THE DOUBLE SESSILE DROP

by

Haley R. King

A thesis submitted to the Graduate College of
Texas State University in partial fulfillment
of the requirements for the degree of
Master of Science
with a Major in Mathematics
May 2017

Committee Members:

Ray Treinen, Chair

Julio Dix

Gregory Passty

FAIR USE AND AUTHOR'S PERMISSION STATEMENT

Fair Use

Duplication Permission

ACKNOWLEDGEMENTS

I would like to thank my wonderful husband for his support, encouragement and patience through the years. I appreciate the sacrifices that he and my family have made for me to pursue my academic career.

I would also like to thank my thesis advisor Dr. Ray Treinen. I appreciate the balance between the knowledge and suggestions he offered and guiding me in the right direction to learn and build things on my own. Most of all I am grateful for all the time spent on this project. I would like to thank my committee members Dr. Gregory Passty and Dr. Julio Dix for their time and feedback but also for the support they have shown me as a graduate student of mathematics during my time here at Texas State University.

TABLE OF CONTENTS

Page

LIST OF FIGURES

Figure **Page**

ABSTRACT

We consider the double sessile drop, which is formed of two connected drops of liquid with prescribed volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ resting in equilibrium on a horizontal plane P in a vertical gravity field directed toward P. We suppose that the plane is made of homogeneous material so that contact angles are constant. The size and shape of each drop for any liquid is determined by the prescribed volume and the solutions for the curves that enclose the liquid.

I. BACKGROUND

In order to explore the double sessile drop, we consider previous findings for the single sessile drop. The mathematics that describes this object are the inspiration for the goal of this project: designing a computer program, that given desired prescribed quantities, will model the double drop for any two liquids.

I.1 Sessile Drop

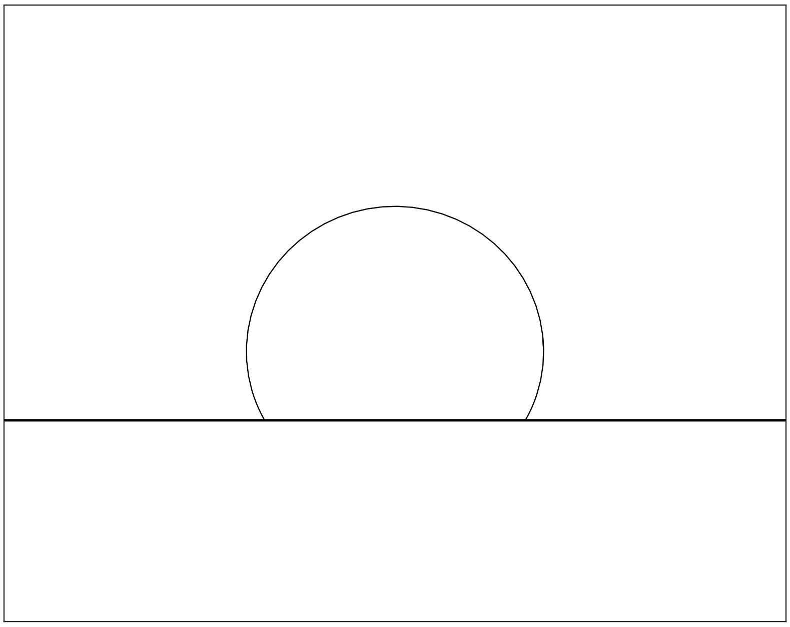

Figure I.1: Sessile drop

The standard reference is a manuscript by Finn [4]. Finn describes a tube of infinite height. The tube rests in a circular container of large diameter, so that the fluid surface level at a large distance provides a reference level $z = 0$ for atmospheric conditions that do not perturb the fluid surface of the tube. With this configuration, he limited his attention to surfaces $z = (x, y)$. These are capillary surfaces. Finn also described the fact that to every symmetric sessile

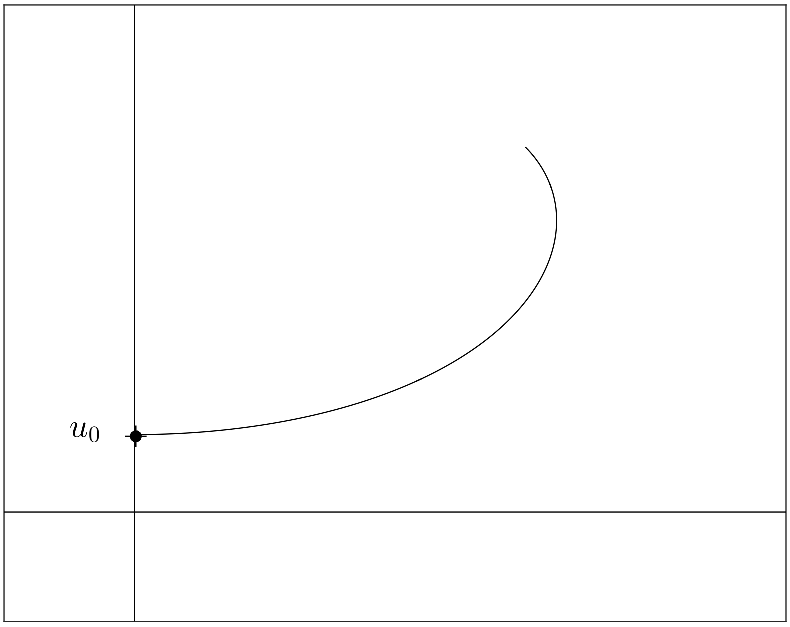

Figure I.2: Continued capillary section

drop, there corresponds a unique capillary surface. See Figures I.1 and I.2. That is, a unique interface of at least two different materials: liquids or gases, positioned adjacent to each other that do not mix such that at least one of those materials is a liquid.

Assuming symmetry, the three-dimensional drop solves the following system of differential equations parameterized by inclination angle

$$\begin{cases} \frac{dr}{d\psi} = \frac{r\cos\psi}{\kappa r u - \sin\psi}, \\ \frac{du}{d\psi} = \frac{r\sin\psi}{\kappa r u - \sin\psi}. \end{cases} \tag{I.1}$$

Finn concluded, among other interesting results, that the set of all capillary surfaces is determined by a one-parameter family of solutions to partial differential equations in terms of center height u_0.

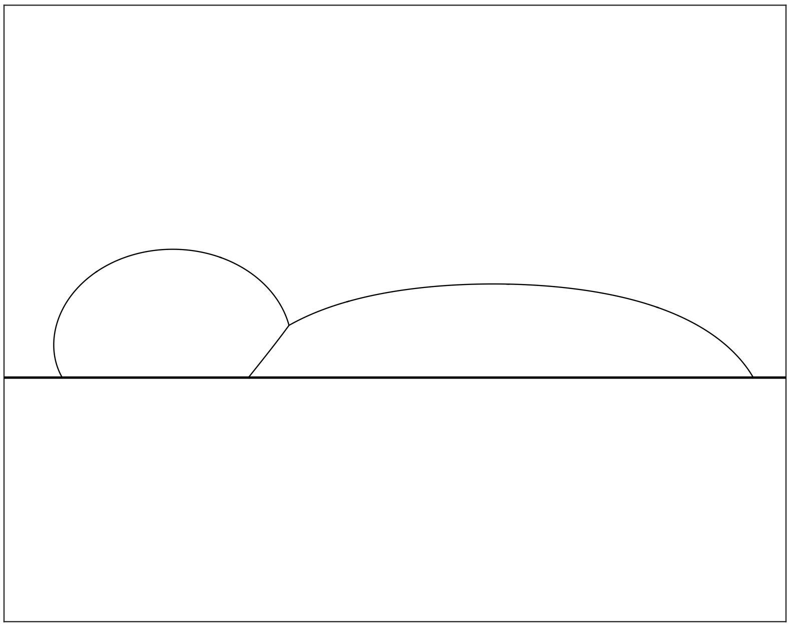

Figure I.3: Double sessile drop

I.2 Double Sessile Drop in 2D

Finn proved in [3] that this family of solutions that solved the set of all capillary surfaces are also solutions for the n-dimensional system. Thus we are able to explore the problem in terms of curvature which is the analog of mean curvature:

$$div\, Tu = \kappa u - \lambda \tag{I.2}$$

where

$$Tu = \frac{1}{\sqrt{1 - |Du|^2}} Du. \tag{I.3}$$

So we study the lower dimension problem and describe the set of all symmetric sessile drops as a one family parameter of curves in the coordinate system (x, u) of two-dimensions using instead a family of ordinary differential equations. With this view, the double sessile drop, shown in Figure I.3 , is composed of enclosed volumes E_1 and E_2 by the three parameterized arcs each of which corresponds to the family of solutions determined by each respective center height u_{ij}, obtaining solutions from a corresponding system of differential equations for the same family of curves, described as continued capillary sections by Finn [4]. We will use this coordinate system with the fluids resting on the plate in order to explore the configurations of the drops.

II. DROP CONFIGURATIONS AND VOLUME COMPUTATIONS

II.1 Drop Configurations

We now begin the steps necessary to construct the volumes of the double drop. We study the problem in a lower dimensional setting. We envision this lower dimensional problem either for its own interests, or as a model of the double sessile drop resting on a plate and trapped between two vertical planes that are a small distance from each other. We assume homogeneous boundary data on these vertical planes. In either case, we retain the intuitive language of volume, though strictly speaking, it is actually an area. Since the double drop is formed of two enclosed volumes, we will need to be able to implement a volume computation in our program. Implementing this computation will allow us to verify that the double drop we are generating matches the prescribed desired quantity. If we can define a formula for the volume contained by a single parameterized arc and its boundary, then we can later modify this result to determine the volume contained by the three arcs and boundary that form the double drop. First, we must understand the possible configurations so that we may verify our formula will work for all drop types.

If we examine the enclosed volumes of the double sessile drop as permutations of horizontal and vertical points along the arc, then we may examine all cases for our configurations. We consider the following illustrations of configurations for the ease of computing the enclosed volumes.

Let the right drop to be of a fixed type, then there are five cases for the left side of the double sessile drop:

Case 1: no horizontal point, no vertical point

Case 2: one horizontal point, no vertical point

Case 3: no horizontal point, one vertical point

Case 4: one horizontal point and one vertical point

Case 5: one horizontal point and two vertical points

See Figures II.1-II.5.

For Case 1, the left side of the drop has no maximum points or vertical points along the arc.

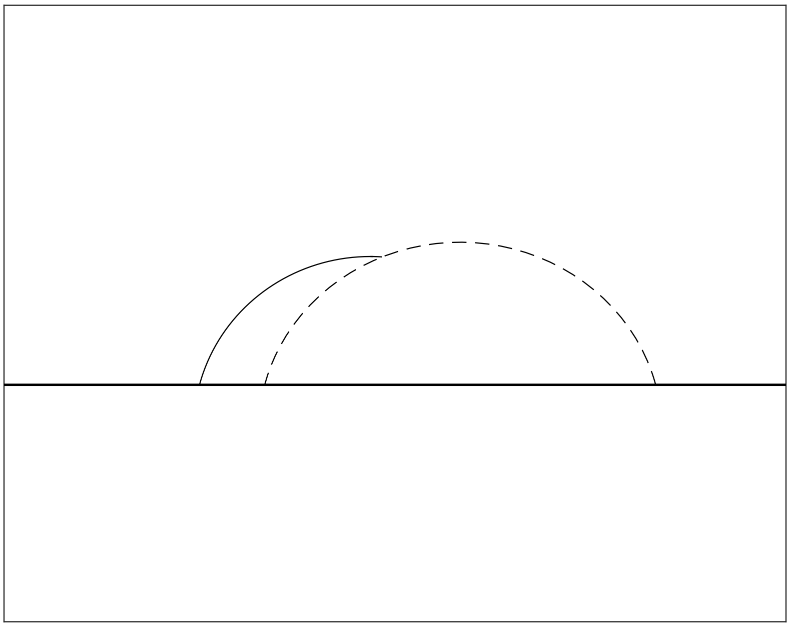

Figure II.1: Case 1

For Case 2, the left side of the drop has a maximum point along the arc in the u direction.

For Case 3, the left side of the drop has a vertical point along the arc in the x direction.

For Case 4, the left side of the drop has a vertical point in the x direction and a maximum point in the u direction.

For Case 5, the left side of the drop has two vertical points in the x direction and one maximum point in the u direction.

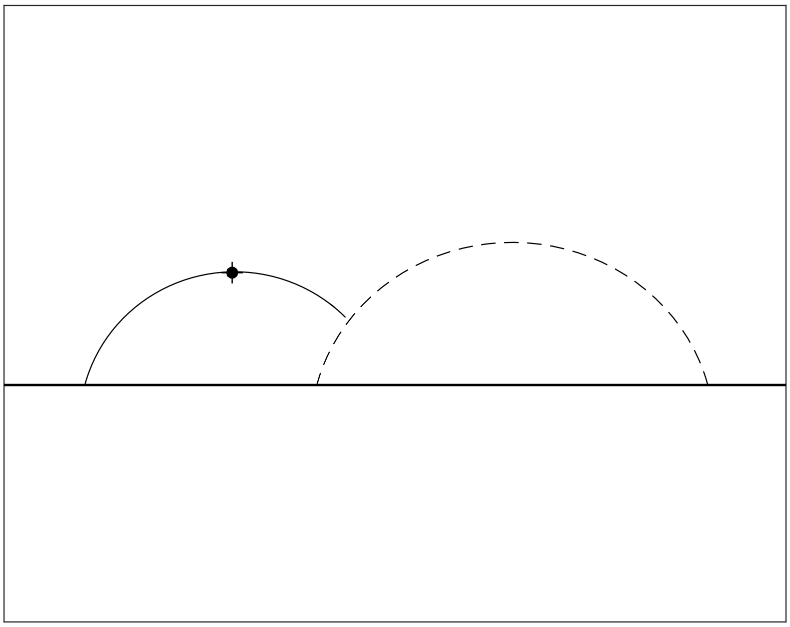

Figure II.2: Case 2

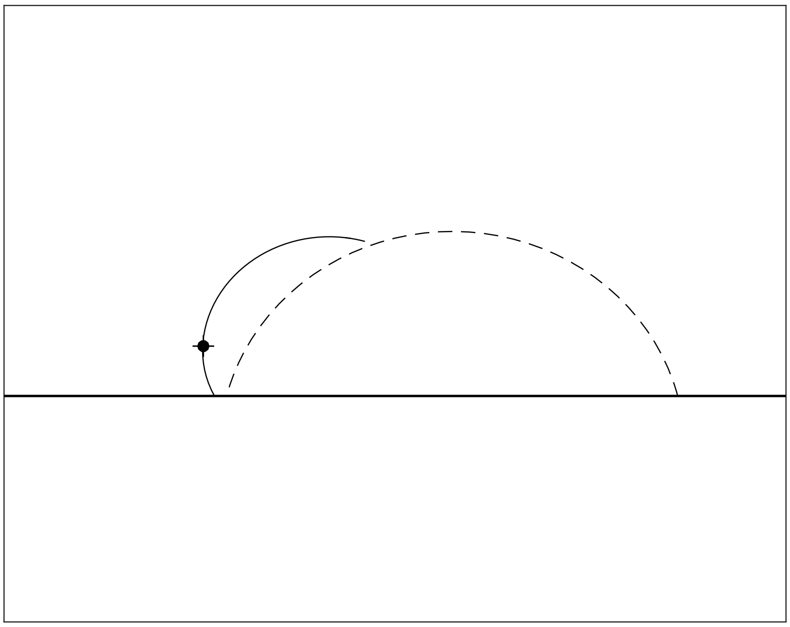

Figure II.3: Case 3

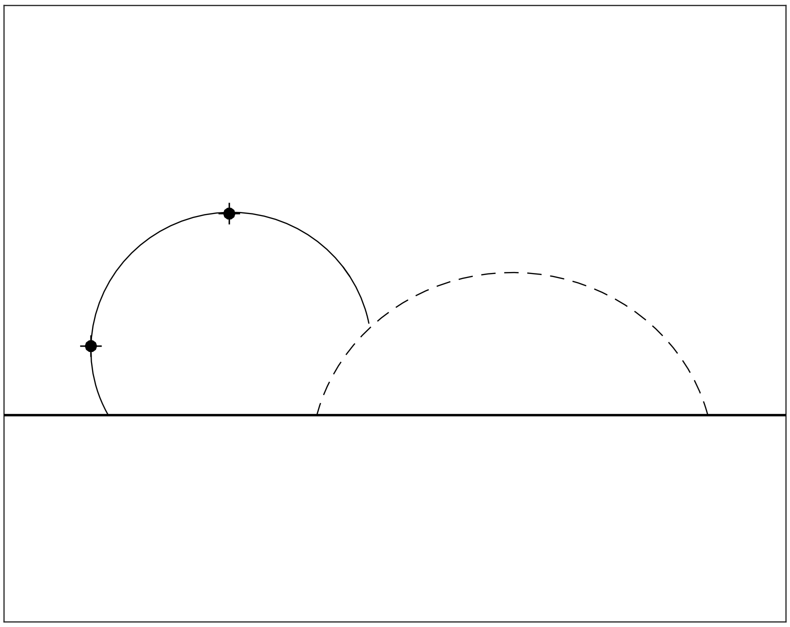

Figure II.4: Case 4

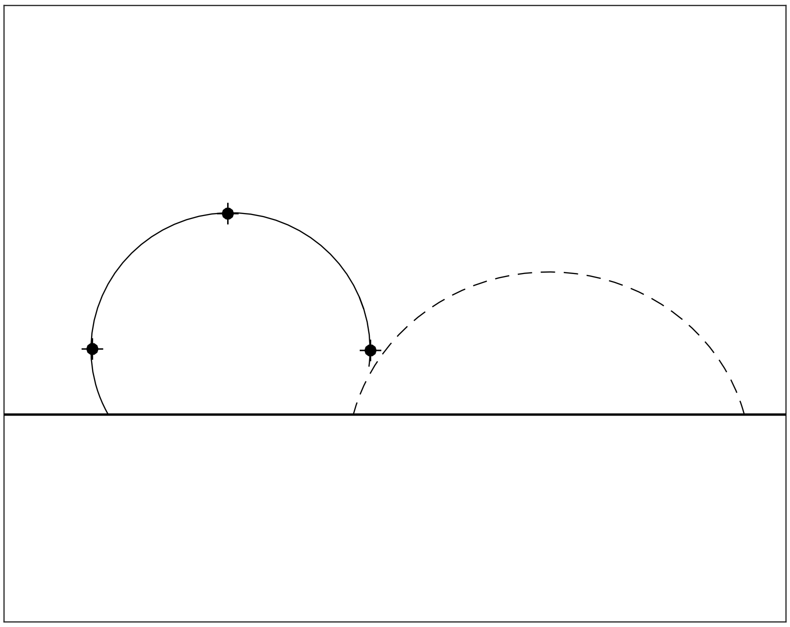

Figure II.5: Case 5

II.2 Computing the Volume

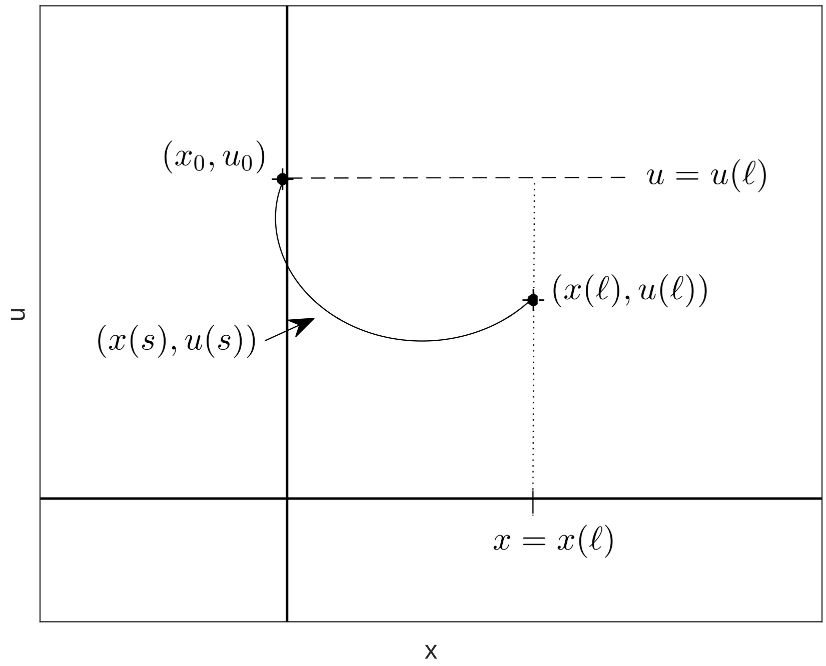

Figure II.6: Enclosed volume of $(x(s), u(s))$

Lemma II.2.1 *Let $(x(s), u(s))$ be a curve parameterized by arclength. Let ψ_0 be the inclination angle at the initial point (x_0, u_0). Let $(x(\ell), u(\ell))$ be the terminal point, at ending arclength ℓ. Then the volume enclosed by the line $x = x(\ell)$, the curve $(x(s), u(s))$ and the line $u = u(\ell)$, that is the volume of the fluid between the air interface and the upper boundary is given by*

$$V = u(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi(\ell) - \sin\psi(0)) \tag{II.1}$$

where the curves satisfy

$$\begin{cases} \frac{dx}{ds} = \cos\psi, \\ \frac{du}{ds} = \sin\psi, \\ \frac{d\psi}{ds} = \kappa u \end{cases} \tag{II.2}$$

with initial conditions

$$\begin{cases} x(0) = x_0, \\ u(0) = u_0, \\ \psi(0) = \psi_0. \end{cases} \tag{II.3}$$

Proof. We will use the following approach to establish (II.1). We will compute the volume for each case described above in terms of component surfaces that describe the curves with no inflection points for the region enclosed by $(x(s), u(s))$ and $u = u(\ell)$ in terms of initial point and ending point of $(x(s), u(s))$ and the curves with inflection points for the region enclosed by $(x(s), u(s))$ and $u = u(\ell)$ in terms of initial point and ending point. We break the curve at inflection points of $(x(s), u(s))$, which occur when $u = 0$ and at this point $\psi = \psi_{max}$. The second type of curve treats the case where an inflection point is present. We describe the points of $u(x)$ in terms of these component surfaces for both cases, where $u(x)$ is the height function of $(x(s), u(s))$ at point x.
For the computations, we use the fact $u = \frac{1}{\kappa}\frac{d\psi}{ds}$ given by (II.3) and the chain rule to obtain (II.6). The following cases will follow similarly.

Case 1, curve with no inflection points: no horizontal point and no vertical point

For Case 1 we have no horizontal points and no vertical points. So to find the volume we integrate our height function $u(x)$ using the initial and terminal points of our arc $x(0)$ and $x(\ell)$. The volume is given by

$$\begin{aligned} \mathcal{V} &= \int_{x(0)}^{x(\ell)} (u(\ell) - u(x))\, dx & \text{(II.4)} \\ &= u(\ell)(x(\ell) - x(0)) - \int_{x(0)}^{x(\ell)} u(x)\, dx & \text{(II.5)} \\ &= u(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}\int_{\psi(0)}^{\psi(\ell)} \cos\psi\, d\psi & \text{(II.6)} \\ &= u(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi(\ell) - \sin\psi(0)). & \text{(II.7)} \end{aligned}$$

Case 1, curve with inflection points: no horizontal points and no

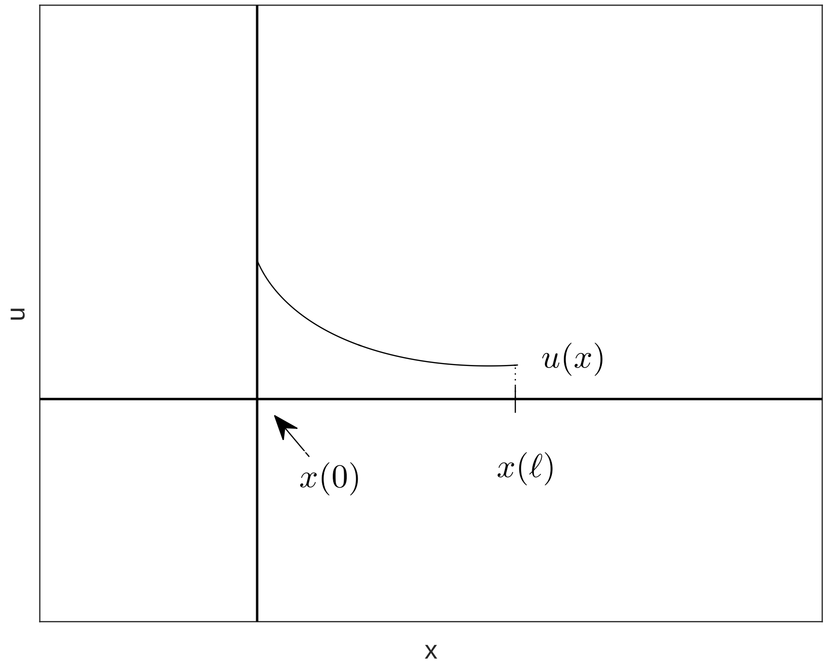

Figure II.7: Case 1 without inflection point

vertical points

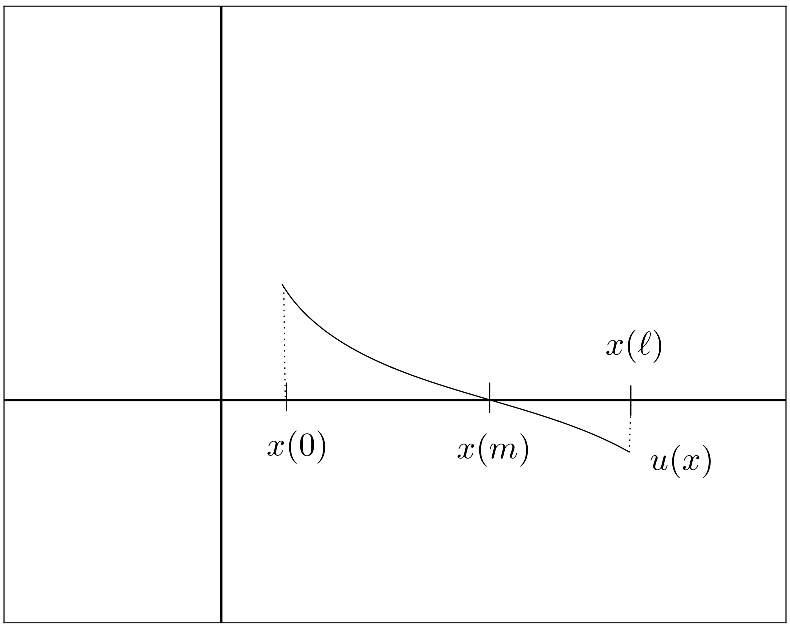

Figure II.8: Case 1 with inflection point

For Case 1 there are no horizontal points and no vertical points. However, we may reach an inflection point. So we perform the same calculation as above except we do not integrate across an inflection point. We know this inflection point will happen when ψ is at its maximum point. Therefore we identify the point where $\frac{d\psi}{dx} = 0$. Since $\frac{d\psi}{dx} = \kappa u$ by (II.3) we know this will happen when $u(x) = 0$. So we partition the integral at the point where the component surface may cross the x-axis and denote this point $(x(m), 0)$. The volume is given by

$$
\begin{aligned}
\mathcal{V} &= \int_{x(0)}^{x(m)} (u(\ell) - u(x))\, dx + \int_{x(m)}^{x(\ell)} (u(\ell) - u(x))\, dx && \text{(II.8)} \\
&= u(\ell)(x(\ell) - x(0)) - \int_{x(0)}^{x(m)} u(x)\, dx - \int_{x(m)}^{x(\ell)} u(x)\, dx && \text{(II.9)} \\
&= u(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}\int_{\psi(0)}^{\psi(m)} \cos\psi\, d\psi - \frac{1}{\kappa}\int_{\psi(m)}^{\psi(\ell)} \cos\psi\, d\psi && \text{(II.10)} \\
&= u(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi(m) - \sin\psi(0)) \\
&\qquad - \frac{1}{\kappa}(\sin\psi(\ell) - \sin\psi(m)) && \text{(II.11)} \\
&= u(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi(\ell) - \sin\psi(0)). && \text{(II.12)}
\end{aligned}
$$

Case 2, curve without inflection points: one horizontal point and no vertical points

For Case 2 there is a horizontal point and no vertical points. This horizontal point $(x(h), u(h))$ occurs as the maximum point of the arc in the u direction. This computation will be the same as Case 1 without inflection points except we

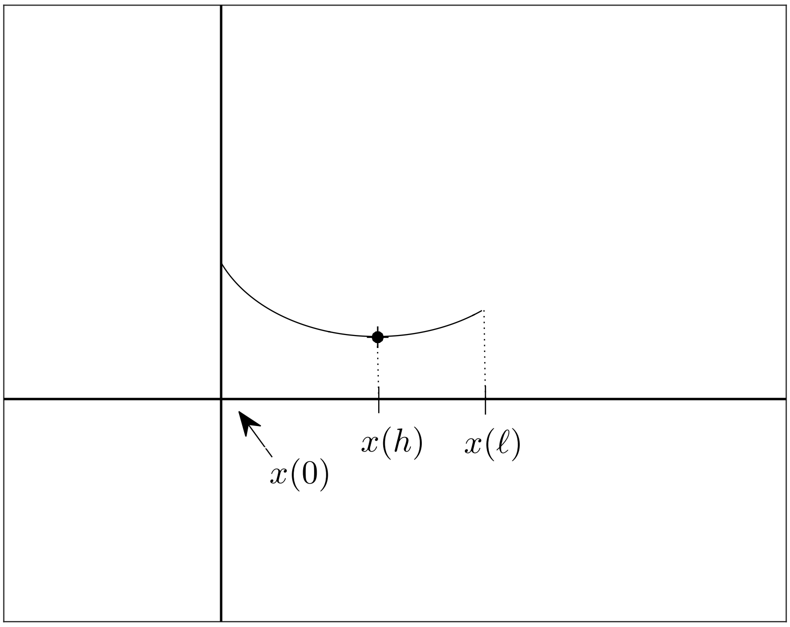

Figure II.9: Case 2 without inflection point

include the point $(x(h), u(h))$. The volume is given by

$$\begin{aligned}
\mathcal{V} &= \int_{x(0)}^{x(h)} (u(\ell) - u(x))\, dx + \int_{x(h)}^{x(\ell)} (u(\ell) - u(x))\, dx && \text{(II.13)}\\
&= u(\ell)(x(\ell) - x(0)) - \int_{x(0)}^{x(h)} u(x)\, dx - \int_{x(h)}^{x(\ell)} u(x)\, dx && \text{(II.14)}\\
&= u(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}\int_{\psi(0)}^{\psi(h)} \cos\psi\, d\psi - \frac{1}{\kappa}\int_{\psi(h)}^{\psi(\ell)} \cos\psi\, d\psi && \text{(II.15)}\\
&= u(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi(h) - \sin\psi(0))\\
&\quad -\frac{1}{\kappa}(\sin\psi(\ell) - \sin\psi(h)) && \text{(II.16)}\\
&= u(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi(\ell) - \sin\psi(0)). && \text{(II.17)}
\end{aligned}$$

In the above computation we have that $x(0) < x(m)$. Note that in the case that $x(0) > x(m)$, the component surface is reflected. Thus by symmetry of the integral the result will be identical.

Case 2, curve with inflection points: one horizontal point and no vertical points

Here $\frac{d\psi}{dx}$ will be strictly increasing along the curve. Thus ψ will not reach a maximum point and our component surface will not include an inflection point. It follows that the computation is identical to the above.

Case 3, curve without inflection points: no horizontal point and a vertical point

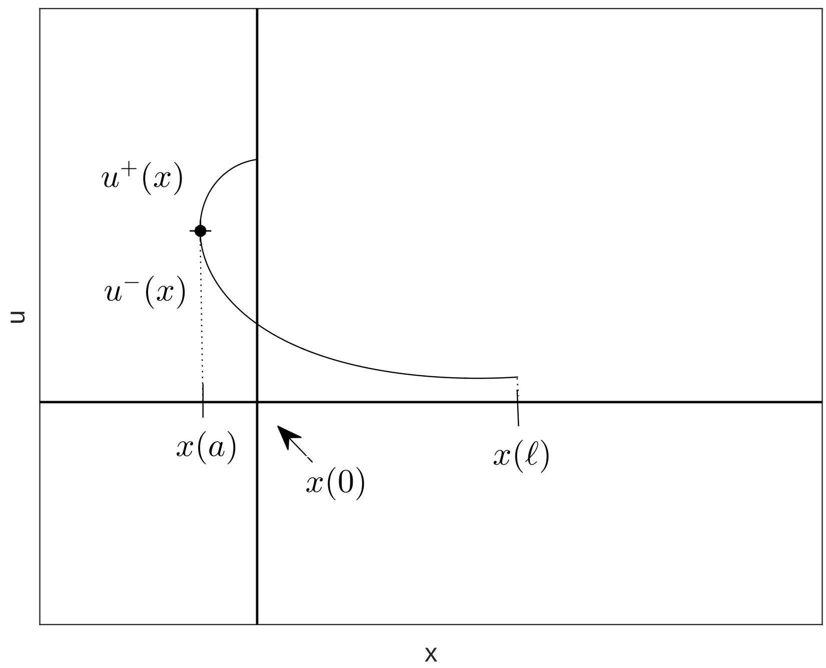

Figure II.10: Case 3 without inflection point

Here we have a vertical point. If the vertical point occurs on the left side of the arc we will denote this point $(x(a), u(a))$ with arclength $s = a$ there and similarly if the vertical point occurs on the right side of the arc we will denote this point $(x(b), u(b))$ with arclength $s = b$ there. Also, for our computations we use the fact that $(x(a), u(a))$ is the point where angle $\psi(a) = -\frac{\pi}{2}$ and $(x(b), u(b))$ is the angle $\psi(b) = \frac{\pi}{2}$.

If there exist a vertical point, then let the height function $u(x)$ be partitioned into $u^+(x)$ and $u^-(x)$ above and below that point. The volume is given by

$$
\begin{aligned}
\mathcal{V} &= \int_{x(a)}^{x(0)} (u^+(x) - u^-(x))\, dx + \int_{x(0)}^{x(\ell)} (u^+(\ell) - u^-(x))\, dx && \text{(II.18)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \int_{x(a)}^{x(0)} u^+(x)\, dx \\
&\quad - \int_{x(a)}^{x(0)} u^-(x)\, dx - \int_{x(0)}^{x(\ell)} u^-(x)\, dx && \text{(II.19)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa} \int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^+\, d\psi \\
&\quad - \frac{1}{\kappa} \int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^-\, d\psi - \frac{1}{\kappa} \int_{\psi(0)}^{\psi(\ell)} \cos\psi^-\, d\psi && \text{(II.20)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa} \left(\sin\psi^+(0) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&\quad - \frac{1}{\kappa} \left(\sin\psi^-(0) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&\quad - \frac{1}{\kappa} (\sin\psi^-(\ell) - \sin\psi^-(0)) && \text{(II.21)} \\
&= u^+(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa} (\sin\psi^-(\ell) - \sin\psi^+(0)). && \text{(II.22)}
\end{aligned}
$$

Case 3, curve with inflection point: no horizontal point and a vertical point

We perform the same computation as above but as in Case 1 for a curve with an inflection point we do not integrate across the inflection point $(x(m), 0)$. The

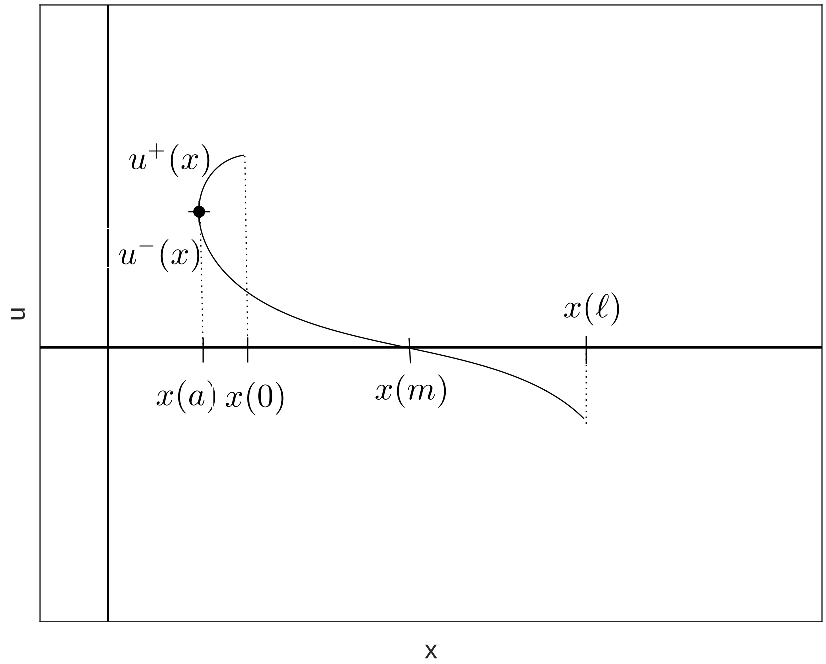

Figure II.11: Case 3 with inflection point

volume is given by

$$
\begin{aligned}
\mathcal{V} &= \int_{x(a)}^{x(0)} (u^+(x) - u^-(x))\, dx + \int_{x(0)}^{x(m)} (u^+(\ell) - u^-(x))\, dx \\
&\quad + \int_{x(m)}^{x(\ell)} (u^+(\ell) - u^-(x))\, dx \qquad \text{(II.23)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \int_{x(a)}^{x(0)} u^+(x)\, dx \\
&\quad - \int_{x(a)}^{x(0)} u^-(x)\, dx - \int_{x(0)}^{x(m)} u^-(x)\, dx - \int_{x(m)}^{x(\ell)} u^-(x)\, dx \qquad \text{(II.24)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa}\int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^+\, d\psi - \frac{1}{\kappa}\int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^-\, d\psi \\
&\quad -\frac{1}{\kappa}\int_{\psi(0)}^{\psi(m)} \cos\psi^-\, d\psi - \frac{1}{\kappa}\int_{\psi(m)}^{\psi(\ell)} \cos\psi^-\, d\psi \qquad \text{(II.25)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa}\left(\sin\psi^+(0) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&\quad -\frac{1}{\kappa}\left(\sin\psi^-(0) - \sin\left(-\frac{\pi}{2}\right)\right) - \frac{1}{\kappa}\left(\sin\left(-\frac{\pi}{2}\right) - \sin\psi^-(0)\right) \\
&\quad -\frac{1}{\kappa}\left(\sin\psi^-(\ell) - \sin\left(-\frac{\pi}{2}\right)\right) \qquad \text{(II.26)} \\
&= u^+(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi^-(\ell) - \sin\psi^+(0)). \qquad \text{(II.27)}
\end{aligned}
$$

Case 4, curve without inflection points: one horizontal point and one vertical point

Without loss of generality, assume we have a left vertical point. Then we have both $(x(h), u(h))$ and $(x(a), u(a))$. We include both of these points in our computation. The volume is given by

$$\begin{aligned}
\mathcal{V} &= \int_{x(a)}^{x(0)} (u^+(x) - u^-(x))\, dx + \int_{x(0)}^{x(h)} (u^+(\ell) - u^-(x)) dx \\
&\quad + \int_{x(h)}^{x(\ell)} (u^+(\ell) - u^-(x))\, dx \qquad \text{(II.28)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \int_{x(a)}^{x(0)} u^+(x)\, dx \\
&\quad - \int_{x(a)}^{x(0)} u^-(x)\, dx - \int_{x(0)}^{x(h)} u^-(x)\, dx - \int_{x(h)}^{x(\ell)} u^-(x)\, dx \qquad \text{(II.29)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa} \int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^+\, d\psi \\
&\quad - \frac{1}{\kappa} \int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^-\, d\psi - \int_{\psi(0)}^{\psi(h)} \cos\psi^-\, d\psi \\
&\quad - \frac{1}{\kappa} \int_{\psi(h)}^{\psi(\ell)} \cos\psi^-\, d\psi \qquad \text{(II.30)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa}\left(\sin\psi^+(0) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&\quad - \frac{1}{\kappa}\left(\sin\psi^-(0) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&\quad - \frac{1}{\kappa}(\sin\psi^-(h) - \sin\psi^-(0)) - \frac{1}{\kappa}(\sin\psi^-(\ell) - \sin\psi^-(h)) \qquad \text{(II.31)} \\
&= u^+(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi^-(\ell) - \sin\psi^+(0)). \qquad \text{(II.32)}
\end{aligned}$$

Case 4, curve with inflection point: one horizontal point and one vertical point

For this case we have a horizontal point $((x(h), u(h))$ and, without loss of generality, a left vertical point $((x(a), u(a))$. So we know that ψ reaches a maximum and our component surface will include an inflection point. We include $(x(a), u(a))$ in our computation and as before we do not integrate across the inflection point $(x(m), 0)$. The volume is given by

$$\begin{aligned}
\mathcal{V} &= \int_{x(a)}^{x(0)} (u^+(x) - u^-(x))\, dx + \int_{x(0)}^{x(h)} (u^+(\ell) - u^-(x))\, dx \\
&\quad + \int_{x(h)}^{x(m)} (u^+(\ell) - u^-(x))\, dx + \int_{x(m)}^{x(\ell)} (u^+(\ell) - u^-(x))\, dx \quad \text{(II.33)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \int_{x(a)}^{x(0)} u^+(x) dx - \int_{x(a)}^{x(0)} u^-(x) dx \\
&\quad - \int_{x(0)}^{x(h)} u^-(x)\, dx - \int_{x(h)}^{x(m)} u^-(x)\, dx - \int_{x(m)}^{x(\ell)} u^-(x)\, dx \quad \text{(II.34)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa} \int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^+\, d\psi \\
&\quad - \frac{1}{\kappa} \int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^-\, d\psi - \int_{\psi(0)}^{\psi(h)} \cos\psi^-\, d\psi \\
&\quad - \int_{\psi(h)}^{\psi(m)} \cos\psi^-\, d\psi - \int_{\psi(m)}^{\psi(\ell)} \cos\psi^-\, d\psi \quad \text{(II.35)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa}\left(\sin\psi^+(0) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&\quad - \frac{1}{\kappa}\left(\sin\psi^-(0) - \sin\left(-\frac{\pi}{2}\right)\right) - \frac{1}{\kappa}(\sin\psi^-(h) - \sin\psi^-(0)) \\
&\quad - \frac{1}{\kappa}(\sin\psi^-(m) - \sin\psi^-(h)) \\
&\quad - \frac{1}{\kappa}(\sin\psi^-(\ell) - \sin\psi^-(m)) \quad \text{(II.36)} \\
&= u^+(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi^-(\ell) - \sin\psi^+(0)). \quad \text{(II.37)}
\end{aligned}$$

Case 5, curve without inflection point: one horizontal point and two vertical points

For this case we have a both a left and right vertical point. Then we have a horizontal maximum point $(x(h), u(h))$, left vertical point $(x(a), u(a))$ and right vertical point $(x(b), u(b))$. We include each of these points in our computation.

The volume is given by

$$
\begin{aligned}
\mathcal{V} &= \int_{x(a)}^{x(0)} (u^+(x) - u^-(x))\,dx + \int_{x(0)}^{x(h)} (u^+(\ell) - u^-(x))\,dx + \\
&\quad + \int_{x(h)}^{x(\ell)} (u^+(\ell) - u^-(x))\,dx + \int_{x(\ell)}^{x(b)} (u^+(x) - u^-(x))\,dx \qquad \text{(II.38)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \int_{x(a)}^{x(0)} u^+(x)\,dx - \int_{x(a)}^{x(0)} u^-(x)\,dx \\
&\quad - \int_{x(0)}^{x(h)} u^-(x)dx \; - \int_{x(h)}^{x(\ell)} u^-(x)\,dx \\
&\quad + \int_{x(\ell)}^{x(b)} u^+(x)\,dx - \int_{x(\ell)}^{x(b)} u^-(x)\,dx \qquad \text{(II.39)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa}\int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^+\,d\psi - \frac{1}{\kappa}\int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^-\,d\psi \\
&\quad - \frac{1}{\kappa}\int_{\psi(0)}^{\psi(h)} \cos\psi^-\,d\psi - \frac{1}{\kappa}\int_{\psi(h)}^{\psi(\ell)} \cos\psi^-\,d\psi \\
&\quad + \frac{1}{\kappa}\int_{\psi(\ell)}^{\frac{\pi}{2}} \cos\psi^+\,d\psi - \frac{1}{\kappa}\int_{\psi(\ell)}^{\frac{\pi}{2}} \cos\psi^-\,d\psi \qquad \text{(II.40)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa}\left(\sin\psi^+(0) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&\quad - \frac{1}{\kappa}\left(\sin\psi^-(0) - \sin\left(-\frac{\pi}{2}\right)\right) - \frac{1}{\kappa}(\sin\psi^-(h) - \sin\psi^-(0)) \\
&\quad - \frac{1}{\kappa}(\sin\psi^-(\ell) - \sin\psi^-(h)) + \frac{1}{\kappa}\left(\sin\left(\frac{\pi}{2}\right) - \sin\psi^+(\ell)\right) \\
&\quad - \frac{1}{\kappa}\left(\sin\left(\frac{\pi}{2}\right) - \sin\psi^-(\ell)\right) \qquad \text{(II.41)} \\
&= u^+(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi^+(\ell) - \sin\psi^+(0)). \qquad \text{(II.42)}
\end{aligned}
$$

Case 5, curve with inflection point: one horizontal point and two vertical points

This case includes both a left and a right vertical point. So we know that ψ reaches a maximum and our component surface will include an inflection point. We include $(x(a), u(a))$ and $(x(b), u(b))$ in our computation and as before split the integration as we cross the inflection point $(x(m), 0)$. The volume is given by

$$
\begin{aligned}
\mathcal{V} &= \int_{x(a)}^{x(0)} (u^+(x) - u^-(x))\,dx + \int_{x(0)}^{x(h)} (u^+(\ell) - u^-(x))\,dx \\
&\quad + \int_{x(h)}^{x(m)} (u^+(\ell) - u^-(x))\,dx + \int_{x(m)}^{x(\ell)} (u^+(x) - u^-(x))\,dx \\
&\quad + \int_{x(\ell)}^{x(b)} (u^+(x) - u^-(x))\,dx \qquad \text{(II.43)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \int_{x(a)}^{x(0)} u^+(x)\,dx - \int_{x(a)}^{x(0)} u^-(x)\,dx \\
&\quad - \int_{x(0)}^{x(h)} u^-(x)\,dx \ - \int_{x(h)}^{x(m)} u^-(x)\,dx \\
&\quad - \int_{x(m)}^{x(\ell)} u^-(x)\,dx + \int_{x(\ell)}^{x(b)} u^+(x)\,dx - \int_{x(\ell)}^{x(b)} u^-(x)\,dx \qquad \text{(II.44)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa}\int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^+\,d\psi \\
&\quad -\frac{1}{\kappa}\int_{-\frac{\pi}{2}}^{\psi(0)} \cos\psi^-\,d\psi - \frac{1}{\kappa}\int_{\psi(0)}^{\psi(h)} \cos\psi^-\,d\psi \\
&\quad -\frac{1}{\kappa}\int_{\psi(h)}^{\psi(m)} \cos\psi^-\,d\psi - \frac{1}{\kappa}\int_{\psi(m)}^{\psi(\ell)} \cos\psi^-\,d\psi \\
&\quad +\frac{1}{\kappa}\int_{\psi(\ell)}^{\frac{\pi}{2}} \cos\psi^+\,d\psi + \frac{1}{\kappa}\int_{\psi(\ell)}^{\frac{\pi}{2}} \cos\psi^-\,d\psi \qquad \text{(II.45)} \\
&= u^+(\ell)(x(\ell) - x(0)) + \frac{1}{\kappa}\left(\sin\psi^+(0) - \sin\left(-\frac{\pi}{2}\right)\right) \\
&\quad -\frac{1}{\kappa}\left(\sin\psi^-(0) - \sin\left(-\frac{\pi}{2}\right)\right) - \frac{1}{\kappa}(\sin\psi^-(h)) - \sin\psi^-(0)) \\
&\quad -\frac{1}{\kappa}(\sin\psi^-(m) - \sin\psi^-(h)) - \frac{1}{\kappa}\left(\sin\psi^-(\ell) - \sin\psi^-(m)\right) \\
&\quad +\frac{1}{\kappa}\left(\sin\left(\frac{\pi}{2}\right) - \sin\psi^+(\ell)\right) - \frac{1}{\kappa}\left(\sin\left(\frac{\pi}{2}\right) - \sin\psi^-(\ell)\right) \qquad \text{(II.46)} \\
&= u^+(\ell)(x(\ell) - x(0)) - \frac{1}{\kappa}(\sin\psi^+(\ell) - \sin\psi^+(0)). \qquad \text{(II.47)}
\end{aligned}
$$

■

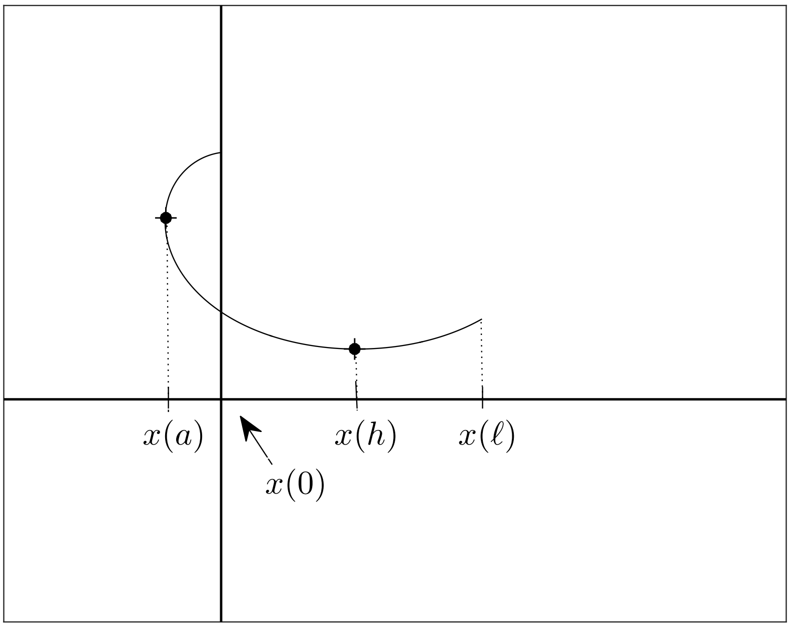

Figure II.12: Case 4 without inflection point

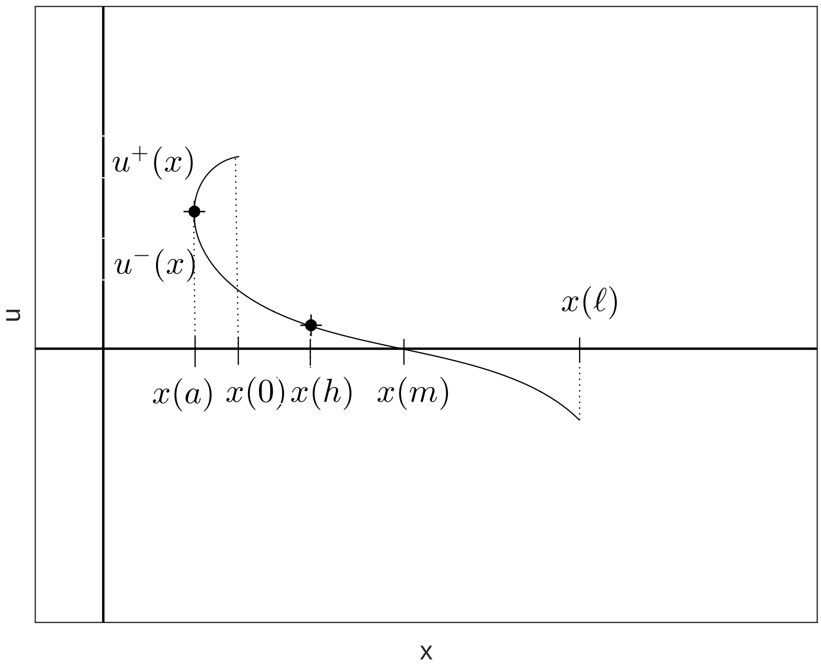

Figure II.13: Case 4 with inflection point

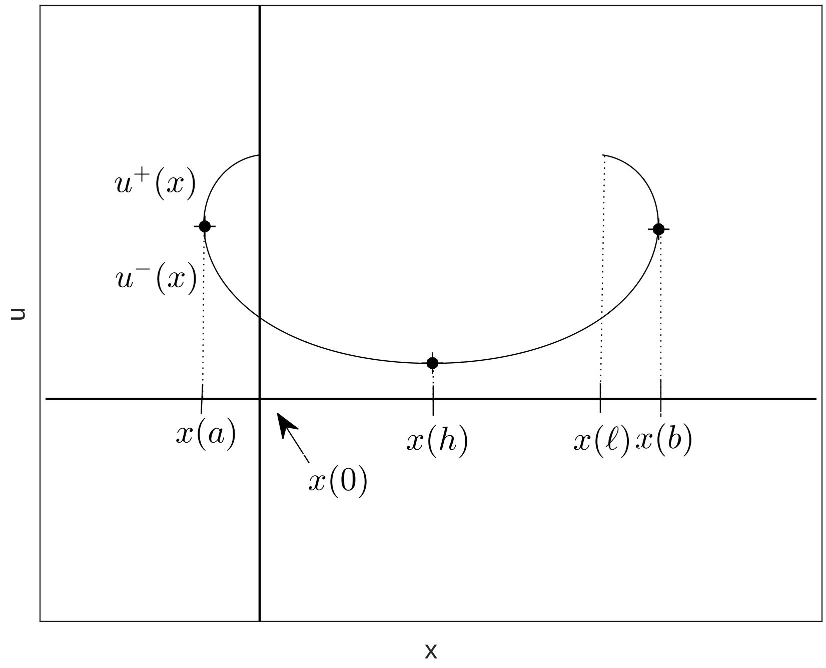

Figure II.14: Case 5 without inflection point

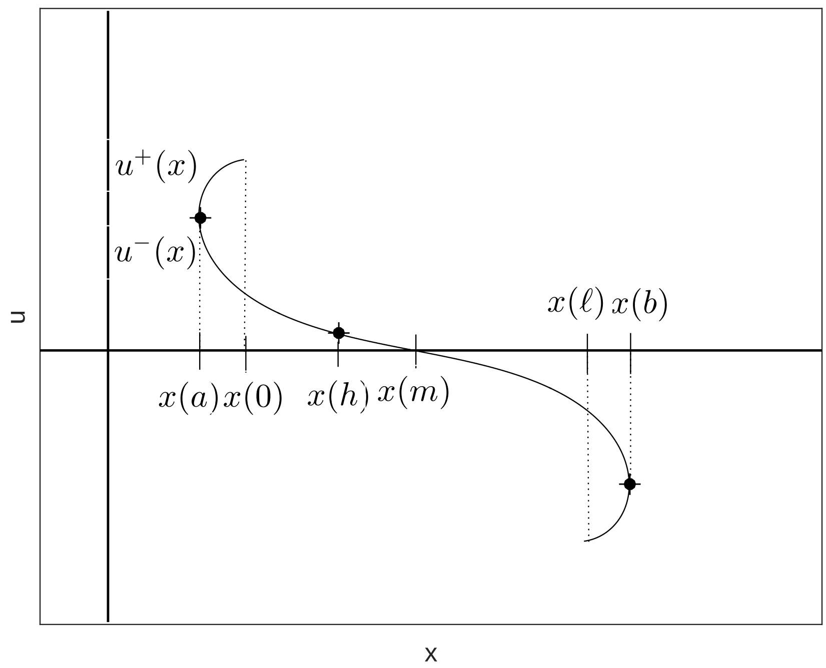

Figure II.15: Case 5 with inflection point

III. CAPILLARY CONSTANTS, SURFACE TENSIONS, CONTACT ANGLES AND ENERGY DENSITY

We have discussed the solutions for the curves that enclose the liquids. The capillary constant κ is a parameter in determining the size and the shape of each drop. Specifically $\kappa = \rho g/\sigma$, where ρ is the density of the fluid, g is the gravity constant and σ is the surface tension of the fluid. Similarly, with multiple fluids, we have multiple capillary constants, which we define as $\kappa_{ij} = (\rho_j - \rho_i)/\sigma_{ij}$ for $i, j = 0, 1, 2$.

Next, consider contact angles γ_{ip}^{j} of the double sessile drop at rest on horizontal plane P. We note that Thomas Young in his 1805 essay [6] established in the existence of the contact angle γ for boundary components in terms of surface tensions σ. Where surface tension is force acting on a surface separating two immiscible fluids in equilibrium. However, in more recent work Finn found in [5] that fluid/fluid interfaces may be described in terms of surface tensions but fluid/solid interfaces are more accurately described in terms of energy density. So in considering the angles at the plate, a fluid/solid interface, we refer instead to the more recently advanced version developed by Finn.

Thus we denote the energy density between the fluid E_i and horizontal plane P as e_{iP}. Where energy density is an attraction or repulsion of molecules between two adjacent media at an interface leads to an areal energy density e on the interface, which is the work per unit area required to form the interface.

Proposition III.0.2 *Let γ_{ip}^{j} denote the contact angle inside E_j, at the triple junction of fluid E_j with fluid E_i and horizontal plane P for fluids E_i, E_j and E_k. Given energy densities we have*

$$\cos \gamma_{ip}^{j} = \frac{e_{iP} - e_{jP}}{e_{ij}}. \tag{III.1}$$

So any two contact angles will determine the third contact angle given by

$$e_{01} \cos \gamma_{0P}^{1} + e_{12} \cos \gamma_{1P}^{2} = e_{02} \cos \gamma_{0P}^{2}. \tag{III.2}$$

Proof. Consider fluids E_0, E_1 and E_2 of the double sessile drop at rest on horizontal plane P. We have the following equalities:

$$e_{01} \cos \gamma_{0P}^{1} = e_{0P} - e_{1P} \tag{III.3}$$

$$e_{12} \cos \gamma_{1P}^{2} = e_{1P} - e_{2P} \tag{III.4}$$

$$e_{02} \cos \gamma_{0P}^{2} = e_{0P} - e_{2P} \tag{III.5}$$

We then have

$$\begin{aligned} e_{01} \cos \gamma_{0P}^{1} + e_{12} \cos \gamma_{1P}^{2} &= e_{0P} - e_{1P} + e_{1P} - e_{2P} && \text{(III.6)} \\ &= e_{0P} - e_{2P} && \text{(III.7)} \\ &= e_{02} \cos \gamma_{0P}^{2} && \text{(III.8)} \end{aligned}$$

■

Next, consider three angles γ_{ij} at the triple junction of fluids E_0, E_1 and E_2. Elcrat, Neel and Siegel established in [1] the contact angles at the triple junction for a floating drop. These are the contact angles measured between fluid E_i and E_j at the triple junction $(x(j), u(j))$. Obtaining the inclination angles at the ending arclength ℓ for each surface ψ_{ij} will be necessary to apply Lemma II.2.1 to the double drop.

Theorem III.0.3 *Let the three contact angles be γ_{ij} at the triple junction of fluids E_0, E_1 and E_2. Define $\bar{\psi}_{ij}$ to be the inclination angle at the ending arclength at the terminal point $(\bar{x}_{ij}, \bar{u}_{ij})$ for each surface S_{ij}. Define $\bar{\psi}_{12} = \bar{\theta}$ for $\bar{\theta} \leq \frac{\pi}{2}$ and $\bar{\psi}_{12} = \pi - \bar{\theta}$ for $\bar{\theta} > \frac{\pi}{2}$. Then for each $\bar{\psi}_{12}$ we can describe each inclination angle at the terminal point in terms of $\bar{\theta}$ and contact angles γ_{ij} given* *by*

$$\bar{\psi}_{01} = \bar{\theta} - \gamma_{02}, \quad \text{(III.9)}$$

$$\bar{\psi}_{02} = \pi - \bar{\theta} - \gamma_{01}, \quad \text{(III.10)}$$

Proof. Consider the inclination angles $\bar{\psi}_{ij}$ at the terminal point $(\bar{x}_{ij}, \bar{u}_{ij})$ for surfaces S_{12}, S_{01}, S_{02}. We have defined $\bar{\psi}_{12} = \bar{\theta}$ for $\bar{\theta} \leq \frac{\pi}{2}$ and $\bar{\psi}_{12} = \pi - \bar{\theta}$ for $\bar{\theta} > \frac{\pi}{2}$.

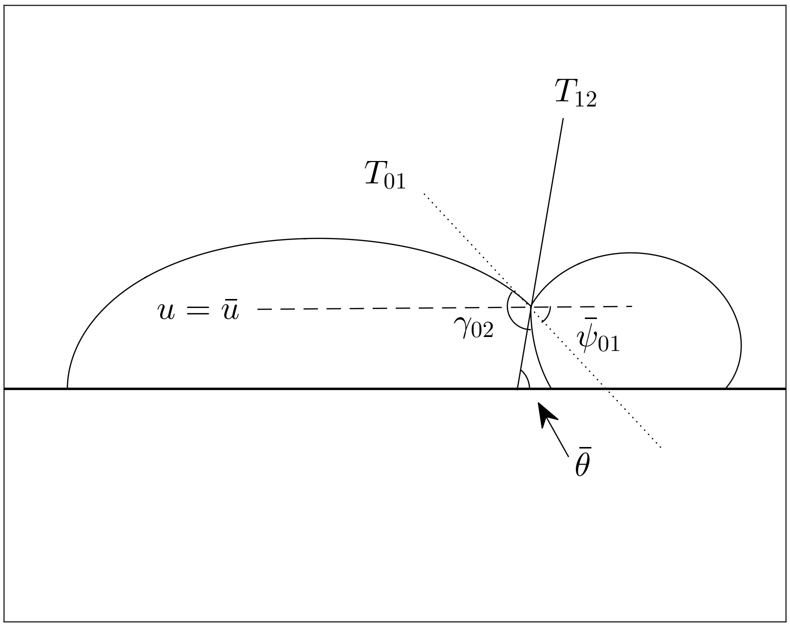

Figure III.1: Inclination Angle of S_{01} at the Terminal Point

Consider $\bar{\psi}_{01}$. We implement the use of the horizontal $u = \bar{u}$, the plane P and the tangent lines T_{01} and T_{12} at the point $(\bar{x}, \bar{u})$. Notice that the acute angle between the horizontal $u = \bar{u}$ and tangent line T_{12} is equivalent to $\bar{\theta}$. Thus the contact angle γ_{02} can be used to establish the equality

$$\gamma_{02} = \bar{\psi}_{01} + \bar{\theta}. \quad \text{(III.11)}$$

See Figure III.1 We then have,

$$\bar{\psi}_{01} = \gamma_{02} - \bar{\theta}. \tag{III.12}$$

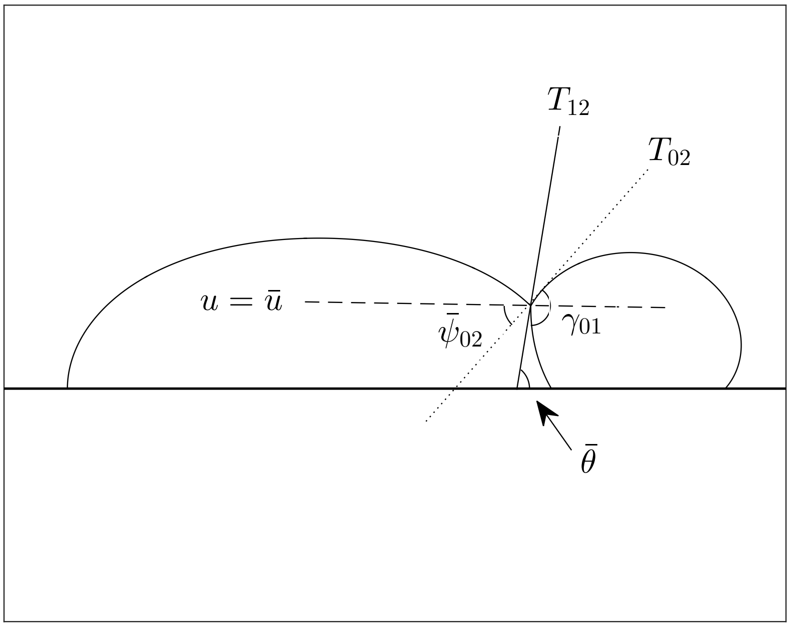

Figure III.2: Inclination Angle of S_{02} at the Terminal Point

Next consider $\bar{\psi}_{02}$. Using the same construction, this time examining the tangent lines T_{02} and T_{12} at the point $(\bar{x}, \bar{u})$ and the contact angle γ_{01}. We have the contact angle γ_{01} as measured between the surfaces S_{02} and S_{12} or equivalently measured between the tangent lines T_{02} and T_{12} near $(\bar{x}, \bar{u})$. Notice that the obtuse angle above the horizontal $u = \bar{u}$ measured to T_{12} is $\pi - \bar{\theta}$. We have that the angles $\pi - \bar{\theta}$ and $\bar{\psi}_{02}$ can also be measured from T_{02} to T_{12}. See Figure III.2. We have

$$\gamma_{01} = \pi - \bar{\theta} + \bar{\psi}_{02}. \tag{III.13}$$

Notice that $\bar{\psi}_{02}$ is an angle of negative magnitude. Solving for $\bar{\psi}_{02}$ gives

$$\bar{\psi}_{02} = \pi - \bar{\theta} - \gamma_{01}. \tag{III.14}$$

The other cases follow similarly. ∎

IV. THE TRANSLATED SYSTEM

From Lemma (II.2.1) we have solutions giving a curve $(x(s), u(s))$ parameterized by arclength. We describe points on the curve using the height function $u(x)$. As in Chapter II, we refer to these curves as the component surfaces. In this chapter we use the component surfaces to construct the physical configuration surfaces of the double sessile drop.

In order to achieve this construction, we will utilize three component surfaces and shift each height function u_{ij}, $i, j = 0, 1, 2$ to satisfy our volume constraints. The result of these shifts are that each curve (x_{ij}, u_{ij}) of the component surfaces will be translated so that we may obtain solutions for each surface S_{ij} of the double drop.

This can be achieved by using the Calculus of Variations with some Lagrange multiplier λ. However, there are consequences to our normalized system of ordinary differential equations (II.2) with initial conditions (II.3).

For the translated system the curves satisfy

$$\begin{cases} \frac{dx_1}{ds} = \cos\psi_1, \\ \frac{du_1}{ds} = \sin\psi_1, \\ \frac{d\psi_1}{ds} = \kappa u_1 - \lambda \end{cases} \tag{IV.1}$$

with initial conditions

$$\begin{cases} x_1(0) = x_{1,0}, \\ u_1(0) = 0, \\ \psi_1(0) = \psi_{1,0} \end{cases} \tag{IV.2}$$

Solutions of the form of horizontal translations $u(x + c)$, $c \in \mathbb{R}$ are also solutions to the normalized system. Also, solutions of the form of vertical reflections solve the system. That is, if $u(x)$ solves the system then $-u(x)$ also solves the system. However, a vertical shift is not a solution to the normalized system. So, we take solutions to (II.2) with initial conditions (II.3), and reflect them about the x-axis, then translate the solutions in the positive u direction so that $u(0) = 0$ is in the new system (IV.1) with initial conditions (IV.2). See Figure IV.1.

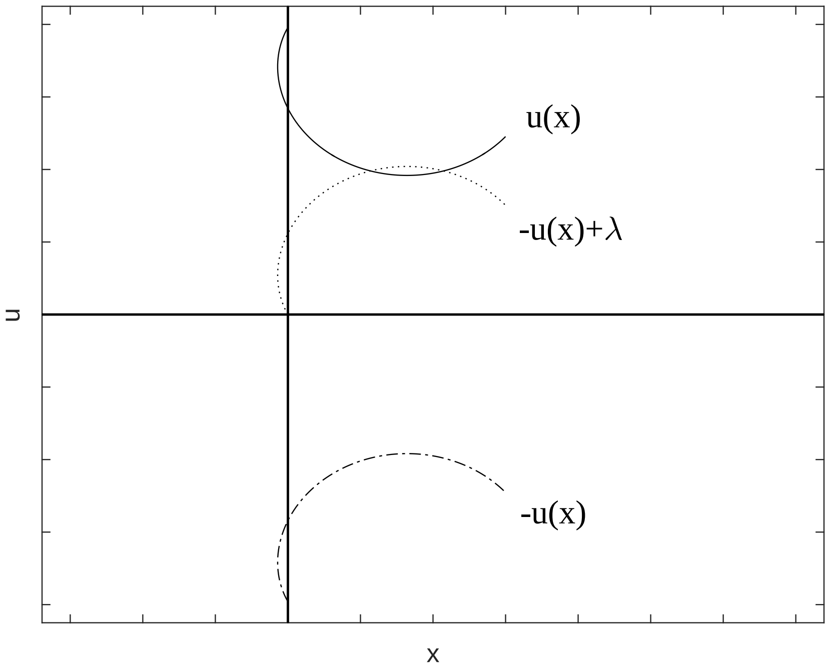

Figure IV.1: u translation

Next, let $u(\ell)$ be the height of u at ending arclength ℓ for our component surface and let $\bar{u}$ be the height u at ending arclength ℓ for our physical configuration surface. Then notice that

$$\kappa \bar{u}_1 - \lambda = \kappa u(\ell) \tag{IV.3}$$

holds between the normalized system and the translated system. Thus λ is the vertical shift between the solutions to the two systems.

V. COMPUTING THE VOLUME OF THE DOUBLE DROP

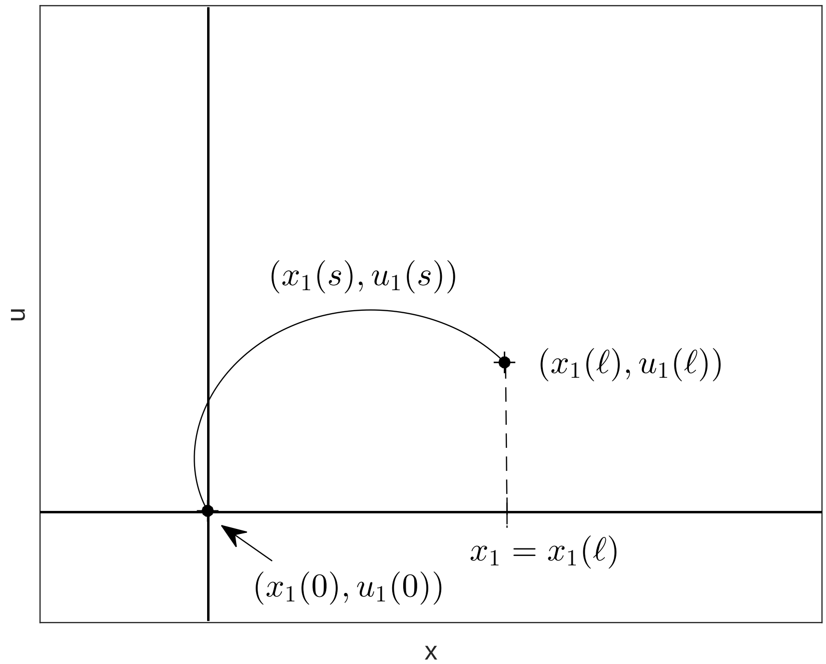

Figure V.1: Enclosed volume of $(x_1(s), u_1(s))$

Lemma V.0.4 *Let $(x_1(s), u_1(s))$ be a curve parameterized by arclength. Let $\psi_{1,0}$ be the inclination angle at the initial point $(x_{1,0}, u_{1,0})$. Let $(x_1(\ell), u_1(\ell))$ be the terminal point, at ending arclength ℓ. Then the volume enclosed by the line $x = x(\ell)$, the curve $(x_1(s), u_1(s))$ and the plate P, that is the volume of the fluid between the air interface and the plate is given by*

$$V = u_1(\ell)\left(x_1(\ell) - x_1(0) - \frac{\lambda}{\kappa}\right) - \frac{1}{\kappa}(\sin\psi_1(\ell) - \sin\psi_1(0)) \tag{V.1}$$

where the curves satisfy

$$\begin{cases} \frac{dx_1}{ds} = \cos\psi_1, \\ \frac{du_1}{ds} = \sin\psi_1, \\ \frac{d\psi_1}{ds} = \kappa u_1 - \lambda \end{cases} \tag{V.2}$$

with initial conditions

$$\begin{cases} x_1(0) = x_{1,0}, \\ u_1(0) = u_{1,0}, \\ \psi_1(0) = \psi_{1,0}. \end{cases} \tag{V.3}$$

See Figure V.1.

Proof. To establish (V.1), consider the following computation. Note that we use the fact $u_1 = \frac{1}{\kappa}(\frac{d\psi_1}{ds} + \lambda)$, given by (IV.1) and the chain rule to obtain (V.6). Also the linearity of the integrals is used to move the $\frac{\lambda}{\kappa}$ constant to the left term.

$$\begin{aligned} \mathcal{V} &= \int_{x_1(0)}^{x_1(\ell)} (u_1(\ell) - u_1(x))\, dx && \text{(V.4)} \\ &= u_1(\ell)(x_1(\ell) - x_1(0)) - \int_{x_1(0)}^{x_1(\ell)} u_1(x)\, dx && \text{(V.5)} \\ &= u_1(\ell)(x_1(\ell) - x_1(0)) - \frac{1}{\kappa}\int_{\psi_1(0)}^{\psi_1(\ell)} \cos\psi_1\, d\psi && \text{(V.6)} \\ &= u_1(\ell)\left(x_1(\ell) - x_1(0) - \frac{\lambda}{\kappa}\right) - \frac{1}{\kappa}(\sin\psi_1(\ell) - \sin\psi_1(0)). && \text{(V.7)} \end{aligned}$$

The computations for the drop configurations follow similarly to the proof of Lemma II.2.1.

■

Next, we apply Lemma V.0.4 to each physical component curve $(x_{1,ij}, u_{1,ij})$, where x_1 and u_1 are from the translated system (IV.1) and ij denotes the surface S_{ij} referenced in the computation.

Theorem V.0.5 *Let the three contact angles γ_{ip}^j be the contact angle inside E_j, at the triple junction $(x_1(\ell), u_1(\ell))$ of fluid E_j with fluid E_i and horizontal plane P for fluids $i, j = 0, 1, 2$. Let the three angles be γ_{ij} at the triple junction of fluids E_0, E_1 and E_2. Let the left and right initial points be $x_{1,01}(0)$ and $x_{1,02}(0)$ where $x_{1,12}(0) = 0$. Let the height of the junction be $\bar{u}_1 = u_1(\ell)$ and λ be a Lagrange multiplier. Then the two volumes in the double sessile drop are given by*

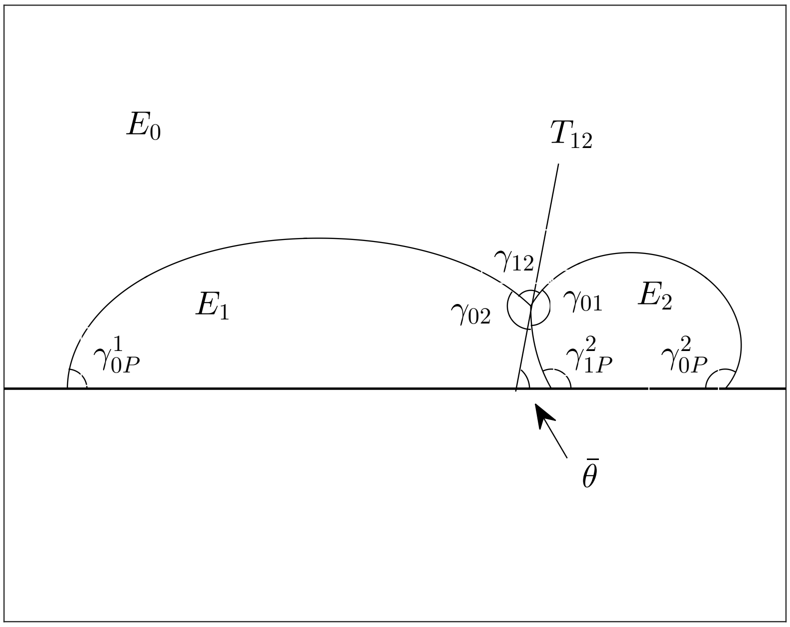

Figure V.2: Angles of the Double Sessile Drop

$$
\begin{aligned}
|E_1| &= \frac{1}{\kappa_{01}}(\sin(\gamma^1_{0P}) - \sin(\bar{\theta} - \gamma_{02})) + \frac{1}{\kappa_{12}}(\sin(\bar{\theta}) - \sin(\gamma^2_{1P})) \\
&\quad -\bar{u}_1 \left(x_{1,01}(0) - \frac{\lambda_{01}}{\kappa_{01}} - \frac{\lambda_{12}}{\kappa_{12}} \right) \qquad \text{(V.8)}
\end{aligned}
$$

and

$$
\begin{aligned}
|E_2| &= \frac{1}{\kappa_{02}}(\sin(-\gamma^2_{0P}) - \sin(\gamma_{01} + \bar{\theta} - \pi)) - \frac{1}{\kappa_{12}}(\sin(\bar{\theta}) - \sin(\gamma^2_{1P})) \\
&\quad +\bar{u}_1 \left(x_{1,02}(0) - \frac{\lambda_{02}}{\kappa_{02}} - \frac{\lambda_{12}}{\kappa_{12}} \right) \qquad \text{(V.9)}
\end{aligned}
$$

in terms of $\bar{\theta}$ where $\bar{\theta}$ is the inclination angle of S_{12} the surface of fluids E_1 and E_2.

See Figure V.2.

Proof. From Lemma II.2.1 we are given a formula for component quantities that gives us $|E_1|$ with quantities for S_{01} and $|E_2|$ with quantities for S_{02}. The cases are $\bar{\theta} < \frac{\pi}{2}$, $\bar{\theta} > \frac{\pi}{2}$ and $\bar{\theta} = \frac{\pi}{2}$. Consider Case 1, when $\bar{\theta} < \frac{\pi}{2}$ E_2 will have two components. Denote these components E_{2a} and E_{2b} so that $|E_2| = |E_{2a}| + |E_{2b}|$. We partition the areas enclosed by the double sessile drop at $x = \bar{x}$, where $(\bar{x}, \bar{u})$ is the triple junction of fluids E_0, E_1 and E_2. Then we have two regions $R_{left} = |E_1| + |E_{2a}|$ and $R_{right} = |E_{2b}|$. When we apply Lemma II.2.1 to the surface S_{01} enclosing R_{left} and the plate P, to the surface S_{12} and P and the surface S_{02} and the plate P.

We then have

$$|E_1| = |E_1 + E_{2a}| - |E_2 a| \tag{V.10}$$

$$|E_2| = |R_{right}| + |E_2 a| = |E_{2b}| + |E_{2a}| \tag{V.11}$$

Similarly for case 2, $\bar{\theta} > \frac{\pi}{2}$ we partition again at $x = x(j)$ and from Lemma II.2.1 we have two components for E_1 so that $E_1 = E_{1a} + E_{1b}$. Now $R_{left} = |E_{1a}|$ and $R_{right} = |E_{1b}| = |E_2|$. We then have

$$|E_1| = R_{left} + |E_{1b}| = |E_{1a}| + |E_{1b}| \tag{V.12}$$

and

$$|E_2| = R_{right} - |E_{1b}| = |E_{1b}| + |E_2| - |E_{1b}| \tag{V.13}$$

For case 3, $\bar{\theta} = \frac{\pi}{2}$ the double sessile drop will be again partitioned at $x = x(j)$ where $R_{left} = |E_1|$ and $R_{right} = |E_2|$. Thus we apply Lemma II.2.1 to each region. Applying Lemma II.2.1 to each component of (V.10) , the computation for $\bar{\theta} < \frac{\pi}{2}$ is:

$$
\begin{aligned}
|E_1| &= u_{1,01}(\ell_{01})\left(x_{1,01}(\ell_{01}) - x_{1,01}(0) - \frac{\lambda_{01}}{\kappa_{01}}\right) \\
&\quad -\frac{1}{\kappa_{01}}(\sin(\gamma_{02} - \bar{\theta}) - \sin(\gamma^1_{0P})) \\
&\quad -[u_{1,12}(\ell_{12})\left(x_{1,12}(\ell_{12}) - x_{1,12}(0) - \frac{\lambda_{12}}{\kappa_{12}}\right) \\
&\quad -\frac{1}{\kappa_{12}}(\sin(\bar{\theta}) - \sin(\gamma^2_{1P}))] \qquad \text{(V.14)}
\end{aligned}
$$

Since at the triple junction $(\bar{x}, \bar{u}) = (x(\ell), u(\ell))$ for each surface we let $\bar{u}_1 = u_{1,01}(\ell_{01}) = u_{1,12}(\ell_{12}) = u_{1,02}(\ell_{02})$ and $\bar{x}_1 = x_{1,01}(\ell_{01}) = x_{1,12}(\ell_{12}) = x_{1,02}(\ell_{02})$. Recall that we chose $x_{12}(0) = 0$. We then have

$$
\begin{aligned}
|E_1| &= \frac{1}{\kappa_{01}}(\sin(\gamma^1_{0P}) - \sin(\bar{\theta}) - \gamma_{02})) + \frac{1}{\kappa_{12}}(\sin(\bar{\theta}) - \sin(\gamma^2_{1P})) \\
&\quad -\bar{u}_1\left(x_{1,01}(0) + \frac{\lambda_{01}}{\kappa_{01}} - \frac{\lambda_{12}}{\kappa_{12}}\right) \qquad \text{(V.15)}
\end{aligned}
$$

Similarly, applying Lemma II.2.1 to each component of (V.11) and inserting the reflected angles for Region E_{2b} the computation for $\bar{\theta} < \frac{\pi}{2}$ is:

$$
\begin{aligned}
|E_2| &= u_{1,02}\left(x_{1,02}(0) - x_{1,02}(\ell_{02}) - \frac{\lambda_{02}}{\kappa_{02}}\right) \\
&\quad -\frac{1}{\kappa_{02}}(\sin(\gamma_{01} + \bar{\theta} - \pi) - \sin(-\gamma^2_{0P})) \\
&\quad +[u_{1,12}(\ell_{12})\left(x_{1,12}(\ell_{12}) - x_{1,12}(0) - \frac{\lambda_{12}}{\kappa_{12}}\right) \\
&\quad -\frac{1}{\kappa_{12}}(\sin(\bar{\theta}) - \sin(\gamma^2_{1P}))] \qquad \text{(V.16)}
\end{aligned}
$$

Using the same equalities as above we have,

$$
\begin{aligned}
|E_2| &= \frac{1}{\kappa_{02}}(\sin(-\gamma^2_{0P}) - \sin(\gamma_{01} + \bar{\theta} + \pi)) - \frac{1}{\kappa_{12}}(\sin(\bar{\theta}) - \sin(\gamma^2_{1P})) \\
&\quad +\bar{u}_1\left(x_{1,02}(0) - \frac{\lambda_{02}}{\kappa_{02}} - \frac{\lambda_{12}}{\kappa_{12}}\right) \qquad \text{(V.17)}
\end{aligned}
$$

Cases 2 and 3 follow similarly. ■

VI. CREATING THE DOUBLE DROP

After exploring the physical properties of the double drop, we are now prepared to solve the problem. That is, given prescribed quantities for a set of drops, to use our program to generate the drops desired. The physical quantities are: the outer contact angles γ_{0P}^1 and γ_{0P}^2; capillary constants for each surface κ_{01}, κ_{02} and κ_{12}; surface tensions σ_{01}, σ_{02} and σ_{12} and volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ it will generate a physical representation for any two fluids. Matlab is the software used to construct this program.

Two solvers were implemented throughout the program: ode45 and fsolve. According to the Matlab Guide [2], the solver ode45 is prescribed for nonstiff differential equations. The algorithm is based on Runge-Kutta formulas. Fsolve is a nonlinear system solver, that uses a trust-region dogleg method.

For our problem we used a shooting method, that is proposed a guess in terms of arclength and height to obtain solutions from ode45 near the actual solution. This guess is used together with a residual function to specify requirements for the solution. The requirements used in the residual function are

$$\psi(\ell) - \bar{\psi} = 0 \tag{VI.1}$$

and

$$x(\ell) - \bar{x} = 0. \tag{VI.2}$$

That is, the difference between the output for ending arclength from ode45 $\psi(\ell)$ and the prescribed $\bar{\psi}$ is near zero. Similarly, the difference between the ending x position $x(\ell)$ computed and the prescribed $\bar{x}$ is near zero. This satisfies our boundary conditions.

To solve the system, solutions were generated for a single curve. Then the curve was replicated and translated in a modular fashion to create the surfaces S_{01}, S_{12} and S_{02}. The residual error is then minimized to implement the boundary conditions. The result is that all three surfaces were joined at the appropriate boundaries to form the double drop.

There is a system of sixteen equations and sixteen unknowns identified that must be solved to receive solutions for the double drop. The boundary conditions for each of the three component surfaces are:

$$\psi_{01}(\ell_{01}) - \bar{\psi}_{01} = 0 \tag{VI.3}$$

$$\psi_{12}(\ell_{12}) - \bar{\psi}_{12} = 0 \tag{VI.4}$$

$$\psi_{02}(\ell_{02}) - \bar{\psi}_{02} = 0 \tag{VI.5}$$

$$x_{01}(\ell_{01}) - \bar{x}_{01} = 0 \tag{VI.6}$$

$$x_{12}(\ell_{12}) - \bar{x}_{12} = 0 \tag{VI.7}$$

$$x_{02}(\ell_{02}) - \bar{x}_{02} = 0. \tag{VI.8}$$

Many equations were of the same form but solved for each of the three surfaces. We implemented this in a modular fashion. For example boundary conditions for angles and x values (VI.1) and (VI.2) were implemented three times each as a larger residual function, minimizing the residual error for each surface. Our first six equations are (VI.3)-(VI.8).

Another condition that must be satisfied is that all three surfaces must end at the triple junction $J = (\bar{x}, \bar{u})$, as in

$$S_{01}(\ell_{01}) = (\bar{x}, \bar{u}) \tag{VI.9}$$

$$S_{12}(\ell_{12}) = (\bar{x}, \bar{u}) \tag{VI.10}$$

$$S_{02}(\ell_{02}) = (\bar{x}, \bar{u}) \tag{VI.11}$$

Solving these equations gives six equations one in x and one in u. These are the next six equations.

Next, we verify the angle conditions at the junction.

$$\frac{\sin \gamma_{01}}{\sigma_{01}} = \frac{\sin \gamma_{02}}{\sigma_{02}} = \frac{\sin \gamma_{12}}{\sigma_{12}} \tag{VI.12}$$

Recall, that the surface tensions σ_{ij} are forces acting on a surface separating two immiscible fluids in equilibrium. With this view we may arrange our vectors σ_{ij} tangential to each surface at J to form a triangle in order to express the contact angles γ_{ij} in terms of the law of cosines.
For γ_{02} we have

$$\sigma_{02}^2 = \sigma_{12}^2 + \sigma_{01}^2 - 2\sigma_{12}\sigma_{01}\cos(\pi - \gamma_{02}) \tag{VI.13}$$

$$\gamma_{02} = \pi - \arccos\left(\frac{\sigma_{12}^2 + \sigma_{01}^2 - \sigma_{02}^2}{2\sigma_{12}\sigma_{01}}\right). \tag{VI.14}$$

The construction for γ_{01} follows similarly and the result is

$$\gamma_{01} = \pi - \arccos\left(\frac{\sigma_{02}^2 + \sigma_{12}^2 - \sigma_{01}^2}{2\sigma_{02}\sigma_{12}}\right). \tag{VI.15}$$

For γ_{12} we use the fact $\gamma_{12} = 2\pi - \gamma_{02} - \gamma_{01}$. We used the law of cosines for this construction but we really wished to solve the law of sines at this location. However, we note that $\cos(\phi) = |\sin(\frac{\pi}{2} - \phi)|$ for any ϕ and so our computation is valid. By solving (VI.12) we have two more of our equations.
We have taken care of all of our requirements but the volume. The final two equations verify that the difference between the prescribed volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ and our computed volumes (V.8) and (V.9), denoted $|E_1|$ and $|E_2|$ are minimized.

$$\mathcal{V}_1 - |E_1| = 0 \tag{VI.16}$$

$$\mathcal{V}_2 - |E_2| = 0 \tag{VI.17}$$

Therefore, using this system of sixteen equations and sixteens and unknowns together for the prescribed quantities desired, we are able to generate a double sessile drop for any two fluids.

The following is a collection of double drop examples. For each example, see Figures VI.1-VI.3, we have included both the prescribed quantities and output values. See Tables VI.1 and VI.2.

Table VI.1: Double drop examples

Prescribed quantities	**Figure VI.1**	**Figure VI.2**	**Figure VI.3**
e_{01}	20	20	20
e_{02}	30	30	30
e_{12}	40	40	40
ρ_0	0	0	0
ρ_1	3	3	3
ρ_2	5	5	5
$\mathcal{V}_1$	0.75	0.75	0.75
$\mathcal{V}_2$	0.50	0.40	0.55
κ_{01}	1.4715	1.4715	1.4715
κ_{02}	1.6350	1.6350	1.6350
κ_{12}	0.4905	0.4905	0.4905
γ_{0P}^1	$2\pi/3$	$\pi/2$	$\pi/2$
γ_{0P}^2	$\pi/3$	$2\pi/3$	$5\pi/12$
γ_{1P}^2	$13\pi/48$	$2\pi/3$	$11\pi/24$

Table VI.2: Double drop examples outputs

Output values	**Figure VI.1**	**Figure VI.2**	**Figure VI.3**
$\bar{\theta}$	$\pi/3$	$\pi/2$	$5\pi/12$
$\bar{x}_{01}$	1.6239	2.9631	2.2580
$\bar{x}_{02}$	3.3251	1.2431	2.2957
$\bar{x}_{12}$	0.2906	-0.1317	0.0671

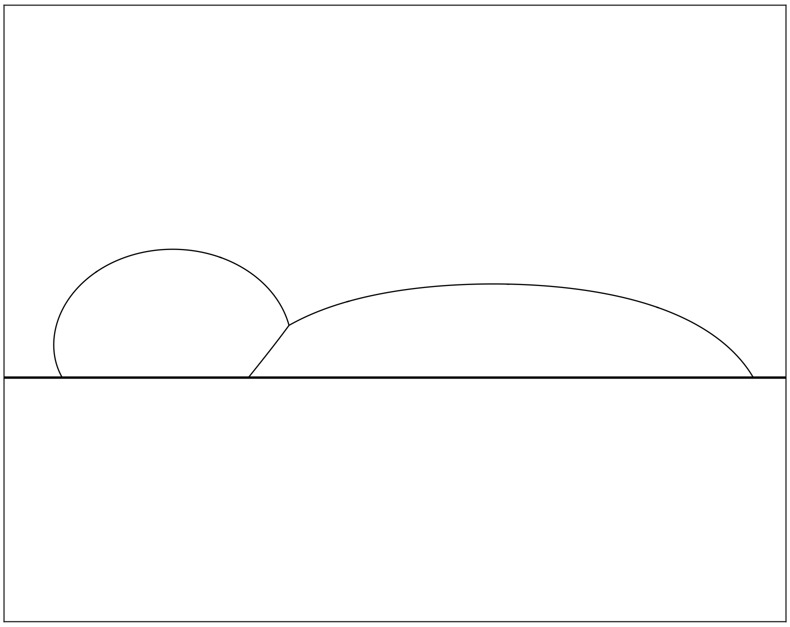

Figure VI.1: Example 1

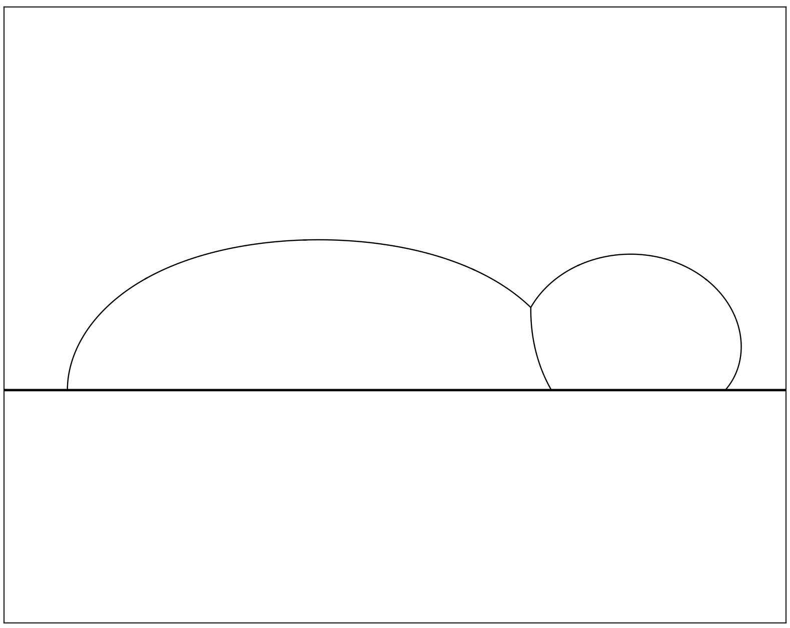

Figure VI.2: Example 2

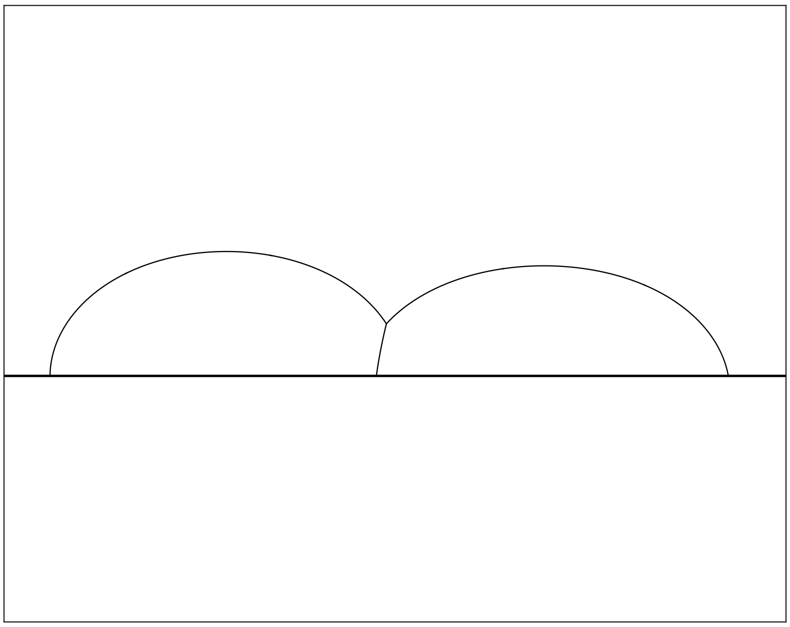

Figure VI.3: Example 3

Bibliography

[1] Alan Elcrat, Robert Neel, and David Siegel, *Equilibrium configurations for a floating drop*, J. Math. Fluid Mech. **6** (2004), no. 4, 405–429.

[2] Desmond J. Higham and Nicholas J. Higham, *MATLAB guide*, 2nd ed., Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.

[3] Robert Finn, *The sessile liquid drop I. Symmetric case*, Pacific J. Math. **88** (1980), no. 2, 541–587.

[4] ———, *Equilibrium capillary surfaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 284, Springer-Verlag, New York, 1986.

[5] ———, *The contact angle in capillarity*, Phys. Fluids **18** (2006), no. 4, 0471027.

[6] Thomas Young, *An essay on the cohesion of fluids*, Philosophical Transactions of the Royal Society of London **95** (1805), 65–87.