existed for continuous functions, and at removable discontinuities. The students who produced general examples, recognized that they could not generate an exhaustive list of all the limits of sequences and functions. Therefore, to encapsulate the most examples possible, they attempted to generate a "base" example that could be transformed to generate a different example with the same "base" properties.

It is not necessarily true that every Real Analysis student who generates an evoked example consisting of graphical representations has no misconceptions. While general graphical representations "may be seen as an indication of mathematical understanding, other general examples may point to deficiencies in understanding" (Zazkis & Leikin, 2007). This was seen in one student's evoked example space. This student had a dominant incorrect example that appeared in both evoked example spaces that illuminated a serious cognitive conflict about limits of functions and sequences and the formal definition.

Rich symbolically-named examples had a variety of types of sequences and functions, such as linear, polynomial, trigonometric, and logarithmic. Symbolically-named example spaces were typically larger but many of the examples were of the same type. Other students systematically generated shorter lists of examples too, where each example was a representative of a different type. For instance, a student's evoked example space of functions, consisted of the four examples: $\lim_{x \to 2} x$, $\lim_{x \to \sqrt{2}} x^2$, $\lim_{x \to e^2} ln(x)$, $\lim_{x \to 0} 2cos(x)$, and $\lim_{x \to 0} \frac{2sin(x)}{x}$. These four examples were categorized as linear, quadratic, and trigonometric functions.

Both types of evoked examples spaces, graphical representations or symbolically named contained errors. Students included examples whose limit was not the specified value, or examples whose limit did not exist. Another error that appeared was the inclusion limits of functions examples in an evoked example space of limits of sequences. Some of the students had not recognized that examples whose input values approached positive infinity could be examples for both limits of sequences and functions. Whereas, students whose examples' input values approached values other than positive infinity were limits of functions examples.

Processes. Prior processes Real Analysis students have learned in perquisite calculus courses, are typically more computational, less formal, and conceptually-based. Their prior experiences with associated processes, algebraically or graphically, do not typically use a formal definition of proof (Dawkins, 2012). Therefore, the process of determining an appropriate $\delta > 0$ that satisfies the formal definition of a limit of a function for a given $\varepsilon > 0$, or determining an appropriate natural number *N*, that satisfies the formal definition of a limit of a sequence for a given $\varepsilon > 0$ is less familiar for students before Real Analysis and a difficult task for students to encapsulate (Cottrill et al., 1996; Cornu, 1991, Davis & Tall, 1986, Dubinsky, Elterman, & Gong, 1988, Kidron & Zehavi, 2002; Pinto & Tall, 1999, 2002). However, by the end of the limits units, Real Analysis students are expected to have mastered not only the process of determining specific *N* or delta values for a given epsilon but understand the process in general terms with respect to the universal and existential quantifiers.

Computing limits algebraically for some Real Analysis students was a natural process. They were able to explain the limit of properties they were applying, and the algebraic methods in detail. Other students had difficulty computing the limits of trigonometric sequences, and were not able to identify when a limit did not exist. Both sections did not dedicate much instructional time or homework exploring limits of sequences such as $\{cos(n) + 3\}_{n=1}^{\infty}$. The emphasis was placed on limits not existing when the limit was positive or negative infinity. One student, who correctly determined that cosine did not have a limit because it did not satisfy the formal definition, had experienced cognitive conflict and could not definitively state that the limit did not exist because the limit value was not positive or negative infinity.

Another process typically taught in a first semester calculus course is determining limits graphically. The students' correct processes of determining if a limit existed was showing that the left- and right-sided limits existed *and* were be equal to each other, and also showing that it satisfied the formal definition. Incorrect reasoning showed some of the Real Analysis students considered removable discontinuities to be instances when limits do not exist.

Computing the corresponding variable for a given epsilon algebraically for each formal definition presented challenges for the students. The process of determining a delta for limits of functions and *N* for limits of sequences, both involve simplifying absolute value inequalities. Less than half of the students even attempted to simplify the absolute value inequalities for both formal definitions. Similarly, to Cottrill et al.'s (1996) findings, students struggled to make sense of the absolute value inequalities. For those who did attempt so simplify the absolute value inequalities, an array of algebraic errors occurred that prohibited students from correctly finding the corresponding variables. There was one student who determined the corresponding variables for a given epsilon correctly for both formal definitions.

The Real Analysis students were more successful at determining the corresponding variable for a given epsilon graphically for each formal definition. For limits of functions, some students were able to correctly reason that, for a given epsilon, the largest corresponding delta was the minimum of the two possible deltas. For example, one student who explained "It's δ_1, because $\delta_1 < \delta_2$, therefore $\delta_1$is the largest appropriate delta." Others had different interpretations, such as Amy who considered the possible range of delta values to be the minimum value $(c - \delta)$ and the maximum value

$(c + \delta)$. The Real Analysis students did not always recognize delta was a positive value that represented a distance.

For limits of sequences, the majority of the students were able to determine the smallest corresponding N for a given epsilon. However, not all students correctly chose N to be a natural number, or were able to determine that any natural number larger than the smallest N would also satisfy the formal definition. The graphical representations of limits of sequences on a Cartesian plane did cause cognitive conflict for one student and uncovered that she had not developed an understanding of the n-values of a sequence, and the role of N.

Properties. In a Real Analysis course, students are taught to understand and recognize which combinations of properties are sufficient for a limit to exist. For instance, it is adequate for a function or a sequence to have the property of being constant to determine that a limit exists. In contrast, the single property of increasing is not sufficient to determine if a limit exists or does not. However, if one combines the property of increasing with the property of bounded above, then one can determine that the limit exists.

Students whose evoked example spaces composed of general graphical representations demonstrated that properties were an important aspect to consider when generating examples. Therefore, to gain insight into what combinations of properties were evoked by Real Analysis students, their evoked example spaces were analyzed for the underlying properties. Overall, most of the students were able to generate examples with a variety of combinations of properties that are sufficient for a limit to exist. There were a few students who generated examples that did not satisfy all properties necessary for a limit to exist, such as one student who misunderstood that the property bounded above to imply the domain of a function was bounded above.

Additionally, the process of computing limits algebraically also illuminates students' knowledge of the properties of limits. The students' algebraic computations of the limits showed that the students were able to correctly apply the explicit and implicit properties of limits. All of the students except for one were able to generate an example for the product property of limits.

Concept definition. Throughout a Real Analysis course, students transition their understanding of the concept of limits from describing to formally defining. The students are expected to internalize the formal definition and adequately apply it in writing and reading proofs. Therefore, it is ideal that Real Analysis courses transform their concept definition from using informal colloquialisms (Tall, 1990) to be more precise language that aligns with the formal definition.

There was a small group of students whose evoked concept definitions of limits of sequences were informal definitions, such as Kayla's evoked concept definition that "$\lim_{n\to\infty} s_n$ (is) the number s_n approaches as $n \to \infty$." The majority of the students had adapted their concept definition to be the formal definition of limits of sequences. There were a few misalignments between some of the students' evoked concept definitions, such as incorrect ordering or not including the universal and existential quantifiers.

There were two students who did not generate evoked concept definitions for limits of functions. One student stated the definition was the "same as limits of sequences" and the other stated that he was unable to provide a definition because he had not memorized it. All of the students who generated a concept definition adopted the formal definition of limits of functions as their concept definition except for one. Her evoked concept definition for limits of functions was “a fundamental concept concerning the behavior of the function near a particular input.” There was only one student who had misalignment with their concept definition and the formal definitions, and misalignment was incorrect ordering.

Potential conflict factors. It was seen that within some of the individuals’ concept images they held factors that conflicted with the formal definitions or with other factors within a particular domain on the concept image, with different factors across the domains, or with factors within their concept definition. Tall and Vinner (1981) defined these factors to be potential conflict areas. In this study some of these factors were subtle and not apparent to the individual, and others caused cognitive conflicts that the individual did not recognize. A “serious type of potential conflict factor is one in the concept image which is at variance not with another part of the concept image but with the formal concept definition itself” (Tall & Vinner, 1981). These potential conflict factors are serious, because if someone experiences cognitive conflict with the formal definition, it may impede someone’s ability to adopt the formal definition, or accurately use it when reading and write proofs.

From the data there emerged two groups of students: Those whose evoked concept images and evoked concept definitions demonstrated serious potential conflict factors that misaligned with either one or both of the formal definitions, and individuals whose evoked concept images and evoked concept definitions did not present potential conflict factors or cognitive conflict with the formal definition. The groups were respectively labeled the cognitive conflict group and cognitive resolution group. There were a variety of serious conflict factors that emerged among the cognitive conflict group. One student in the cognitive conflict group held the perception of a dominant image that displayed his cognitive conflict with the two-dimensional images of the formal definitions of limits of functions and limits of sequence with the one-dimensional image of the formal definition of limits of sequences. Other serious cognitive conflicts were students' misinterpretations of sequences' *n*-values, and incorrect encapsulations of the process of determining a corresponding *N* value for a given positive epsilon value for limits of sequences. Similarly, students held potential conflict factors with the process of determining a corresponding delta for a given epsilon value for limits of functions. The evoked concept image and concept definitions of the cognitive resolution group did not present these serious potential conflict factors.

Proof comprehension. Students from one Real Analysis section were given an assessment on their comprehension of limit proofs. The assessment was designed using Mejia-Ramos et al.'s (2012) proof comprehension model. The assessment analyzed two dimensions of the students' local comprehension and two dimensions of their holistic understanding. Overall the cognitive resolution group was more successful with evaluating the local aspects of the proofs. The cognitive resolution group tended to be more precise with defining and explaining terms and statements within the proof. The cognitive resolution group was also able to correctly identify how assertions relate within the proof and to the conclusion of the proof. Some of the members of the cognitive conflict group were successful in some of the parts of assessing the local aspects of a proof. However, they were more likely to have errors, or not demonstrate an understanding of how assertions were logically connected.

Students in the cognitive conflict group are more likely to not acquire the necessary surface level understanding required for Real Analysis courses. Not all of the cognitive conflict students adopted formal definitions of limits and were not familiar with the meaning of the terms and notation discussed in class. Similarly, the students in the cognitive conflict group had more difficulty deductively reasoning about assertions in the proof and justifying claims. This finding does not necessarily mean the serious conflict factor is the reason why students have difficulty understanding the local aspects of the proofs, but their serious conflict factor could be a contributing reason. Future research would be needed to explore this relationship further.

The relationship between the students' conceptual understanding of limits with their holistic comprehension was seen through the activation of their mental images and concept definition. If students were experiencing conflict generating a mental image or concept definition, the students thus experienced difficulty producing summaries. The cognitive conflict student's additions and omissions of properties to their mental images also appeared in their summaries. In general, the members of the cognitive resolution generated more accurate summaries of the proofs, with the exception of one student who included an additional property in one proof summary.

Both groups of students had difficulties generating correct examples. Only two students were able to generate a correct example for the second proof: one was a symbolically named example and the other was a general graphical representation. One student was able to generate a correct example for the second proof that was a general graphical representation. The students were more successful at generating an example for the third proof. The students seemed to be more successful at generating an example for the third proof because the symbolic structure of the example was provided. Such was not the case for the other two proofs, the examples were to be generated based on the specified properties. The symbolic structure was something the students were more familiar with, which they used as a tool to help generate their evoked example space. Generating examples for the first two proofs activated two concept image domains, example space and properties and it was seen that the students' two domains were not strongly connected enough for them to generate correct examples.

Implications

Mathematics majors encounter the difficult challenge of advancing their informal conception of limits to the formal limit conception with abstract symbolism (Cornu, 1991; Davis & Vinner, 1986; Pinto & Tall, 1999, 2002; Tall & Vinner, 1981; Williams, 1991). As seen in this study, there are students who successfully adopted the formal definition and were able to apply their concept definition when understanding proofs. However, there were students, who resorted to rote memorization and experienced cognitive conflict between the formal definitions. This is further evidence supporting Tall & Vinner's (1981) findings.

The cognitive conflict group also experienced difficulty understanding and manipulating the formal definition. Similar to Cornu's (1991) study, the cognitive conflict group had difficulty encapsulating the process given a positive epsilon, to find the corresponding variable for each of the formal definitions. Research has shown that as students learn the formal definition, they struggle with understanding the relationship of the epsilon with delta in the limits of functions definition, and epsilon with *N* in the limits of sequences definition required students to understand both the universal and existential quantifiers and the dependency of the relationship (Cottrill et al., 1991; Davis & Vinner, 1986; Kidron & Zehavi, 2002; Dubinsky, Elterman, & Gong, 1988, Pinto & Tall, 1999, 2002). Consistent with the literature, these difficulties persisted for some of cognitive conflict students at the end of a Real Analysis course.

A struggle that impacted both groups was manipulating absolute-value inequalities. The algebraic process was something that most of the students were not able to correctly execute. The difference between the two cognitive groups, was that students in the cognitive resolution group had a conceptual understanding that they were simplifying absolute-value inequalities in the formal definition of limits of functions to solve for positive value that represented a distance. Whereas, students in the cognitive conflict group display difficulty correctly interpreting the absolute-value inequalities, which has been seen in prior studies (Cottrill et al., 1996).

While prior research places a large emphasis for more attention on the role of visualization of learning limits in calculus (Dreyfus, 1990; Eisenberg & Dreyfus, 1991; Ferrini-Mundy & Graham, 1994; Tall, 1991; Vinner, 1989; Zimmerman, 1991), this emphasis was observed to be carried throughout one of the sections of the Real Analysis course. It was observed that throughout the semester that one instructor encouraged students to generate mathematical drawing to understand definitions, theorems, and proofs. Toward the end of the semester, the cognitive resolution group appeared to be more forthcoming with generating mathematical drawings during their presentations. In contrast, students in the cognitive conflict group still heavily relied on the instructor’s guidance to generate images. These were the students who were more reluctant to adopt the formal definition, and whose concept image and definition had misalignment with the formal definition.

Therefore, it may be appropriate to incorporate dynamic software in a Real Analysis course to aid students who are struggling to build connections between the formal definitions, their concept definition, and their mental images. Studies have shown that using dynamic software has helped students develop their understanding of the formal definitions of limits and build a connection between the abstract symbolism and visualizations (Cory and Garofalo, 2011; Kidron & Zehavi, 2002; Parks, 1995). The dynamic software could also be used to help students visually explore their example spaces.

A majority of the Real Analysis students generated symbolically-named evoked example spaces with little to, no emphasis on the properties of the sequences and functions, or the formal definition. Thus, when the students were asked to generate an example that adhered to specified properties on the proof comprehension assessment, the students were not able to generate correct examples for each of the proofs. Students' example spaces were not strongly connected to the other domains of their concept image. This was seen with students in both groups who generated a symbolically-named example that was intended to be an increasing sequence that meant to be bounded above but was actually a decreasing sequence that was bounded below.

Incorporating dynamic software in a Real Analysis course could allow students strengthen the connection between their concept image domains by exploring their symbolically-named examples. The dynamic software would allow the students to build a mental image of the example, discover the properties of the sequence or function, determine whether it satisfies the formal definition and practice the process of determining a corresponding *N* or delta for a given positive epsilon. Not only is aligning an individual's concept definition to formal definition important, strengthening the four domains of students' concept image with their concept definition is necessary to improve students' holistic understanding of a limit proof. To increase student's limit proof comprehension, connections between the concept image domains, concept definition and the formal definition need to be fostered.

Implications for practice. One of the important implications from this study is for instructors to formalize students' conceptualization limits in a manner that addresses the individual's serious potential conflicts. An individual is not always aware of their serious potential conflict factors, or if the individual is aware of their confusion, they may not know how to create cohesion and connections between their concept image and the formal definition. Therefore, it is important for the instructor to provide assignments that not only challenge students to confront their potential conflicts but helps the student reach cognitive resolution.

The assignments could include students using dynamic software to explore sequences or functions that have been shown throughout literature to cause cognitive conflict (Cornu, 1991; Cory & Garofalo, 2011; Davis & Vinner, 1986; Pinto & Tall, 2002; Williams, 1991). For instance, the sequence constant sequence $a_n = \{5\}_{n=1}^{\infty}$ can be used to address students' conception that limits are unreachable. Students should also be allowed to explore whether their dominant images satisfy the formal definition. Allowing these different explorations will either strengthen their mental images and align it with the formal definition, or help eliminate unproductive images by visually showing how their dominant images do not satisfy the formal definition.

An aspect that all the students struggled with was generating examples that depicted the main idea of a proof. Therefore, it is important for instructors to not only model this but develop students' ability to generate such examples. An assignment that could potentially connect students' holistic understanding of a proof with their example space would be for them to identify which examples satisfy the big idea of the proof from a mixture of general examples and symbolically named examples. Such tasks allow students to consider the properties within the examples, and connect the formality of the mathematical language of the proof to the general or specific examples. Initially, providing the examples for the students is important since "most examples come to students from authorities" (Watson & Mason, 2002). Therefore, such assignments can potentially enrich students' example spaces that are associated with proofs and strengthen their concept images. Eventually the instructor would want to transition from providing the example to having the students generate their own examples that demonstrate the main ideas of a proof.

Limitations

This study used two sections from a single university in central Texas, and there was low participation in one section. From one section, only three students volunteered to participate and only two of the participants completed all of the components of the study. Therefore, the concept images and concept definitions presented were predominately formed by students in one section of a Real Analysis course.

A major issue was that not all participants completed both limit surveys. With the consent of both instructors, the surveys were given as a take-home review at the end of the respective limit units. Both instructors advertised the limit surveys as reviews for their unit exams. One instructor offered that the completion of a survey would replace their lowest homework score. However, that instructor was already experiencing difficulty with students completing his homework assignments. Therefore, to increase completion of the surveys it would have been better to have had the students complete them in class. However, completing the surveys in class would have taken away from valuable instruction.

The interview participants were selected to capture a diverse scope of concept images and concept definitions held by Real Analysis students. However, if it had been feasible, interviewing all participants would have provided a more in depth understanding of all of the students' conceptualizations of limits. The interviews of participants showed that students who had potential conflict factors were able to better articulate their confusions and misunderstandings during the interviews rather than on the surveys. The interview participants were also able to provide more detail into their thought processes and responses. Overall, the interviews greatly enriched the data.

The original design of the study incorporated a third limit survey that would have had the students compare and contrast both limits of sequences and limits of functions at the end of the semester, and explain their interpretation of what it means for a limit not to exist. It would have been intriguing to gain insight into students' understanding of a limit not existing since one instructor discussed this topic in depth and the other instructor only discussed the two instances of a limit not existing when the limit was positive or negative infinity. However, both instructors ended the limits of functions survey during the last week of classes. Therefore, there was no time between the limits of functions survey and the proof comprehension assessment to give a third survey.

Another limitation of the study was that in order to get class time for the proof comprehension assessment, each assessment had to align with the section's final exam. Therefore, each instructor provided different proofs that were appropriate for their section. Hence, cross-section analysis of the assessments was not deemed appropriate. Also, the section that had low participation in completing the limit surveys was also reluctant to respond to the in class proof comprehension. Most students in that section left more than 75% of the assessment blank.

Lastly, the researcher solely conducted the observations, interviews, and analyzed the data. As with any qualitative data analysis, there was some level of bias due to the researcher making decisions on the interpretation of the data.

Future Research

This study brought to light how weak connections between an individual's concept image domains and concept definitions of limits impacted their holistic comprehension of limit proofs. However, this study investigated the domains of concept image more than the connections between the domains. Thus, future research needs to investigate the connections between the domains specifically and how the connections impacts proof comprehension.

Future research is also needed to determine appropriate practices that will help dismantle serious conflict factors that negatively impact student's proof comprehension. Prior research has shown dynamic software has aided students understanding of the formal definitions of limits (Cory and Garofalo, 2011; Kidron & Zehavi, 2002; Parks, 1995). However, future research will be needed to determine whether it will have the same effect in a Real Analysis course of improving their concept image, concept definition, and ultimately their proof comprehension. More exploration is needed about students' holistic proof comprehension. It is unclear why some students were able to generate accurate proof summaries but not able to generate correct examples. As such, more researcher is needed to understand the disconnect between students' example spaces and proof comprehension.

APPENDIX SECTION

APPENDIX A
PILOT STUDY IRB EXEMPTION

TEXAS STATE UNIVERSITY
SAN MARCOS
The rising STAR of Texas

Institutional Review Board

Request For Exemption

Certificate of Approval

Applicant: Christine Herrera

Request Number : EXP2015N528092E

Date of Approval: 02/04/15

Assistant Vice President for Research and Federal Relations

Chair, Institutional Review Board

APPENDIX B
PILOT STUDY CONSENT FORM

TEXAS STATE UNIVERSITY
PARTICIPANT INFORMED CONSENT STATEMENT
Pilot Dissertation Study

I, ___________________________________, agree to participate in the IRB approved, pilot study for a dissertation conducted by Christine Herrera from the Department of Mathematics at Texas State University. I understand that my participation is voluntary. I can stop taking part without giving any reason, and without penalty. I can ask to have all of the information about me returned to me, removed from the research records, or destroyed.

PURPOSE

A general overarching goal of this project is to capture a student's evoked concept image on a mathematical concept that is explored in Math 3380, Analysis I.

PROCEDURES

I will be recorded during the interview.

BENEFITS

The benefits to me of participating in this research include the opportunity to understand a mathematical concept deeply.

DISCOMFORTS & RISKS

There are no discomforts or stresses anticipated as a result of my participation in this interview. This interview will be confidential and have no impact on my grade in Math 3380, Analysis I.

CONFIDENTIALITY

Any reports of this research will use pseudonyms. No information that identifies me will be shared with others without my written permission except as required by law.

Any data gathered, audiotapes and copies of my written drawings and examples will be stored in a locked office. And audiotapes will be used for research purposes. No information that identifies me will be shared with school officials.

FURTHER QUESTIONS

The researchers will answer any further questions I have about this research, now or during the course of the dissertation. The contact person is Christine Herrera (cah221@txstate.edu).

CONSENT

My signature below indicates that the researchers have answered all of my questions to my satisfaction and that I consent to participate in this interview. I have been given a copy of this form.

Participant's signature______________________________________

Researcher's signature_________________________________ Date: _________

APPENDIX C
PILOT STUDY INTERVIEW PROTOCOL

INTRODUCTION

Hello, my name is Christine. Thank you so much for taking the time to be here. This is an interview; in which I will ask you questions about limits. The purpose is to help understand how you conceptualize and think about limits. This is not a test; I would like you to feel comfortable with saying whatever comes to minds. This isn't about being right or wrong. The first part of our interview will consist of me asking you some open-ended questions about limits. Then, I'll ask you to consider some graphs and have a discussion with me about those. At any time in the interview you can ask me questions to clarify or explain something more.

LIVESCRIBE RECORDER INSTRUCTIONS

If it is okay with you, I will be recording our conversation (with a LiveScribe). The purpose of this is so that I can get all the details but at the same time be able to carry on an attentive conversation with you. I assure you that all your comments will remain confidential. I will be compiling a report which will contain all students' responses without any reference to individuals.

CONSENT FORM INSTRUCTIONS

Before we get started, please take a few minutes to read and sign the consent form.

APPENDIX D
PILOT STUDY INTERVIEW QUESTIONS

Student Information

- Name:
- Major:
- Have you taken Calculus 1?
  - How many times did you take Calculus 1?
  - Where did you take Calculus 1?
- Have you taken Calculus 2?
  - How many times did you take Calculus 2?
  - Where did you take Calculus 2?
- Is this your first time take Math 3380, Analysis 1? If not, when and where did you take it before?
- Do you use technology when doing math? (i.e. graphing calculator, applets, software, etc.)

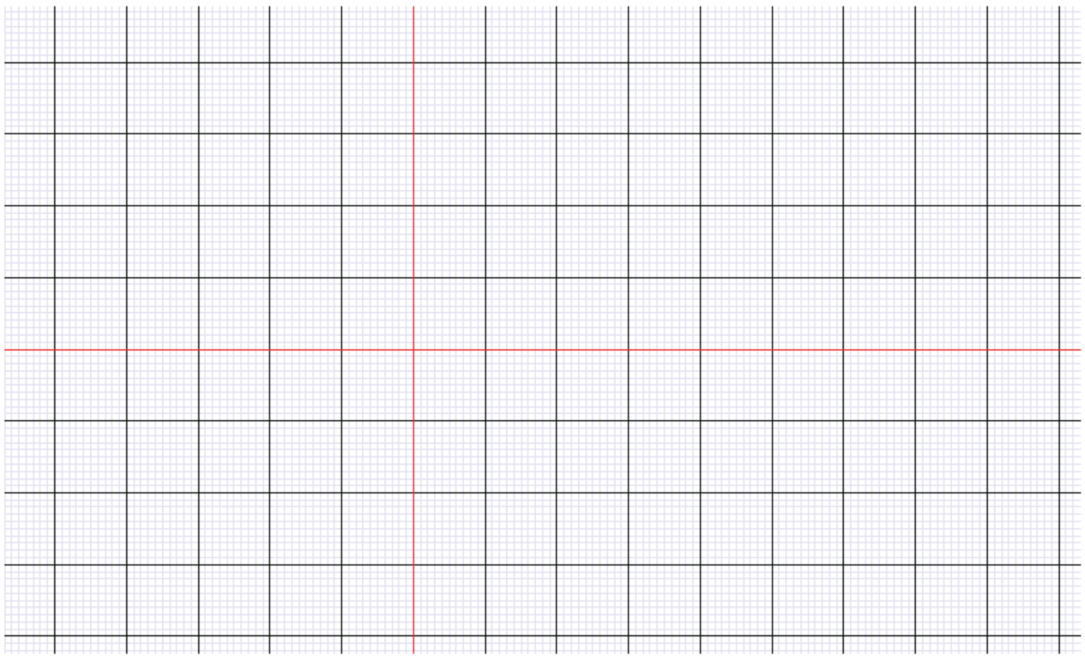
Limits Questions

1. How do you personally think about limits?
2. How do you personally visualize limits?
3. Can you state the formal definition for the limit of a function?
4. Find algebraically $\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$.
5. Consider $\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$.

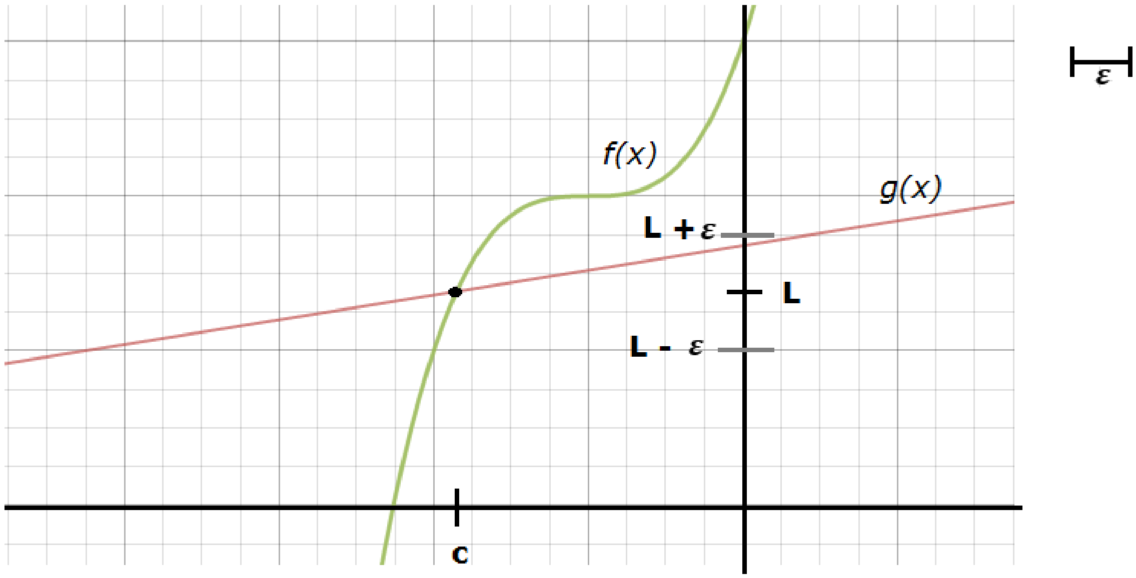
$$\lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5} = \lim_{x \to -5} \frac{(x + 5)(x - 2)}{x + 5} = \lim_{x \to -5} x - 2 = (-5) - 2 = -7$$

The precise definition of a limit of a function states that $\lim_{x \to c} f(x) = L$ if and only if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $x, 0 \, , |x - c| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$. For the function above and for $\varepsilon = 1$, it can be shown that a corresponding $\delta = 1$.

Locate and label on your graph the values of $\boldsymbol{L}$, $\boldsymbol{c}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\delta}$.



6. Consider the graph below, where $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = L$. Let ε be the indicated distance on the graph.



a. For the given ε, is the range of possible $\delta' s$ for the *f(x)* function the same range of possible δ's for the *g(x)* function?

b. Why or why not? (Either explain why they would be the same or how they would be different.)

7. Consider the following theorem:
If $\lim_{x \to c} f(x) = f(c)$, then there is a number $M > 0$ and $\delta > 0$ so that for each $x \in (c - \delta, c + \delta)$, $|f(x)| < M$.

Can you explain what this theorem is saying in your own words and if you believe it to be true or not? You may draw or write anything to help in your explanation.

APPENDIX E

IRB EXEMPTION

TEXAS STATE UNIVERSITY
SAN MARCOS
The rising STAR *of Texas*

Institutional Review Board

Request For Exemption

Certificate of Approval

Applicant: Christine Herrera

Request Number : EXP2015T703765N

Date of Approval: 05/19/15

Assistant Vice President for Research and Federal Relations

Chair, Institutional Review Board

APPENDIX F

INFORMED CONSENT FORM

INVESTIGATORS

Christine Herrera, Doctoral Teaching Assistant Texas State University, Department of Mathematics, 601 University Drive, San Marcos, TX 78666, (512) 245-4749

TITLE OF STUDY

The Effect of the Conceptualization of Limits on Proof Comprehension

This study will be conducted in the Department of Mathematics at Texas State University.

PURPOSE

The purpose of this research study is to Analysis I student's conceptualization of limits of sequences and limits of functions and how their understanding effects their proof comprehension.

PROCEDURE

I agree to participate and complete five surveys throughout the semester. A select few will be asked to participate in follow-up interviews in a safe, quiet place, such as a classroom, during a session that will last no longer than an hour. The interviews will allow for further explanation of one's answers and the opportunity for the researcher to ask follow-up questions. The interview will be audio-recorded.

VOLUNTARY PARTICIPATION

I understand that my participation in this study is voluntary. I understand that I may discontinue my participation in this study at any time without any penalty or prejudice. Upon my request, my responses will not be included in the study.

RISKS AND DISCOMFORTS

Risks and discomforts to the participants will be minimal, the surveys will serve as a reflection. It is not the investigators' intention, however, to criticize your work, but rather to examine the individual's understanding.

BENEFITS/RESULTS

First, my participation allows me to reflect on my understanding of limits. Second, this study will contribute to the small but growing body of research on the teaching and learning of mathematical proof at the college level and has the potential to improve how proof comprehension of limits in an Analysis I course is taught.

CONFIDENTIALITY

I understand that my identity in this study will not be disclosed in any published document. Only the investigators will have access to the data that link my name to my responses.

REQUEST FOR MORE INFORMATION

The study has been explained to me, and I have had an opportunity to ask questions. If any other questions should arise during this study about my participation or the rights and welfare of human participants in research, I understand that I can contact the investigator listed above, my department chair, or the Texas State University Institutional Review Board Office at 512-245-2314, ospirb@txstate.edu.

My signature below acknowledges my consent to voluntarily participate in this research project and to be audio-recorded. Such participation does not release the investigator or university from their professional and ethical responsibility to me. I acknowledge receiving a copy of this consent form.

_________________________________ Participant's signature & date

________________________________ Investigator's signature & date

________________________________Witness's signature & date

APPENDIX G

PARTICIPANT INFORMED CONSENT STATEMENT

I, ________________________________, agree to participate in the IRB approved, interviews for a dissertation conducted by Christine Herrera from the Department of Mathematics at Texas State University. I understand that my participation is voluntary. I can stop taking part without giving any reason, and without penalty. I can ask to have all of the information about me returned to me, removed from the research records, or destroyed.

PURPOSE

A general overarching goal of this project is to capture a student’s evoked concept image on a mathematical concept that is explored in Math 3380, Analysis I.

PROCEDURES

I will be audio recorded during the interview.

BENEFITS

The benefits to me of participating in this research include the opportunity to understand a mathematical concept deeply.

DISCOMFORTS & RISKS

There are no discomforts or stresses anticipated as a result of my participation in this interview. This interview will be confidential and have no impact on my grade in Math 3380, Analysis I.

CONFIDENTIALITY

Any reports of this research will use pseudonyms. No information that identifies me will be shared with others without my written permission except as required by law.

Any data gathered, audiotapes and copies of my written drawings and examples will be stored in a locked office. And audiotapes will be used for research purposes. No information that identifies me will be shared with school officials.

FURTHER QUESTIONS

The researchers will answer any further questions I have about this research, now or during the course of the dissertation. The contact person is Christine Herrera (cah221@txstate.edu).

CONSENT

My signature below indicates that the researchers have answered all of my questions to my satisfaction and that I consent to participate in this interview. I have been given a copy of this form.

Participant's signature______________________________________

Researcher's signature___________________________________ Date: _________

APPENDIX H

FIRST WEEK SURVEY

Dear Analysis I Student,

Thank you for agreeing to participate in this research study. We ask you to please take about 30 – 40 minutes to complete the survey below. Your responses will greatly help our research.

Answer the following questions as honestly as you can.

Name:

Major:

Classification (Circle One):

Freshman Sophomore Junior Senior Post-Bachelor Degree

Gender: _____________

Is this your first time taking this course?

Yes No

If not, please state when you first took this course.

<u>**Prior Math Courses**</u>

Please check all the math courses you have taken at Texas State University or another institution.

☐	**Pre-College Algebra**	☐	**Number Systems**
☐	Basic Mathematics	☐	Introduction to Advanced Mathematics
☐	College Algebra	☐	Deterministic Operations Research
☐	A Survey of Contemporary Mathematics	☐	Calculus III
☐	Plane Trigonometry	☐	Engineering Mechanics
☐	Mathematics for Business and Economics I	☐	Linear Algebra
☐	Mathematics for Business and Economics II	☐	Analysis I
☐	Principle of Mathematics I	☐	Discrete Mathematics II
☐	Informal Geometry	☐	Principles of Mathematics II
☐	Calculus for Life Sciences I	☐	Capstone Mathematics for Middle School Teachers
☐	Elementary Statistics	☐	Math Understandings
☐	Calculus for Life Sciences II	☐	Probability and Statistics
☐	Discrete Mathematics I	☐	Fourier Series and Boundary Value Problems
	Pre-Calculus Mathematics		Modern Algebra
☐	Calculus I	☐	Introduction to the History of Mathematics
☐	Calculus II	☐	Analysis II
☐	Introduction to Probability and Statistics	☐	General Topology
☐	Modern Geometry	☐	Studies in Applied Mathematics
☐	Differential Equations	☐	The Literature and Modern History of Mathematics and Its Applications

Limits

A. Please mark the following six statements about limits as being true or false:

1. T F A limit describes how a function moves as *x* moves toward a certain point.
2. T F A limit is a number or a point past which a function cannot go.
3. T F A limit is a number that the *y*-values of a function can be made arbitrarily close to by restricting *x*-values.
4. T F A limit is a number or point the function gets close to but never reaches.
5. T F A limit is an approximation that can be made as accurate as you wish.
6. T F A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

B. Which of the above statements best describes a limit as you understand it? (Circle One)

1 2 3 4 5 6 None

C. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function f as $x \to c$ is number L. (Use can draw a picture to aid in your description.)

APPENDIX I

LIMITS OF SEQUENCES SURVEY

<u>LIMITS OF SEQUENCES</u>

Please answer the questions to the best of your ability. Any and all of your responses are greatly values so please do not leave questions blank. If you have any questions please contact me at cah221@txstate.edu. ***Thank you in advance for completing this survey!*** **Your submission of the survey enters you for the chance to win a gift card!**

1. How do you personally think about limits of sequences?

2. How do you visualize limits of sequences? If you can, please provide a picture with your description.

3. *Without using any resources* please state the formal definition for the limit of a sequence.

4. Please provide as many examples as possible of sequences with the limit of 5.

5. Determine the limits of the following sequences. Please show all work and provide explanations.
   a. $\left\{\frac{7\sin(n)}{n} - 2\right\}_{n=1}^{\infty}$

   b. $\{\cos(n) + 3\}_{n=1}^{\infty}$

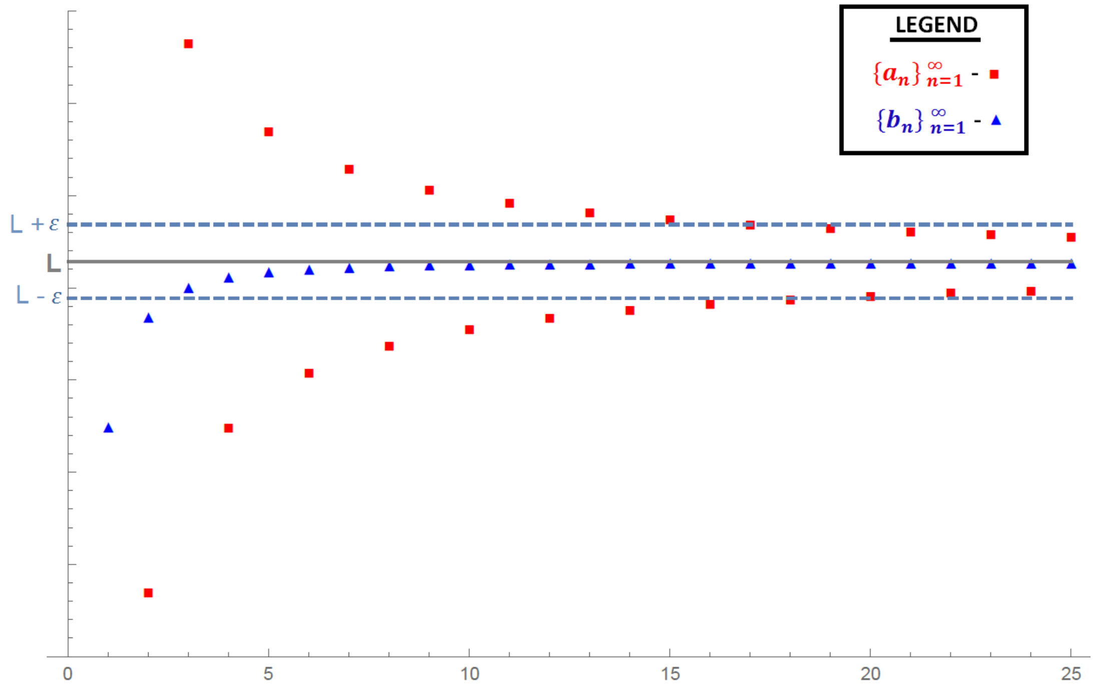
   c. $\{-\frac{1}{2}\}_{n=1}^{\infty}$

   d. $\{\frac{1-5n^4}{n^4+8n^3}\}_{n=1}^{\infty}$

6. Consider $\lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1}$.

The precise definition of a limit of a sequence states that $\lim_{n\to\infty} a_n = A$, if for any positive number ε, there is a natural number N such that $|a_n - A| < \varepsilon$ for all $n \geq N$. The limit of a sequence is also referred to as convergence. For the sequence above and for $\varepsilon = \frac{1}{10}$, it can be shown that the corresponding $N = 6$. **Locate and label on your graph the values of *A*, $\boldsymbol{\varepsilon}$, and *N*.**

7. Consider the graph below, where both sequences, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to the real number L. Let ε be the indicated distance on the graph.



a. Using the graph above of the $\{a_n\}_{n=1}^{\infty}$ sequence and the indicated $\varepsilon > 0$ on the graph, what is the smallest N?

b. Using the graph above of the $\{b_n\}_{n=1}^{\infty}$ sequence and the indicated $\varepsilon > 0$ on the graph, what is the smallest N?

c. For the given $\varepsilon > 0$, is the smallest of possible $N's$ for the $\{a_n\}_{n=1}^{\infty}$ sequence, the same as the smallest of possible $N's$ for the $\{b_n\}_{n=1}^{\infty}$ sequence ?

Yes or No

d. Why or why not? (Either explain why they would be the same or how they would be different.)

e. How do you determine the range of possible $N's$ for the $\{a_n\}_{n=1}^{\infty}$ sequence and the range of possible $N's$ for the $\{b_n\}_{n=1}^{\infty}$ sequence?

8. Consider the $\lim_{n\to\infty} \frac{1-2n}{1+4n} = -\frac{1}{2}$.
   a. Let $\varepsilon = 0.1$, and find a N.

   b. Is there another *N* possible? If so how do the two *N* relate?

9. Consider the following statement:

   Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Then $\lim_{n\to\infty} s_n t_n = st$.

   a. Is the statement above true or false?

   b. If the statement is true provide an example, or if the statement is false provide a counterexample.

   c. Prove or disprove the above statement. Please attach all scratch work.

10. Consider the following theorem:

Suppose $\{x_n\}$ is a sequence such that for each natural number n, $x_n \leq x_{n+1}$, and there exists a number M so that for each n, $x_n \leq M$, then $\lim_{n\to\infty} x_n$ exists.

Can you explain what this theorem is saying in your own words and if you consider it to be true or not? You may draw or write anything to better clarify your explanation.

THANK YOU FOR COMPLETING THIS SURVEY!!

APPENDIX J

LIMITS OF FUNCTIONS SURVEY

<u>LIMITS OF FUNCTIONS</u>

Please answer the questions to the best of your ability. Any and all of your responses are greatly values so please do not leave questions blank. If you have any questions, please contact me at cah221@txstate.edu. ***Thank you in advance for completing this survey!*** **Your submission of the survey enters you for the chance to win a gift card!**

1. How do you think about limits of functions?

2. How do you visualize limits of functions? If you can, please provide a picture with your description.

3. *Without using any resources* please state the formal definition for the limit of a function.

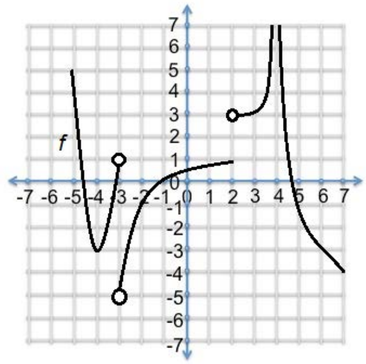
4. Please provide as many examples as possible of functions with the limit of 2.

5. Find algebraically the following limits:

   a. $\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$

   b. $\lim_{x \to -\infty} \frac{x}{(x-3)(x+2)}$

6. Use the graph below to answer the following questions:



a. Does the limit of *f(*c*)* exist for the following c values? Explain why or why not.

i. c = -4

ii. c = -3

iii. c = 2

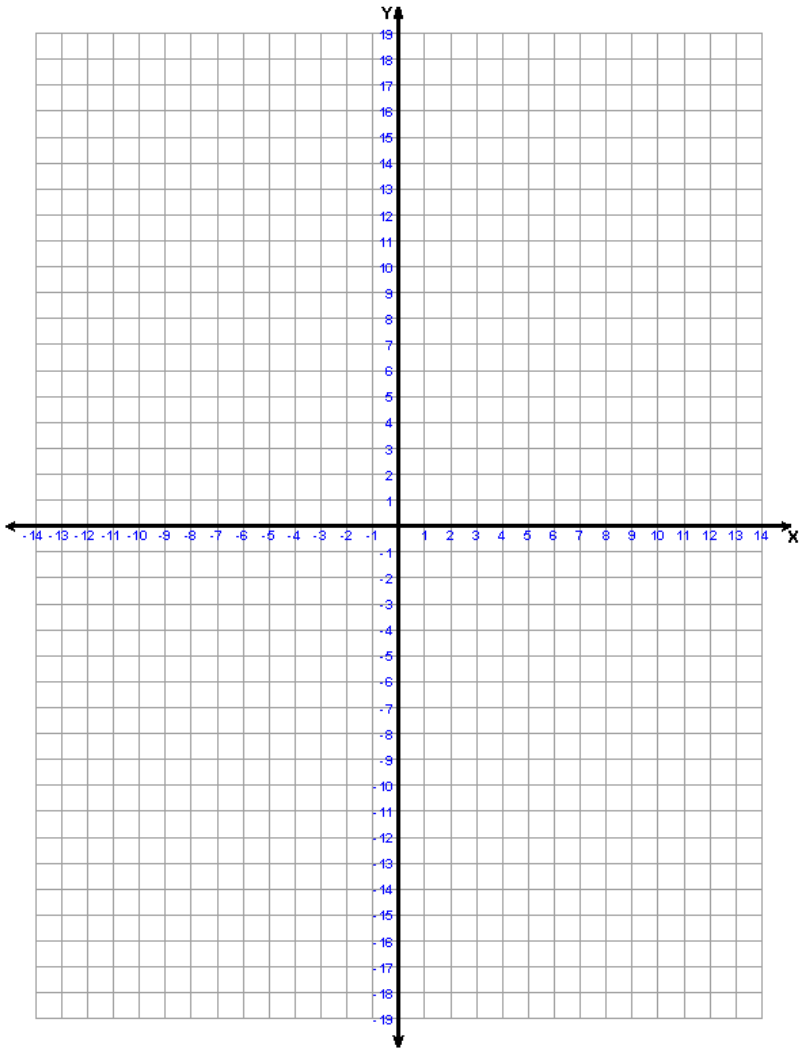
7. Consider $\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$.

$$\lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5} = \lim_{x \to -5} \frac{(x + 5)(x - 2)}{x + 5} = \lim_{x \to -5} x - 2 = (-5) - 2 = -7$$

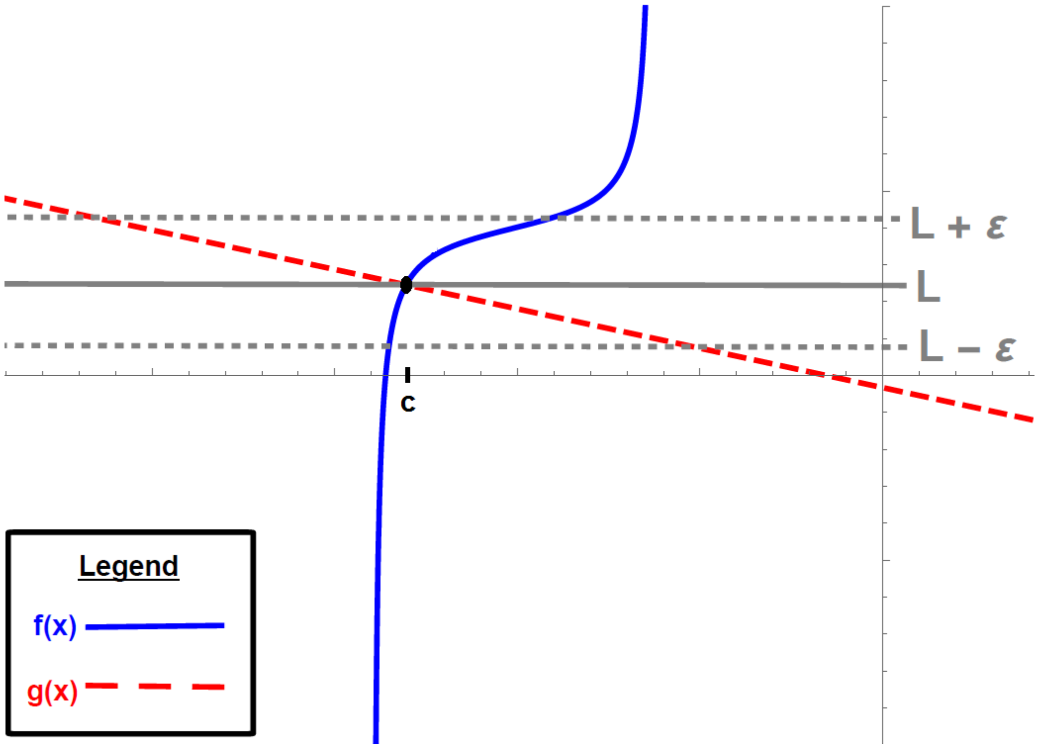
The precise definition of a limit of a function states that $\lim_{x \to c} f(x) = L$ if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $x <$ $0\,, |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

For the function above and for $\varepsilon = 1$, it can be shown that the corresponding $\delta = 1$.

Locate and label on your graph the values of L, c, ε, and δ.



8. Consider the graph below, where $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = L$. Let ε be the indicated distance on the graph.



a. Using the graph above of the function *f(x)* and the given $\varepsilon > 0$ on the graph, what is the largest possible δ?

b. Using the graph above of the function *g(x)* and the given $\varepsilon > 0$ on the graph, what is the largest possible δ?

c. Explain why the largest possible δ for the *f(x)* function is different than the largest δ for the g*(x)* function for the given $\varepsilon > 0$.

9. Consider the $\lim_{x \to 0} \sqrt{x+1} = 1$.
    a. Let $\varepsilon = 0.1$, and find a δ.

    b. Is there another delta possible? If so how do the two deltas relate?

10. Consider the following statement:

Suppose that $\lim_{x \to c} f(x) = K$ and $\lim_{x \to c} g(x) = L$ exist. Then $\lim_{x \to c} f(x) + g(x) = K + L$.

a. Is the statement above true or false?

b. If the statement is true provide an example, or if the statement is false provide a counterexample.

c. Prove or disprove the above statement. Please attach all scratch work.

11. Consider the following theorem:

    If $\lim_{x \to c} f(x) = f(c)$, then there is a number $M > 0$ and $\delta > 0$ so that for each $x \in (c - \delta, c + \delta)$, $|f(x)| < M$.

    Can you explain what this theorem is saying in your own words and if you believe it to be true or not? You may draw or write anything to better clarify your explanation.

APPENDIX K

PROOF COMPREHENSION ASSESSMENT

REAL ANALYSIS FINAL REVIEW – FALL 2015

Name: ___________________________

Major: ___________________________ Seeking Teacher Certification: Yes or No

Suppose M is the least upper bound of the set S and $p < M$, then there exists a number $x \in S$ so that $p < x \leq M$.

Proof:

(Line 1) Let S be a set with the least upper bound M.

(Line 2) Suppose p is a number such that $p < M$.

(Line 3) p is not an upper bound for S.

(Line 4) Thus, there exists $x \in S$ such that $p < x$.

(Line 5) Since M is the least upper bound for S and $x \in S$, then $x \leq M$.

(Line 6) Hence, $p < x \leq M$.

Use the mathematical statement and corresponding proof above to answer the following questions:

1. State the mathematical definition of *least upper bound*.

2. Explain why the proof-writer knew *there existed $x \in S$ in Line 4*.

3. Provide an example that illustrates the mathematical statement.

If $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence that is bounded above, then $\lim_{n\to\infty} a_n$ exists.

Proof:

(Line 1) Suppose $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence that is bounded above by some real number M.

(Line 2) Let $\varepsilon > 0$.

(Line 3) Since, $\{a_n\}_{n=1}^{\infty}$ is bounded, there exists some least upper bound of $\{a_n\}_{n=1}^{\infty}$, say L.

(Line 4) Since $L - \varepsilon < L$ then $L - \varepsilon$ is not an upper bound of $\{a_n\}_{n=1}^{\infty}$.

(Line 5) There exists a natural number N such that $L - \varepsilon < a_N \leq L$.

(Line 6) Since $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence, for every $n \geq N$, $L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon$.

(Line 7) Thus, for every $n \geq N$, $-\varepsilon < a_n - L < \varepsilon$.

(Line 8) Hence, $\lim_{n\to\infty} a_n$ exists.

Use the mathematical statement and corresponding proof above to answer the following questions:

1. State the mathematical definition of *limit of a sequence*.

2. Which line(s) depend on *Line 1*?

3. Referring to *Line 4*, explain what it means to not be an upper bound of $\{a_n\}_{n=1}^{\infty}$.

4. State the contrapositive of the mathematical statement being proved.

5. Would proving the contrapositive be an appropriate proof method for the mathematical statement above, why or why not?

6. Explain the logical ordering of the key ideas in the proof.

7. Provide an example of the mathematical statement.

Suppose $\boldsymbol{f}$ is an increasing function and $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence such that $\lim_{n\to\infty} x_n = p$. Then $\{f(x_n)\}_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to\infty} f(x_n)$ exists.

<u>*Proof:*</u>

(Line 1) Suppose f is an increasing function and $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence such that $\lim_{n\to\infty} x_n = p$.

(Line 2) Since $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence and f is an increasing function then $\{f(x_n)\}_{n=1}^{\infty}$ is an increasing sequence.

(Line 3) Let $\varepsilon > 0$.

(Line 4) Since $\lim_{n\to\infty} x_n = p$, there exists a natural number *N* such that for all $n \geq N$ $x_n \in (p-\varepsilon, p + \varepsilon)$.

(Line 5) Therefore, the increasing sequence $\{x_n\}_{n=1}^{\infty}$ is bounded above by $(p + \varepsilon)$.

(Line 6) Since x_n is bounded above for all natural numbers *n*, then $f(x_n)$ is bounded above for all natural numbers *n*.

(Line 7) $\{f(x_n)\}_{n=1}^{\infty}$ is an increasing and bounded above sequence hence, $\lim_{n\to\infty} f(x_n)$ exists.

1. State the mathematical definition of *limit of a sequence*.

2. Why was *Line 2* included in the proof?

3. Explain the logical ordering of the key ideas in the proof.

4. Provide a short summary of the proof.

5. Provide an example of the mathematical statement.

Suppose $\boldsymbol{f(x) = mx + b}$ and $\boldsymbol{p}$ is a real number, then $\boldsymbol{\lim_{x \to p} f(x) = f(p)}$.

Proof:

(Line 1) Suppose $f(x) = mx + b$ and p is a real number.

(Line 2) Let $\varepsilon > 0$.

(Line 3) Case i.) $m = 0.$ Let $\delta = \varepsilon$.

(Line 4) Assume $|x - p| < \delta$, then $|f(x) - f(p)| = |(mx + b) - (mp + b)| = |m(x - p)| = |m||x - p| = 0 < \varepsilon$.

(Line 5) Case ii.) $|m| > 0.$ Let $\delta = \frac{\varepsilon}{|m|}$.

(Line 6) Assume $|x - p| < \delta$, then $|f(x) - f(p)| = |(mx + b) - (mp + b)| = |m(x - p)| = |m||x - p| < |m|\delta = \varepsilon$.

(Line 7) Hence, $\lim_{x \to p} f(x) = f(p)$.

Use the mathematical statement and corresponding proof above to answer the following questions:

1. Explain what $\lim_{x \to p} f(x) = f(p)$ means?

2. Explain the logical ordering of the key ideas in the proof.

3. Using the ideas in the proof, provide a specific example for both cases.

REFERENCES

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. In F. Hitt, D. Holton, & P. W. Thompson (Eds.), Research in collegiate mathematics education VII (pp. 63–91). Washington, DC: MAA.

Alcock, L., Bailey, T., Inglis, M., & Docherty, P. (2014). The ability to reject invalid logical inferences predicts proof comprehension and mathematics performance.

Alcock, L., & Simpson, A. (2002). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics*, 28-34.

Alcock, L., & Simpson, A. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, *57*(1), 1-32.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *The Journal of Mathematical Behavior*, *24*(2), 125-134.

Arzarello, F., Paola, D., & Sabena, C. (2009). Proving in early calculus. In *Proceedings of the ICMI Study 19 conference* (pp. 35-40).

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics*, *33*(3), 301-317.

Bills, L., & Tall, D. (1998). Operable definitions in advanced mathematics: The case of the least upper bound. In *Proceedings of the Conference of the International Group for* (p. 111).

Biza, I., Souyoul, A., & Zachariades, T. (2005). Conceptual change in advanced mathematical thinking. In *Proceedings of the Fourth Conference on European Research in Mathematics Education* (pp. 1727-1736).

Black, M. (1962). *Models and metaphors.* Ithaca, NY: Cornell University Press.

Bloch, E. D. (2011). *Proofs and fundamentals: a first course in abstract mathematics*. Springer.

Borwein, P., & Jörgenson, L. (2001). Visible structures in number theory. *American Mathematical Monthly*, 897-910.

Cai, J., & Cirillo, M. (2014). What do we know about reasoning and proving? Opportunities and missing opportunities from curriculum analyses. *International Journal of Educational Research*, *64*, 132-140.

Cappetta, R., & Zollman, A. (2009). Creating a discourse-rich classroom on the concept of limits in calculus: Initiating shifts in discourse to promote reflective abstraction. *The role of mathematics discourse in producing leaders of discourse*, 17-39.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), Research in Collegiate Mathematics Education, 1, (7), pp. 115-162. Providence, RI: American Mathematical Society.

Carlson, M., Larsen, S., & Jacobs, S. (2001). An investigation of covariational reasoning and its role in learning the concepts of limit and accumulation. In *Proceedings of the 23rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 145-153).

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.

Conradie, J., & Frith, J. (2000). Comprehension tests in mathematics. *Educational Studies in Mathematics*, *42*(3), 225-235.

Cornu, B. (1991). Limits. In D. Tall (Ed.), Advanced mathematical thinking (pp. 153–166). Dordrecht, The Netherlands: Kluwer.

Cory, B. L., & Garofalo, J. (2011). Using dynamic sketches to enhance preservice secondary mathematics teachers' understanding of limits of sequences. *Journal for Research in Mathematics Education*, *42*(1), 65-97.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *The Journal of Mathematical Behavior*, *15*(2), 167-192.

Creswell, J. W. (2013). *Qualitative inquiry and research design: Choosing among five approaches*. Sage.

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *The Journal of Mathematical Behavior*.

Dawkins, P. C. (2012). Metaphor as a possible pathway to more formal understanding of the definition of sequence convergence. *The Journal of Mathematical Behavior*, *31*(3), 331-343.

Dewey, J. (1915). *The school and society* (2nd ed.). Chicago: University of Chicago Press.

Domingos, A. (2010). LEARNING ADVANCED MATHEMATICAL CONCEPTS: THE CONCEPT OF LIMIT. In *Proceedings of CERME* (Vol. 6, p. 28).

Dreyfus, T. (1999). Why Johnny can't prove. *Educational studies in mathematics*, *38*(1-3), 85-109.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), Advanced mathematical thinking (pp. 95-123). Dordrecht: Kluwer.

Dubinsky, E., Elterman, F., & Gong, C. (1988). The student's construction of quantification. *For the learning of mathematics*, 44-51.

Dubinsky, E., Hawks, J., & Nichols, D. (1989). Development of the process conception of function by pre-service teachers in a discrete mathematics course. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), Proceedings of the 13th PME International Conference, 1, 291-298.

Edwards, B., & Ward, M. B. (2008). Undergraduate Mathematics Courses. *Making the connection: Research and teaching in undergraduate mathematics education,* (73), 223.

Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding of limits, derivatives, and integrals. In J. Kaput & E. Dubinsky (Eds.), Research issues in undergraduate mathematics learning: Preliminary analyses and results (MAA Notes, No. 33, pp. 31–45). Washington, DC: Mathematical Association of America.

Fukawa-Connelly, T. P., & Newton, C. (2014). Analyzing the teaching of advanced mathematics courses via the enacted example space. *Educational Studies in Mathematics*, *87*(3), 323-349.

Gray, E., & Tall, D. (1991). Duality, ambiguity and flexibility in successful mathematical thinking. In F. Furinghetti (Ed.), Proceedings of the 15th PME International Conference, 2, 72-79.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational studies in mathematics*, *44*(1-2), 5-23.

Harel, G., Selden, A., & Selden, J. O. H. N. (2006). Advanced mathematical thinking. *Handbook of research on the psychology of mathematics education: past, present and future*, 147-172.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *Research in collegiate mathematics education III*, *7*, 234-282.

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical thinking and learning*, *7*(1), 27-50.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. *Second handbook of research on mathematics teaching and learning*, *2*, 805-842.

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-Explanation Training Improves Proof Comprehension. *Journal for Research in Mathematics Education*, *45*(1), 62-101.

Hoyles, C. (1997). The curricular shaping of students' approaches to proof. *For the learning of mathematics*, 7-16.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, *43*(4), 358-390.

Inglis, M., & Mejia-Ramos, J. P. (2008). How persuaded are you? A typology of responses. *Research in Mathematics Education 10*(2), 119-133.

Inglis, M., & Simpson, A. (2009, July). The defective and material conditionals in mathematics: does it matter?. In *Psychology of mathematics education.* (Vol. 3, pp. 225-232). International Group for the Psychology of Mathematics Education.

Keene, K. A., Hall, W., & Duca, A. (2014). Sequence limits in calculus: using design research and building on intuition to support instruction. *ZDM*, *46*(4), 561-574.

Kidron, I., & Zehavi, N. (2002). The Role of Animation in Teaching the Limit Concept. *International Journal of Computer Algebra in Mathematics Education*, *9*(3), 205-27.

Laborde, C. (1990). Language and mathematics. *Mathematics and cognition*, 53-69.

Mamona-Downs, J. (2001). Letting the intuitive bear on the formal; a didactical approach for the understanding of the limit of a sequence. *Educational studies in mathematics*, *48*(2-3), 259-288.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 41-51.

Mason, J. and Pimm, D. (1984). Generic examples: seeing the general in the particular. *Educational Studies in Mathematics*, 277-289.

Mason, J., & Watson, A. (2008). Mathematics as a constructive activity: Exploiting dimensions of possible variation. In M. Carlson & C. Rasmussen (Eds.), Making the connection: Research and practice in undergraduate mathematics (pp. 189–202). Washington, DC: MAA.

McGee, V. (1997). How we learn mathematical language. *Philosophical Review*, 35-68.

Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, *79*(1), 3-18.

Mejía-Ramos, J. P., & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In *Proceedings of the ICMI study 19 conference: Proof and proving in mathematics education* (Vol. 2, pp. 88-93).

Monaghan, J. (1991). Problems with the language of limits. *For the learning of mathematics*, 20-24.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in mathematics*, *27*(3), 249-266.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, 396-426.

Parks, V. W. (1995). Impact of a laboratory approach supported by Mathematica on the conceptualization of limit in a first calculus course.

Patel, R. M., McCombs, P., & Zollman, A. (2014). Metaphor Clusters: Characterizing Instructor Metaphorical Reasoning on Limit Concepts in College Calculus. *School Science and Mathematics*, *114*(5), 236-245.

Pinto, M. M. F. (1998). Students' understanding of real analysis. *Unpublished PhD Thesis, Warwick University*.

Pinto, M. F., & Tall, D. (1999). Student constructions of formal theory: giving and extracting meaning. In *PME CONFERENCE* (Vol. 4, pp. 4-65).

Pinto, M., & Tall, D. (2002). Building formal mathematics on visual imagery: A case study and a theory. *For the learning of mathematics*, 2-10.

Raman, M. (2003). Key ideas: what are they and how can they help us understand how people view proof?. *Educational Studies in Mathematics*, *52*(3), 319-325.

Richardson, K., Carter, T., & Berenson, S. (2010). Connected tasks: The building blocks of reasoning and proof. *Australian Primary Mathematics Classroom*, *15*(4), 17.

Segal, J. (1999). Learning about mathematical proof: Conviction and validity. *The Journal of Mathematical Behavior*, *18*(2), 191-210.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(2), 123-151.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem?. *Journal for research in mathematics education*, 4-36.

Sfard, A. (1987). The conceptions of mathematical notions: Operational and structural. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), Proceedings of the 11th PME International Conference, 3, 162-169.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), Proceedings of the 13 PME International Conference, 3, 151-158.

Sfard, A. (1991). On the dual nature of mathematical conceptions, Educational Studies in Mathematics, 22, 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and quandary of reification - The case of function. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy.

Sierpińska, A. (1987). Humanities students and epistemological obstacles related to limits. Educational Studies in Mathematics, 18, 371–397.

Simpson, A. (1995). Developing a proving attitude. *Conference proceedings: Justifying and Proving in School Mathematics,* Institute of Education, University of London, London, pp. 39-46.

Stylianides, G. J., & Stylianides, A. J. (2008). Proof in school mathematics: Insights from psychological research into students' ability for deductive reasoning. *Mathematical Thinking and Learning*, *10*(2), 103-133.

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education*, 314-352.

Swinyard, C. (2011). Reinventing the formal definition of limit: The case of Amy and Mike. *The Journal of Mathematical Behavior*, *30*(2), 93-114.

Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education*, 258-276.

Tall, D. (1980). Mathematical intuition, with special reference to limiting processes. In *Proceedings of the Fourth International Conference for the Psychology of Mathematics Education* (pp. 170-176).

Tall, D. (1990). Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in mathematics, 12*(3), 49-63.

Tall, D. (1991). Intuition and rigour: the role of visualization in the calculus. *Visualization in teaching and learning mathematics*, 105-119.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. *Handbook of research on mathematics teaching and learning*, 495-511.

Tall, D. (1993). Students' difficulties in calculus. In *proceedings of working group* (Vol. 3, pp. 13-28).

Tall, D. (1995, July). Cognitive growth in elementary and advanced mathematical thinking. In *PME conference* (Vol. 1, pp. 1-61). THE PROGRAM COMMITTEE OF THE 18TH PME CONFERENCE.

Tall, D. (2004, July). Thinking through three worlds of mathematics. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 281-288).

Tall, D. (2013). *How Humans Learn to Think Mathematically: Exploring the three worlds of mathematics*. Cambridge University Press.

Tall, D., McGowen, M., & DeMarois, P. (2000) The Function Machine as a Cognitive Root for Building a Rich Concept Image of the Function Concept. *Proceedings of PME-NA, 1, 255-261*.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics*, *12*(2), 151-169.

Toulmin, S. E. (2003). *The uses of argument*. Cambridge University Press.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In *Advanced mathematical thinking* (pp. 65-81). Springer Netherlands.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), Proceedings of the 4th PME International Conference , 177-184.

Yang, K. L., & Lin, F. L. (2008). A model of reading comprehension of geometry proof. *Educational Studies in Mathematics*, *67*(1), 59-76.

Yoo, S. (2008). *Effects of traditional and problem-based instruction on conceptions of proof and pedagogy in undergraduates and prospective mathematics teachers* (Doctoral Dissertation). Retrieved from the University of Texas Libraries Digital Repository (http://repositories.lib.utexas.edu/handle/2152/17834)

Watson, A., & Mason, J. (2002). Extending example spaces as a learning/teaching strategy in mathematics. In *PME CONFERENCE* (Vol. 4, pp. 4-377).

Watson, A., & Mason, J. (2005). Mathematics as a constructive activity: Learners generating examples. Mahwah, NJ: Erlbaum.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101-119.

Weber, K. (2002). Beyond proving and explaining: Proofs that justify the use of definitions and axiomatic structures and proofs that illustrate technique. *For the learning of Mathematics*, 14-17.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor’s lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior*, *23*(2), 115-133.

Weber, K. (2010). Mathematics' majors perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning 12*, 306-336.

Weber, K. (2014). What is Proof? A linguistic answer to an educational question.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational studies in mathematics*, *56*(2-3), 209-234.

Weber, K., & Alcock, L. (2005). Using warranted implications to understand and validate proofs. *For the Learning of Mathematics 25*(1), 34–38.

Weber, K., & Mejia-Ramos, J. P. (2014). Mathematics majors' beliefs about proof reading. *International Journal of Mathematical Education in Science and Technology*, *45*(1), 89-103.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for research in Mathematics Education*, 219-236.

Williams, S. R. (2001). Predictions of the limit concept: An application of repertory grids. *Journal for Research in Mathematics Education*, *32*, 341-367.

Wheeler, D. (1990). Aspects of mathematical proof. *Interchange*, *21*(1), 1-5.

Zazkis, R., & Leikin, R. (2007). Generating examples: From pedagogical tool to a research tool. *For the learning of mathematics*, 15-21.