Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 108, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF INFINITELY MANY SOLUTIONS FOR SINGULAR SEMILINEAR PROBLEMS ON EXTERIOR DOMAINS

JOSEPH A. IAIA

Abstract. In this article we prove the existence of infinitely many radial solutions of $\Delta u+K(r)f(u)=0$ on the exterior of the ball of radius $R>0$, B_R, centered at the origin in $\mathbb{R}^N$ with $u=0$ on ∂B_R and $\lim_{r\to\infty}u(r)=0$ where $N>2$, f is odd with $f<0$ on $(0,\beta)$, $f>0$ on (β,∞), f is superlinear for large u, $f(u)\sim -1/(|u|^{q-1}u)$ with $0<q<1$ for small u, and $0<K(r)\le K_1/r^\alpha$ with $N+q(N-2)<\alpha<2(N-1)$ for large r.

1. Introduction

In this article we study radial solutions of

$$\Delta u+K(r)f(u)=0 \quad \text{in } \mathbb{R}^N\backslash B_R, \tag{1.1}$$

$$u=0 \quad \text{on } \partial B_R, \tag{1.2}$$

$$u\to 0 \quad \text{as } |x|\to\infty \tag{1.3}$$

where B_R is the ball of radius $R>0$ centered at the origin in $\mathbb{R}^N$ and $K(r)>0$. We assume that

(H1) $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is locally Lipschitz, f is odd, $f<0$ on $(0,\beta)$, $f>0$ on (β,∞),

$$f(u)=-\frac{1}{|u|^{q-1}u}+g(u)$$

with $0<q<1$ and $g(0)=0$.

(H2) there exists p with $p>1$ such that

$$f(u)=|u|^{p-1}u+g_1(u), \quad \text{where } \lim_{u\to\infty}\frac{|g_1(u)|}{|u|^p}=0.$$

We let $F(u)=\int_0^u f(s)\,ds$. Since f is odd it follows that F is even and from (H1) it follows that F is bounded below by $-F_0<0$, F has a unique positive zero, γ, with $0<\beta<\gamma$, and

(H3) $-F_0<F<0$ on $(0,\gamma)$, and $F>0$ on (γ,∞).

2010 *Mathematics Subject Classification.* 34B40, 35B05.
Key words and phrases. Exterior domain; semilinear; singular; superlinear; radial solution.

Submitted March 9, 2018. Published September 23, 2019.

Since we are interested in radial solutions of (1.1)-(1.3), we assume that $u(x) = u(|x|) = u(r)$ where $x \in \mathbb{R}^N$ and $r = |x| = \sqrt{x_1^2 + \dots + x_N^2}$ with $r > R > 0$ so that u satisfies

$$u'' + \frac{N-1}{r} u' + K(r) f(u) = 0 \quad \text{on } (R, \infty), \tag{1.4}$$

$$u(R) = 0, \lim_{r \to \infty} u(r) = 0. \tag{1.5}$$

We also assume K is continuously differentiable and $K(r) > 0$ on $[R, \infty)$. In addition, we assume there exist positive constants α and C_1 such that

(H4) $0 < K(r) \le C_1 / r^\alpha$ on $[R, \infty)$ where $\alpha > N + q(N-2)$,
(H5) $2(N-1) + \frac{rK'}{K} \ge 0$.

We note that solutions of (1.4)-(1.5) will not be twice differentiable at any points where $u = 0$ because of the singularity of f at $u = 0$. Therefore multiplying (1.4) by r^{N-1} and integrating on (R, r) gives

$$r^{N-1} u' = R^{N-1} u'(R) - \int_R^r t^{N-1} K(t) f(u) \, dt. \tag{1.6}$$

So in this article by a solution of (1.4) we mean a $u \in C^1[R, \infty) \cap C^0[R, \infty)$ that satisfies (1.6). In this article we prove the following result.

Theorem 1.1. *Let $N > 2$ and assuming* (H1)–(H5). *Then there exist infinitely many radial functions $u \in C^1[R, \infty) \cap C^0[R, \infty)$ which satisfy* (1.5)-(1.6) *on* $[R, \infty)$.

A number of papers have been written on this and similar topics. Some have used sub/super solutions, degree theory, or critical point theory to prove existence of a positive solution [5, 6, 12, 13, 15]. Here we prove the existence of an *infinite* number of solutions as in [1, 2, 7, 8, 9, 10, 11, 14, 16].

In section two we prove the main lemmas for this paper. In particular, we show that if a particular parameter $a > 0$ is sufficiently small then u_a stays positive on (R, ∞). And we also show that if a is sufficiently large then u_a has a large number of zeros on (R, ∞). We use these facts in section three to prove the main theorem.

2. Preliminaries

We begin by first making the substitution $t = r^{2-N}$ and letting $u(r) = v(r^{2-N})$ in (1.4)-(1.5). This gives

$$v'' + h(t) f(v) = 0 \quad \text{on } (0, R^{2-N}), \tag{2.1}$$

$$\lim_{t \to 0^+} v(t) = 0, \quad v(R^{2-N}) = 0, \tag{2.2}$$

where

$$h(t) = \frac{t^{-\frac{2(N-1)}{N-2}} K(t^{-\frac{1}{N-2}})}{(N-2)^2}. \tag{2.3}$$

It follows from (H4) and (H5) that

$$h > 0 \text{ and } h' \le 0 \quad \text{on } (0, R^{2-N}]. \tag{2.4}$$

We now consider the initial value problem

$$v_a'' + h(t) f(v_a) = 0 \quad \text{for } t > 0, \tag{2.5}$$

$$\lim_{t \to 0^+} v_a(t) = 0, \quad \lim_{t \to 0^+} v_a'(t) = a > 0. \tag{2.6}$$

We attempt to find values of $a > 0$ for which $v_a(R^{2-N}) = 0$ for then $u_a(r) = v_a(r^{2-N})$ solves (1.5)-(1.6).

Assuming there is a solution of (2.5)-(2.6) then integrating (2.5) on $(0, t)$ and using (2.6) gives

$$v_a'(t) = a - \int_0^t h(x) f(v_a(x))\, dx. \tag{2.7}$$

Integrating again gives

$$v_a(t) = at - \int_0^t \int_0^s h(x) f(v_a(x))\, dx\, ds. \tag{2.8}$$

Letting $v_a(t) = t y_a(t)$, (2.8) becomes

$$y_a(t) = a - \frac{1}{t} \int_0^t \int_0^s h(x) f(x y_a(x))\, dx\, ds. \tag{2.9}$$

We will show that there is a continuously differentiable solution of (2.9) (and thus of (2.8)) on $[0, \epsilon]$ for some $\epsilon > 0$.

Lemma 2.1. *Let $N > 2$ and assume* (H1)–(H5) *hold. Then there exists an $\epsilon > 0$ and a unique solution of* (2.8) *on* $[0, \epsilon]$.

Proof. Let $\epsilon > 0$ and $a > 0$. Also let

$$A = \{y \in C[0, \epsilon] : y(0) = a \text{ and } \|y - a\| < \frac{a}{2}\} \tag{2.10}$$

where $C[0, \epsilon]$ is the set of continuous functions on $[0, \epsilon]$ with the supremum norm, $\|\cdot\|$. Next using (2.9) we define $X : A \to C[0, \epsilon]$ by

$$Xy(t) = \begin{cases} a & \text{for } t = 0 \\ a - \frac{1}{t} \int_0^t \int_0^s h(x) f(x y(x))\, dx\, ds & \text{for } t > 0. \end{cases} \tag{2.11}$$

Let

$$\tilde{\alpha} = \frac{2(N-1) - \alpha}{N-2}. \tag{2.12}$$

By (H4) we have $K(r) \le \frac{C_1}{r^\alpha}$ on $[R, \infty)$ then by (2.3) and (2.12) it follows that

$$h(t) \le \frac{C_2}{t^{\tilde{\alpha}}} \quad \text{on } (0, R^{2-N}] \tag{2.13}$$

where $C_2 = \frac{C_1}{(N-2)^2}$. Then since $\alpha > N + q(N-2)$ (by (H4)) we see that

$$q + \tilde{\alpha} < 1 \quad \text{and} \quad \int_0^t x^{-q} h(x)\, dx \le C_3 t^{1-q-\tilde{\alpha}} \quad \text{on } (0, R^{2-N}] \tag{2.14}$$

where $C_3 = \frac{C_2}{1-q-\tilde{\alpha}}$.

Assuming $0 \le t \le 1$ we let L be the Lipschitz constant for g on $[-2a, 2a]$ and let $y_a \in A$. Next using (2.11)-(2.14) and (H1) we have

$$\begin{aligned} |Xy(t) - a| &\le \frac{1}{t} \int_0^t \int_0^s \left(x^{-q} h(x) y_a^{-q}(x) + h(x) |g(x y_a(x))| \right) dx\, ds \\ &\le \int_0^t \left(\frac{2}{a}\right)^q x^{-q} h(x)\, dx + \int_0^t 2aLx h(x)\, dx \\ &\le \left(\frac{2}{a}\right)^q C_3 t^{1-q-\tilde{\alpha}} + \frac{2aC_2L}{2-\tilde{\alpha}} t^{2-\tilde{\alpha}} \end{aligned}$$

$$
\begin{aligned}
&\le (\frac{2}{a})^q C_3 \epsilon^{1-q-\tilde{\alpha}} + \frac{2aC_2L}{2-\tilde{\alpha}}\epsilon^{2-\tilde{\alpha}} \\
&< \frac{a}{2} \quad \text{if } \epsilon \text{ is sufficiently small.}
\end{aligned}
$$

Thus $X : A \to A$ if ϵ is sufficiently small. Suppose next that $y_1, y_2 \in A$ and $0 \le t \le 1$. Then

$$
Xy_1 - Xy_2 = -\frac{1}{t}\int_0^t \int_0^s h(x)\,(f(xy_1(x)) - f(xy_2(x))\;dx\,ds \tag{2.15}
$$

and therefore by (H1),

$$
|Xy_1 - Xy_2| \le \int_0^t x^{-q}h(x)|y_1^{-q} - y_2^{-q}|\,dx + \int_0^t 2aLxh(x)|y_1 - y_2|\,dx. \tag{2.16}
$$

By the mean value theorem and the fact that $y_1, y_2 \in A$ we see that

$$
|y_1^{-q} - y_2^{-q}| \le q\big(\frac{2}{a}\big)^{q+1}|y_1 - y_2|.
$$

Thus

$$
|Xy_1 - Xy_2| \le \|y_1 - y_2\| \int_0^t \Big(\big(\frac{2}{a}\big)^{q+1} q x^{-q}h(x) + 2aLxh(x)\Big)\,dx. \tag{2.17}
$$

Since $x^{-q}h(x)$ and $xh(x)$ are integrable near $t = 0$ (by (2.13)-(2.14)) then we see the integral term in (2.17) gets arbitrarily small as $t \to 0^+$ and so there exists an $\epsilon > 0$ and $0 \le c < 1$ such that for $0 \le t \le \epsilon$ and ϵ sufficiently small we have

$$
|Xy_1 - Xy_2| \le c\|y_1 - y_2\|.
$$

Thus we see X is a contraction. Hence by the contraction mapping principle [3] there is a unique fixed point y_a of (2.11) and thus a solution $v_a(t) = ty_a(t)$ of (2.8) on $[0, \epsilon]$. □

Lemma 2.2. *Let $N > 2$ and assume* (H1)–(H5) *hold. Then the solution v_a of* (2.8) *exists on* $(0, R^{2-N}]$.

Proof. Consider

$$
E_a = \frac{1}{2}\frac{v_a'^2}{h} + F(v_a). \tag{2.18}
$$

Using (2.1) and (2.4) we see that

$$
E_a' = -\frac{v_a'^2 h'}{h^2} \ge 0. \tag{2.19}
$$

From (2.6) we see $\lim_{t\to 0^+} E_a(t) \ge 0$ thus

$$
E_a > 0 \quad \text{for } t > 0. \tag{2.20}
$$

Similarly it follows using (2.1) and (2.6) that

$$
\frac{1}{2}v_a'^2 + hF(v_a) = \frac{1}{2}a^2 + \int_0^t h'(x)F(v_a)\,dx. \tag{2.21}
$$

Now for $t \ge \epsilon$ (where ϵ is from Lemma 2.1) we have

$$
\frac{1}{2}v_a'^2 + hF(v_a) = \frac{1}{2}v_a'^2(\epsilon) + h(\epsilon)F(v_a(\epsilon)) + \int_\epsilon^t h'(x)F(v_a)\,dx.
$$

Then since $F \geq -F_0$ by (H3) and $h' \leq 0$ by (2.4) we see that

$$\begin{aligned}\frac{1}{2}v_a'^2 - hF_0 \leq \frac{1}{2}v_a'^2 + hF(v_a) &= \frac{1}{2}v_a'^2(\epsilon) + h(\epsilon)F(v_a(\epsilon)) + \int_\epsilon^t h'(x)F(v_a)\,dx \\ &\leq \frac{1}{2}v_a'^2(\epsilon) + h(\epsilon)F(v_a(\epsilon)) - F_0(h - h(\epsilon)).\end{aligned}$$

Thus

$$\frac{1}{2}v_a'^2 \leq \frac{1}{2}v_a'^2(\epsilon) + h(\epsilon)[F(v_a(\epsilon)) + F_0] \quad \text{for } t \geq \epsilon. \tag{2.22}$$

It follows from Lemma 2.1 that $v_a(\epsilon)$ and $v_a'(\epsilon)$ are finite and so we see by (2.22) that v_a and v_a' are uniformly bounded on $[\epsilon, R^{2-N}]$ from which it follows that v_a and v_a' are defined on $[\epsilon, R^{2-N}]$. Combining this with Lemma 2.1 it follows that v_a and v_a' are defined on all of $[0, R^{2-N}]$ for all $a > 0$. This completes the proof. □

Note that if v_a is a solution of (2.8) and there exists a $z_a \in (0, R^{2-N}]$ such that $v_a(z_a) = 0$, then it follows from (2.20) that

$$0 < E_a(z_a) = \frac{1}{2}\frac{v_a'^2(z_a)}{h(z_a)}$$

and therefore $v_a'(z_a) \neq 0$.

Lemma 2.3. *Let $N > 2$ and assume* (H1)–(H5) *hold. Suppose v_a solves* (2.8). *Then the functions $\{v_a\}$ vary continuously with $a > 0$ on* $[0, R^{2-N}]$.

Proof. Let $0 < \underline{a} < \overline{a}$. We consider the set of solutions y_a of (2.9) such that $\|y_a - a\| < \frac{\underline{a}}{2}$ and $0 < \underline{a} \leq a \leq \overline{a}$. From (2.17) it follows that for all a with $\underline{a} \leq a \leq \overline{a}$ there is a common $\epsilon > 0$ such that the corresponding mapping X_a from Lemma 2.1 is a contraction on $[0, \epsilon]$. Then for $0 \leq t \leq 1$ and for $\underline{a} \leq a_1 < a_2 \leq \overline{a}$ it follows from (2.8),

$$y_{a_1} - y_{a_2} = a_1 - a_2 - \frac{1}{t}\int_0^t \int_0^s h(x)[f(xy_{a_1}) - f(xy_{a_2})]\,dx\,ds.$$

Estimating as we did in (2.17) we see

$$|y_{a_1} - y_{a_2}| \leq |a_1 - a_2| + \int_0^t \Big(\Big(\frac{2}{\underline{a}}\Big)^{q+1} x^{-q}h(x) + 2\overline{a}Lxh(x)\Big)|y_{a_1} - y_{a_2}|\,dx.$$

Using the Gronwall inequality [5] we then obtain

$$|y_{a_1} - y_{a_2}| \leq |a_1 - a_2|\Big(\Big(\frac{2}{\underline{a}}\Big)^{q+1}\frac{C_2}{1 - \tilde{\alpha} - q}e^{t^{1-\tilde{\alpha}-q}} + 2\overline{a}Le^{t^{1-\tilde{\alpha}}}\Big) \text{ on } [0, \epsilon]$$

and therefore

$$|v_{a_1} - v_{a_2}| \leq |a_1 - a_2|t\Big(\Big(\frac{2}{\underline{a}}\Big)^{q+1}\frac{C_2}{1 - \tilde{\alpha} - q}e^{t^{1-\tilde{\alpha}-q}} + 2\overline{a}Le^{t^{1-\tilde{\alpha}}}\Big) \quad \text{on } [0, \epsilon]. \tag{2.23}$$

Thus we see the $\{v_a\}$ varies continuously on $[0, \epsilon]$ for all $a \in [\underline{a}, \overline{a}]$.

More generally now let $a^* > 0$. We want to show that $v_a \to v_{a^*}$ uniformly on $[0, R^{2-N}]$ as $a \to a^*$. So suppose not. Then there exists an $\epsilon_1 > 0$, a sequence $x_j \in [0, R^{2-N}]$, and a subsequence v_{a_j} such that

$$|v_{a_j}(x_j) - v_{a^*}(x_j)| \geq \epsilon_1 \text{ for all } j. \tag{2.24}$$

However it follows from comments at the beginning of the proof of this lemma that the v_{a_j} and v'_{a_j} are uniformly bounded on $[0,\epsilon]$ for all a_j sufficiently close to a^* and then from (2.22) we see that the v_{a_j} and v'_{a_j} are uniformly bounded on $[0,R^{2-N}]$ for all a_j sufficiently close to a^*. Then by the Arzela-Ascoli theorem there is a subsequence of the v_{a_j}, say $v_{a_{j_k}}$, such that $v_{a_{j_k}} \to v^*$ uniformly on $[0,R^{2-N}]$ which contradicts (2.24). This completes the proof. $\square$

Lemma 2.4. *Let $N > 2$ and assume* (H1)–(H5) *hold. Then v_a has only have a finite number of local extrema on $[0,R^{2-N}]$. In addition, $\|v_a\| = \max_{[0,R^{2-N}]} |v_a| \to \infty$ as $a \to \infty$. Further, if v_a has a local maximum, M_a, with $v'_a > 0$ on $(0,M_a)$ then $v_a(M_a) \to \infty$ as $a \to \infty$.*

Proof. First, if $M_n \in (0,R^{2-N}]$ were distinct local extrema for v_a then a subsequence (still labeled M_n) would converge to some $M^* \in [0,R^{2-N}]$ and it would follow that $v'_a(M^*) = 0$. Since $\lim_{t\to 0^+} v'_a(t) = a > 0$ then $M^* > 0$. Also by the mean value theorem

$$0 = v'_a(M_k) - v'_a(M_{k+1}) = v''_a(c_k)(M_k - M_{k+1})$$

with c_k between M_k and M_{k+1} (and in particular $c_k \neq 0$) and thus $v''_a(c_k) = 0$ so by (2.1) we see $f(v_a(c_k)) = 0$. Since $M_k \to M^*$ then we also have $c_k \to M^*$ and thus $f(v_a(M^*)) = 0$ so $v_a(M^*) = 0$ or $\pm\beta$. This along with $v'_a(M^*) = 0$ implies by (H3) and (2.20) that $0 < E(M^*) = F(\beta) < 0$ or $0 < E(M^*) = F(0) = 0$ so in either case we get a contradiction. Thus v_a has only a finite number of extrema on $[0,R^{2-N}]$.

Next we show that

$$\|v_a\| = \max_{[0,R^{2-N}]} |v_a| \to \infty \quad \text{as } a \to \infty. \tag{2.25}$$

We assume by the way of contradiction that $|v_a| \le Q$ on $[0,R^{2-N}]$.

First we rewrite (2.1) as $(tv'_a - v_a)' = -th(t)f(v_a)$ and so integrating on $(0,t)$ gives $tv'_a - v_a = -\int_0^t xh(x)f(v_a)\,dx$. Thus $(\frac{v_a}{t})' = -\frac{1}{t^2}\int_0^t xh(x)f(v_a)\,dx$ and so

$$v_a = at - t\int_0^t \frac{1}{t^2}\int_0^s xh(x)f(v_a)\,dx\,ds \tag{2.26}$$

Case 1: $v_a > 0$ on $(0,R^{2-N}]$. It follows from (H1) that $|g(v)| \le C_4|v|^p + C_5$ for all v for some constants C_4 and C_5. After rewriting and estimating (2.26) using (H1) and that $v_a > 0$ gives

$$\begin{aligned} at &= v_a + t\int_0^t \frac{1}{s^2}\int_0^s xh(x)f(v_a)\,dx\,ds \\ &\le v_a + t\int_0^t \frac{1}{s^2}\int_0^s xh(x)g(v_a)\,dx\,ds \\ &\le Q + t\int_0^t \frac{1}{s^2}\int_0^s xh(x)(C_4Q^p + C_5)\,dx\,ds \\ &\le Q + \frac{C_2(C_4Q^p + C_5)}{(1-\tilde\alpha)(2-\tilde\alpha)}t^{2-\tilde\alpha}. \end{aligned} \tag{2.27}$$

Now let $t = R^{2-N}$ in (2.27) and we obtain

$$aR^{2-N} \leq Q + \frac{C_2(C_4Q^p + C_5)}{(1-\tilde{\alpha})(2-\tilde{\alpha})} R^{(2-N)(2-\tilde{\alpha})} \tag{2.28}$$

which gives a contradiction because the right-hand side is bounded but the left-hand side goes to ∞ as $a \to \infty$. This completes Case 1.

Case 2: There exists z_a with $0 < z_a < R^{2-N}$ such that $v_a(z_a) = 0$ and $v_a > 0$ on $(0, z_a)$. In this case we see v_a has a local maximum, M_a, with $0 < M_a < z_a \leq R^{2-N}$ and letting $t = M_a$ in (2.27) we obtain

$$aM_a \leq Q + \frac{C_2(C_4Q^p + C_5)}{(1-\tilde{\alpha})(2-\tilde{\alpha})} M_a^{2-\tilde{\alpha}} \leq Q + \frac{C_2(C_4Q^p + C_5)}{(1-\tilde{\alpha})(2-\tilde{\alpha})} R^{(2-N)(2-\tilde{\alpha})}. \tag{2.29}$$

If $M_a \geq d_0 > 0$ for all sufficiently large a then left-hand side of (2.29) goes to infinity as $a \to \infty$ but the right-hand side does not. Thus $\max_{[0,R^{2-N}]} |v_a| \geq v_a(M_a) \to \infty$ as $a \to \infty$.

Thus the only case left to consider is if $M_a \to 0$ as $a \to \infty$. So by way of contradiction suppose that the $v_a(M_a)$ are bounded by some constant Q and that $M_a \to 0$ as $a \to \infty$. Then integrating (2.5) on $[t, M_a]$ gives

$$v_a'(t) = \int_t^{M_a} h(x) f(v_a(x))\,dx \leq \int_t^{M_a} h(x) g(v_a(x))\,dx.$$

Integrating on $[0, M_a]$ and using the Lipschitz constant L_2 for $g(v)$ on $[0, Q]$ gives

$$\begin{aligned} v_a(M_a) &= \int_0^{M_a} \int_t^{M_a} h(x) f(v_a(x))\,dx\,dt \\ &\leq \int_0^{M_a} \int_t^{M_a} h(x) g(v_a(x))\,dx\,dt \\ &\leq L_2 v_a(M_a) \int_0^{M_a} \int_t^{M_a} h(x)\,dx\,dt. \end{aligned}$$

Then using (2.13) and that $v_a(M_a) > 0$ we obtain

$$1 \leq L_2 \int_0^{M_a} \int_t^{M_a} h(x)\,dx\,dt \leq \frac{L_2 C_2}{2-\tilde{\alpha}} M_a^{2-\tilde{\alpha}}. \tag{2.30}$$

Thus since $\tilde{\alpha} < 1$ (by (2.14)) then the right-hand side of (2.30) goes to zero (since we are assuming $M_a \to 0$) but the left-hand side does not. Thus we obtain a contradiction and so in Case 2 we see as well that $\max_{[0,R^{2-N}]} |v_a| \geq v_a(M_a) \to \infty$ as $a \to \infty$.

Thus in all cases we see that $\|v_a\| = \max_{[0,R^{2-N}]} |v_a| \to \infty$ as $a \to \infty$. This completes the proof. □

Lemma 2.5. *Let $N > 2$ and assume* (H1)–(H5) *hold. Then if $a > 0$ is sufficiently large then v_a has a local maximum, M_a, with $v_a' > 0$ on $(0, M_a)$. In addition, $M_a \to 0$ as $a \to \infty$.*

Proof. We first define t_a as the smallest value of t (if one exists) such that $v_a(t_a) = \beta$ and $0 < v_a < \beta$. We see then that $f(v_a) \leq 0$ on $(0, t_a)$ and thus $v_a'' \geq 0$ on $(0, t_a)$. It then follows that $v_a \geq at$ here. Thus we see v_a gets larger than β on $[0, R^{2-N}]$ if a is sufficiently large. Then letting $t = t_a$ in this inequality we see $\beta \geq at_a$ and therefore

$$t_a \to 0 \quad \text{as } a \to \infty. \tag{2.31}$$

Next we show v_a has a local maximum if a is sufficiently large. So suppose not. Then v_a is increasing on $[0, R^{2-N}]$ for sufficiently large a and since $v_a(0) = 0$ it also follows that $v_a > 0$ on $(0, R^{2-N}]$. From (2.25) we see that $v_a(R^{2-N}) = \max_{[0,R^{2-N}]} |v_a| \to \infty$ as $a \to \infty$. Then from (2.5) it follows that $v_a'' \leq 0$ on $[t_a, R^{2-N}]$ thus v_a is concave down here and therefore

$$v_a\Big(\frac{t_a + R^{2-N}}{2}\Big) \geq \frac{v_a(R^{2-N}) + \beta}{2} \to \infty \quad \text{as } a \to \infty. \tag{2.32}$$

Now let

$$A_a = \min_{[\frac{t_a+R^{2-N}}{2},\, R^{2-N}]} \frac{h(t)f(v_a)}{v_a}. \tag{2.33}$$

Since $h(t) > 0$ is continuous on $[\frac{1}{2}R^{2-N}, R^{2-N}] \supset [\frac{t_a+R^{2-N}}{2}, R^{2-N}]$ it follows that $h(t)$ is bounded from below by a positive constant on $[\frac{1}{2}R^{2-N}, R^{2-N}]$. Also from (H1) we see that $f(v)$ is superlinear and so by (2.32)-(2.33) and the fact that v_a is increasing on $[\frac{t_a+R^{2-N}}{2},\, R^{2-N}]$ we see $\frac{f(v_a)}{v_a} \to \infty$ uniformly for $t \in [\frac{t_a+R^{2-N}}{2},\, R^{2-N}]$. Thus

$$\lim_{a\to\infty} A_a = \infty. \tag{2.34}$$

Next we apply the Sturm comparison theorem [4]. We consider

$$v_a'' + \Big(\frac{h(t)f(v_a)}{v_a}\Big)v_a = 0 \tag{2.35}$$

and

$$z'' + A_a z = 0 \tag{2.36}$$

where

$$v_a\Big(\frac{t_a + R^{2-N}}{2}\Big) = z\Big(\frac{t_a + R^{2-N}}{2}\Big) > \beta, \quad v_a'\Big(\frac{t_a + R^{2-N}}{2}\Big) = z'\Big(\frac{t_a + R^{2-N}}{2}\Big) > 0.$$

By way of contradiction we assume now that $v_a > 0$ on $(0, R^{2-N}]$. Since $z'' + A_a z = 0$ and $z \not\equiv 0$ then we know z is a linear combination of $\sin(\sqrt{A_a}t)$ and $\cos(\sqrt{A_a}t)$. In particular, any interval of length $\frac{\pi}{\sqrt{A_a}}$ contains a zero of $z(t)$. Thus there exists a $z_0 > 0$ with $z(z_0) = 0$, $z(t) > 0$ on $[\frac{t_a+R^{2-N}}{2}, z_0)$, and

$$\frac{t_a + R^{2-N}}{2} < z_0 < \frac{t_a + R^{2-N}}{2} + \frac{\pi}{\sqrt{A_a}}.$$

Since $\frac{1}{\sqrt{A_a}} \to 0$ by (2.34) and $t_a \to 0$ by (2.31) as $a \to \infty$ it follows that $z_0 < R^{2-N}$ if a is sufficiently large. Now multiplying (2.35) by z, (2.36) by v_a, and subtracting gives

$$(v_a'z - v_a z')' + \Big(\frac{h(t)f(v_a)}{v_a} - A_a\Big)v_a z = 0. \tag{2.37}$$

By assumption $\big(\frac{h(t)f(v_a)}{v_a} - A_a\big)v_a z \geq 0$ on $[\frac{t_a+R^{2-N}}{2}, z_0]$ and so $(v_a'z - v_a z')' \leq 0$ on $[\frac{t_a+R^{2-N}}{2}, z_0]$. Integrating on $[\frac{t_a+R^{2-N}}{2}, t]$ with $t \leq z_0$ gives

$$v_a'z - v_a z' \leq 0 \quad \text{on } \big[\frac{t_a + R^{2-N}}{2}, z_0\big] \tag{2.38}$$

which implies $(\frac{z}{v_a})' \geq 0$ on $[\frac{t_a+R^{2-N}}{2}, z_0]$ and so after integrating we obtain $v_a \leq z$ on $[\frac{t_a+R^{2-N}}{2}, z_0]$. In particular, $v_a(z_0) \leq z(z_0) = 0$ which contradicts that $v_a > 0$ on $(0, R^{2-N}]$. Therefore if a is sufficiently large then our assumption that v_a is increasing is false and so v_a has a positive local maximum, M_a, with $t_a < M_a < R^{2-N}$ and v_a increasing on $[0, M_a)$. It then follows as in the proof of Lemma 2.3 that

$$v_a(M_a) \to \infty \quad \text{as } a \to \infty. \tag{2.39}$$

Next we show $M_a \to 0$ as $a \to \infty$. Using (2.39) and the fact that $v_a'' \le 0$ on $[\frac{t_a+M_a}{2}, M_a]$ gives

$$v_a\Big(\frac{t_a+M_a}{2}\Big) \ge \frac{v_a(M_a)+\beta}{2} \to \infty \text{ as } a \to \infty. \tag{2.40}$$

Thus we see $v_a \to \infty$ uniformly on $[\frac{t_a+M_a}{2}, M_a]$.

Next notice from (H1) and (H3) that

$$f(v) \ge c_0 v^p \quad \text{for } v \ge \gamma \text{ for some } c_0 > 0. \tag{2.41}$$

Thus

$$v'' + c_0 h(t) v^p \le v'' + h(t) f(v) = 0 \quad \text{when } v \ge \gamma. \tag{2.42}$$

It then follows that

$$\Big(\frac{v'}{v^p}\Big)' + c_0 h(t) \le 0 \quad \text{when } v \ge \gamma. \tag{2.43}$$

Integrating this on $[t, M_a]$ then integrating on $[\frac{t_a+M_a}{2}, M_a]$ and estimating gives

$$c_0 \int_{\frac{t_a+M_a}{2}}^{M_a} \int_t^{M_a} h(x)\,dx\,dt \le \frac{1}{(p-1)v^{p-1}(\frac{t_a+M_a}{2})}. \tag{2.44}$$

From (2.39)-(2.40) and since $p > 1$ (by (H1)) the right-hand side of (2.44) goes to 0 as $a \to \infty$. Also since $t_a \to 0$ as $a \to \infty$ by (2.31) it follows that

$$M_a \to 0 \quad \text{as } a \to \infty. \tag{2.45}$$

This completes the proof. □

Lemma 2.6. *Let $N > 2$ and assume* (H1)–(H5) *hold. Let n be a positive integer. If $a > 0$ is sufficiently large then v_a has n zeros on $(0, R^{2-N}]$ such that $0 < z_{1,a} < z_{2,a} < \cdots < z_{n,a}$ and $z_{n,a} \to 0$ as $a \to \infty$.*

Proof. Since $E_a(t)$ is nondecreasing we have

$$\frac{1}{2}\frac{v_a'^2}{h} + F(v_a) = E_a(t) \ge E_a(M_a) = F(v_a(M_a)). \tag{2.46}$$

Now we have $v_a > 0$ and $v_a' < 0$ on (M_a, t) for t close to M_a. We notice now that v_a cannot have a positive local minimum, m_a, on (M_a, R^{2-N}) with v_a decreasing on (M_a, m_a) for at such a point we would have $0 < v_a(m_a) < v_a(M_a)$ and since E_a is nondecreasing it follows that $F(v_a(m_a)) = E(m_a) \ge E(M_a) = F(v_a(M_a)) > 0$ and so $v_a(m_a) > \gamma$ but F is increasing (by (H1)-(H3)) for $v > \gamma$ and thus $F(v_a(m_a)) < F(v_a(M_a)$. Hence we get a contradiction.

Thus we see either v_a is decreasing and positive on $[M_a, R^{2-N}]$ or v_a has a zero on $[M_a, R^{2-N}]$. Let us suppose the former. Then rewriting (2.46) and integrating on (M_a, R^{2-N}) gives

$$\begin{aligned}
&\int_0^{v_a(M_a)} \frac{1}{\sqrt{2}\sqrt{F(v_a(M_a))-F(s)}}\,ds \\
&\geq \int_{v_a(R^{2-N})}^{v_a(M_a)} \frac{1}{\sqrt{2}\sqrt{F(v_a(M_a))-F(s)}}\,ds \\
&= \int_{M_a}^{R^{2-N}} \frac{-v_a'(t)}{\sqrt{2}\sqrt{F(v_a(M_a))-F(v_a(t))}}\,dt \\
&\geq \int_{M_a}^{R^{2-N}} \sqrt{h}\,dt.
\end{aligned} \tag{2.47}$$

Since f is superlinear and $v_a(M_a) \to \infty$ as $a \to \infty$ (by Lemma 2.5) it follows that the left-hand side of (2.47) goes to 0 as $a \to \infty$ but the right-hand side of (2.47) does not and so we obtain a contradiction. Therefore if a is sufficiently large then v_a has a zero, z_a, on (M_a, z_a). Now rewriting (2.46) and integrating on (M_a, z_a) we obtain

$$\int_0^{v_a(M_a)} \frac{1}{\sqrt{2}\sqrt{F(v_a(M_a))-F(t)}}\,dt \geq \int_{M_a}^{z_a} \sqrt{h}\,dt. \tag{2.48}$$

And again the left-hand side goes to 0 as $a \to \infty$ so therefore must the right-hand side and since we know $M_a \to 0$ from Lemma 2.5 it follows that $z_a \to 0$ as well when $a \to \infty$.

Repeating this process it follows that given any positive integer n if a is sufficiently large then v_a will have n zeros, $0 < z_1 < z_2 < \cdots < z_{n-1} < z_n < R^{2-N}$, and $z_n \to 0$ as $a \to \infty$. This completes the proof. $\square$

3. Proof of Theorem 1.1

Let

$$S_n = \{a > 0 : v_a \text{ has exactly } n \text{ zeros on } (0, R^{2-N})\}.$$

Then S_n is nonempty for some smallest value of n, say n_0, by Lemma 2.5 and S_n is bounded above by Lemma 2.6. Therefore we let

$$a_{n_0} = \sup S_{n_0}.$$

We claim that $v_{a_{n_0}}$ has exactly n_0 zeros on $(0, R^{2-N})$ and $v_{a_0}(R^{2-N}) = 0$.

First, if $v_{a_{n_0}}$ has an (n_0+1)st zero on $(0, R^{2-N})$ then by the continuous dependence on initial parameters of the $\{v_a\}$ (Lemma 2.3) and since $v'_{a_{n_0}}(z) \neq 0$ at each zero, z, of $v_{a_{n_0}}$ (by the note after Lemma 2.2) it follows that v_a will have an (n_0+1)st zero on $(0, R^{2-N})$ for a slightly smaller than a_{n_0} contradicting the definition of S_{n_0}. Similarly, if $v_{a_{n_0}}$ has fewer than n_0 zeros on $(0, R^{2-N})$ then so would v_a for a slightly larger than a_{n_0} contradicting the definition of supremum. Thus $v_{a_{n_0}}$ must have exactly n_0 zeros on $(0, R^{2-N})$. Similarly it follows that $v_{a_{n_0}}(R^{2-N}) = 0$ for if $v_{a_{n_0}}(R^{2-N}) > 0$ then by continuous dependence $v_a(R^{2-N}) > 0$ for a slightly smaller than a_{n_0} contradicting the definition of S_{n_0} and if $v_{a_{n_0}}(R^{2-N}) < 0$ then $v_a(R^{2-N}) < 0$ for a slightly larger than a_{n_0} contradicting the definition of supremum. Thus $v_{a_{n_0}}(R^{2-N}) = 0$.

Now for a slightly larger than a_{n_0}, due to continuous dependence and that $v_a'(z) \neq 0$ at each zero of v_a then v_a will have exactly $n_0 + 1$ zeros on $(0, R^{2-N})$

and therefore S_{n_0+1} will be nonempty. Again by Lemma 2.6 it follows that S_{n_0+1} will be bounded above thus we can define

$$a_{n_0+1} = \sup S_{n_0+1}$$

and similarly we show that $v_{a_{n_0+1}}$ has exactly $n_0 + 1$ zeros on $(0, R^{2-N})$ and $v_{a_{n_0+1}}(R^{2-N}) = 0$. Continuing in this way we can obtain an infinite number of solutions of (1.4)-(1.5), one with any number, n, of zeros on $(0, R^{2-N})$ for $n \geq n_0$. This completes the proof of the main theorem.

References

[1] H. Berestycki, P. L. Lions; Non-linear scalar field equations I, *Arch. Rational Mech. Anal.*, Volume 82, 313-347, 1983.

[2] H. Berestycki, P. L. Lions; Non-linear scalar field equations II, *Arch. Rational Mech. Anal.*, Volume 82, 347-375, 1983.

[3] M. Berger; *Nonlinearity and functional analysis*, Academic Free Press, New York, 1977.

[4] G. Birkhoff, G. C. Rota; *Ordinary differential equations*, Ginn and Company, 1962.

[5] F. Bauer, J. Noel; *The qualitative theory of differential equations: an introduction*, Dover, 1969.

[6] A. Castro, L. Sankar, R. Shivaji; Uniqueness of nonnegative solutions for semipositone problems on exterior domains, *Journal of Mathematical Analysis and Applications*, Volume 394, Issue 1, 432-437, 2012.

[7] M. Chhetri, L. Sankar, R. Shivaji; Positive solutions for a class of superlinear semipositone systems on exterior domains, *Boundary Value Problems,* 198-207, 2014.

[8] J. Iaia; Existence and nonexistence for semilinear equations on exterior domains, *Journal of Partial Differential Equations*, Volume 30, No. 4, 1-17, 2017.

[9] J. Iaia; Existence and nonexistence of solutions for sublinear equations on exterior domains, *Electronic Journal of Differential Equations*, No. 181, 1-14, 2018.

[10] J. Iaia; Existence of solutions for semilinear problems with prescribed number of zeros on exterior domains, *Journal of Mathematical Analysis and Applications*, 446, 591-604, 2017.

[11] C. K. R. T. Jones, T. Kupper; On the infinitely many solutions of a semi-linear equation, *SIAM J. Math. Anal.*, Volume 17, 803-835, 1986.

[12] J. Joshi; Existence and nonexistence of solutions of sublinear problems with prescribed number of zeros on exterior domains, *Electronic Journal of Differential Equations*, No. 133, 1-10, 2017.

[13] E. K. Lee, R. Shivaji, B. Son; Positive radial solutions to classes of singular problems on the exterior of a ball, *Journal of Mathematical Analysis and Applications,* 434, No. 2, 1597-1611, 2016.

[14] E. Lee, L. Sankar, R. Shivaji; Positive solutions for infinite semipositone problems on exterior domains, *Differential and Integral Equations*, Volume 24, Number 9/10, 861-875, 2011.

[15] K. McLeod, W. C. Troy, F. B. Weissler; Radial solutions of $\Delta u + f(u) = 0$ with prescribed numbers of zeros, *Journal of Differential Equations*, Volume 83, Issue 2, 368-373, 1990.

[16] L. Sankar, S. Sasi, R. Shivaji; Semipositone problems with falling zeros on exterior domains, *Journal of Mathematical Analysis and Applications*, Volume 401, Issue 1, 146-153, 2013.

[17] W. Strauss; Existence of solitary waves in higher dimensions, *Comm. Math. Phys.*, Volume 55, 149-162, 1977.

Joseph A. Iaia

Department of Mathematics, University of North Texas, P.O. Box 311430, Denton, TX 76203-1430, USA

Email address: iaia@unt.edu