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STATISTICAL MECHANICS OF THE N-POINT VORTEX SYSTEM WITH RANDOM INTENSITIES ON $\mathbb{R}^2$

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Abstract. The system of N-point vortices on $\mathbb{R}^2$ is considered under the hypothesis that vortex intensities are independent and identically distributed random variables with respect to a law P supported on $(0,1]$. It is shown that, in the limit as N approaches ∞, the 1-vortex distribution is a minimizer of the free energy functional and is associated to (some) solutions of the following non-linear Poisson Equation:

$$-\Delta u(x) = C^{-1}\int_{(0,1]} r\mathrm{e}^{-\beta r u(x)-\gamma r|x|^2}P(\mathrm{d}r), \quad \forall x\in\mathbb{R}^2,$$

where $C = \int_{(0,1]}\int_{\mathbb{R}^2}\mathrm{e}^{-\beta r u(y)-\gamma r|y|^2}\mathrm{d}yP(\mathrm{d}r)$.

1. Introduction

In a previous work [27] we have studied the system of N point vortices on a bounded domain of $\mathbb{R}^2$ with random-vortice intensities identically distributed with respect to a law P. We generalized some results of Cagliotti et al. [3] in which all the vortices have intensity equal to 1, and thus, P is a Dirac measure concentrated on 1.

Here we will study the same problem on the whole plane. We shall have some technical difficulties which did not arise on the case of bounded domain [27], since $\mathbb{R}^2$ has infinite Lebesgue measure. However, the presence of factors like $\mathrm{e}^{-r|x|^2}$ inside integrals are sufficient to fix most of these problems. Often, the proofs will be analogous to those in [27] just replacing $\mathrm{d}x$ by $\mathrm{e}^{-r|x|^2}\mathrm{d}x$. Related to this, we should suppose also that vortex intensities (which correspond to r in $\mathrm{e}^{-r|x|^2}$) are positive and the law P "decreases" fast enough near 0.

The phase space of this Hamiltonian system is, essentially, $\mathbb{R}^2$. But despite its infinity Lebesgue measure $\mathrm{d}x$, the exponential term acts in such a way that the phase space has $\mathrm{e}^{-r|x|^2}\mathrm{d}x$ finite measure. Therefore, similar to the bounded case, we shall find for this system, negative temperature states as noticed by Onsager [28]. These states have been studied by several authors [1, 2, 3, 4, 10, 13, 14, 18,

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21, 22, 23, 24, 25, 27, 28] since they arises naturally on some physical systems. We emphasize the work of Lundgren and Pointin [22] which also considered the system on the plane. In their work all the vortices have the same intensities. We weaked this assumption by modeling intensities as random variables. But, as explained before, we consider only positive intensities.

Our strategy is the following: we introduce the Gibbs measure μ^N (where N is the number of vortices) and its marginal density of the first k coordinates μ_k^N. Taking the limit as $N \to \infty$, we observe the same factorization property (the so called "propagation of chaos") found in the bounded case. Hence, μ_k^N behaves like product measures $\mu^{\otimes k}$ (or in better terms, as an average of product measures) of k copies of the 1-vortex distribution μ. Since the Gibbs measure is, naturally, a solution of a variational problem, we can characterize the 1-vortex distributions as a solution of a limit variational problem. The Newtonian potentials associated to this 1-vortex distributions are solutions of

$$\begin{gathered} -\Delta u(x) = C^{-1} \int_{(0,1]} r \mathrm{e}^{-\beta r u(x) - \gamma r |x|^2} P(\mathrm{d}r), \quad \forall x \in \mathbb{R}^2, \\ C = \int_{(0,1]} \int_{\mathbb{R}^2} \mathrm{e}^{-\beta r u(y) - \gamma r |y|^2} \mathrm{d}y P(\mathrm{d}r). \end{gathered} \tag{1.1}$$

(the Mean Field Equation, MFE, for short). The propagation of chaos is related to the uniqueness of solution for MFE. Even in the easiest case of positive temperature states the functional minimized by the 1-vortex distribution is not convex and hence we do not have general results of uniqueness.

Notation. We introduce some notation which will be used in the sequel. Set $\Omega = \mathbb{R}^2$ and $\tilde{\Omega} = \Omega \times (0,1]$. $\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_N)$ denotes an arbitrary point in $\tilde{\Omega}^N$, where $\tilde{x}_i = (x_i, r_i)$ ($x_i \in \Omega$ and $r_i \in (0,1]$). All r_i's are random variables identically distributed with respect to a Borelian probability measure P on $(0,1]$. On $\tilde{\Omega}$ we consider the product measure *Lebesgue*$\times P$. By a.e. we mean *almost everywhere* with respect to *Lebesgue*, P, or *Lebesgue*$\times P$ measures without precising which one we are considering.

For $\tilde{X} \in \tilde{\Omega}^N$ and $1 \leq k \leq n$ we set $X = (x_1, \dots x_N)$ and define $\tilde{X}_k = (\tilde{x}_1, \dots, \tilde{x}_k)$ and $\tilde{X}^{N-k} = (\tilde{x}_{k+1}, \dots, \tilde{x}_N)$ (X_k and X^{N-k} are analogous defined.)

For the purpose of integration we set $\mathrm{d}\tilde{x}_i = \mathrm{d}x_i P(\mathrm{d}r_i)$, $\mathrm{d}\tilde{X} = \mathrm{d}\tilde{x}_1 \cdots \mathrm{d}\tilde{x}_N$, and $\mathrm{d}X = \mathrm{d}x_1 \cdots \mathrm{d}x_N$. In an obvious way we define $\mathrm{d}\tilde{X}_k$, $\mathrm{d}\tilde{X}^{N-k}$, $\mathrm{d}X_k$, and $\mathrm{d}X^{N-k}$.

The Hamiltonian of the N-point vortex system is given by

$$H^N(\tilde{X}) = \frac{1}{2} \sum_{i \neq j}^{N} r_i r_j V(x_i, x_j),$$

where V is the Green function of the Poisson equation in $\mathbb{R}^2$, that is,

$$V(x_1, x_2) = -\frac{1}{2\pi} \log |x_1 - x_2|. \tag{1.2}$$

For this system we have other integrals beyond H^N named the center of vorticity M^N and the moment of inertia defined by

$$M^N(\tilde{X}) = \sum_{i=1}^{N} r_i x_i$$

which is supposed to be null, and $I^N(\tilde{X}) = \sum_{i=1}^N r_i|x_i|^2$.

Given $\beta \in \mathbb{R}$ and $\gamma > 0$ we define the canonical Gibbs measure, with inverse temperature β/N, by

$$\mu^N(\tilde{X}) = \frac{1}{Z(N,\beta,\gamma)} \mathrm{e}^{-\frac{\beta}{N}H^N(\tilde{X})-\gamma I^N(\tilde{X})},$$

where Z is the partition function given by

$$Z(N,\beta,\gamma) = \int_{\tilde{\Omega}^N} \mathrm{e}^{-\frac{\beta}{N}H^N(\tilde{X})-\gamma I^N(\tilde{X})} \mathrm{d}\tilde{X}.$$

For simplicity, we denote $H = H^2$ and $I = I^1$.

For $\rho \in L^1(\tilde{\Omega}^N)$ symmetric, that is, for which

$$\rho(\tilde{x}_1, \dots, \tilde{x}_i, \dots, \tilde{x}_j, \dots, \tilde{x}_N) = \rho(\tilde{x}_1, \dots, \tilde{x}_j, \dots, \tilde{x}_i, \dots, \tilde{x}_N),$$

we define the family of correlation functions of ρ, $(\rho_k)_{1\le k<N}$, by

$$\rho_k(\tilde{X}_k) = \int_{\tilde{\Omega}^{N-k}} \rho(\tilde{X}), \mathrm{d}\tilde{X}^{N-k} \quad \forall \tilde{X}_k \in \tilde{\Omega}^k.$$

When $\|\rho\|_{L^1} = 1$, ρ is a probability density for the distribution of N vortices in $\tilde{\Omega}$. Thus, ρ_k is the marginal probability density for the distribution of k vortices (chosen among the N ones) in $\tilde{\Omega}$.

For $t > 0$ and $x_1 \in \tilde{\Omega}$ we set

$$\tilde{B}_t(x_1) = \{\tilde{x}_2 \in \tilde{\Omega} : |x_2 - x_1| < t\} = B_t(x_1) \times (0,1].$$

Finally C, with or without indices, denotes several positive constants, and 1_A denotes the characteristic function of a set A.

We note the presence of factors like $\mathrm{e}^{-\gamma I(\tilde{x}_i)}$ inside integrals (for example, in the definition of the partition function). The decay of these factors at infinity makes the problem very similar to the bounded case. Of course, it works only if $\gamma I > 0$. For that reason we suppose that vortex intensities and γ are strictly positive. Moreover, we shall suppose that the "decay" of P near 0 is fast enough. More precisely, we assume

$$\int_{\tilde{\Omega}} \mathrm{e}^{-\gamma I(\tilde{x}_1)} \mathrm{d}\tilde{x}_1 = \frac{\pi}{\gamma} \int_{(0,1]} \frac{1}{r_1} P(\mathrm{d}r_1) < \infty. \tag{1.3}$$

In the sequel we set

$$|\tilde{\Omega}|_\gamma = \int_{\tilde{\Omega}} \mathrm{e}^{-\gamma I(\tilde{x}_1)} \mathrm{d}\tilde{x}_1.$$

2. Bounds for the partition function

We start with a proposition giving the range of β and γ for which the partition function is well defined (and thus, also the Gibbs measure.)

Proposition 2.1. *Let $\beta > -8\pi$ and $\gamma > 0$. There exists some constant $C = C(\beta,\gamma)$ such that*

$$Z(N,\beta,\gamma) \le C^N.$$

Moreover, C is bounded in β for β on compact subsets of $(-8\pi,\infty)$. In particular, the Gibbs measure μ^N is well defined for $\beta > -8\pi$ and $\gamma > 0$.

Proof. We take a and b such that $-8\pi < a \le \beta \le b$. If $\beta > 0$, then we have

$$
\begin{aligned}
Z(N,\beta,\gamma) &= \int_{\tilde\Omega^N} \Big[\prod_{i\neq j}^{N} |x_i - x_j|^{\beta r_i r_j/4\pi N}\Big] \mathrm{e}^{-\gamma I^N(\tilde X)} \mathrm{d}\tilde X \\
&\le \int_{\tilde\Omega^N} \Big[\prod_{i\neq j}^{N} (|x_i|+1)^{b r_i/4\pi N}(|x_j|+1)^{b r_j/4\pi N}\Big] \mathrm{e}^{-\gamma I^N(\tilde X)} \mathrm{d}\tilde X \\
&= \int_{\tilde\Omega^N} \prod_{i=1}^{N} (|x_i|+1)^{b r_i (N-1)/2\pi N} \mathrm{e}^{-\gamma I(\tilde x_i)} \mathrm{d}\tilde X \\
&\le \Big[\int_{\tilde\Omega} (|x_1|+1)^{b r_1/2\pi} \mathrm{e}^{-\frac{\gamma}{2} I(\tilde x_1)} e^{-\frac{\gamma}{2} I(\tilde x_1)} \mathrm{d}\tilde x_1\Big]^N .
\end{aligned}
$$

Since the map $\tilde x_1 \ni \tilde\Omega \mapsto (|x_1|+1)^{b r_1/2\pi} \mathrm{e}^{-\frac{\gamma}{2} I(\tilde x_1)}$ is bounded from above by some constant $C = C(b,\gamma)$, the conclusion follows from (1.3).

Now, if $-8\pi < \beta \le 0$, then we have

$$
\begin{aligned}
Z(N,\beta,\gamma) &= \int_{\tilde\Omega^N} \prod_{i=1}^{N} \mathrm{e}^{-\frac{\gamma}{N} I(\tilde x_i)} \prod_{\substack{j=1\\ j\neq i}}^{N} |x_i - x_j|^{\beta r_i r_j/4\pi N} \mathrm{e}^{-\frac{\gamma}{N} I(\tilde x_j)} \mathrm{d}\tilde X \\
&\le \prod_{i=1}^{N} \Big[\int_{\tilde\Omega^N} \mathrm{e}^{-\gamma I(\tilde x_i)} \prod_{\substack{j=1\\ j\neq i}}^{N} |x_i - x_j|^{\beta r_i r_j/4\pi} \mathrm{e}^{-\gamma I(\tilde x_j)} \mathrm{d}\tilde X\Big]^{1/N} \\
&= \int_{\tilde\Omega} \mathrm{e}^{-\gamma I(\tilde x_1)} \Big[\int_{\tilde\Omega} |x_1 - x_2|^{\beta r_1 r_2/4\pi} \mathrm{e}^{-\gamma I(\tilde x_2)} \mathrm{d}\tilde x_2\Big]^{N-1} \mathrm{d}\tilde x_1 .
\end{aligned}
$$

Hence it is enough to show that there exists some constant $C = C(a,\gamma)$ which is an upper bound for the integral inside the brackets. We have,

$$
\int_{\tilde B_1(\tilde x_1)} |x_1 - x_2|^{\beta r_1 r_2/4\pi} \mathrm{e}^{-\gamma I(\tilde x_2)} \mathrm{d}\tilde x_2 \le \int_{B_1(x_1)} |x_1 - x_2|^{a/4\pi} dx_2 = \frac{8\pi^2}{8\pi + a}
$$

and

$$
\int_{\tilde\Omega\setminus\tilde B_1(\tilde x_1)} |x_1 - x_2|^{\beta r_1 r_2/4\pi} \mathrm{e}^{-\gamma I(\tilde x_2)} \mathrm{d}\tilde x_2 \le \int_{\tilde\Omega\setminus\tilde B_1(x_1)} \mathrm{e}^{-\gamma I(\tilde x_2)} \mathrm{d}\tilde x_2 \le |\tilde\Omega|_\gamma .
$$

□

Remark 2.2. From Proposition 2.1 with $N = 2$ and $\beta = \pm 2$ it follows that the function $\mathrm{e}^{\pm H - \gamma I^2}$ is in $L^1(\tilde\Omega^2)$. Hence, $He^{-\gamma I^2} \in L^p(\tilde\Omega^2)$, for all $p \in [1,\infty)$, which follows from the fact that there exists some constant $C = C(p)$ such that $|t|^p \le C(\mathrm{e}^t + \mathrm{e}^{-t})$.

Lemma 2.3. *Let $\beta > -8\pi$ and $\gamma > 0$. There exists some constant $C = C(\beta,\gamma)$ such that*

$$C^N \le Z(N,\beta,\gamma).$$

Moreover, C is bounded in β for β on bounded sets of $(-8\pi,\infty)$.

Proof. Let $\alpha > |\beta|$. By Jensen's inequality we have

$$
Z(N,\beta,\gamma) \ge |\tilde\Omega|_\gamma^N \exp\Big(-\frac{\beta}{2N|\tilde\Omega|_\gamma^N} \sum_{i\neq j}^{N} \int_{\tilde\Omega^N} H(\tilde x_i,\tilde x_j) \mathrm{e}^{-\gamma I^N(\tilde X)} \mathrm{d}\tilde X\Big)
$$

$$\begin{aligned}&\geq C^N \exp\Big(-\frac{\alpha(N-1)}{2|\tilde{\Omega}|_\gamma^2}\int_{\tilde{\Omega}^2}|H(\tilde{X}_2)|\mathrm{e}^{-\gamma I^2(\tilde{X}_2)}\mathrm{d}\tilde{X}_2\Big)\\&\geq C^N\exp(-C(N-1))\geq C^N\end{aligned}$$

with $C=C(\alpha,\gamma)$. □

For the rest of this article, β and γ will be fixed in $(-8\pi,\infty)$ and $(0,\infty)$, respectively.

3. Existence of weak cluster points of Gibbs measures

The elements of the Gibbs sequence $(\mu^N)_{N>1}$ are functions defined on different domains. They are points in different functional spaces. This leads to a problem when looking for limits of this sequence. To overcome this problem we proceed as in [27] by introducing the family of correlation functions $(\rho_k)_{1\leq k\leq N}$ of a function $\rho\in L^1(\tilde{\Omega}^N)$, defined by

$$\rho_k(\tilde{X}_k)=\int_{\tilde{\Omega}^{N-k}}\rho(\tilde{X})\mathrm{d}\tilde{X}^{N-k}.$$

Now, for each $k\in\mathbb{N}$, $(\mu_k^N)_{N>k}$ is a sequence on $L^1(\tilde{\Omega}^k)$ and thus we can look for its cluster points. Before finding L^p estimates for these sequences we find pointwise ones. First we have the following lemma.

Lemma 3.1. *There exists some constant $C=C(\beta,\gamma)$ such that*

$$Z\Big(k,\frac{\beta k}{N},\gamma\Big)\leq C^{N-k}Z(N,\beta,\gamma)\quad\forall N>k.$$

Moreover, C is bounded in β for β on bounded subsets of $(-8\pi,\infty)$.

Proof. Let $N>k$ and fix $a>\beta$. It is easy too see that

$$Z\Big(k+1,\frac{\beta(k+1)}{N},\gamma\Big)=\int_{\tilde{\Omega}^k}\mathrm{e}^{-\frac{\beta}{N}H^k(\tilde{X}_k)-\gamma I^k(\tilde{X}_k)}f(\tilde{X}_k)\mathrm{d}\tilde{X}_k,\tag{3.1}$$

where

$$f(\tilde{X}_k)=\int_{\tilde{\Omega}}\mathrm{e}^{-\frac{\beta}{N}\sum_{i=1}^k r_ir_{k+1}V(x_i,x_{k+1})-\gamma I(\tilde{x}_{k+1})}\mathrm{d}\tilde{x}_{k+1}.$$

It follows from Jensen's inequality that

$$f(\tilde{X}_k)\geq|\tilde{\Omega}|_\gamma\exp\Big(\frac{\beta}{2\pi|\tilde{\Omega}|_\gamma N}\sum_{i=1}^k\int_{\tilde{\Omega}}r_ir_{k+1}\log|x_i-x_{k+1}|\mathrm{e}^{-\gamma I(\tilde{x}_{k+1})}\mathrm{d}\tilde{x}_{k+1}\Big).\tag{3.2}$$

Consider $\beta\geq 0$. From (3.2) it follows that

$$\begin{aligned}f(\tilde{X}_k)&\geq C\exp\Big(\frac{\beta}{2\pi|\tilde{\Omega}|_\gamma N}\sum_{i=1}^k\int_{B_1(x_i)}\log|x_i-x_{k+1}|\mathrm{d}x_{k+1}\Big)\\&\geq C\exp\Big(-\frac{aCk}{N}\Big)\geq C\exp(-aC)=C,\end{aligned}$$

and thus, from (3.1), we conclude that

$$Z\Big(k,\frac{\beta k}{N},\gamma\Big)\leq CZ\Big(k+1,\frac{\beta(k+1)}{N},\gamma\Big),$$

with $C=C(a,\gamma)$. The result follows by induction on k.

Now, we suppose $-8\pi < \beta < 0$. From

$$r_i r_{k+1} \log |x_i - x_{k+1}| \leq r_i r_{k+1} |x_i - x_{k+1}|^2 \leq 2(r_i|x_i|^2 + r_{k+1}|x_{k+1}|^2)$$

we conclude that

$$\int_{\tilde{\Omega}} r_i r_{k+1} \log |x_i - x_{k+1}| \mathrm{e}^{-\gamma I(\tilde{x}_{k+1})} \mathrm{d}\tilde{x}_{k+1} \leq 2I(\tilde{x}_i)|\tilde{\Omega}|_\gamma + \theta(\gamma),$$

where

$$\theta(\gamma) = 2\int_{\tilde{\Omega}} I(\tilde{x}_{k+1}) \mathrm{e}^{-\gamma I(\tilde{x}_{k+1})} \mathrm{d}\tilde{x}_{k+1} = 2\pi\gamma^{-1}|\tilde{\Omega}|_\gamma.$$

From (3.2) and (1.3) it follows that

$$\begin{aligned}
f(\tilde{X}_k) &\geq |\tilde{\Omega}|_\gamma \exp\Big(\frac{\beta}{2\pi|\tilde{\Omega}|_\gamma N} \sum_{i=1}^{k} \Big[2I(\tilde{x}_i)|\tilde{\Omega}|_\gamma + 2\pi\gamma^{-1}|\tilde{\Omega}|_\gamma\Big]\Big) \\
&\geq C\gamma^{-1} \exp\Big(\frac{\beta}{\pi N} I^k(\tilde{X}_k)\Big) \exp\Big(\frac{\beta k}{\gamma N}\Big) \\
&\geq C\gamma^{-1} \exp\Big(-\frac{8}{N} I^k(\tilde{X}_k)\Big) \exp\Big(-\frac{8\pi}{\gamma}\Big).
\end{aligned}$$

Note that the constant C depends neither on β nor on γ, and thus, (3.1) yields

$$Z\Big(k, \frac{\beta k}{N}, \gamma + \frac{8}{N}\Big) \leq \varphi(\gamma) Z\Big(k+1, \frac{\beta(k+1)}{N}, \gamma\Big),$$

where $\varphi(\gamma) = C\gamma \mathrm{e}^{8\pi/\gamma}$ is continuous from $(0,\infty)$ in $(0,\infty)$. By repeating $N-k$ times this argument and replacing γ by

$$\gamma + \frac{8(N-k-1)}{N}, \dots, \quad \gamma + \frac{8}{N}, \quad \gamma, \tag{3.3}$$

we obtain

$$Z\Big(k, \frac{\beta k}{N}, \gamma + \frac{8(N-k)}{N}\Big) \leq \varphi\Big(\gamma + \frac{8(N-k-1)}{N}\Big) \cdots \varphi(\gamma) Z(N, \beta, \gamma).$$

All numbers in (3.3) are in $[\gamma, \gamma+8]$. Since φ is continuous, it is bounded from above on this interval by some constant $C = C(\gamma)$. Therefore,

$$Z\Big(k, \frac{\beta k}{N}, \gamma + 8\Big) \leq Z\Big(k, \frac{\beta k}{N}, \gamma + \frac{8(N-k)}{N}\Big) \leq C^{N-k} Z(N, \beta, \gamma).$$

The sequence $(\beta k/N)_{N>k}$ is in a compact subset of $(-8\pi, \infty)$. Hence, Proposition 2.1 and Lemma 2.3 give the existence of constants $C_1 = C_1(\alpha, \gamma)$ and $C_2 = C_2(\alpha, \gamma)$ such that

$$C_1^k \leq Z\Big(k, \frac{\beta k}{N}, \gamma + 8\Big) \leq Z\Big(k, \frac{\beta k}{N}, \gamma\Big) \leq C_2^k.$$

Hence,

$$Z\Big(k, \frac{\beta k}{N}, \gamma\Big) \leq \Big[\frac{C_2}{C_1}\Big]^k Z\Big(k, \frac{\beta k}{N}, \gamma + 8\Big) \leq C^{N-k} Z(N, \beta, \gamma).$$

□

Proposition 3.2. *There exists some constant $C = C(\beta, \gamma)$ such that for N large enough,*

$$\mu_k^N(\tilde{X}_k) \leq C^k \mathrm{e}^{-\frac{\beta}{N} H^k(\tilde{X}_k) - \frac{\gamma}{2} I^k(\tilde{X}_k)}.$$

Proof. Let $k \geq 2$, $N_0, N \in \mathbb{N}$, and $r, p, p' \in \mathbb{R}$ such that

- $r > 1$ with $\beta r \in (-8\pi, \infty)$;

- $N_0 = \min\{N \in \mathbb{N} : N > 2k \text{ and } N/(N-2k) < r\}$ and $N \geq N_0$;
- $p = N/(N-2k)$ and $p' = N/2k$;

We have

$$-\frac{\beta}{N} H^N(\tilde{X}) = -\frac{\beta}{N} H^k(\tilde{X}_k) - \frac{\beta}{N} H^{N-k}(\tilde{X}^{N-k}) - \frac{\beta}{N} \sum_{\substack{1 \leq i \leq k \\ k < j \leq N}} H(\tilde{x}_i, \tilde{x}_j).$$

Thus,

$$\begin{aligned} \mu_k^N(\tilde{X}_k) = \frac{1}{Z(N,\beta,\gamma)} \, & \mathrm{e}^{-\frac{\beta}{N} H^k(\tilde{X}_k) - \gamma I^k(\tilde{X}_k)} \\ & \times \int_{\tilde{\Omega}^{N-k}} \mathrm{e}^{-\frac{\beta}{N} H^{N-k}(\tilde{X}^{N-k}) - \gamma I^{N-k}(\tilde{X}^{N-k}) - \frac{\beta}{N} \sum_{i=1}^{k} \sum_{j=k+1}^{N} H(\tilde{x}_i, \tilde{x}_j)} \mathrm{d}\tilde{X}^{N-k}. \end{aligned}$$

From Hölder's inequality, it follow that the last integral is bounded from above by

$$\Big[\int_{\tilde{\Omega}^{N-k}} \mathrm{e}^{-\frac{\beta p}{N} H^{N-k}(\tilde{X}^{N-k}) - \gamma I^{N-k}(\tilde{X}^{N-k})} \mathrm{d}\tilde{X}^{N-k} \Big]^{1/p} \times f(\tilde{X}_k)$$

where

$$f(\tilde{X}_k) = \Big[\int_{\tilde{\Omega}^{N-k}} \mathrm{e}^{-\frac{\beta p'}{N} \sum_{i=1}^{k} \sum_{j=k+1}^{N} H(\tilde{x}_i, \tilde{x}_j)} \mathrm{e}^{-\gamma I^{N-k}(\tilde{X}^{N-k})} \mathrm{d}\tilde{X}^{N-k} \Big]^{1/p'}.$$

Now, we look for bounds on $f(\tilde{X}_k)$. It is easy to see that

$$\begin{aligned} f(\tilde{X}_k) &= \Big[\int_{\tilde{\Omega}^{N-k}} \prod_{i=1}^{k} \prod_{j=k+1}^{N} |x_i - x_j|^{\beta p' r_i r_j / 2\pi N} \mathrm{e}^{-\frac{\gamma}{k} I(\tilde{x}_j)} \mathrm{d}\tilde{X}^{N-k} \Big]^{1/p'} \\ &\leq \prod_{i=1}^{k} \Big[\int_{\tilde{\Omega}} |x_i - x_N|^{\beta r_i r_N / 4\pi} \mathrm{e}^{-\gamma I(\tilde{x}_N)} \mathrm{d}\tilde{x}_N \Big]^{(N-k)/kp'}. \end{aligned}$$

Recall that $(N-k)/kp' < 2$. By an argument similar to the proof of Proposition 2.1 we can show that there exists a constant $C = C(\beta,\gamma)$ such that

$$f(\tilde{X}_k) \leq \prod_{i=1}^{k} C(|x_i| + 1)^{\beta r_i / 2\pi}.$$

Since the function $\tilde{x}_i \in \tilde{\Omega} \mapsto (|x_i| + 1)^{\beta r_i / 2\pi} e^{-\frac{\gamma}{2} I(\tilde{x}_i)}$ is bounded from above by a constant $C = C(\beta,\gamma)$ we have

$$f(\tilde{X}_k) \mathrm{e}^{-\frac{\gamma}{2} I^k(\tilde{X}_k)} \leq \prod_{i=1}^{k} C(|x_i| + 1)^{\beta r_1 / 2\pi} \mathrm{e}^{-\frac{\gamma}{2} I(\tilde{x}_i)} \leq C^k.$$

It remains to show that there exists some constant $C = C(\beta,\gamma)$ such that

$$\begin{aligned} &\frac{1}{Z(N,\beta,\gamma)} \Big[\int_{\tilde{\Omega}^{N-k}} \mathrm{e}^{-\frac{\beta p}{N} H^{N-k}(\tilde{X}^{N-k}) - \gamma I^{N-k}(\tilde{X}^{N-k})} \mathrm{d}\tilde{X}^{N-k} \Big]^{1/p} \\ &= \frac{Z\Big(N-k, \frac{\beta p(N-k)}{N}, \gamma\Big)^{1/p}}{Z(N,\beta,\gamma)} \leq C^k. \end{aligned}$$

By Lemma 3.1 (notice that $\beta p \in (-8\pi, |\beta| r))$ there exists a constant $C = C(\beta, \gamma)$ such that

$$\frac{Z\Big(N-k, \frac{\beta p(N-k)}{N}, \gamma\Big)}{Z(N,\beta,\gamma)}^{1/p} \le C^{k/p} \frac{Z(N,\beta p,\gamma)}{Z(N,\beta,\gamma)}^{1/p} \le C^k \frac{Z(N,\beta p,\gamma)}{Z(N,\beta,\gamma)}^{1/p}.$$

Applying Hölder's inequality, we have

$$Z(N,\beta p,\gamma)^{1/p} \le Z(N,\beta r,\gamma)^{\theta/r} Z(N,\beta,\gamma)^{1-\theta},$$

where $\theta \in (0,1)$ is such that

$$\frac{1}{p} = \frac{\theta}{r} + \frac{1-\theta}{1} \quad i.e. \quad \theta = \frac{2kr}{N(r-1)}.$$

Therefore,

$$\frac{Z(N,\beta p,\gamma)}{Z(N,\beta,\gamma)}^{1/p} \le Z(N,\beta r,\gamma)^{\theta/r} Z(N,\beta)^{-\theta}.$$

Since $\beta r > -8\pi$, from Proposition 2.1 and Lemma 2.3, it follows that there exist constants $C_1 = C_1(\beta,\gamma)$ and $C_2 = C_2(\beta,\gamma)$ such that

$$Z(N,\beta r,\gamma)^{\theta/r} \le C_2^{N\theta/r} \le C^k \quad \text{and} \quad Z(N,\beta,\gamma)^{-\theta} \le C_1^{-N\theta} \le C^k.$$

$\square$

Corollary 3.3. *Let $p \in [1,\infty)$. Thus $\mu_k^N \in L^p(\tilde\Omega^k)$ for all $k \in \mathbb{N}$ and for all N large enough. Moreover, there exists a constant $C = C(\beta,\gamma,p)$ such that*

$$\|\mu_k^N\|_{L^p} \le C^k, \quad \forall k \in \mathbb{N}, \quad \text{for } N \text{ large enough.}$$

Hence, if $p > 1$, then there exists $\mu_k \in L^p(\tilde\Omega^k)$ and a subsequence $(\mu_k^{N_j})_{j\in\mathbb{N}}$ such that $\mu_k^{N_j} \rightharpoonup \mu_k$ weakly in $L^p(\tilde\Omega^k)$.

For the proof of the above corollary, see [27, Corollary 4, p. 386.].

Remark 3.4. A priori the index choice $(N_j)_{j\in\mathbb{N}}$ depends on p and k. But we can always, by a diagonalization process, suppose that

$$\mu_k^{N_j} \rightharpoonup \mu_k \quad \text{weakly in } L^p(\tilde\Omega^k), \quad \forall k \in \mathbb{N}, \quad \forall p \in [1,\infty).$$

This holds even for $p = 1$ by Proposition 3.6 (by taking $f \in L^\infty(\tilde\Omega^k)$.) In the sequel, we shall say that $\mu_* = (\mu_k)_{k\in\mathbb{N}}$ is a weak cluster point of $(\mu^N)_{N>1}$ in that sense and we shall denote the index sequence always by $(N_j)_{j\in\mathbb{N}}$.

Lemma 3.5. *There exists some constant C such that*

$$\int_{B_r^N} \mathrm{d}\tilde X \le C^N r^N, \quad \forall r > 0,\ \forall N \in \mathbb{N},$$

where $B_r^N = \{\tilde X \in \tilde\Omega^N : I^N(\tilde X) < r\}$. In particular, $1 \in L^1(B_r^N)$.

Proof. Let $r > 0$. We proceed by induction on N. We have

$$\begin{aligned}\int_{B_r^1} \mathrm{d}\tilde x_1 &= \int_{(0,1]} \Big[\int_{\{|x_1|^2 < r/r_1\}} \mathrm{d}x_1\Big] P(\mathrm{d}r_1) = 2\pi \int_{(0,1]} \Big[\int_0^{\sqrt{r/r_1}} s\ \mathrm{d}s\Big] P(\mathrm{d}r_1)\\ &= \pi r \int_{(0,1]} \frac{1}{r_1} P(\mathrm{d}r_1) = Cr.\end{aligned}$$

Let $N \geq 2$ and suppose the result is true for $N-1$. Then,

$$\begin{aligned}\int_{B_r^N} \mathrm{d}\tilde{X} &= \int_{B_r^1} \int_{B_{r-I(\tilde{x}_1)}^{N-1}} \mathrm{d}\tilde{X}^{N-1} d\tilde{x}_1 \leq \int_{B_r^1} C^{N-1}(r - I(\tilde{x}_1))^{N-1} \mathrm{d}\tilde{x}_1 \\ &\leq C^{N-1} r^{N-1} \int_{B_r^1} \mathrm{d}\tilde{x}_1 = C^N r^N.\end{aligned}$$

□

Proposition 3.6. *Let $f : \tilde{\Omega}^k \to \mathbb{R}$ be a measurable function such that $|f| \leq C e^{\frac{\gamma}{4} I^k}$, for some constant $C > 0$. Then, $f\mu_k^N \in L^1(\tilde{\Omega}^k)$ for N large enough. Moreover, if $\mu_k^{N_j} \rightharpoonup \mu_k$ weakly in $L^2(\tilde{\Omega}^k)$, then $f\mu_k \in L^1(\tilde{\Omega}^k)$ and*

$$\int_{\tilde{\Omega}^k} f(\tilde{X}_k) \mu_k^{N_j}(\tilde{X}_k) \mathrm{d}\tilde{X}_k \to \int_{\tilde{\Omega}^k} f(\tilde{X}_k) \mu_k(\tilde{X}_k) \mathrm{d}\tilde{X}_k.$$

Proof. Let $r > 1$ be such that $\beta r > -8\pi$ and let $\phi \in L^\infty(\tilde{\Omega})$ with $0 \leq \phi \leq 1$. From the bound on f and Proposition 3.2, there exists a constant $C = C(\beta, \gamma, k)$ such that

$$\phi |f| \mu_k^N \leq C \phi \mathrm{e}^{-\frac{\beta}{N} H^k - \frac{\gamma}{4} I^k}.$$

In particular, by taking $\phi = 1$, we have $f\mu_k^N \in L^1(\tilde{\Omega}^k)$ for N large enough. By Hölder's inequality we have

$$\begin{aligned}\int_{\tilde{\Omega}^k} \phi |f| \mu_k^N \mathrm{d}\tilde{X}_k &\leq C \Big[\int_{\tilde{\Omega}^k} \phi \mathrm{e}^{-\frac{\gamma}{4} I^k} \mathrm{d}\tilde{X}_k \Big]^{1/r'} \Big[\int_{\tilde{\Omega}^k} \mathrm{e}^{-\frac{\beta r}{N} H^k - \frac{\gamma}{4} I^k} \mathrm{d}\tilde{X}_k \Big]^{1/r} \\ &= C Z\Big(k, \frac{\beta r k}{N}, \frac{\gamma}{4}\Big)^{1/r} \Big[\int_{\tilde{\Omega}^k} \phi \mathrm{e}^{-\frac{\gamma}{4} I^k} \mathrm{d}\tilde{X}_k \Big]^{1/r'}.\end{aligned}$$

Since $(\beta r k / N_j)_{N_j > k}$ is in a compact subset of $(-8\pi, \infty)$, Proposition 2.1 yields a constant $C = C(k, \beta, \gamma)$ such that

$$\int_{\tilde{\Omega}^k} \phi |f| \mu_k^N \mathrm{d}\tilde{X}_k \leq C \Big[\int_{\tilde{\Omega}^k} \phi \ \mathrm{e}^{-\frac{\gamma}{4} I^k} \mathrm{d}\tilde{X}_k \Big]^{1/r'}. \tag{3.4}$$

We have shown that there is a constant $C = C(\beta, \gamma, k)$ such that

$$\int_{\tilde{\Omega}^k} \phi |f| \mu_k^N \mathrm{d}\tilde{X}_k \leq C, \quad \forall \phi \in L^\infty(\tilde{\Omega}^k) \text{ such that } 0 \leq \phi \leq 1. \tag{3.5}$$

For $r > 0$ we set $g_r = 1 - f_r$, where f_r is given by

$$f_r(\tilde{X}_k) = \begin{cases} 1, & \text{if } I^k(\tilde{X}_k) < r, \\ 0, & \text{otherwise.} \end{cases}$$

From Lemma 3.5 it follows that $f_r f \in L^2(\tilde{\Omega}^k)$. Thus, by the weakly convergence of $\mu_k^{N_j}$ to μ_k in $L^2(\tilde{\Omega}^k)$, we have

$$\int_{\tilde{\Omega}^k} f_r f \mu_k^{N_j} \mathrm{d}\tilde{X}_k \to \int_{\tilde{\Omega}^k} f_r f \mu_k \mathrm{d}\tilde{X}_k \quad \text{when } j \to \infty, \quad \forall r > 0. \tag{3.6}$$

In the same way,

$$\int_{\tilde{\Omega}^k} f_r |f| \mu_k^{N_j} \mathrm{d}\tilde{X}_k \to \int_{\tilde{\Omega}^k} f_r |f| \mu_k \mathrm{d}\tilde{X}_k \quad \text{when } j \to \infty, \quad \forall r > 0.$$

By taking $\phi = f_r$ in (3.5) we conclude that the above sequence is bounded from above by a constant $C = C(\beta, \gamma, k)$. By taking limits we find

$$\int_{\tilde{\Omega}^k} f_r |f| \mu_k \mathrm{d}\tilde{X}_k \le C \quad \forall r > 0.$$

But $f_r|f|\mu_k \nearrow |f|\mu_k$ when $r \to \infty$ thus, Monotone Convergence Theorem yields $|f|\mu_k \in L^1(\tilde{\Omega}^k)$. Hence $(f_r f \mu_k)_{r>0} \subset L^1(\tilde{\Omega}^k)$ is bounded from above, in absolute value, by $|f|\mu_k \in L^1(\tilde{\Omega}^k)$. From Dominated Convergence Theorem it follows that

$$\int_{\tilde{\Omega}^k} f_r f \mu_k \mathrm{d}\tilde{X}_k \to \int_{\tilde{\Omega}^k} f \mu_k \mathrm{d}\tilde{X}_k, \quad \text{when } r \to \infty. \tag{3.7}$$

It is not difficult to see that $(g_r e^{-\frac{\gamma}{4} I^k})_{r>0} \subset L^1(\tilde{\Omega}^k)$ is convergent a.e. to 0 and is bounded from above, in absolute value, by $\mathrm{e}^{-\frac{\gamma}{4} I^k} \in L^1(\tilde{\Omega}^k)$. Again, by Dominated Convergence Theorem, this sequence converges to 0 in $L^1(\tilde{\Omega}^k)$. Hence, by taking $\phi = g_r$ in (3.4), we show that

$$\int_{\tilde{\Omega}^k} g_r |f| \mu_k^{N_j} \mathrm{d}\tilde{X}_k \to 0 \quad \text{when } r \to \infty, \quad \text{uniformly on } j. \tag{3.8}$$

By writing $f = f_r f + g_r f$ we have

$$\begin{aligned}
&\Big| \int_{\tilde{\Omega}^k} f \mu_k^{N_j} \mathrm{d}\tilde{X}_k - \int_{\tilde{\Omega}^k} f \mu_k \mathrm{d}\tilde{X}_k \Big| \\
&\le \Big| \int_{\tilde{\Omega}^k} f_r f \mu_k^{N_j} \mathrm{d}\tilde{X}_k - \int_{\tilde{\Omega}^k} f_r f \mu_k \mathrm{d}\tilde{X}_k \Big| \\
&+ \Big| \int_{\tilde{\Omega}^k} f_r f \mu_k \mathrm{d}\tilde{X}_k - \int_{\tilde{\Omega}^k} f \mu_k \mathrm{d}\tilde{X}_k \Big| + \Big| \int_{\tilde{\Omega}^k} g_r f \mu_k^{N_j} \mathrm{d}\tilde{X}_k \Big|.
\end{aligned}$$

Finally, the result follows from (3.6), (3.7) and (3.8). $\square$

4. Variational problems

For $N \in \mathbb{N}$ we set

$$D(F^N) = \{\rho \in L^1(\tilde{\Omega}^N) : \rho \log \rho \in L^1(\tilde{\Omega}^N), I^N \rho \in L^1(\tilde{\Omega}^N)\}.$$

For $\rho \in D(F^N)$ we define the following functionals

$$\begin{gathered}
S^N(\rho) = \int_{\tilde{\Omega}^N} \rho(\tilde{X}) \log \rho(\tilde{X}) \mathrm{d}\tilde{X} \quad \text{(entropy)}, \\
E^N(\rho) = \frac{1}{N} \int_{\tilde{\Omega}^N} H^N(\tilde{X}) \rho(\tilde{X}) \mathrm{d}\tilde{X} \quad \text{(energy)}, \\
J^N(\rho) = \int_{\tilde{\Omega}^N} I^N(\tilde{X}) \rho(\tilde{X}) \mathrm{d}\tilde{X} \quad \text{(moment of inertia)}, \\
F^N(\rho) = S^N(\rho) + \beta E^N(\rho) + \gamma J^N(\rho) \quad \text{(free energy)}.
\end{gathered}$$

We shall see that $D(F^N)$ is convex. The functional F^N is convex, since S^N is convex and E^N and J^N are linear.

Lemma 4.1. *Let $N \ge 2$ and $\rho \in D(F^N)$. Then $H^N \rho \in L^1(\tilde{\Omega}^N)$.*

Proof. We have

$$-\frac{1}{N} \rho(\tilde{X}) H^N(\tilde{X}) = \frac{1}{4\pi N} \rho(\tilde{X}) \sum_{i \ne j}^{N} r_i r_j \log |x_i - x_j|$$

$$\leq \frac{1}{2\pi N}\rho(\tilde{X})\sum_{i\neq j}^{N}(r_i|x_i|^2 + r_j|x_j|^2) \leq \frac{1}{\pi}\rho(\tilde{X})I^N(\tilde{X}).$$

Hence, $H^N\rho$ is bounded from below by some function in $L^1(\tilde{\Omega}^N)$. Apply the following inequality

$$sr \leq r\log r + \frac{1}{\mathrm{e}}\mathrm{e}^s, \quad \forall r \geq 0,\ \forall s \in \mathbb{R}, \tag{4.1}$$

with $r = \rho\mathrm{e}^{I^N}$ and $s = \frac{1}{N}H^N$ and, then multiply by e^{-I^N} to find

$$\frac{1}{N}\rho H^N \leq \rho\log\rho + \rho I^N + \frac{1}{\mathrm{e}}\mathrm{e}^{\frac{1}{N}H^N - I^N}.$$

Since all the terms on the right hand side are in $L^1(\tilde{\Omega}^N)$ we have $H^N\rho \in L^1(\tilde{\Omega}^N)$. ☐

Remark 4.2. Let $N \geq 2$ and $\rho \in D(F^N)$. If ρ is symmetric, then we have simpler expressions for the energy and moment of inertia, which are

$$E^N(\rho) = \frac{N-1}{2}\int_{\tilde{\Omega}}\int_{\tilde{\Omega}} H(\tilde{x}_1,\tilde{x}_2)\rho_2(\tilde{x}_1\tilde{x}_2)\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2,$$

$$J^N(\rho) = N\int_{\tilde{\Omega}} I(\tilde{x}_1)\rho_1(\tilde{x}_1)\mathrm{d}\tilde{x}_1.$$

Proposition 4.3. *Let $1 \leq k < N$ and let $\rho \in D(F^N)$ be symmetric and such that $\|\rho\|_{L^1} = 1$. We have*

$$S^k(\rho_k) + S^{N-k}(\rho_{N-k}) \leq S(\rho).$$

Proof. See [27, Proposition 6 on page 387]. Note that in that proof, we have not used the boundedness of Ω. Thus, the proof works also in the present case. ☐

Lemma 4.4. *There exists measurable functions $f,\ g : \tilde{\Omega}^2 \to \mathbb{R}$ such that $H = g + f$ with $|f| \leq \mathrm{e}^{\frac{\gamma}{4}I^2}$ and $g \in L^p(\tilde{\Omega}^2)$ for all $p \in [1,\infty)$.*

Proof. We set

$$A = \{\tilde{X}_2 \in \tilde{\Omega}^2 : H(\tilde{x}_1,\tilde{x}_2) > 0\} = \{\tilde{X}_2 \in \tilde{\Omega}^2 : |x_1 - x_2| < 1\}.$$

We write $H = g + f$, where

$$g = \frac{1}{2}1_A|H|^2\mathrm{e}^{-\frac{\gamma}{4}I^2} \quad \text{and} \quad f = 1_A H - g + 1_{A^{\complement}}H.$$

We shall show that f and g satisfy the stated properties.

By Remark 2.2, we have $g \in L^p(\tilde{\Omega}^2)$ for all $p \in [1,\infty)$. From Young's inequality, it follows that

$$1_A H = 1_A H\mathrm{e}^{-\frac{\gamma}{8}I^2}\mathrm{e}^{\frac{\gamma}{8}I^2} \leq \frac{1}{2}1_A|H|^2\mathrm{e}^{-\frac{\gamma}{4}I^2} + \frac{1}{2}\mathrm{e}^{\frac{\gamma}{4}I^2} = g + \frac{1}{2}\mathrm{e}^{\frac{\gamma}{4}I^2},$$

and thus, $1_A H - g \leq \frac{1}{2}\mathrm{e}^{\frac{\gamma}{4}I^2}$. Finally, for $\tilde{X}_2 \in \tilde{\Omega}^2$ we have

$$\begin{aligned}|1_{A^{\complement}}(\tilde{X}_2)H(\tilde{X}_2)| &= 1_{A^{\complement}}(\tilde{X}_2)\frac{1}{2\pi}r_1r_2\log|x_1 - x_2| \leq \frac{1}{2\pi}r_1r_2|x_1 - x_2|^2\\ &\leq \frac{1}{\pi}r_1r_2(|x_1|^2 + |x_2|^2) \leq \frac{1}{\pi}(r_1|x_1|^2 + r_2|x_2|^2)\\ &= \frac{1}{\pi}I^2(\tilde{X}_2) \leq C\mathrm{e}^{\frac{\gamma}{4}I^2(\tilde{X}_2)},\end{aligned}$$

for some constant $C = C(\gamma)$. ☐

Remark 4.5. If N is large enough, then $\mu^N \in D(F^N)$. Indeed,

$$\mu^N \log \mu^N = -\frac{\beta}{N} H^N \mu^N - \gamma I^N \mu^N - \mu^N \log Z(N,\beta,\gamma).$$

Hence, it suffices to show that $H^N \mu^N,\ I^N \mu^N \in L^1(\tilde{\Omega}^N)$. By symmetries of H^N and μ^N, it is enough to show that $H\mu_2^N \in L^1(\tilde{\Omega}^2)$ and $I\mu_1^N \in L^1(\tilde{\Omega})$. Using the decomposition of H (Lemma 4.4) we write $H\mu_2^N = g\mu_2^N + f\mu_2^N$, where $g \in L^2(\tilde{\Omega}^2)$ and $|f| \le C\mathrm{e}^{\frac{\gamma}{4}I^2}$. Hence, $g\mu_2^N \in L^1(\tilde{\Omega}^2)$ since $\mu_2^N \in L^2(\tilde{\Omega}^2)$ (Corollary 3.3.) By Proposition 3.6 we have $f\mu_2^N \in L^1(\tilde{\Omega}^2)$ and $I\mu_1^N \in L^1(\tilde{\Omega})$ since $I \le C\mathrm{e}^{\frac{\gamma}{4}I}$ for some constant $C = C(\gamma)$.

Lemma 4.6. *Let $\rho : \tilde{\Omega}^N \to \mathbb{R}$ be a positive measurable function such that $I^N\rho \in L^1(\tilde{\Omega}^N)$. Then $[\rho\log\rho]^- \in L^1(\tilde{\Omega}^N)$ and there exists a constant $C = C(N)$ such that*

$$\int_{\tilde{\Omega}^N} \Big[\rho(\tilde{X}) \log \rho(\tilde{X})\Big]^- \mathrm{d}\tilde{X} \le C + J^N(\rho).$$

Proof. We write

$$\begin{aligned}
&\int_{\tilde{\Omega}^N} \Big[\rho(\tilde{X}) \log \rho(\tilde{X})\Big]^- \mathrm{d}\tilde{X} \\
&= -\int_{\{\rho \le \mathrm{e}^{-I^N}\}} \rho(\tilde{X}) \log \rho(\tilde{X}) \mathrm{d}\tilde{X} - \int_{\{\mathrm{e}^{-I^N} < \rho \le 1\}} \rho(\tilde{X}) \log \rho(\tilde{X}) \mathrm{d}\tilde{X}.
\end{aligned}$$

Since $-t\log t \le C\sqrt{t}$ for all $t \ge 0$ and for some constant $C > 0$, we have

$$-\int_{\{\rho \le \mathrm{e}^{-I^N}\}} \rho(\tilde{X}) \log \rho(\tilde{X}) \mathrm{d}\tilde{X} \le C \int_{\tilde{\Omega}^N} \mathrm{e}^{-\frac{1}{2}I^N(\tilde{X})} \mathrm{d}\tilde{X} = C.$$

We have also

$$-\int_{\{\mathrm{e}^{-I^N} < \rho \le 1\}} \rho(\tilde{X}) \log \rho(\tilde{X}) d\tilde{X} \le \int_{\tilde{\Omega}^N} I^N(\tilde{X}) \rho(\tilde{X}) d\tilde{X} = J^N(\rho)$$

which completes the proof. ◻

It follows immediately from Lemma 4.6 that

$$D(F^N) = \{\rho \in L^1(\tilde{\Omega}^N) : [\rho\log\rho]^+ \in L^1(\tilde{\Omega}^N),\ I^N\rho \in L^1(\tilde{\Omega}^N)\}.$$

Hence $D(F^N)$ is convex since the map $t \in [0,\infty) \mapsto [t\log t]^+$ is convex and J^N is linear.

Lemma 4.7. *For $C > 0$, the set*

$$M_C = \{\rho \in D(F^N) : \int_{\tilde{\Omega}^N} [\rho(\tilde{X}) \log \rho(\tilde{X})]^+ \mathrm{d}\tilde{X} \le C \text{ and } J^N(\rho) \le C\}$$

is weakly compact on $L^1(\tilde{\Omega}^N)$.

Proof. We shall show that M_C is closed in the strong topology of $L^1(\tilde{\Omega}^N)$. Since M_C is convex, it will follow that M_C is weakly closed on $L^1(\tilde{\Omega}^N)$.

Let $(\rho_n)_{n\in\mathbb{N}}$ be a strongly convergent sequence on M_C to $\rho \in L^1(\tilde{\Omega}^N)$. We can take a subsequence $(\rho_{n_j})_{j\in\mathbb{N}}$ such that $\rho_{n_j} \to \rho$ almost everywhere on $\tilde{\Omega}^N$. The sequences $([\rho_{n_j}\log\rho_{n_j}]^+)_{j\in\mathbb{N}}$ and $(I^N\rho_{n_j})_{n\in\mathbb{N}}$ are bounded on $L^1(\tilde{\Omega}^N)$, almost everywhere convergent to $[\rho\log\rho]^+$ and $I^N\rho$, respectively, and composed by positive functions. From Fatou's Lemma we conclude that $\rho \in M_C$.

We show now that every sequence on M_C has a weakly convergent subsequence on $L^1(\tilde{\Omega}^N)$. Let $(\rho_n)_{n\in\mathbb{N}} \subset M_C$. Given $\varepsilon > 0$, take $r > 0$ and $M > 1$ such that

$$\frac{1}{r}J^N(\rho_n) \le \varepsilon \quad \text{and} \quad \frac{1}{\log M}\int_{\tilde{\Omega}^N}\Big[\rho_n(\tilde{X})\log\rho_n(\tilde{X})\Big]^+ \mathrm{d}\tilde{X} \le \varepsilon, \quad \forall n \in \mathbb{N}.$$

We set

$$B = \{\tilde{X} \in \tilde{\Omega}^N : I^N(\tilde{X}) < r\} \quad \text{and} \quad C_n = \{\tilde{X} \in \tilde{\Omega}^N : \rho_n(\tilde{X}) \ge M\}.$$

Let $(E_j)_{j\in\mathbb{N}}$ be a decreasing sequence of measurable subsets of $\tilde{\Omega}^N$ with empty intersection. For $n,\ j \in \mathbb{N}$ we have

$$\int_{E_j\setminus B}\rho_n(\tilde{X})\mathrm{d}\tilde{X} \le \frac{1}{r}\int_{E_j\setminus B} I^N(\tilde{X})\rho_n(\tilde{X})\mathrm{d}\tilde{X} \le \frac{1}{r}J^N(\rho_n) \le \varepsilon$$

and

$$\begin{aligned}\int_{E_j\cap C_n}\rho_n(\tilde{X})\mathrm{d}\tilde{X} &\le \frac{1}{\log M}\int_{E_j\cap C_n}\rho_n(\tilde{X})\log\rho_n(\tilde{X})\mathrm{d}\tilde{X}\\ &\le \frac{1}{\log M}\int_{\tilde{\Omega}^N}\Big[\rho_n(\tilde{X})\log\rho_n(\tilde{X})\Big]^+\mathrm{d}\tilde{X} \le \varepsilon.\end{aligned}$$

The set $E_1 \cap B$ has finite measure by Lemma 3.5. Therefore, the sequence of measures of $E_j \cap B$ goes to 0 as $j \to \infty$. It follows that

$$\int_{E_j\cap(B\setminus C_n)}\rho_n(\tilde{X})\mathrm{d}\tilde{X} \le \int_{E_j\cap B} Md\tilde{X} \le M\int_{E_j\cap B}\mathrm{d}\tilde{X} \le \varepsilon$$

for all $n \in \mathbb{N}$ and j large enough. Since $E_j = [E_j \setminus B] \cup [E_j \cap C_n] \cup [E_j \cap (B \setminus C_n)]$ we have

$$\int_{E_j}\rho_n(\tilde{X})\mathrm{d}\tilde{X} \le 3\varepsilon \quad \forall n \in \mathbb{N},\ \forall j \text{ large enough.}$$

We conclude the proof by applying the Dunford-Petis Theorem (see [9, theorem IV.8.9]). $\square$

Theorem 4.8. *For $N \in \mathbb{N}$ large enough, μ^N is the unique solution of*

$$\min\left\{F^N(\rho) : \rho \in D(F^N),\ \|\rho\|_{L^1} = 1\right\}.$$

Proof. (This proof is similar to Theorem 8 of [27] with minor changes.)

We split the proof into two steps: in the first one we shall show that the problem has a solution $\tilde{\mu}$, and in the second step we shall prove that $\tilde{\mu} = \mu^N$. Let N be such that $\mu^N \in D(F^N)$.

Step 1: Let $\rho \in D(F^N)$ and $t \ge 1$ such that $\beta t > -8\pi$. From inequality (4.1), applied to $r = \frac{1}{t}\rho$ and $s = -(\beta t/N)H^N - (\gamma t/2)I^N$, it follows that

$$-\frac{\beta}{N}H^N\rho - \frac{\gamma}{2}I^N\rho \le \frac{1}{t}\rho\log\Big(\frac{\rho}{t}\Big) + \frac{1}{\mathrm{e}}\mathrm{e}^{-\frac{\beta t}{N}H^N - \frac{\gamma t}{2}I^N}.$$

Therefore,

$$\rho\log\rho + \frac{\beta}{N}H^N\rho + \gamma I^N\rho \ge \Big(1 - \frac{1}{t}\Big)\rho\log\rho + \frac{1}{t}\rho\log t - \frac{1}{\mathrm{e}}\mathrm{e}^{-\frac{\beta t}{N}H^N - \frac{\gamma t}{2}I^N} + \frac{\gamma}{2}I^N\rho. \tag{4.2}$$

In particular, for $t = 1$ one has

$$\rho\log\rho + \frac{\beta}{N}H^N\rho + \gamma I^N\rho + \frac{1}{\mathrm{e}}\mathrm{e}^{-\frac{\beta}{N}H^N - \frac{\gamma}{2}I^N} \ge \frac{\gamma}{2}I^N\rho \ge 0. \tag{4.3}$$

Let us show that F^N is a l.s.c. in the strong topology of $L^1(\tilde\Omega^N)$. Hence, by convexity, F^N is also l.s.c. in the weakly topology of $L^1(\tilde\Omega^N)$. Let $(\rho_n)_{n\in\mathbb{N}} \subset D(F^N)$ be a convergent sequence to $\rho \in L^1(\tilde\Omega^N)$ in the strong topology. We can take a subsequence $(\rho_{n_j})_{j\in\mathbb{N}}$ such that

$$\begin{gathered} \rho_{n_j} \to \rho \quad \text{a.e. on } \tilde\Omega^N \\ F^N(\rho_{n_j}) \to \liminf_{n\to\infty} F^N(\rho_n) \quad \text{(which is supposed to be finite.)} \end{gathered}$$

The sequence $(h_j)_{j\in\mathbb{N}}$, given by

$$h_j = \rho_{n_j}\log\rho_{n_j} + \frac{\beta}{N}H^N\rho_{n_j} + \gamma I^N\rho_{n_j} + \frac{1}{e}e^{-\frac{\beta}{N}H^N - \frac{\gamma}{2}I^N},$$

satisfies

- $h_j \in L^1(\tilde\Omega^N)$, $\quad \forall j \in \mathbb{N}$;
- $\|h_j\|_{L^1} = F^N(\rho_{n_j}) + \frac{1}{\mathrm{e}}Z\left(N,\beta,\frac{\gamma}{2}\right) \le C$, $\quad \forall j\in\mathbb{N}$;
- $h_j \ge 0$ (by (4.3)), $\quad \forall j \in \mathbb{N}$;
- $h_j \to \rho\log\rho + \frac{\beta}{N}H^N\rho + \gamma I^N\rho + \frac{1}{\mathrm{e}}\mathrm{e}^{-\frac{\beta}{N}H^N - \frac{\gamma}{2}I^N}$ $\quad$ a.e. on $\tilde\Omega^N$.

From Fatou's Lemma, it follows that

$$F^N(\rho) \le \liminf_{j\to\infty} F^N(\rho_{n_j}) = \liminf_{n\to\infty} F^N(\rho_n).$$

Now, suppose that $(\rho_n)_{n\in\mathbb{N}}$ is a minimizing sequence for the problem. Taking $t > 1$ in (4.2) and integrating on $\tilde\Omega^N$ we obtain

$$C \ge F^N(\rho_n) \ge \Big(1 - \frac{1}{t}\Big)S^N(\rho_n) + \frac{1}{t}\log t + \frac{\gamma}{2}J^N(\rho_n) - \frac{1}{\mathrm{e}}Z\Big(N,\beta t,\frac{\gamma t}{2}\Big).$$

Hence,

$$S^N(\rho_n) \le C, \quad \forall n \in \mathbb{N},$$

and from (4.3) it follows that

$$J^N(\rho_n) \le C, \quad \forall n\in\mathbb{N}.$$

From the last two estimates and from Lemma 4.6, we conclude that there exists C such that

$$\int_{\tilde\Omega^N} [\rho_n(\tilde X)\log\rho_n(\tilde X)]^+ \mathrm{d}\tilde X \le C, \quad \forall n \in \mathbb{N}.$$

Hence, we have shown that there exists $C > 0$ such that $(\rho_n)_{n\in\mathbb{N}}$ is in a set M_C as in Lemma 4.7 and thus it has a subsequence weakly convergent to $\tilde\mu \in D(F^N)$. It is clear that $\|\tilde\mu\|_{L^1} = 1$. Hence, by the lower semi-continuity of F^N in the weak topology of $L^1(\tilde\Omega^N)$, $\tilde\mu$ is a solution for the problem.

Step 2: We are going to show that $\tilde\mu = \mu^N$. For $\delta > 0$ we set

$$\Lambda_\delta = \{\tilde X \in \tilde\Omega^N : \tilde\mu(\tilde X) > \delta\} \quad \text{and} \quad U_\delta = \{\varphi \in C_c(\tilde\Omega^N) : \|\varphi\|_{L^\infty} < \frac{\delta}{2}\}.$$

Consider the following functionals

$$\begin{array}{rcl} J_\delta : U_\delta & \to & \mathbb{R} \\ \varphi & \mapsto & F^N(\tilde\mu + 1_{\Lambda_\delta}\varphi) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G_\delta : U_\delta & \to & \mathbb{R} \\ \varphi & \mapsto & \int_{\tilde\Omega^N} 1_{\Lambda_\delta}\varphi. \end{array}$$

Take $\varphi \in U_\delta$ and $\rho = \tilde\mu + 1_{\Lambda_\delta}\varphi$. First, we can easily see that $0 \le \rho \le 2\tilde\mu$. Thus $I^N\rho \in L^1(\tilde\Omega^N)$ and $[\rho\log\rho]^+ \in L^1(\tilde\Omega^N)$. From Lemma 4.6 we deduce that $[\rho\log\rho]^- \in L^1(\tilde\Omega^N)$. Hence, $\rho \in D(F^N)$ and thus J_δ is a real valued functional defined on U_δ.

Since $\tilde{\mu}$ is a minimizer of F^N under the constraint $\|\rho\|_{L^1} = 1$ we know that

$$J_\delta(0) = \min_{G_\delta(\varphi)=0} J_\delta(\varphi).$$

By the Lagrange Multiplier Theorem, there exists λ_δ such that $J'_\delta(0) = \lambda_\delta G'_\delta(0)$, that is, for all $\varphi \in C_c(\tilde{\Omega}^N)$ we have

$$\int_{\tilde{\Omega}^N} [\log \tilde{\mu} + 1] 1_{\Lambda_\delta} \varphi + \frac{\beta}{N} \int_{\tilde{\Omega}^N} H^N 1_{\Lambda_\delta} \varphi + \gamma \int_{\tilde{\Omega}^N} I^N 1_{\Lambda_\delta} \varphi = \lambda_\delta \int_{\tilde{\Omega}^N} 1_{\Lambda_\delta} \varphi.$$

Therefore,

$$\log \tilde{\mu} + 1 + \frac{\beta}{N} H^N + \gamma I^N = \lambda_\delta \quad \text{a.e. on } \Lambda_\delta.$$

It follows that $\tilde{\mu} = C_\delta \mathrm{e}^{-\frac{\beta}{N} H^N - \gamma I^N}$ almost everywhere on Λ_δ.

If $\delta_1 < \delta_2$, then $\Lambda_{\delta_2} \subset \Lambda_{\delta_1}$. Since $\tilde{\mu} = C_{\delta_2} \mathrm{e}^{-\frac{\beta}{N} H^N - \gamma I^N}$ on Λ_{δ_2} and $\tilde{\mu} = C_{\delta_1} \mathrm{e}^{-\frac{\beta}{N} H^N - \gamma I^N}$ on Λ_{δ_1} we have $C_{\delta_1} = C_{\delta_2} = C$ (independent on δ). We set

$$\Lambda = \{\tilde{X} \in \tilde{\Omega}^N : \tilde{\mu} > 0\} = \bigcup_{\delta > 0} \Lambda_\delta.$$

Hence, $\tilde{\mu} = 0$ on $\Lambda^{\complement}$ and $\tilde{\mu} = C \mathrm{e}^{-\frac{\beta}{N} H^N - \gamma I^N}$ on Λ, where

$$C = \Big[\int_\Lambda \mathrm{e}^{-\frac{\beta}{N} H^N(\tilde{X}) - \gamma I^N(\tilde{X})} \mathrm{d}\tilde{X} \Big]^{-1}.$$

A simple calculus shows that

$$F^N(\tilde{\mu}) = -\log \Big(\int_\Lambda e^{-\frac{\beta}{N} H^N - \gamma I^N} \Big)$$
$$F^N(\mu^N) = -\log \Big(\int_{\tilde{\Omega}^N} e^{-\frac{\beta}{N} H^N - \gamma I^N} \Big).$$

Since $F^N(\tilde{\mu}) \le F^N(\mu^N)$ we have $|\Lambda^{\complement}| = 0$, and thus $\tilde{\mu} = \mu^N$. □

Remark 4.9. We emphasize that in the last proof we have shown that F^N is a l.s.c. functional on the weak topology of $L^1(\tilde{\Omega}^N)$.

We consider now the limit problem. We define the set $D(F^*)$ of all $\rho_* = (\rho_k)_{k \in \mathbb{N}} \in \prod_{k=1}^\infty D(F^k)$ which verify, for all $k \in \mathbb{N}$,

(i) $\|\rho_k\|_{L^1} = 1$;
(ii) ρ_k is symmetric;
(iii) $\rho_k(\tilde{X}_k) = \int_{\tilde{\Omega}} \rho_{k+1}(\tilde{X}_{k+1}) \mathrm{d}\tilde{x}_{k+1}$;
(iv) there exits $C = C(\rho_*)$ such that $\|\rho_k\|_{L^\infty} \le C^k$.

Remark 4.10. If μ_* is a weak cluster point of $(\mu^N)_{N>1}$, then $\mu_* \in D(F^*)$. Indeed, the first three properties are easily verified. The fourth property follows from Proposition 3.2. To verify that $\mu_k \in D(F^k)$ we note, again by Proposition 3.6, that $I^k \mu_k \in L^1(\tilde{\Omega}^k)$. Hence, from Lemma 4.6 it follows that $[\mu_k \log \mu_k]^- \in L^1(\tilde{\Omega}^k)$. Finally, from *(iv)* we obtain $[\mu_k \log \mu_k]^+ \le [\mu_k \log C^k]^+ = k[\log C]^+ \mu_k \in L^1(\tilde{\Omega}^k)$.

For $\rho_* \in D(F^*)$ we define the following functionals

$$S^*(\rho_*) = \lim_{k\to\infty} \frac{1}{k} \int_{\tilde{\Omega}^k} \rho_k(\tilde{X}_k) \log \rho_k(\tilde{X}_k) \mathrm{d}\tilde{X}_k = \lim_{k\to\infty} \frac{1}{k} S^k(\rho_k),$$
$$E^*(\rho_*) = \frac{1}{2} \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} H(\tilde{x}_1, \tilde{x}_2) \rho_2(\tilde{x}_1, \tilde{x}_2) \mathrm{d}\tilde{x}_1 \mathrm{d}\tilde{x}_2,$$

$$
\begin{aligned}
J^*(\rho_*) &= \int_{\tilde\Omega} I(\tilde x_1)\rho_1(\tilde x_1)\mathrm{d}\tilde x_1,\\
F^*(\rho_*) &= S^*(\rho_*) + \beta E^*(\rho_*) + \gamma J^*(\rho_*).
\end{aligned}
$$

For $\rho_* \in D(F^*)$ is not difficult to see that $E^*(\rho_*) \in \mathbb{R}$ (apply Lemma 4.1 with $N = 2$) and $J^*(\rho_*) \in \mathbb{R}$ (since $\rho_1 \in D(F^1)$). From property (iii), by induction, it follows that

$$
\rho_k(\tilde X_k) = \int_{\tilde\Omega^{N-k}} \rho_N(\tilde X)\mathrm{d}\tilde X^{N-k}.
$$

By Proposition 4.3, $(S^k(\rho_k))_{k\in\mathbb{N}}$ is sub-additive. Thus the limit which defines S^* exists but it can be infinity. However, by property (iv), he have the following bounds

$$
\frac{1}{k}\int_{\tilde\Omega^k} \rho_k(\tilde X_k)\log\rho_k(\tilde X_k)\mathrm{d}\tilde X_k \le \frac{1}{k}\int_{\tilde\Omega^k} \rho_N(\tilde X_k)\log C^k\mathrm{d}\tilde X_k = \log C.
$$

Therefore, $S^*(\rho_*) \in \mathbb{R}$ and F^* are real valued.

Proposition 4.11. *Let $\rho_* \in D(F^*)$ and μ_* be a weak cluster point of $(\mu^N)_{N>1}$. We have*

(i) $\frac{1}{N}E^N(\rho_N) \to E^*(\rho_*)$ *as* $N \to \infty$;
(ii) $\frac{1}{N}J^N(\rho_N) = J^*(\rho_*)$ $\forall N \in \mathbb{N}$;
(iii) $\frac{1}{N}S^N(\rho_N) \to S^*(\rho_*)$ *as* $N \to \infty$;
(iv) $\frac{1}{N}F^N(\rho_N) \to F^*(\rho_*)$ *as* $N \to \infty$;
(v) $\frac{1}{N_j}E^{N_j}(\mu^{N_j}) \to E^*(\mu_*)$ *as* $j \to \infty$;
(vi) $\frac{1}{N_j}J^{N_j}(\mu^{N_j}) \to J^*(\mu_*)$ *as* $j \to \infty$;
(vii) $S^k(\mu_k) \le \liminf_{j\to\infty} S^k(\mu_k^{N_j})$;
(viii) $\frac{1}{N_j}F^{N_j}(\mu^{N_j}) \to F^*(\mu_*)$ *as* $j \to \infty$;
(ix) $\frac{1}{N_j}S^{N_j}(\mu^{N_j}) \to S^*(\mu_*)$ *as* $j \to \infty$;
(x) $\frac{1}{k}S^k(\mu_k) \le \liminf_{j\to\infty}\frac{1}{k}S^k(\mu_k^{N_j}) \le \limsup_{j\to\infty}\frac{1}{k}S^k(\mu_k^{N_j}) \le S^*(\mu_*)$.

Proof. (i), (ii) and (iii) have trivial proofs. (iv). It follows from (i), (ii) and (iii). Now we prove (v). By the symmetry of μ^{N_j} (see Remark 4.2) it is sufficient to show that

$$
\int_{\tilde\Omega}\int_{\tilde\Omega} H(\tilde x_1,\tilde x_2)\mu_2^{N_j}(\tilde x_1,\tilde x_2)\mathrm{d}\tilde x_1\mathrm{d}\tilde x_2 \to \int_{\tilde\Omega}\int_{\tilde\Omega} H(\tilde x_1,\tilde x_2)\mu_2(\tilde x_1,\tilde x_2)\mathrm{d}\tilde x_1\mathrm{d}\tilde x_2,
$$

as $j \to \infty$. But this is a consequence of Proposition 3.6, Lemma 4.4 and the weak convergence $\mu_2^{N_j} \rightharpoonup \mu_2$ in $L^2(\tilde\Omega^2)$.

(vi). This point is a consequence of the symmetry of μ^{N_j} (see Remark 4.2) and Proposition 3.6 (with $k = 1$ and $f = I$.)

(vii). By hypothesis, $\mu_k^{N_j} \rightharpoonup \mu_k$ weakly in $L^1(\tilde\Omega^k)$. Here, we suppose $\beta = 0$. Thus, from the weak lower semicontinuity of F^k in $L^1(\tilde\Omega^k)$ (Remark 4.9) it follows that

$$
S^k(\mu_k) + \gamma J^k(\mu_k) \le \liminf_{j\to\infty}\Big(S^k(\mu_k^{N_j}) + \gamma J^k(\mu_k^{N_j})\Big).
$$

Hence, it is enough to show that $J^k(\mu_k^{N_j}) \to J^k(\mu_k)$ as $j \to \infty$. We note (see Remark 4.2) that

$$
\frac{1}{k}J^k(\mu_k^{N_j}) = \int_{\tilde\Omega} I(\tilde x_1)\mu_1^{N_j}(\tilde x_1)\mathrm{d}\tilde x_1 = \frac{1}{N_j}J^{N_j}(\mu_k^{N_j}).
$$

The result follows now from (ii) and (vi).
$(viii)$. Fix $k \in \mathbb{N}$. For each $j \in \mathbb{N}$, large enough, we find integers m_j and n_j such that $N_j = m_j k + n_j$ and $0 < n_j \leq k$. By Proposition 4.3 we have

$$\frac{m_j}{N_j} S^k(\mu_k^{N_j}) + \frac{1}{N_j} S^{n_j}(\mu_{n_j}^{N_j}) \leq \frac{1}{N_j} S^{N_j}(\mu^{N_j}).$$

But (vii) implies that each one of the following k sequences

$$\Big(S^1(\mu_1^{N_j})\Big)_{j\in\mathbb{N}}, \ \dots \ , \ \Big(S^k(\mu_k^{N_j})\Big)_{j\in\mathbb{N}}$$

is bounded from below, and thus, there exists $C = C(k)$ such that $S^{n_j}(\mu_{n_j}^{N_j}) \geq -C$, $\forall j \in \mathbb{N}$. It follows that

$$\frac{m_j}{N_j} S^k(\mu_k^{N_j}) - \frac{C}{N_j} \leq \frac{1}{N_j} S^{N_j}(\mu^{N_j}). \tag{4.4}$$

By adding $\frac{\beta}{N_j} E^{N_j}(\mu^{N_j}) + \frac{\gamma}{N_j} J^{N_j}(\mu^{N_j})$ to (4.4) and considering that μ^N minimizes F^N (see Theorem 4.8) we obtain

$$\begin{aligned} \frac{m_j}{N_j} S^k(\mu_k^{N_j}) + \frac{\beta}{N_j} E^{N_j}(\mu^{N_j}) + \frac{\gamma}{N_j} J^{N_j}(\mu^{N_j}) - \frac{C}{N_j} &\leq \frac{1}{N_j} F^{N_j}(\mu^{N_j}) \\ &\leq \frac{1}{N_j} F^{N_j}(\mu_{N_j}). \end{aligned}$$

By taking the limit $j \to \infty$, from (iv), (v), (vi), (vii) and the fact that $m_j/N_j \to 1/k$ we conclude

$$\begin{aligned} \frac{1}{k} S^k(\mu_k) + \beta E^*(\mu_*) + \gamma J^*(\mu_*) &\leq \liminf_{j\to\infty} \frac{1}{N_j} F^{N_j}(\mu^{N_j}) \\ &\leq \limsup_{j\to\infty} \frac{1}{N_j} F^{N_j}(\mu^{N_j}) \leq F^*(\mu_*). \end{aligned}$$

Finally, we take $k \to \infty$ and we use (iii) to complete the proof of $(viii)$.
(ix). Follows form (v), (vi) and $(viii)$.
(x). The first inequality is just (vii). The second one is trivial. The last one follows from (ix) by taking limits in (4.4). ☐

Theorem 4.12. *Let μ_* be a weak cluster point of $(\mu^N)_{N>1}$. Then μ_* is a solution of*

$$\min\{F^*(\rho_*) : \rho_* \in D(F^*)\}.$$

Proof. Take $\rho_* \in D(F^*)$ and $j \in \mathbb{N}$. By Theorem 4.8, we have that μ^{N_j} minimizes F^{N_j} and thus

$$F^{N_j}(\mu^{N_j}) \leq F^{N_j}(\rho_{N_j}).$$

The result follows from Proposition 4.11 (iv) and $(viii)$. ☐

Definition 4.13. By $\mathcal{P}(\tilde{\Omega})$ we denote the space of Borelian probabilities on $\tilde{\Omega}$ endowed with the weak topology. We denote $\mathcal{Q}(\tilde{\Omega})$ the set of all Borelian probabilities ν on $\mathcal{P}(\tilde{\Omega})$ such that for ν-almost all ρ in the support of ν we have

- $\rho \in L^\infty(\tilde{\Omega}) \cap D(F)$ (where $D(F) = D(F^1)$);
- There exists $C = C(\nu)$ such that $\|\rho\|_{L^\infty} \leq C$.

Remark 4.14. In the previous definition, we can take the set $N(\tilde{\Omega}) = \Big\{\rho \in L^\infty(\tilde{\Omega}) : I\rho \in L^1(\tilde{\Omega})\Big\}$ instead of $L^\infty(\tilde{\Omega}) \cap D(F)$. Indeed, it is clear that $L^\infty(\tilde{\Omega}) \cap D(F) \subset N(\tilde{\Omega})$. On the other hand, if $\rho \in N(\tilde{\Omega}) \cap \mathcal{P}(\tilde{\Omega})$, then $\rho \in L^1(\tilde{\Omega})$ and $[\rho \log \rho]^+ \le \rho \log(1 + \|\rho\|_{L^\infty}) \in L^1(\tilde{\Omega})$. From Lemma 4.6 we deduce that $\rho \log \rho \in L^1(\tilde{\Omega})$. Hence $\rho \in D(F)$.

Theorem 4.15. *The application which maps $\nu \in \mathcal{Q}(\tilde{\Omega})$ to $\rho_* \in D(F^*)$ given by*

$$\rho_k(\tilde{X}_k) = \int_{\mathcal{P}}(\tilde{\Omega}) \rho(\tilde{x}_1) \cdots \rho(\tilde{x}_k)\nu(\mathrm{d}\rho) \quad \forall k \in \mathbb{N},$$

or, equivalently, by

$$\int_{\tilde{\Omega}^k} f(\tilde{X}_k)\rho_k(\tilde{X}_k)\mathrm{d}\tilde{X}_k = \int_{\mathcal{P}}(\tilde{\Omega}) \int_{\tilde{\Omega}^k} f(\tilde{X}_k)\rho(\tilde{x}_1)\cdots\rho(\tilde{x}_k)\mathrm{d}\tilde{X}_k \nu(\mathrm{d}\rho), \tag{4.5}$$

for all f such that $f\rho_k \in L^1(\tilde{\Omega}^k)$, is onto.

Proof. Let $\rho_* \in D(F^*)$. By Hewitt-Savage's Theorem (see [17], Theorem 7.4) there exists a (unique) Borelian probability ν on $\mathcal{P}(\tilde{\Omega})$ such that

$$\int_{\tilde{\Omega}^k} f(\tilde{X}_k)\rho_k(\tilde{X}_k)\mathrm{d}\tilde{X}_k = \int_{\mathcal{P}(\tilde{\Omega})} \int_{\tilde{\Omega}^k} f(\tilde{X}_k)\rho(\mathrm{d}\tilde{x}_1)\cdots\rho(\mathrm{d}\tilde{x}_k)\nu(\mathrm{d}\rho) \tag{4.6}$$

for all f such that $f\rho_k \in L^1(\tilde{\Omega}^k)$. By taking $f(\tilde{X}_k) = g(\tilde{x}_1)\cdots g(\tilde{x}_k)$ ($g \in L^1(\tilde{\Omega})$) and recalling that $\|\rho_k\|_{L^\infty} \le C^k$ we deduce

$$\int_{\mathcal{P}}(\tilde{\Omega}) \Big[\int_{\tilde{\Omega}} g(\tilde{x}_1)\rho(\mathrm{d}\tilde{x}_1)\Big]^k \nu(\mathrm{d}\rho) \le C^k \|g\|_{L^1}^k, \quad \forall g \in L^1(\tilde{\Omega}).$$

Hence,

$$\Big|\int_{\tilde{\Omega}} g(\tilde{x}_1)\rho(\mathrm{d}\tilde{x}_1)\Big| \le C\|g\|_{L^1} \quad \nu-\text{a.e. } \rho \in \mathcal{P}(\tilde{\Omega}).$$

This means that the support of ν is included in the ball of $L^\infty(\tilde{\Omega})$ with center 0 and radius C. Thus, the relation (4.6) becomes (4.5). To conclude, we should show that $\nu \in \mathcal{Q}(\tilde{\Omega})$.

Since $I\rho_1 \in L^1(\tilde{\Omega})$, by taking $f = I$ in (4.5), Fubini's Theorem gives

$$I\rho \in L^1(\tilde{\Omega}) \quad \nu-\text{a.e. } \rho \in \mathcal{P}(\tilde{\Omega}),$$

that is, ν is supported in $\{\rho \in L^\infty(\tilde{\Omega}) : I\rho \in L^1(\tilde{\Omega})\}$. Therefore, $\nu \in \mathcal{Q}(\tilde{\Omega})$ (see Remark 4.14). $\square$

Let $\rho_* \in D(F^*)$ and $\nu \in \mathcal{Q}(\tilde{\Omega})$ for which (4.5) holds. We apply this relation with $f = H$ ($H\rho_2 \in L^1(\tilde{\Omega}^2)$ by Lemma 4.1) and also with $f = I$ to obtain

$$E^*(\rho_*) = \int_{\mathcal{P}}(\tilde{\Omega}) E(\rho)\nu(\mathrm{d}\rho), \quad \text{where} \quad E(\rho) = \frac{1}{2}\int_{\tilde{\Omega}}\int_{\tilde{\Omega}} H(\tilde{x}_1, \tilde{x}_2)\rho(\tilde{x}_1)\rho(\tilde{x}_2)\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2,$$

and

$$J^*(\rho_*) = \int_{\mathcal{P}}(\tilde{\Omega}) J(\rho)\nu(\mathrm{d}\rho), \quad \text{where} \quad J(\rho) = \int_{\tilde{\Omega}} I(\tilde{x}_1)\rho(\tilde{x}_1)\mathrm{d}\tilde{x}_1.$$

In [29] it is shown that

$$S^*(\rho_*) = \int_{\mathcal{P}}(\tilde{\Omega}) S(\rho)\nu(\mathrm{d}\rho), \quad \text{where} \quad S(\rho) = \int_{\tilde{\Omega}} \rho(\tilde{x}_1)\log\rho(\tilde{x}_1)\mathrm{d}\tilde{x}_1.$$

Hence, by setting $F = S + \beta E + \gamma J$ we find

$$F^*(\rho_*) = \int_{\mathcal{P}} (\tilde{\Omega}) F(\rho)\nu(\mathrm{d}\rho). \tag{4.7}$$

All the functionals E, J, S and F are defined on $D(F)$.

For the rest of this article, ξ stands for an element of $\mathcal{Q}(\tilde{\Omega})$ associated (as stated in Theorem 4.15) to a weak cluster point $\mu_* = (\mu_k)_{k\in\mathbb{N}}$ of $(\mu^N)_{N>1}$. That is, μ_* and ξ are related by

$$\mu_k(\tilde{X}_k) = \int_{\mathcal{P}} (\tilde{\Omega})\mu(\tilde{x}_1)\cdots\mu(\tilde{x}_k)\xi(\mathrm{d}\mu), \quad \forall k \in \mathbb{N}. \tag{4.8}$$

Using (4.7) one can rewrite the claim of Theorem 4.12 to find that ξ is a solution of

$$\min\Big\{ \int_{\mathcal{P}} (\tilde{\Omega}) F(\rho) d\nu(\rho), \nu \in \mathcal{Q}(\tilde{\Omega})\Big\}.$$

Thus we obtain easily the following theorem.

Theorem 4.16. *The functional F is ξ-almost everywhere constant on the support of ξ and equal to its minimum value. In other words, ξ-almost all $\mu \in \operatorname{supp}\xi$ is a solution of*

$$\min\Big\{F(\rho) : \rho \in \mathcal{P}(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega}) \cap D(F)\Big\}.$$

Any solution of this problem will be called minimizer of F.

5. The mean field equation

For each cluster point μ_* of $(\mu^N)_{N>1}$ we have a measure $\xi \in \mathcal{Q}(\tilde{\Omega})$ such that (4.8) holds. Theorem 4.16, essentially, says that the minimality of μ_* is carried to ξ in the sense that in the support of ξ we should have only minimizers of F. Thus to each weak cluster point of $(\mu^N)_{N>1}$ corresponds an "average" (with respect to ξ) of minimizers of F. In this section we look for such minimizers and we shall see that they are solutions of a certain partial differential equation.

We recall that $N(\tilde{\Omega}) = \Big\{\rho \in L^\infty(\tilde{\Omega}) : I\rho \in L^1(\tilde{\Omega})\Big\}$. The potential of $\rho \in L^1(\tilde{\Omega}) \cap N(\tilde{\Omega})$, given by

$$v(x_1) = \int_{\tilde{\Omega}} r_2 V(x_1, x_2)\rho(\tilde{x}_2)\mathrm{d}\tilde{x}_2 = -\frac{1}{2\pi}\int_{\tilde{\Omega}} r_2 \log|x_1 - x_2|\rho(\tilde{x}_2)\mathrm{d}\tilde{x}_2,$$

is in $L^\infty_{\mathrm{loc}}(\tilde{\Omega})$. Indeed, for $|x_1| < R$ we have

$$|v(x_1)| \le \frac{\|\rho\|_{L^\infty}}{2\pi}\int_{|x_2|<1} -\log|x_2|\mathrm{d}x_2 + C\int_{\tilde{\Omega}} r_2(R^2 + |x_2|^2)|\rho(\tilde{x}_2)|\mathrm{d}\tilde{x}_2.$$

Hence, v is a solution (in the distribution sense) of

$$-\Delta v(x_1) = \int_{(0,1]} r_2\rho(\tilde{x}_2)P(\mathrm{d}r_2).$$

If ρ is a minimizer of F, then the corresponding Euler-Lagrange equation gives us a way to write ρ in terms of its potential v. We can insert this relation into the last equation and, by a boot-strap argument, show the regularity of v (and consequently of ρ.)

Proposition 5.1. *Let μ be a minimizer of F and u its potential. Then we have*

$$\mu(\tilde{x}_1) = \Big[\int_{\tilde{\Omega}} \mathrm{e}^{-\beta r_2 u(x_2) - \gamma I(\tilde{x}_2)} \mathrm{d}\tilde{x}_2\Big]^{-1} \mathrm{e}^{-\beta r_1 u(x_1) - \gamma I(\tilde{x}_1)}.$$

Proof. First, recall that $\mu \in \mathcal{P}(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega}) \cap D(F) \subset L^1(\tilde{\Omega}) \cap N(\tilde{\Omega})$. We proceed as in the first step in the proof of Theorem 4.8 to show that

$$\mu(\tilde{x}_1) = \begin{cases} C\mathrm{e}^{-\beta r_1 u(x_1) - \gamma I(\tilde{x}_1)} & \text{on } \Lambda, \\ 0 & \text{on } \Lambda^{\complement}, \end{cases}$$

where $\Lambda = \{\tilde{x}_1 \in \tilde{\Omega} : \mu(\tilde{x}_1) > 0\}$ and $C = \Big[\int_{\tilde{\Omega}} \mathrm{e}^{-\beta r_2 u(x_2) - \gamma I(\tilde{x}_2)} \mathrm{d}\tilde{x}_2\Big]^{-1}$. We should show that $|\Lambda^{\complement}| = 0$. Suppose, by contradiction, that $|\Lambda^{\complement}| > 0$. Then there exists a bounded measurable set $A \subset \Lambda^{\complement}$ such that $|A| = a > 0$. For $\delta > 0$ we set

$$\rho = \frac{\mu + \delta 1_A}{1 + \delta a}.$$

It is easy to see that $\rho \in \mathcal{P}(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega}) \cap D(F)$, $\|\rho\|_{L^1} = 1$ and, by simple but tedious computations, we find a constant C, not depending on δ, such that

$$F(\rho) \le F(\mu) + C\delta(1 + \delta + \log\delta).$$

Hence, for δ small enough we have $F(\rho) < F(\mu)$, which contradicts the minimality of μ. $\square$

In view of this theorem we can perform the boot-strap argument to show one of our main results:

Theorem 5.2. *For ξ-almost all $\mu \in \operatorname{supp}\xi$, its potential u is in $C^\infty(\mathbb{R}^2)$ and verifies the following equation (called Mean Field Equation, or MFE for short):*

$$-\Delta u(x_1) = \Big[\int_{\tilde{\Omega}} \mathrm{e}^{-\beta r_2 u(x_2) - \gamma I(\tilde{x}_2)} \mathrm{d}\tilde{x}_2\Big]^{-1} \int_{(0,1]} r_1 \mathrm{e}^{-\beta r_1 u(x_1) - \gamma I(\tilde{x}_1)} P(\mathrm{d}r_1). \tag{5.1}$$

Proposition 5.3. *If, for $\beta > -8\pi$ and $\gamma > 0$, the MFE has a unique solution u, then $(\mu_k^N)_{N>k}$ converges in $L^p(\tilde{\Omega}^k)$ to $\mu^{\otimes k}$ for all $k \in \mathbb{N}$ and for all $p \in [1, \infty)$, where μ is the distribution associated to u.*

Proof. Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. From Theorem 5.2, we have that for ξ-almost all $\mu \in \operatorname{supp}\xi$ the associated potential is a solution of MFE, ant thus, by uniqueness, equals to u. By Proposition 5.1 μ is the distribution associated to u. Thus, ξ is a Dirac measure concentrated on μ. It follows that $\mu_k(\tilde{X}_k) = \mu(\tilde{x}_1) \cdots \mu(\tilde{x}_k)$. Hence,

$$\frac{1}{k} S^k(\mu_k) = \frac{1}{k} \int_{\tilde{\Omega}^k} \mu_k(\tilde{X}_k) \log \mu_k(\tilde{X}_k) \mathrm{d}\tilde{X}_k = \int_{\tilde{\Omega}} \mu(\tilde{x}_1) \log \mu(\tilde{x}_1) \mathrm{d}\tilde{x}_1 = S(\mu).$$

We know also that

$$S^*(\mu_*) = \int_{\mathcal{P}} (\tilde{\Omega}) S(\rho) \mathrm{d}\xi(\rho) = S(\mu).$$

From Proposition 4.11, item (x), we have $S^k(\mu_k^{N_j}) \to S^k(\mu_k)$. Since S is strictly convex we conclude that $\mu_k^{N_j} \to \mu_k$ strongly in $L^p(\tilde{\Omega}^k)$. We have shown that every weakly cluster point of $(\mu_k^N)_{N>k}$ is a strongly one and unique. $\square$

6. An alternative to the study of MFE

One may think that it is straightforward to proceed as we did in [27] by introducting a functional G on $H^1(\mathbb{R}^2)$ and for which the Euler-Lagrange equation is the MFE. Thus changing our point of view to a variational problem on potentials. But there are many technical difficulties which arises. For example:

(i) We need an inequality of Poincaré type over $\mathbb{R}^2$. In fact, this is not a problem because we work with the measure $\mathrm{e}^{-\gamma r_1|x_1|^2}\mathrm{d}x_1$.

(ii) We need also an $\mathbb{R}^2$ version of Trudinger-Moser inequality. It may exist by the same reason as in (i).

(iii) The potential associated to $\mu \in D(F)$ is not always on $H^1(\mathbb{R}^2)$! Indeed, even, for example, if μ is compactly supported we have that $|\nabla u(x_1)|$ decreases at infinity as fast as $|x_1|^{-1}$ which is not in $L^2(\mathbb{R}^2)$.

Another inconvenience to study the MFE: in general, the hypothesis of Proposition 5.3 does not hold! Take, for example, $P = \delta_1$. The MFE becomes

$$-\Delta u(x_1) = \Big[\int_{\mathbb{R}^2} \mathrm{e}^{-\beta u(x_2)-\gamma|x_2|^2}\mathrm{d}x_2\Big]^{-1}\mathrm{e}^{-\beta u(x_1)-\gamma|x_1|^2}.$$

It is clear that, if u is a solution, then $u+C$ is also a solution. However the conclusion of Proposition 5.3 holds by modifying the argument. The fundamental idea of the proof of Proposition 5.3 was to follow the statistical approach backwards: each weakly cluster point of $(\mu^N)_{N>1}$ gives a solution of MFE. Hence, if this equation has a unique solution, then we have the uniqueness of minimizers of F. We conclude that $(\mu^N)_{N>1}$ has a unique weak cluster point and the convergence is strong. Now we do not have the uniqueness of MFE's solution anymore, but we can start our argument from the uniqueness of minimizers of F.

The previous remarks show us that it might be more convenient to forget the potentials and MFE and study the problem by means of distributions and minimizers of F. Theorem 4.16 gave us the existence now we have a uniqueness result.

Proposition 6.1. *If $P = \delta_1$ and $\beta > 0$, then F has a unique minimizer.*

Proof. Considering this measure P, we can identify $\tilde{\Omega} = \mathbb{R}^2 \times (0, 1]$ to $\mathbb{R}^2 \times \{1\}$ and thus to $\mathbb{R}^2$. Let μ be a minimizer of F. We shall show that

$$\int_{\mathbb{R}^2} x_1\mu(x_1)\mathrm{d}x_1 = 0.$$

First note that the integral above is convergent since $\mu \in L^1(\mathbb{R}^2)$ and $|x_1|^2\mu(x_1) \in L^1(\mathbb{R}^2)$. Suppose, by contradiction, that the integral above equals to $x_0 \neq 0$. Consider the function $\tilde{\mu}(x_1) = \mu(x_1 + x_0)$.

It is easy to see that $\tilde{\mu} \in \mathcal{P}(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega}) \cap D(F)$, $S(\tilde{\mu}) = S(\mu)$ and $E(\tilde{\mu}) = E(\mu)$. By a change of variables we have

$$\begin{aligned} J(\tilde{\mu}) &= \int_{\mathbb{R}^2} |x_1|^2\mu(x_1 + x_0)\mathrm{d}x_1 = \int_{\mathbb{R}^2} |x_1 - x_0|^2\mu(x_1)\mathrm{d}x_1 \\ &= \int_{\mathbb{R}^2} |x_1|^2\mu(x_1)\mathrm{d}x_1 - 2x_0 \cdot \int_{\mathbb{R}^2} x_1\mu(x_1)\mathrm{d}x_1 + |x_0|^2 \int_{\mathbb{R}^2} \mu(x_1)\mathrm{d}x_1 \\ &= J(\mu) - 2|x_0|^2 + |x_0|^2 = J(\mu) - |x_0|^2 < J(\mu). \end{aligned}$$

Hence, $F(\rho) < F(\mu)$, which contradicts the minimality of $F(\mu)$. To complete the proof, it suffices to show that F is strictly convex on

$$\Big\{\rho \in \mathcal{P}(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega}) \cap D(F) : \int_{\mathbb{R}^2} x_1\rho(x_1)\mathrm{d}x_1 = 0\Big\}.$$

Since S is strictly convex, J is linear, $\beta > 0$ and E is quadratic, it is enough to show that $E(\rho)$ is positive if $\rho \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $|x_2|^2\rho(x_2) \in L^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \rho(x_2)\mathrm{d}x_2 = 0 \quad \text{and} \quad \int_{\mathbb{R}^2} x_2\rho(x_2)\mathrm{d}x_2 = 0. \tag{6.1}$$

Take a such ρ and let $v \in L^\infty_{\rm loc}(\mathbb{R}^2)$ be its potential. We shall show that $v \in L^2(\mathbb{R}^2)$. For $x_1 \in \mathbb{R}^2$, we denote $A = A(x_1) = \{x_2 \in \mathbb{R}^2 : |x_2| \le |x_1|/2\}$ and write $v = -\frac{1}{2\pi}(v_1 + v_2)$, with

$$v_1(x_1) = \int_A \log|x_1 - x_2|\rho(x_2)\mathrm{d}x_2 \quad \text{and} \quad v_2(x_1) = \int_{A^\complement} \log|x_1 - x_2|\rho(x_2)\mathrm{d}x_2.$$

Hence, it suffices to show that $v_1, v_2 \in L^2(\mathbb{R}^2)$. From Taylor's expansion of $\log|x|$ we have

$$\log|x + y| = x \cdot y + \mathcal{O}(|y|^2) \quad \text{if } |x| = 1, \quad |y| \le \frac{1}{2},$$

uniformly on x since $\log|x|$ is C^2 on the compact $\{0.5 \le |x| \le 1\}$. Hence, take $x = x_1/|x_1|$ and $y = -x_2/|x_1|$ to obtain

$$\log|x_1 - x_2| = \log|x_1| - \frac{x_1}{|x_1|^2} \cdot x_2 + \frac{1}{|x_1|^2}\mathcal{O}(|x_2|^2) \quad \text{if } x_2 \in A.$$

Thus, from (6.1) it follows that

$$\begin{aligned}
&v_1(x_1)\\
&= \log|x_1| \int_A \rho(x_2)\mathrm{d}x_2 - \frac{x_1}{|x_1|^2} \cdot \int_A x_2\rho(x_2)\mathrm{d}x_2 + \frac{1}{|x_2|}\int_A \mathcal{O}(|x_2|^2)\rho(x_2)\mathrm{d}x_2\\
&= -\log|x_1| \int_{A^\complement} \rho(x_2)\mathrm{d}x_2 + \frac{x_1}{|x_1|^2} \cdot \int_{A^\complement} x_2\rho(x_2)\mathrm{d}x_2 + \frac{1}{|x_2|}\int_A \mathcal{O}(|x_2|^2)\rho(x_2)\mathrm{d}x_2.
\end{aligned}$$

Since $|x_2|^2\rho(x_2) \in L^1(\rho)$, for $|x_1| > 1$, we have

$$\begin{aligned}
|v_1(x_1)| &\le \log|x_1| \int_{A^\complement} |\rho(x_2)|\mathrm{d}x_2 + \frac{1}{|x_1|}\int_{A^\complement} |x_2||\rho(x_2)|\mathrm{d}x_2 + \mathcal{O}\Big(\frac{1}{|x_1|^2}\Big)\\
&\le \frac{4\log|x_1|}{|x_1|^2}\int_{\mathbb{R}^2} |x_2|^2|\rho(x_2)|\mathrm{d}x_2 + \frac{2}{|x_1|^2}\int_{\mathbb{R}^2} |x_2|^2|\rho(x_2)|\mathrm{d}x_2 + \mathcal{O}\Big(\frac{1}{|x_1|^2}\Big)\\
&= \mathcal{O}\Big(\frac{\log|x_1|}{|x_1|^2}\Big).
\end{aligned}$$

Therefore $v_1 \in L^2(\mathbb{R}^2)$.

To show that $v_2 \in L^2(\mathbb{R}^2)$, we write $v_2 = w_1 + w_2$, where $w_1 = v_2 1_{B_1(x_1)}$ and $w_2 = v_2 1_{B_1(x_1)^\complement}$. Holder's Inequality yields

$$\begin{aligned}
|w_1(x_1)| &\le \Big[\int_{A^\complement \cap B_1(x_1)} \Big|\log|x_1 - x_2|\Big|^3 |\rho(x_2)|\mathrm{d}x_2\Big]^{\frac{1}{3}} \Big[\int_{A^\complement \cap B_1(x_1)} |\rho(x_2)|\mathrm{d}x_2\Big]^{\frac{2}{3}}\\
&\le \|\rho\|_{L^\infty}\Big[\int_{B_1(0)} \Big|\log|x_2|\Big|^3 \mathrm{d}x_2\Big]^{\frac{1}{3}} \Big[4\int_{A^\complement} \frac{|x_2|^2}{|x_1|^2}\,|\rho(x_2)|\mathrm{d}x_2\Big]^{\frac{2}{3}} \le \frac{C}{|x_1|^{4/3}}.
\end{aligned}$$

Therefore, $w_1 \in L^2(\mathbb{R}^2)$. We take $C > 0$ such that $\log r \le Cr^{1/4}$ for all $r > 1$. It follows that

$$\begin{aligned}|w_2(x_1)| &\le C \int_{A^{\complement} \cap B_1(x_1)^{\complement}} |x_1 - x_2|^{1/4} |\rho(x_2)| \mathrm{d}x_2 \\ &\le C \int_{A^{\complement}} |x_2|^{1/4} \frac{|x_2|^{7/4}}{|x_1|^{7/4}} |\rho(x_2)| \mathrm{d}x_2 \le \frac{C}{|x_1|^{7/4}}.\end{aligned}$$

Therefore, $w_2 \in L^2(\mathbb{R}^2)$. From Plancherel's Theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \hat{v}(\eta)\overline{\hat{\rho}(\eta)} \mathrm{d}\eta &= \int_{\mathbb{R}^2} v(x_1)\rho(x_1) \mathrm{d}x_1 \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} V(x_1, x_2)\rho(x_1)\rho(x_2) \mathrm{d}x_1 \mathrm{d}x_2 = 2E(\rho),\end{aligned}$$

where $\hat{v}$ and $\hat{\rho}$ are the Fourier transforms of v and ρ, resp. But, $-\Delta v = \rho$ and thus $|\eta|^2 \hat{v}(\eta) = \hat{\rho}(\eta)$, hence

$$E(\rho) = \frac{1}{2} \int_{\mathbb{R}^2} |\eta|^2 |\hat{v}(\eta)|^2 \mathrm{d}\eta \ge 0.$$

□

The uniqueness of minimizer, as often, has followed from strict convexity of F. The previous proof can be adapted to a Dirac measure supported on any point. Unfortunately, without the strict convexity we don't know whether uniqueness holds or not. As a bad news, we have that the convexity is a particularity of the case of a Dirac measure.

Proposition 6.2. *If P is not a Dirac measure and $\beta > 0$, then the functional F is not convex on*

$$M = \Big\{\rho \in \mathcal{P}(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega}) \cap D(F) : \int_{\mathbb{R}^2} x_1 \rho(x_1) \mathrm{d}x_1 = 0\Big\}.$$

Proof. Since P is not a Dirac measure, we can take $0 < a_0 \le b_0 < a \le b$ such that $P([a_0, b_0]) > 0$ and $P([a, b]) > 0$. Taking a and b close enough, we may suppose $b_0 b < a^2$. Let N be an even number such that

$$b_0 b < a^2 \frac{(N-1)}{N}.$$

We set $B_0 = B_1(0)$. Take $t > 2$ (to be chosen later) and consider a regular polygon of N sides and radius t centered at origin. We denote by B_i the unity ball centered at its i-th vertex. Let $A = \bigcup_{i=1}^N B_i$ (t is large enough to have a disjoint union). We define also $\tilde{B}_0 = B_0 \times [a_0, b_0]$, $\tilde{B}_i = B_i \times [a, b]$ ($i = 1, \dots, N$) and $\tilde{A} = \bigcup_{i=1}^N \tilde{B}_i$.

Finally, we define $\rho_0 = \alpha_0 1_{\tilde{B}_0}$ and $\rho_1 = \alpha_1 1_{\tilde{A}}$, where $\alpha_0, \alpha_1 > 0$ are such that

$$\alpha_0 \int_{\tilde{B}_0} \mathrm{d}\tilde{x}_1 = \int_{\tilde{\Omega}} \rho_0(\tilde{x}_1) \mathrm{d}\tilde{x}_1 = 1$$

and

$$N\alpha_1 \int_{\tilde{B}_i} \mathrm{d}\tilde{x}_1 = \alpha_1 \int_{\tilde{A}} \mathrm{d}\tilde{x}_1 = \int_{\tilde{\Omega}} \rho_1(\tilde{x}_1) \mathrm{d}\tilde{x}_1 = 1.$$

By the symmetries of $\tilde{B}_0$ and $\tilde{A}$ with respect to origin, it is easy too see that $\rho_0, \rho_1 \in M$. Simple computations give

$$S\left(\frac{1}{2}(\rho_0+\rho_1)\right) = \frac{1}{2}S(\rho_0) + \frac{1}{2}S(\rho_1) - \log 2$$

and

$$J\left(\frac{1}{2}(\rho_0+\rho_1)\right) = \frac{1}{2}J(\rho_0) + \frac{1}{2}J(\rho_1).$$

Since E is quadratic, we have

$$E\left(\frac{1}{2}(\rho_0+\rho_1)\right) = \frac{1}{2}E(\rho_0) + \frac{1}{2}E(\rho_1) - \frac{1}{4}E(\rho_0-\rho_1).$$

Hence,

$$F\left(\frac{1}{2}(\rho_0+\rho_1)\right) = \frac{1}{2}F(\rho_0) + \frac{1}{2}F(\rho_1) - \log 2 - \frac{\beta}{4}E(\rho_0-\rho_1).$$

We shall show that $E(\rho_0-\rho_1) \to -\infty$ as $t \to \infty$. Therefore, for t large enough, we shall have

$$F\left(\frac{1}{2}(\rho_0+\rho_1)\right) > \frac{1}{2}F(\rho_0) + \frac{1}{2}F(\rho_1).$$

The result follows.

Since $\tilde{B}_0, \tilde{B}_1 \dots, \tilde{B}_N$ are disjoints and ρ_0 and ρ_1 are constants and supported on these balls, we have

$$\begin{aligned}
E(\rho_0-\rho_1) = &-\frac{\alpha_0^2}{4\pi}\int_{\tilde{B}_0}\int_{\tilde{B}_0} r_1r_2\log|x_1-x_2|\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2 \\
&-\frac{\alpha_1^2}{4\pi}\sum_{i=1}^{N}\int_{\tilde{B}_i}\int_{\tilde{B}_i} r_1r_2\log|x_1-x_2|\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2 \\
&+\frac{\alpha_0\alpha_1}{4\pi}\sum_{i=1}^{N}\int_{\tilde{B}_0}\int_{\tilde{B}_i} r_1r_2\log|x_1-x_2|\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2 \\
&-\frac{\alpha_1^2}{4\pi}\sum_{i\neq j}^{N}\int_{\tilde{B}_i}\int_{\tilde{B}_j} r_1r_2\log|x_1-x_2|\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2.
\end{aligned}$$

Clearly, the first term on the RHS does not depend on t. The second neither, since by a translation on x variables, the integrand does not change and the domain becomes, for example, $(B_0 \times [a,b])^2$. For t large enough the last two terms behave like

$$\Phi(t) = \frac{\alpha_0\alpha_1}{4\pi}\sum_{i=1}^{N}\int_{\tilde{B}_0}\int_{\tilde{B}_i} r_1r_2\log t\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2 - \frac{\alpha_1^2}{4\pi}\sum_{i\neq j}^{N}\int_{\tilde{B}_i}\int_{\tilde{B}_j} r_1r_2\log(\theta t)\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2,$$

where θ is a constant (the ratio side/radius for the regular polygon of N sides). But, $\log(\theta t) = \log\theta + \log t$ and, by the same translation on x variables as before, we conclude that the factor multiplying $\log\theta$ does not depend on t. Therefore, we may suppose $\theta = 1$. Then

$$\Phi(t) = \frac{\log t}{4\pi}\Big[\alpha_0\alpha_1\sum_{i=1}^{N}\int_{\tilde{B}_0}\int_{\tilde{B}_i} r_1r_2\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2 - \alpha_1^2\sum_{i\neq j}^{N}\int_{\tilde{B}_i}\int_{\tilde{B}_j} r_1r_2\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2\Big]$$

$$
\begin{aligned}
&\le \frac{\log t}{4\pi}\Big[\alpha_0\alpha_1 bb_0 \sum_{i=1}^{N}\int_{\tilde B_0}\int_{\tilde B_i} \mathrm{d}\tilde x_1 \mathrm{d}\tilde x_2 - \alpha_1^2 a^2 \sum_{i\neq j}^{N}\int_{\tilde B_i}\int_{\tilde B_j} \mathrm{d}\tilde x_1 \mathrm{d}\tilde x_2\Big]\\
&= \frac{\log t}{4\pi}\Big[bb_0 - a^2\frac{(N-1)}{N}\Big]
\end{aligned}
$$

We conclude that $\Phi(t) \to -\infty$ as $t \to \infty$. □

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