

**LOCAL $W^{1,p}$ -REGULARITY ESTIMATES FOR WEAK
SOLUTIONS OF PARABOLIC EQUATIONS WITH SINGULAR
DIVERGENCE-FREE DRIFTS**

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ABSTRACT. We study weighted Sobolev regularity of weak solutions of non-homogeneous parabolic equations with singular divergence-free drifts. Assuming that the drifts satisfy some mild regularity conditions, we establish local weighted L^p -estimates for the gradients of weak solutions. Our results improve the classical one to the borderline case by replacing the L^∞ -assumption on solutions by solutions in the John-Nirenberg BMO space. The results are also generalized to parabolic equations in divergence form with small oscillation elliptic symmetric coefficients and therefore improve many known results.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

We study local weighted L^p -estimates for the gradients of weak solutions of parabolic equations with low regularity of the divergence-free drifts. A typical example is the parabolic equation

$$u_t - \Delta u - b \cdot \nabla u = 0, \quad \mathbb{R}^n \times (0, \infty), \quad (1.1)$$

where the drift $b : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}^n$ is of divergence-free, i.e. $\operatorname{div}(b(\cdot, t)) = 0$ in the sense of distribution for a.e. t . Due to its relevance in many applications such as in fluid dynamics, and biology, the equation (1.1) has been investigated by many mathematicians (for example [15, 16, 28, 34]). Local boundedness, Harnack's inequality, and Hölder's regularity are established in [15, 24, 28, 32, 34] with possible singular drifts. Many other classical results with regular drifts can be found in [14, 17, 18, 19]. Hölder's regularity for the fractional Laplace type equations of the form (1.1) are extensively studied recently (see [7, 13, 29]).

Unlike the mentioned work, this note investigates the Sobolev regularity of weak solutions of (1.1) in weighted spaces. Our goal is to establish local weighted estimates of Calderón-Zygmund type for weak solutions of (1.1) with some mild requirements on the regularity of the drifts b . We study the following parabolic equation that is more general than (1.1):

$$u_t - \operatorname{div}[a(x, t)\nabla u] - b \cdot \nabla u = \operatorname{div}(F), \quad (1.2)$$

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where $a = (a^{ij})_{i,j=1}^n$ is a given symmetric $n \times n$ matrix of bounded measurable functions, and F, b are given vector fields with $\operatorname{div}(b) = 0$ in distribution sense. The exact required regularity conditions of a, b, F will be specified.

To state our results, we introduce some notation. For each $r > 0$, and $z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, we denote $Q(z_0)$ the parabolic cylinder in \mathbb{R}^{n+1}

$$Q_r(z) = B_r(x_0) \times \Gamma_r(t_0),$$

where

$$\Gamma_r(t_0) = (t_0 - r^2, t_0 + r^2), \quad B_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}.$$

When $z_0 = (0, 0)$, we also write

$$Q_r = Q_r(0, 0), \quad \text{for } 0 < r < \infty.$$

As we are interested in the local regularity, we reduce our study to the equation

$$u_t - \operatorname{div}[a(x, t)\nabla u] - b \cdot \nabla u = \operatorname{div}(F), \quad \text{in } Q_2, \quad (1.3)$$

for given $a : Q_2 \rightarrow \mathbb{R}^{n \times n}$, $b, F : Q_2 \rightarrow \mathbb{R}^n$, and

$$\operatorname{div}(b(\cdot, t)) = 0, \quad \text{in the sense of distribution in } B_2 \text{ for a.e. } t \in \Gamma_2. \quad (1.4)$$

For the coefficient matrix a , we assume that

$$\begin{aligned} a &= (a^{ki})_{k,i=1}^n : Q_2 \rightarrow \mathbb{R}^{n \times n} \text{ is symmetric, measurable,} \\ &\text{and there exists } \Lambda \text{ such that } \Lambda^{-1}|\xi|^2 \leq \langle a(x, t)\xi, \xi \rangle \leq \Lambda|\xi|^2 \\ &\text{for a.e. } (x, t) \in Q_2 \text{ and all } \xi \in \mathbb{R}^n. \end{aligned} \quad (1.5)$$

We also require that the matrix a have a small oscillation. Therefore, we need the following definition.

Definition 1.1. Let $a : Q_2 \rightarrow \mathbb{R}^{n \times n}$ be a measurable matrix valued function. We define

$$[a]_{\operatorname{BMO}(Q_1)} = \sup_{0 < \rho \leq 1} \sup_{(y, s) \in \overline{Q_1}} \frac{1}{|Q_\rho(y, s)|} \int_{Q_\rho(y, s)} |a(x, t) - \bar{a}_{B_\rho(y)}(t)|^2 dx dt,$$

where $\bar{a}_U(t) = \int_U a(x, t) dx$ is the average of a in the set $U \subset B_2$.

For the regularity of the vector field b , we need the following function space, which was introduced in [20, 25].

Definition 1.2. For $x_0 \in \mathbb{R}^n$ and $r > 0$, a locally square integrable function f defined in a neighborhood of $B_r(x_0)$ is said to be in $\mathcal{V}^{1,2}(B_r(x_0))$ if there is $k \in [0, \infty)$ such that

$$\int_{B_r(x_0)} |f(x)|^2 \varphi(x)^2 dx \leq k \int_{B_r(x_0)} |\nabla \varphi(x)|^2 dx, \quad \forall \varphi \in C_0^\infty(B_r(x_0)). \quad (1.6)$$

We denote

$$\|f\|_{\mathcal{V}^{1,2}(B_r(x_0))}^2 = \inf\{k \in [0, \infty) \text{ such that (1.6) holds}\}.$$

In this article, the numbers $s, s' \in (1, \infty)$, α , and λ are fixed and satisfy

$$\frac{1}{s'} + \frac{1}{s} = 1, \quad -(n+2) \leq \lambda \leq s', \quad \alpha = \lambda(s-1). \quad (1.7)$$

We also denote $L^p(Q, \omega)$ the weighted Lebesgue space with weight ω :

$$L^p(Q, \omega) = \left\{ f : Q \rightarrow \mathbb{R} : \|f\|_{L^p(Q, \omega)} := \left(\int_Q |f(x, t)|^p \omega(x, t) dx dt \right)^{1/p} < \infty \right\},$$

for $1 < p < \infty$. At this moment, we refer the readers to Section 2 for the definition of weak solutions of (1.3), the definition of Muckenhoupt A_q weights, and the definition of fractional Hardy-Littlewood maximal functions \mathcal{M}_α . Our main result is the following theorem on local weighted $W^{1,p}$ -regularity estimates for weak solutions of (1.3).

Theorem 1.3. *Let Λ, M_0 be positive numbers, $p \in (2, \infty)$, and $\omega \in A_{p/2}$. Let s, s', λ, α be as in (1.7). Then, there exists $\delta = \delta(\Lambda, M_0, s, \lambda, [\omega]_{A_{p/2}}, p, n) > 0$ sufficiently small such that the following holds: Suppose that a satisfies (1.5), $F \in L^2(Q_2)$, and $b \in L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))$ such that (1.4) holds, and*

$$[a]_{BMO(Q_1)} < \delta, \quad \|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} \leq M_0.$$

Then for every weak solution u of (1.3), the following estimate holds

$$\begin{aligned} \int_{Q_1} |\nabla u|^p \omega(z) dz \leq C \left[\|F\|_{L^p(Q_2, \omega)}^p + [u]_{s', \lambda, Q_1}^p \|\mathcal{M}_{\alpha, Q_2}(|b|^s)^{2/s}\|_{L^{p/2}(Q_2, \omega)}^{p/2} \right. \\ \left. + \omega(Q_1) \|\nabla u\|_{L^2(Q_2)}^p \right], \end{aligned} \quad (1.8)$$

as long as its right hand side is finite. Here, $[u]_{s', \lambda, Q_1}$ is the parabolic semi-Campanato's norm of u on Q_1 ,

$$[u]_{s', \lambda, Q_1} = \sup_{0 < \rho < 1, z \in Q_1} \left[\rho^{-\lambda} \int_{Q_\rho(z)} |u(x, t) - \bar{u}_{Q_\rho(z)}|^{s'} dx dt \right]^{1/s'},$$

and $C > 0$ is a constant depending only on $\Lambda, M_0, s, \lambda, p, n$ and $[\omega]_{A_{p/2}}$.

We now point out a few remarks regarding Theorem 1.3. Firstly, observe that the standard Calderón-Zygmund theory can be applied directly to (1.3) to obtain

$$\|\nabla u\|_{L^p(Q_1)} \leq C [\|u\|_{L^\infty(Q_2)} \|b\|_{L^p(Q_2)} + \dots],$$

as long as $u \in L^\infty(Q_2)$. Theorem 1.3 improves this Calderón-Zygmund estimate theory for the equation (1.3) to the borderline case, replacing the assumption $u \in L^\infty(Q_2)$ by $u \in BMO(Q_1)$. Indeed, if we take $\lambda = 0$ (and then $\alpha = 0$), then the estimate (1.8) reduces to

$$\|\nabla u\|_{L^p(Q_1, \omega)} \leq C [\|u\|_{BMO(Q_1)} \|b\|_{L^p(Q_2, \omega)} + \dots]. \quad (1.9)$$

Secondly, the weighted $W^{1,p}$ -regularity estimates are useful in some applications. For example, in [2, 3], the weighted $W^{1,p}$ -regularity estimates are key ingredients for proving the existence and uniqueness of very weak solutions of some classes of elliptic equations. Moreover, with some specific choice of ω , the weighted estimate (1.8) is known to produce the regularity estimates for ∇u in Morrey spaces, see for example [1, 4, 10, 21]. Lastly, when $\alpha > 0$, because $\mathcal{M}_\alpha \leq \mathbf{I}_\alpha$, the Riesz potential of order α of b , i.e. $\mathcal{M}_\alpha(|b|^s)^{2/s}$, is more regular than b . This fact enables the estimate (1.8) to be useful in some applications. To see this, we just simply consider the stationary case (i.e. u is time independent), $n \geq 3$ and $s = 2$. Assume, for example, that $b \in L^{n, \infty}(B_2) \subset \mathcal{V}^{1,2}(B_2)$, where $L^{n, \infty}$ is the weak L^n -space, and assume also that F is regular enough. Then, it is proved in [28, 34] that u is Hölder. Therefore,

$[u]_{2,\lambda,B_1} < \infty$ with some $\lambda > 0$. From this, and (1.7), we see that $\alpha > 0$, and we then can find some small constant $\varepsilon_0 > 0$ such that

$$\|\mathcal{M}_{\alpha,B_2}(|b|)\|_{L^p(B_2)} \leq C\|b\|_{L^{n,\infty}(B_2)} < \infty, \quad \text{for all } p < n + \varepsilon_0.$$

Therefore, (1.8) gives the estimate of $\|\nabla u\|_{L^p(B_1)}$ with some $p \in [2, n + \varepsilon_0]$. This estimate with $p > n$ is useful in [12] to prove the regularity, and uniqueness of very weak $W^{1,q}$ -solution of the stationary equation of (1.3), with $1 < q < 2$. Details of this discussion and its application can be also found in [26].

We use perturbation approach introduced in [6] to prove Theorem 1.3. Our approach is also influenced by [5, 11, 21, 23, 27, 33]. To implement the approach, we introduce the function $B(x, t) = ([u]_{s',\lambda,Q_1}|b(x, t)|)^s$ in (5.1), which is invariant under the standard dilation, and translation. This function also captures the cancellation due to the divergence-free of the vector field b , which is the main reason so that the estimate (1.9) holds in the borderline case. The results on the doubling property and reverse Hölder's inequality for the Muckenhoupt weights due to Coifman, and Fefferman in [8] are also used frequently to derive the weighted estimates.

We conclude the section by introducing the organization of the paper. Section 2 gives definitions, notations, and some preliminaries results needed in the paper. Some simple energy estimates for weak solutions of (1.3) is given in Section 3. The main step in the perturbation technique, the approximation estimates, is carried out in Section 4. Section 5.1 is about the proof of Theorem 1.3.

2. DEFINITIONS OF WEAK SOLUTIONS, AND PRELIMINARIES ON WEIGHTED INEQUALITIES

2.1. Definitions of weak solutions. For each $z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, and for any parabolic cylinder $Q_R(z_0)$, we denote $\partial_p Q_R(z_0)$ the parabolic boundary of $Q_R(z_0)$, i.e.

$$\partial_p Q_R(z_0) = (B_R(x_0) \times \{t_0 - R^2\}) \cup (\partial B_R(x_0) \times [t_0 - R^2, t_0 + R^2]).$$

The following standard definitions of weak solutions are also recalled.

Definition 2.1. Let Q_r be a parabolic cube. For every $f \in L^2(Q_r)$, $F, b \in L^2(Q_r)^n$, we say that u is a weak solution of

$$u_t - \operatorname{div}[a\nabla u] - b \cdot \nabla u = \operatorname{div}(F) + f, \quad \text{in } Q_r,$$

if $u \in L^2(\Gamma_r, H^1(B_r))$, $u_t \in L^2(\Gamma_r, H^{-1}(B_r))$, and

$$\begin{aligned} & \int_{\Gamma_r} \langle u_t, \varphi \rangle_{H^{-1}(B_r), H_0^1(B_r)} dt + \int_{Q_r} [\langle a\nabla u, \nabla \varphi \rangle - b \cdot \nabla u \varphi] dx dt \\ & = \int_{Q_r} [f\varphi - \langle F, \nabla \varphi \rangle] dx dt, \end{aligned}$$

for all $\varphi \in \{\phi \in C^\infty(\overline{Q_r}) : \phi = 0 \text{ on } \partial_p Q_r\}$.

The following definition of weak solution is also needed.

Definition 2.2. Let Q_r be a parabolic cube. For every $f \in L^2(Q_r)$, $F, b \in L^2(Q_r)^n$, and for $g \in L^2(\Gamma_r, H^1(B_r))$, we say that u is a weak solution of

$$\begin{aligned} u_t - \operatorname{div}[a\nabla u] - b \cdot \nabla u &= \operatorname{div}(F) + f, \quad \text{in } Q_r, \\ u &= g, \quad \text{on } \partial_p Q_r, \end{aligned}$$

if u is a weak solution of

$$u_t - \operatorname{div}[a\nabla u] - b \cdot \nabla u = \operatorname{div}(F) + f, \quad \text{in } Q_r,$$

in the sense of Definition 2.1 and $u - g \in \{\phi \in L^2(\Gamma_r, H^1(B_r)) : \phi = 0 \text{ on } \partial_p Q_r\}$.

2.2. Muckenhoupt weights and Hardy-Littlewood maximal functions.

For each $1 \leq q < \infty$, a non-negative, locally integrable function $\mu : \mathbb{R}^{n+1} \rightarrow [0, \infty)$ is said to be in the class of parabolic A_q of Muckenhoupt weights if

$$[\mu]_{A_q} := \sup_{r>0, z \in \mathbb{R}^{n+1}} \left(\int_{Q_r(z)} \mu(x, t) \, dx \, dt \right) \left(\int_{Q_r(z)} \mu(x, t)^{\frac{1}{1-q}} \, dx \, dt \right)^{q-1} < \infty,$$

if $q > 1$, and

$$[\mu]_{A_1} := \sup_{r>0, z \in \mathbb{R}^{n+1}} \left(\int_{Q_r(z)} \mu(x, t) \, dx \, dt \right) \|\mu^{-1}\|_{L^\infty(Q_r(z))} < \infty \quad \text{if } q = 1.$$

It is well known that the class of A_p -weights satisfies the reverse Hölder’s inequality and the doubling properties, see for example [8, 9, 30]. In particular, a measure with an A_p -weight density is, in some sense, comparable with the Lebesgue measure.

Lemma 2.3 ([8]). *For $1 < q < \infty$, the following two statements hold*

- (i) *If $\mu \in A_q$, then for every parabolic cube $Q \subset \mathbb{R}^{n+1}$ and every measurable set $E \subset Q$, $\mu(Q) \leq [\mu]_{A_q} \left(\frac{|Q|}{|E|}\right)^q \mu(E)$;*
- (ii) *If $\mu \in A_q$, then there is $C = C([\mu]_{A_q}, n)$ and $\beta = \beta([\mu]_{A_q}, n) > 0$ such that $\mu(E) \leq C \left(\frac{|E|}{|Q|}\right)^\beta \mu(Q)$, for every parabolic cube $Q \subset \mathbb{R}^{n+1}$ and every measurable set $E \subset Q$.*

Let us also recall the definition of the parabolic fractional Hardy-Littlewood maximal operators which will be needed later.

Definition 2.4. Let $\alpha \in \mathbb{R}$, the parabolic Hardy-Littlewood fractional maximal function of order α of a locally integrable function f on \mathbb{R}^{n+1} is defined by

$$(\mathcal{M}_\alpha f)(x, t) = \sup_{\rho>0} \rho^\alpha \int_{Q_\rho(x, t)} |f(y, s)| \, dy \, ds.$$

If f is defined in a region $U \subset \mathbb{R}^n \times \mathbb{R}$, then we denote

$$\mathcal{M}_{\alpha, U} f = \mathcal{M}_\alpha(\chi_U f).$$

Moreover, when $\alpha = 0$, we write

$$\mathcal{M} f = \mathcal{M}_0 f, \quad \mathcal{M}_U f = \mathcal{M}_{0, U} f.$$

The following boundedness of the Hardy-Littlewood maximal operator is due to Muckenhoupt [22]. For the proof of this lemma can be found in [9, 30].

Lemma 2.5. *Assume that $\mu \in A_q$ for some $1 < q < \infty$. Then, the following statements hold:*

- (i) *Strong (q, q) : There exists a constant $C = C([\mu]_{A_q}, n, q)$ such that*

$$\|\mathcal{M}\|_{L^q(\mathbb{R}^{n+1}, \mu) \rightarrow L^q(\mathbb{R}^{n+1}, \mu)} \leq C.$$

- (ii) *Weak $(1, 1)$: There exists a constant $C = C(n)$ such that for any $\lambda > 0$, we have*

$$|\{(x, t) \in \mathbb{R}^{n+1} : \mathcal{M}(f) > \lambda\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^{n+1}} |f(x, t)| \, dx \, dt.$$

2.3. Some useful measure theory lemmas. We collect some results needed in the paper. Our first lemma is the standard result in measure theory.

Lemma 2.6. *Assume that $g \geq 0$ is a measurable function in a bounded subset $U \subset \mathbb{R}^{n+1}$. Let $\theta > 0$ and $\varpi > 1$ be given constants. If μ is a weight in $L^1_{loc}(\mathbb{R}^{n+1})$, then for any $1 \leq p < \infty$*

$$g \in L^p(U, \mu) \Leftrightarrow S := \sum_{j \geq 1} \varpi^{pj} \mu(\{x \in U : g(x) > \theta \varpi^j\}) < \infty.$$

Moreover, there exists a constant $C > 0$ such that

$$C^{-1}S \leq \|g\|_{L^p(U, \mu)}^p \leq C(\mu(U) + S),$$

where C depends only on θ, ϖ and p .

The following lemma is commonly used, and it is a consequence of the Vitali's covering lemma; its proof can be found in [21, Lemma 3.8].

Lemma 2.7. *Let μ be an A_q weight for some $q \in (1, \infty)$ be a fixed number. Assume that $E \subset K \subset Q_1$ are measurable sets for which there exists $\epsilon, \rho_0 \in (0, 1/4)$ such that*

- (i) $\mu(E) < \epsilon \mu(Q_1(z))$ for all $z \in \overline{Q_1}$, and
- (ii) for all $z \in Q_1$ and $\rho \in (0, \rho_0]$, if $\mu(E \cap Q_\rho(z)) \geq \epsilon \mu(Q_\rho(z))$, then $Q_\rho(z) \cap Q_1 \subset K$.

Then with $\epsilon_1 = \epsilon(20)^{(n+2)q}[\mu]_{A_q}^2$, the following estimate holds

$$\mu(E) \leq \epsilon_1 \mu(K).$$

3. CACCIOPPOLI'S TYPE ESTIMATES

Suppose that a satisfies (1.5), and $b \in L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))^n \cap L^2(Q_2)^n$ with $\operatorname{div}(b) = 0$. In this section, let u be a weak solution of

$$u_t - \operatorname{div}[a(x, t)\nabla u] - b(x, t) \cdot \nabla u = \operatorname{div}(F), \quad \text{in } Q_2.$$

Also, let v be the weak solution of

$$\begin{aligned} v_t - \operatorname{div}[\bar{a}_{B_{7/4}}(t)\nabla v] &= 0, & Q_{7/4}, \\ v &= u, & \partial_p Q_{7/4}. \end{aligned}$$

The meanings for weak solutions of these equations are given in Definition 2.1 and Definition 2.2, respectively. We will derive some fundamental estimates for u and v .

Lemma 3.1. *Let $w = u - v$, then there exists a constant C depending on only Λ, n such that*

$$\begin{aligned} & \sup_{t \in \Gamma_{7/4}} \int_{B_{7/4}} w^2(x, t) dx + \int_{Q_{7/4}} |\nabla w|^2 dx dt \\ & \leq C \left[(\|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))}^2 + 1) \int_{Q_{7/4}} |\nabla u|^2 dx dt + \int_{Q_{7/4}} |F|^2 dx dt \right]. \end{aligned}$$

Proof. Note that w is a weak solution of

$$\begin{aligned} w_t - \operatorname{div}[\bar{a}_{B_{7/4}}(t)\nabla w + (a - \bar{a}_{B_{7/4}}(t))\nabla u] - b \cdot \nabla u &= \operatorname{div}(F), & \text{in } Q_{7/4}, \\ w &= 0, & \text{on } \partial_p Q_{7/4}. \end{aligned}$$

Multiplying this equation by w , and using the integration by parts in x , we see that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{B_{7/4}} w^2(x, t) dx + \int_{B_{7/4}} \langle \bar{a}_{B_{7/4}}(t) \nabla w, \nabla w \rangle dx \\ &= - \int_{B_{7/4}} \langle (a - \bar{a}_{B_{7/4}}(t)) \nabla u, \nabla w \rangle dx + \int_{B_{7/4}} [b \cdot \nabla u] w dx - \int_{B_{7/4}} F \cdot \nabla w dx. \end{aligned}$$

Then, by integrating this equality in time and using the ellipticity condition (1.5), we obtain

$$\begin{aligned} & \frac{1}{2} \sup_{\Gamma_{7/4}} \int_{B_{7/4}} w^2 dx + \Lambda^{-1} \int_{Q_{7/4}} |\nabla w|^2 dx dt \\ & \leq \int_{Q_{7/4}} | \langle (a - \bar{a}_{B_{7/4}}(t)) \nabla u, \nabla w \rangle | dx dt + \int_{Q_{7/4}} |b \cdot \nabla u w| dx dt \quad (3.1) \\ & \quad + \int_{Q_{7/4}} [|F \cdot \nabla w|] dx dt. \end{aligned}$$

We now estimate term by term the right-hand side of (3.1). From Hölder's inequality, and the Young's inequality, and the fact that $w = 0$ on $\partial_p Q_{7/4}$, the second term in the right hand side of (3.1) can be estimated as

$$\begin{aligned} & \int_{Q_{7/4}} |b| |w| |\nabla u| dx dt \\ & \leq \left\{ \int_{Q_{7/4}} |b|^2 w^2 dx dt \right\}^{1/2} \left\{ \int_{Q_{7/4}} |\nabla u|^2 dx dt \right\}^{1/2} \quad (3.2) \\ & \leq \|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} \left\{ \int_{Q_{7/4}} |\nabla w|^2 dx dt \right\}^{1/2} \left\{ \int_{Q_{7/4}} |\nabla u|^2 dx dt \right\}^{1/2} \\ & \leq \frac{\Lambda^{-1}}{6} \int_{Q_{7/4}} |\nabla w|^2 dx dt + C(\Lambda) \|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))}^2 \int_{Q_{7/4}} |\nabla u|^2 dx dt. \end{aligned}$$

On the other hand, by the boundedness of a in (1.5), and the Hölder's inequality, we conclude that

$$\int_{Q_{7/4}} | \langle (a - \bar{a}_{B_{7/4}}) \nabla u, \nabla w \rangle | dx dt \leq C(\Lambda) \int_{Q_{7/4}} |\nabla u|^2 dx dt + \frac{\Lambda^{-1}}{6} \int_{Q_{7/4}} |\nabla w|^2 dx dt,$$

and

$$\int_{Q_{7/4}} |F \cdot \nabla w| dx dt \leq C(\Lambda) \int_{Q_{7/4}} |F|^2 dx dt + \frac{\Lambda^{-1}}{6} \int_{Q_{7/4}} |\nabla w|^2 dx dt.$$

Collecting all of the estimates, from (3.1) we obtain

$$\begin{aligned} & \frac{1}{2} \sup_{\Gamma_{7/4}} \int_{B_{7/4}} w^2(x, t) dx + \Lambda^{-1} \int_{Q_{7/4}} |\nabla w|^2 dx dt \\ & \leq \frac{\Lambda^{-1}}{2} \int_{Q_{7/4}} |\nabla w|^2 dx dt + C \left(\left[\|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))}^2 + 1 \right] \int_{Q_{7/4}} |\nabla u|^2 dx dt \right. \\ & \quad \left. + \int_{Q_{7/4}} |F|^2 dx \right). \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup_{t \in \Gamma_{7/4}} \int_{B_{7/4}} w^2(x, t) dx + \int_{Q_{7/4}} |\nabla w|^2 dx dt \\ & \leq C(\Lambda) \left(\left[\|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))}^2 + 1 \right] \int_{Q_{7/4}} |\nabla u|^2 dx dt + \int_{Q_{7/4}} |F|^2 dx dt \right). \end{aligned}$$

The proof is complete. \square

The following version of local energy estimate for $w = u - v$ is also needed.

Lemma 3.2. *There exists a constant $C = C_0$ depending only on Λ, n such that for $w = u - v$, and for every smooth, non-negative cut-off function $\varphi \in C_0^\infty(Q_r)$ with $0 < r \leq 7/4$, it holds*

$$\begin{aligned} & \sup_{t \in \Gamma_r} \int_{B_r} w^2 \varphi^2 dx + \int_{Q_r} |\nabla w|^2 \varphi^2 dx dt \\ & \leq C_0 \left\{ \left[\|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))}^2 + 1 \right] \int_{Q_r} w^2 \left[\varphi^2 + |\partial_t \varphi|^2 + |\nabla \varphi|^2 \right] dx dt + \int_{Q_r} |F|^2 \varphi^2 dx dt \right. \\ & \quad + \|\nabla v \varphi\|_{L^\infty(Q_{7/4})} \|b\|_{L^\infty(\Gamma_2, \mathcal{V}(B_2))} \|\nabla \varphi\|_{L^2(Q_{7/4})} \left\{ \int_{Q_r} |w|^2 \varphi^2 dx dt \right\}^{1/2} \\ & \quad \left. + \|\nabla v \varphi\|_{L^\infty(Q_r)}^2 \int_{Q_r} |a - \bar{a}_{B_{7/4}}(t)|^2 dx dt \right\}. \end{aligned}$$

Proof. We write $Q = Q_r$, $B = B_r$, and $\Gamma = \Gamma_r$. Note that w is a weak solution of

$$w_t - \operatorname{div}[a \nabla w + (a - \bar{a}_{B_{7/4}}) \nabla v] - b \cdot \nabla w - b \cdot \nabla v = \operatorname{div}(F), \quad \text{in } Q_{7/4}.$$

By using $w \varphi^2$ as a test function of the equation of w , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_B w^2(x, t) \varphi^2(x, t) dx + \int_B \langle a \nabla w, \nabla w \rangle \varphi^2 dx \\ & = - \int_B \langle a \nabla w, \nabla(\varphi^2) \rangle w dx - \int_B \langle (a - \bar{a}_{B_{7/4}}(t)) \nabla v, \varphi^2 \nabla w + 2w \varphi \nabla \varphi \rangle dx \\ & \quad + \int_B [b \cdot \nabla w] w \varphi^2 dx + \int_B [b \cdot \nabla v] w \varphi^2 dx \\ & \quad - \int_B \langle F, \nabla(w \varphi^2) \rangle + \int_B w^2 \varphi \varphi_t dx. \end{aligned} \tag{3.3}$$

Note again that the second term in the left hand side of (3.3) can be estimated using (1.5) as

$$\int_Q \langle a \nabla w, \nabla w \rangle \varphi^2 dx dt \geq \Lambda^{-1} \int_Q |\nabla w|^2 \varphi^2 dx dt.$$

Also, from the integration by parts in x , and $\operatorname{div}(b) = 0$, we also have

$$\int_B [b \cdot \nabla w] w \varphi^2 dx = \frac{1}{2} \int_B [b \cdot \nabla(w^2)] \varphi^2 dx = - \int_B [b \cdot \nabla \varphi] \varphi w^2 dx.$$

Hence, (3.3) implies

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_B w^2(x, t) \varphi^2(x, t) dx + \Lambda^{-1} \int_B |\nabla w|^2 \varphi^2 dx \\ & \leq \int_B |\langle a \nabla w, \nabla(\varphi^2) \rangle w| dx + \int_B |\langle (a - \bar{a}_{B_{7/4}}(t)) \nabla v, \varphi^2 \nabla w + 2w \varphi \nabla \varphi \rangle| dx \\ & \quad + \int_B |[b \cdot \nabla \varphi] w^2 \varphi| dx + \int_B |[b \cdot \nabla v] w \varphi^2| dx \\ & \quad + \int_B \left[|\langle F, \nabla(w \varphi^2) \rangle| + 2w^2 |\varphi \varphi_t| \right] dx. \end{aligned}$$

By integrating this inequality in time, and using the L^∞ -bound of a from (1.5), we infer that

$$\begin{aligned} & \frac{1}{2} \sup_{t \in \Gamma_r} \int_B w^2(x, t) \varphi^2(x, t) dx + \Lambda^{-1} \int_Q |\nabla w|^2 \varphi^2 dx dt \\ & \leq 2 \int_Q |\nabla w| |\nabla \varphi| |\varphi w| dx dt \\ & \quad + \int_Q |a - \bar{a}_{B_{7/4}}| |\nabla v| \left[\varphi^2 |\nabla w| + 2|w \varphi| |\nabla \varphi| \right] dx dt \quad (3.4) \\ & \quad + \int_Q |b| |\nabla \varphi| w^2 \varphi dx dt + \int_Q |b| |\nabla v| |w \varphi^2| dx dt \\ & \quad + \int_Q \left[|\langle F, \nabla(w \varphi^2) \rangle| + 2w^2 |\varphi \varphi_t| \right] dx dt. \end{aligned}$$

We now pay particular attention to the terms in the right hand side of (3.4) involving b , as other terms can be estimated exactly as in Lemma 3.1. By using the Hölder's inequality and Young's inequality, we see that

$$\begin{aligned} & \int_Q w^2 \varphi |b| |\nabla \varphi| dx dt \\ & \leq \left\{ \int_Q |b|^2 w^2 \varphi^2 \right\}^{1/2} \left\{ \int_Q w^2 |\nabla \varphi|^2 dx dt \right\}^{1/2} \\ & \leq \|b\|_{L^\infty(\Gamma_{7/4}, \mathcal{V}^{1,2}(B_{7/4}))} \left\{ \int_Q |\nabla(w \varphi)|^2 dx dt \right\}^{1/2} \left\{ \int_Q w^2 |\nabla \varphi|^2 dx dt \right\}^{1/2} \\ & \leq \epsilon \int_Q |\nabla w|^2 \varphi^2 dx dt + C(\epsilon) \|b\|_{L^\infty(\Gamma_{7/4}, \mathcal{V}^{1,2}(B_{7/4}))}^2 \int_Q w^2 |\nabla \varphi|^2 dx dt, \end{aligned}$$

for any arbitrary $\epsilon > 0$. Similarly, we obtain

$$\begin{aligned} & \int_Q |b| |\nabla v| |w \varphi^2| dx dt \\ & \leq \|\nabla v \varphi\|_{L^\infty(Q)} \left\{ \int_Q |b|^2 \varphi^2 dx dt \right\}^{1/2} \left\{ \int_Q |w|^2 \varphi^2 dx dt \right\}^{1/2} \\ & \leq \|\nabla v \varphi\|_{L^\infty(Q)} \|b\|_{L^\infty(\Gamma_{7/4}, \mathcal{V}^{1,2}(B_{7/4}))} \|\nabla \varphi\|_{L^2(Q_{7/4})} \left\{ \int_Q |w|^2 \varphi^2 dx dt \right\}^{1/2}. \end{aligned}$$

The other terms can be estimated similarly. Then, collecting all the estimates and choose ϵ sufficiently small, we obtain the desired result. \square

4. APPROXIMATION ESTIMATES

We apply the “freezing coefficient” technique to establish the regularity estimates for weak solutions of (1.3). To do this, we approximate the weak solution u of the equation

$$u_t - \operatorname{div}[a\nabla u] - b \cdot \nabla u = \operatorname{div}(F) \quad \text{in } Q_2, \quad (4.1)$$

by the weak solution v of the equation

$$\begin{aligned} v_t - \operatorname{div}[\bar{a}_{B_{7/4}}(t)\nabla v] &= 0, \quad \text{in } Q_{7/4}, \\ v &= u, \quad \text{on } \partial_p Q_{7/4} \end{aligned} \quad (4.2)$$

Again, the meanings for weak solutions of equations (4.1)-(4.2) are given in Definition 2.1 and Definition 2.2, respectively. We essentially follow the method in our recent work [11, 23], which in turn is influenced by [5, 6, 27, 33]. We first begin with the standard result on the regularity of weak solution of the constant coefficient equation (4.2).

Lemma 4.1. *There exists a constant C depending only on the ellipticity constant Λ and n such that if v is a weak solution of*

$$v_t - \operatorname{div}[\bar{a}_{B_{7/4}}(t)\nabla v] = 0 \quad \text{in } Q_{7/4},$$

then

$$\|\nabla v\|_{L^\infty(Q_{\frac{3}{2}})} \leq C \left(\int_{Q_{7/4}} |\nabla v(x, t)|^2 dx dt \right)^{1/2}.$$

Our next lemma confirms that we can approximate in $L^2(Q_{7/4})$ the solution u of (4.1) by the solution v of (4.2) if the coefficients and the data are sufficiently close to each others.

Lemma 4.2. *Let $M_0, \Lambda > 0$ and $s > 1$, be fixed. Then, for every $\epsilon > 0$, there exists $\delta > 0$ depending on only $\epsilon, \Lambda, n, M_0, s$ such that the following statement holds: For every a, b, F such that if (1.5) holds, $\|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} \leq M_0$, and*

$$\begin{aligned} & \left\{ \int_{Q_{7/4}} |a - \bar{a}_{B_{7/4}}(t)|^2 dx dt \right\}^{1/2} + \left\{ \int_{Q_2} |F|^2 dx dt \right\}^{1/2} \\ & + \left\{ \int_{Q_2} |b|^s dx dt \right\}^{1/s} \left\{ \int_{Q_2} |\hat{u}|^{s'} dx dt \right\}^{1/s'} \leq \delta \end{aligned} \quad (4.3)$$

with $\hat{u} = u - \bar{u}_{Q_2}$, then every weak solution u of (4.1) with

$$\int_{Q_2} |\nabla u|^2 dx dt \leq 1,$$

the weak solution v of (4.2) satisfies

$$\int_{Q_{7/4}} |u - v|^2 dx dt \leq \epsilon, \quad \int_{Q_{7/4}} |\nabla v|^2 dx dt \leq C(\Lambda, M_0, n).$$

Proof. Note that once the existence of v is proved, it follows from Lemma 3.1 and the assumption (4.3) that

$$\int_{Q_{7/4}} |\nabla w|^2 dx dt \leq C[M_0 + 1], \quad \text{where } w = u - v.$$

From this, and using (4.3), we infer that

$$\int_{Q_{7/4}} |\nabla v|^2 dx dt \leq \int_{Q_{7/4}} |\nabla w|^2 dx dt + \int_{Q_{7/4}} |\nabla u|^2 dx dt \leq C(\Lambda, M_0, n).$$

Therefore, we only need to prove the existence of δ . We use the contradiction argument as this method works well for nonlinear equations, and non-smooth domains. Assume that there exist $M_0, \Lambda > 0, s, s', \lambda$, and $\epsilon_0 > 0$ be as in the assumption such that for every $k \in \mathbb{N}$, there are F_k, a_k, b_k , such that

$$\begin{aligned} & \left\{ \int_{Q_{7/4}} |a_k - \bar{a}_{k,B_{7/4}}(t)|^2 dx dt \right\}^{1/2} + \left\{ \int_{Q_2} |F_k|^2 dx dt \right\}^{1/2} \\ & + \left\{ \int_{Q_2} |b_k|^s dx dt \right\}^{1/s} \left\{ \int_{Q_2} |\hat{u}_k|^{s'} dx dt \right\}^{1/s'} \leq \frac{1}{k}, \quad \text{for } \hat{u}_k = u_k - \bar{u}_{k,Q_2} \end{aligned} \tag{4.4}$$

and a weak solution u_k of

$$\partial_t u_k - \operatorname{div}[a_k \nabla u_k] - b_k \cdot \nabla u_k = \operatorname{div}(F_k), \quad Q_2, \tag{4.5}$$

satisfying

$$\int_{Q_2} |\nabla u_k|^2 dx dt \leq 1, \tag{4.6}$$

but for the weak solution v_k of

$$\begin{aligned} \partial_t v_k - \operatorname{div}[\bar{a}_{k,B_{7/4}}(t) \nabla v_k] &= 0, \quad \text{in } Q_{7/4}, \\ v_k &= u_k, \quad \text{on } \partial_p Q_{7/4}, \end{aligned} \tag{4.7}$$

we have

$$\int_{Q_{7/4}} |u_k - v_k|^2 dx dt \geq \epsilon_0. \tag{4.8}$$

Since $\bar{a}_{k,B_{7/4}}(t)$ is a bounded sequence in $L^\infty(\Gamma_{7/4}, \mathbb{R}^{n \times n})$, we can also assume that there is $\bar{a}(t)$ in $L^\infty(\Gamma_{7/4}, \mathbb{R}^{n \times n})$ such that $\bar{a}_{k,B_{7/4}} \rightharpoonup \bar{a}$ weakly- $*$ in $L^\infty(\Gamma_{7/4}; \mathbb{R}^{n \times n})$. This means that for each vector $\xi \in \mathbb{R}^n$, and for all function $\phi \in L^1(\Gamma_{7/4})$, we have

$$\int_{\Gamma_{7/4}} \langle \bar{a}(t) \xi, \xi \rangle \phi(t) dt = \lim_{k \rightarrow \infty} \int_{\Gamma_{7/4}} \langle \bar{a}_{k,B_{7/4}}(t) \xi, \xi \rangle \phi(t) dt. \tag{4.9}$$

Also, for each $k \in \mathbb{N}$, let $w_k = u_k - v_k$, we see that w_k is a weak solution of

$$\begin{aligned} \partial_t w_k - \operatorname{div}[\bar{a}_{k,B_{7/4}} \nabla w_k + (a_k - \bar{a}_{k,B_{7/4}}) \nabla u_k] - b_k \cdot \nabla u_k \\ = \operatorname{div}[F_k], \quad Q_{7/4}, \\ w_k = 0, \quad \partial_p Q_{7/4}. \end{aligned} \tag{4.10}$$

From (4.4), and (4.6), we can apply Lemma 3.1 to yield

$$\sup_{\Gamma_{7/4}} \int_{B_{7/4}} |w_k|^2 dx + \int_{Q_{7/4}} |\nabla w_k|^2 dx dt \leq C, \quad \forall k \in \mathbb{N}. \tag{4.11}$$

This estimate, together with (4.4), (4.6), and the PDE in (4.10), we conclude that $\{w_k\}_k$ is a bounded sequence in $\mathcal{E}(Q_{7/4})$, where

$$\mathcal{E}(Q_{7/4}) = \{g \in L^2(\Gamma_{7/4}, H^1(B_{7/4})) : g_t \in L^2(\Gamma_{7/4}, H^{-1}(B_{7/4})), g = 0 \text{ on } \partial_p Q_{7/4}\}.$$

By the Aubin-Lions theorem that the imbedding $\mathcal{E}(Q_{7/4}) \hookrightarrow C(\bar{\Gamma}_{7/4}, L^2(B_{7/4}))$ is continuous and $\mathcal{E}(Q_{7/4}) \hookrightarrow L^2(Q_{7/4})$ is compact, see for example [31]. Then, by passing through a subsequence, we can assume that there is $w \in \mathcal{E}(Q_{7/4})$ such that

$$\begin{aligned} w_k \rightarrow w & \text{ strongly in } L^2(Q_{7/4}), \quad \nabla w_k \rightharpoonup \nabla w \text{ weakly in } L^2(Q_{7/4}), \\ \partial_t w_k \rightharpoonup \partial_t w & \text{ weakly-}^* \text{ in } L^2(\Gamma_{7/4}; H^{-1}(B_{7/4})), \\ w_k \rightarrow w & \text{ a.e. in } Q_{7/4}. \end{aligned} \quad (4.12)$$

From (4.8) and (4.12), it follows that

$$\int_{Q_{7/4}} w^2 dx dt \geq \epsilon_0. \quad (4.13)$$

Moreover, from the boundary condition $w_k = 0$ on $\partial_p Q_{7/4}$, and (4.12), we also conclude that, in the trace sense,

$$w = 0, \quad \partial_p Q_{7/4}. \quad (4.14)$$

We claim that w is a weak solution of

$$\begin{aligned} w_t - \operatorname{div}[\bar{a}(t)\nabla w] &= 0, \quad Q_{7/4}, \\ w &= 0, \quad \partial_p Q_{7/4} \end{aligned} \quad (4.15)$$

From this, and by the uniqueness of the weak solution of this equation, we infer that $w = 0$ and this contradicts to (4.13). Thus, it remains to prove that w is a weak solution of (4.15). To prove this, we pass the limit as $k \rightarrow \infty$ of (4.10). By (4.14), we only need to find the limits as $k \rightarrow \infty$ for each term in the weak form of the equation (4.10). Let us fix a test function $\varphi \in C^\infty(\bar{Q}_{7/4})$ with $\varphi = 0$ on $\partial_p Q_{7/4}$. Then, it is easy to see from (4.4), and (4.6) that

$$\lim_{k \rightarrow \infty} \int_{Q_{7/4}} F_k \cdot \nabla \varphi dx dt = 0, \quad \lim_{k \rightarrow \infty} \int_{Q_{7/4}} \langle (a_k - \bar{a}_{k, B_{7/4}}(t)) \nabla u_k, \nabla \varphi \rangle dx dt = 0.$$

Further more, from (4.12), we also find that

$$\lim_{k \rightarrow \infty} \int_{\Gamma_{7/4}} \langle \partial_t w_k, \varphi \rangle_{H^{-1}(B_{7/4}), H_0^1(B_{7/4})} dt = \int_{\Gamma_{7/4}} \langle \partial_t w, \varphi \rangle_{H^{-1}(B_{7/4}), H^1(B_{7/4})} dt.$$

For the term involving b_k , since $\operatorname{div}(b_k) = 0$, we can use the integration by parts in x to write

$$\int_{Q_{7/4}} [b_k \cdot \nabla u_k] \varphi dx dt = - \int_{Q_{7/4}} [b \cdot \nabla \varphi] \hat{u}_k dx dt, \quad \hat{u}_k = u_k - \bar{u}_{k, Q_2}.$$

Then, by Hölder's inequality and (4.6), see that

$$\begin{aligned} \left| \int_{Q_{7/4}} [b_k \cdot \nabla u_k] \varphi dx dt \right| &\leq \|\nabla \varphi\|_{L^\infty(Q_{7/4})} \left\{ \int_{Q_2} |b_k|^s dx dt \right\}^{1/s} \left\{ \int_{Q_2} |\hat{u}_k|^{s'} dx dt \right\}^{1/s'} \\ &\leq \frac{|Q_{7/4}|}{k^{1/2}} \|\nabla \varphi\|_{L^\infty(Q_{7/4})} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Finally, since $\bar{a}_{k, B_{7/4}}$ and \bar{a} are independent on x , by integrating by parts in x , we have

$$\int_{Q_{7/4}} \left[\langle \bar{a}_{k, B_{7/4}}(t) \nabla w_k, \nabla \varphi \rangle - \langle \bar{a}(t) \nabla w, \nabla \varphi \rangle \right] dx dt$$

$$\begin{aligned}
 &= - \sum_{i,j=1}^n \int_{Q_{7/4}} \left[w_k \bar{a}_{k,B_{7/4}}^{ij}(t) \partial_{x_i x_j} \varphi - w \bar{a}^{ij}(t) \partial_{x_i x_j} \varphi \right] dx dt \\
 &= - \sum_{i,j=1}^n \int_{Q_{7/4}} \left\{ \bar{a}_{k,B_{7/4}}^{ij}(t) \partial_{x_i x_j} \varphi [w_k - w] + w \partial_{x_i x_j} \varphi [\bar{a}_{k,B_{7/4}}^{ij}(t) - \bar{a}^{ij}(t)] \right\} dx dt
 \end{aligned}$$

Hence, it follows from (4.9) and (4.12) that

$$\lim_{k \rightarrow \infty} \int_{Q_{7/4}} \left[\langle \bar{a}_{k,B_{7/4}}(t) \nabla w_k, \varphi \rangle - \langle \bar{a}(t) \nabla w, \nabla \varphi \rangle \right] dx dt = 0.$$

Collecting the results, we obtain

$$\int_{\Gamma_{7/4}} \langle w_t, \varphi \rangle_{H^{-1}(B_{7/4}), H_0^1(B_{7/4})} dt + \int_{Q_{7/4}} \langle \bar{a}(t) \nabla w, \nabla \varphi \rangle dx dt = 0,$$

for all $\varphi \in C^\infty(\bar{Q}_{7/4}) : \varphi = 0$ on $\partial_p Q_{7/4}$. Thus, w is a weak solution of (4.15). The proof is then complete. \square

Lemma 4.3. *Let $M_0, s > 0$, and $\Lambda > 0$ be fixed. Then, for every $\epsilon > 0$, there exists $\delta > 0$ depending on n, M_0, Λ, s , and ϵ such that the following statement holds true: For every a, b, F such that if (1.5) holds, $\|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} \leq M_0$, and*

$$\begin{aligned}
 &\left\{ \int_{Q_{7/4}} |a - \bar{a}_{B_{7/4}}(t)|^2 dx dt \right\}^{1/2} + \left\{ \int_{Q_2} |F|^2 dx dt \right\}^{1/2} \\
 &+ \left\{ \int_{Q_2} |b|^s dx dt \right\}^{1/s} \left\{ \int_{Q_2} |\hat{u}|^{s'} dx dt \right\}^{1/s'} \leq \delta
 \end{aligned}$$

then, for every weak solution u of (4.1) with $\int_{Q_2} |\nabla u|^2 dx dt \leq 1$, the weak solution v of (4.2) satisfies

$$\int_{Q_{3/2}} |\nabla u - \nabla v|^2 dx dt \leq \epsilon.$$

Moreover, there is $C = C(\Lambda, M_0, n)$ such that

$$\|\nabla v\|_{L^\infty(Q_{3/2})} \leq C(n, \Lambda, M_0). \tag{4.16}$$

Proof. Let $\mu > 0$ to be determined. By Lemma 4.2, there exists $\delta_1 > 0$ such that if $\|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} \leq M_0$, and

$$\begin{aligned}
 &\left\{ \int_{Q_{7/4}} |a - \bar{a}_{B_{7/4}}(t)|^2 dx dt \right\}^{1/2} + \left\{ \int_{Q_2} |F|^2 dx dt \right\}^{1/2} \\
 &+ \left\{ \int_{Q_2} |b|^s dx dt \right\}^{1/s} \left\{ \int_{Q_2} |\hat{u}|^{s'} dx dt \right\}^{1/s'} \leq \delta_1,
 \end{aligned}$$

then

$$\int_{Q_{7/4}} |u - v|^2 dx dt \leq \mu, \quad \int_{Q_{7/4}} |\nabla v|^2 dx dt \leq C(\Lambda, M_0). \tag{4.17}$$

where u is a weak solution of (4.1), and v is a weak solution of (4.2) and

$$\int_{Q_2} |\nabla u|^2 dx dt \leq 1.$$

From (4.17) and Lemma 4.1, we can conclude that

$$\|\nabla v\|_{L^\infty(Q_{3/2})} \leq C(n, \Lambda, M_0).$$

Note that without loss of generality, we can assume that $\delta_1 \leq \mu$. Therefore, applying Lemma 3.2, we obtain

$$\int_{Q_{3/2}} |\nabla u - \nabla v|^2 dx dt \leq C(\Lambda, M_0, n)\mu^{1/2}.$$

Therefore, if we choose μ such that $\mu^{1/2} = \epsilon/C(\Lambda, M_0, n)$, the proof is complete. \square

We in fact need a localized version of Lemma 4.3. For each $r > 0$ and $z_0 = (x_0, t_0) \in Q_1$, we approximate a weak solution of the equation

$$u_t - \operatorname{div}[a\nabla u] - b \cdot \nabla u = \operatorname{div}(F), \quad \text{in } Q_{2r}(z_0), \tag{4.18}$$

by the weak solution of

$$\begin{aligned} v_t - \operatorname{div}[\bar{a}_{B_{7r/4}(x_0)}(t)\nabla v] &= 0, \quad \text{in } Q_{7r/4}(z_0), \\ v &= u, \quad \text{on } \partial_p Q_{7r/4}(z_0). \end{aligned} \tag{4.19}$$

We then have the following lemma, which is the main result of the section.

Lemma 4.4. *Let $\Lambda > 0, s > 1$, and $M_0 > 0$ be fixed. Then, for any $\epsilon > 0$, there exists $\delta > 0$ depending only on $\epsilon, \Lambda, M_0, s$ and n such that the following statement holds true: For every $z_0 \in Q_1, 0 < r \leq 1$, and for every a, b, F such that (1.5) holds for a , and*

$$\begin{aligned} &\left\{ \int_{Q_{7r/4}(z_0)} |a - \bar{a}_{B_{7r/4}(x_0)}|^2 dx dt \right\}^{1/2} + \left\{ \int_{Q_{2r}(z_0)} |F|^2 dx dt \right\}^{1/2} \\ &+ \left\{ \int_{Q_{2r}(z_0)} |u - \bar{u}_{Q_{2r}(z_0)}|^{s'} \right\}^{1/s'} \left\{ \int_{Q_{2r}(z_0)} |b|^s dx dt \right\}^{1/s} \leq \delta \end{aligned} \tag{4.20}$$

then for every weak solution u of (4.1) with

$$\int_{Q_{2r}(z_0)} |\nabla u|^2 dx dt \leq 1, \tag{4.21}$$

the weak solution v of (4.19) satisfies

$$\int_{Q_{3r/2}(z_0)} |\nabla u - \nabla v|^2 dx dt \leq \epsilon, \quad \text{and} \quad \|\nabla v\|_{L^\infty(Q_{3r/2}(z_0))} \leq C(\Lambda, M_0, n). \tag{4.22}$$

Proof. Given any $\epsilon > 0$, let $\delta = \delta(\epsilon, \Lambda, M_0, s, n) > 0$ be defined as in Lemma 4.3. We now show that Lemma 4.4 holds with this δ . Let u, v satisfy the conditions in of Lemma 4.4. Without loss of generality, we can assume that $z_0 = (0, 0)$. Let us define

$$u'(x, t) = \frac{u(rx, r^2t)}{r}, \quad v'(x, t) = \frac{v(rx, r^2t)}{r}, \quad a'(x, t) = a(rx, r^2t).$$

Also, let us denote

$$F'(x, t) = F(rx, r^2t), \quad b'(x, t) = rb(rx, r^2t).$$

Then u' is a weak solution of

$$u'_t - \operatorname{div}[a'\nabla u'] - b' \cdot \nabla u' = \operatorname{div}(F') \quad \text{in } Q_2$$

and v' is a weak solution of

$$\begin{aligned} v'_t &= \operatorname{div}[\bar{a}'_{B_{7/4}}(t)\nabla v'] \quad \text{in } Q_{7/4}, \\ v' &= u' \quad \text{on } \partial_p Q_{7/4}. \end{aligned}$$

We now check that the conditions in Lemma 4.3 hold with a', b', u', F' and v' . A simple calculation shows

$$\begin{aligned} \|b'\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} &= \|b\|_{L^\infty(\Gamma_{2r}, \mathcal{V}^{1,2}(B_{2r}))} \leq M_0, \\ \int_{Q_{7/4}} \left[|a' - \bar{a}'_{B_3}|^2 dx dt\right] &= \int_{Q_{7r/4}} |a - \bar{a}_{B_{7r/4}}|^2 dx dt, \\ \int_{Q_2} |F'|^2 dx dt &= \int_{Q_{2r}} |F|^2 dx dt. \end{aligned}$$

Also,

$$\begin{aligned} \int_{Q_2} |\nabla u'|^2 dx dt &= \int_{Q_{2r}} |\nabla u|^2 dx dt \leq 1, \\ \left\{ \int_{Q_2} |u' - \bar{u}'_{Q_2}|^{s'} \right\}^{1/s'} \left\{ \int_{Q_2} |b'|^s dx dt \right\}^{1/s} \\ &= \left\{ \int_{Q_{2r}} |u - \bar{u}_{Q_{2r}}|^{s'} \right\}^{1/s'} \left\{ \int_{Q_{2r}} |b|^s dx dt \right\}^{1/s}. \end{aligned}$$

Therefore, if (4.20) and (4.21) hold, then all conditions in Lemma 4.3 are met. Hence, we have

$$\int_{Q_{3/2}} |\nabla u'(x, t) - \nabla v'(x, t)|^2 dx dt \leq \varepsilon, \quad \text{and} \quad \|v'\|_{L^\infty(Q_{3/2})} \leq C(\Lambda, n).$$

Integrating by substitution, we obtain

$$\int_{Q_{3r/2}} |\nabla u(x, t) - \nabla v(x, t)|^2 dx dt \leq \varepsilon, \quad \text{and} \quad \|\nabla v\|_{L^\infty(Q_{3r/2})} \leq C(\Lambda, n).$$

The proof is complete. □

5. WEIGHTED DENSITY ESTIMATES AND WEIGHTED $W^{1,p}$ -REGULARITY ESTIMATES

5.1. Weighted density estimates. We will derive the estimate of $\|\nabla u\|_{L^p(Q_1, \omega)}$ for solution u of (4.1) by estimating the distribution functions of the maximal function of $|\nabla u|^2$. Our first lemma gives a density estimate for the distribution of $\mathcal{M}_{Q_2}(|\nabla u|^2)$, where the maximal operator \mathcal{M}_{Q_2} is defined in Definition 2.4. From now let us fix $s, s' \in (1, \infty), \alpha > 0$ and λ satisfying (1.7). If u is a weak solution of (4.1), we define

$$B(x, t) = ([u]_{s', \lambda, Q_1} |b(x, t)|)^s, \tag{5.1}$$

where $[u]_{s', \lambda, Q_1}$ is the parabolic semi-Campanato's norm of u on Q_1 ,

$$[u]_{s', \lambda, Q_1} = \sup_{0 < \rho < 1, z \in Q_1} \left[\rho^{-\lambda} \int_{Q_\rho(z)} |u(x, t) - \bar{u}_{Q_\rho(z)}|^{s'} dx dt \right]^{1/s'}.$$

Lemma 5.1. *Let $\Lambda > 0, M_0 > 0$ be fixed, and $\omega \in A_q$ for some $1 < q < \infty$. Let s, λ, α be as in (1.7). Then, there exists a constant $N > 1$ depending only on Λ, M_0, s, λ and n such that for every $\varepsilon > 0$, we find $\delta = \delta(\varepsilon, M_0, \Lambda, s, \lambda, [\omega]_{A_q}, n) > 0$*

such that the following statement holds: Suppose that (1.5) holds for the matrix a , $\operatorname{div}(b) = 0$, and $|F|, |b| \in L^2(Q_2)$.

$$\sup_{0 < \rho \leq 1} \sup_{(y,s) \in Q_1} \int_{Q_\rho(y,s)} |a(x,t) - \bar{a}_{B_\rho(y)}(t)|^2 dx dt \leq \delta, \quad (5.2)$$

$$\|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} \leq M_0,$$

and for a weak solution u of (4.1) and for every $z = (y, \tau) \in Q_1$, and $0 < r \leq 1/2$ if the set

$$Q_r(z) \cap Q_1 \cap \{Q_2 : \mathcal{M}_{Q_2}(|\nabla u|^2) \leq 1\} \cap \{Q_2 : \mathcal{M}_{Q_2}(|F|^2) + \mathcal{M}_{\alpha, Q_2}(B)^{2/s} \leq \delta\},$$

is non-empty, then

$$\omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N\} \cap Q_r(z)) \leq \varepsilon \omega(Q_r(z)).$$

Proof. Let $\eta > 0$ depending only on $\epsilon, \Lambda, M_0, s, [\omega]_{A_q}$ and λ to be determined. Then, let $\delta = \delta(\eta, \Lambda, M_0, s, n)$ be the number defined in Lemma 4.4. We prove our lemma with this choice of δ . By the condition on the non-empty intersection, there exists a point $z_0 = (x_0, t_0) \in Q_r(z) \cap Q_1$ such that

$$\mathcal{M}_{Q_2}(|\nabla u|^2)(z_0) \leq 1, \quad \mathcal{M}_{Q_2}(|F|^2)(z_0) + \mathcal{M}_{\alpha, Q_2}(B)(z_0)^{2/s} \leq \delta. \quad (5.3)$$

Notice that with $r \in (0, 1/2)$, $Q_{2r}(z) \subset Q_2$. Since $Q_{2r}(z) \subset Q_{3r}(z_0) \cap Q_2$, it follows from (5.3) that

$$\int_{Q_{2r}(z)} |\nabla u|^2 dx dt \leq \frac{|Q_{3r}(z_0)|}{|Q_{2r}(y, s)|} \frac{1}{|Q_{3r}(z_0)|} \int_{Q_{3r}(z_0) \cap Q_2} |\nabla u|^2 dx dt \leq \left(\frac{3}{2}\right)^{n+2}.$$

Similarly,

$$\begin{aligned} & \left\{ \int_{Q_{2r}(z)} |u - \bar{u}_{Q_{2r}(z)}|^{s'} dx dt \right\}^{2/s'} \left\{ \int_{Q_{2r}(z)} |b|^s dx dt \right\}^{2/s} \\ &= \left\{ (2r)^{-\lambda} \int_{Q_{2r}(z)} |u - \bar{u}_{Q_{2r}(z)}|^{s'} dx dt \right\}^{2/s'} \left\{ (2r)^\alpha \int_{Q_{2r}(z)} |b|^s dx dt \right\}^{2/s} \\ &\leq [u]_{s', \lambda, Q_1}^2 \left\{ (2r)^\alpha \frac{|Q_{3r}(z_0)|}{|Q_{2r}(z)|} \int_{Q_{3r}(z_0) \cap Q_2} |b|^s dx dt \right\}^{2/s} \\ &\leq (3/2)^{2(n+2-\alpha)/s} \left\{ (3r)^\alpha \int_{Q_{3r}(z_0) \cap Q_2} |B|^s dx dt \right\}^{2/s} \\ &\leq (3/2)^{2(n+2-\alpha)/s} \mathcal{M}_{\alpha, Q_2}(B)^{2/s} \\ &\leq (3/2)^{2(n+2-\alpha)/s} \delta, \end{aligned}$$

where we have used (1.7) in our second step in the above estimate. Also we have

$$\int_{Q_{3r}(z)} |F|^2 dx dt \leq \frac{|Q_{3r}(z_0)|}{|Q_{2r}(z)|} \frac{1}{|Q_{3r}(z_0)|} \int_{Q_{3r}(z_0) \cap Q_2} |F|^2 dx dt \leq \left(\frac{3}{3}\right)^{n+2} \delta.$$

From assumption (5.2), and since $Q_{7r/4}(z) \subset Q_2$, we have

$$\int_{Q_{7r/4}(z)} |a(x,t) - \bar{a}_{B_{7r/4}(y)}(t)|^2 dx dt \leq \delta,$$

$$\|b\|_{L^\infty(\Gamma_{2r}(\tau), \mathcal{V}^{1,2}(B_{2r}(y)))} \leq \|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} \leq M_0.$$

These estimates allow us to use Lemma 4.4 with a suitable scaling to obtain

$$\int_{Q_{3r/2}(z)} |\nabla u - \nabla v|^2 dx dt \leq \kappa \eta, \quad \|\nabla v\|_{L^\infty(Q_{3r/2}(z))} \leq C_0 := \kappa C(\Lambda, M_0, n). \quad (5.4)$$

where

$$\kappa = \max \left\{ (3/2)^{n+2}, (3/2)^{2(n+2-\alpha)/s} \right\},$$

and v is the unique weak solution of

$$\begin{aligned} v_t &= \nabla \cdot [\bar{a}_{B_{7r/4}(y)}(t) \nabla v] \quad \text{in } Q_{7r/4}(z), \\ v &= u \quad \text{on } \partial_p Q_{7r/4}(z) \end{aligned}$$

We claim that (5.3), and (5.4) yield

$$\{Q_r(z) : \mathcal{M}_{Q_{3r/2}(z)}(|\nabla u - \nabla v|^2) \leq C_0\} \subset \{Q_r(z) : \mathcal{M}_{Q_2}(|\nabla u|^2) \leq N\} \quad (5.5)$$

with $N = \max \{4C_0, 5^{n+2}\}$. Indeed, let (x, t) be a point in the set on the left hand side of (5.5), and consider $Q_\rho(x, t)$. If $\rho \leq r/2$, then $Q_\rho(x, t) \subset Q_{\frac{3r}{2}}(z) \subset Q_2$ and hence

$$\begin{aligned} & \frac{1}{|Q_\rho(x, t)|} \int_{Q_\rho(x, t) \cap Q_2} |\nabla u|^2 dx dt \\ & \leq \frac{2}{|Q_\rho(x, t)|} \left[\int_{Q_\rho(x, t) \cap Q_2} |\nabla u - \nabla v|^2 dx dt + \int_{Q_\rho(x, t) \cap Q_2} |\nabla v|^2 dx dt \right] \\ & \leq 2\mathcal{M}_{Q_{3r/2}(z)}(|\nabla u - \nabla v|^2)(x, t) + 2\|\nabla v\|_{L^\infty(Q_{\frac{3r}{2}}(z))}^2 \leq 4C_0(\Lambda, M_0, n). \end{aligned}$$

On the other hand if $\rho > r/2$, then $Q_\rho(x, t) \subset Q_{5\rho}(z_0)$. This and the first inequality in (5.3) imply

$$\frac{1}{|Q_\rho(x, t)|} \int_{Q_\rho(x, t) \cap Q_2} |\nabla u|^2 dx dt \leq \frac{5^{n+2}}{|Q_{5\rho}(x_0, t_0)|} \int_{Q_{5\rho}(z_0) \cap Q_2} |\nabla u|^2 dx dt \leq 5^{n+2}.$$

Therefore, $\mathcal{M}_{Q_2}(|\nabla u|^2)(x, t) \leq N$ and the claim (5.5) is proved. Note that (5.5) is equivalent to

$$E := \{Q_r(z) : \mathcal{M}_{Q_2}(|\nabla u|^2) > N\} \subset \{Q_r(z) : \mathcal{M}_{Q_{3r/2}(z)}(|\nabla u - \nabla v|^2) > C_0\}.$$

From this it follows the weak type 1 – 1 estimate of the Hardy-Littlewood maximal function, and from (5.4) that

$$\begin{aligned} |E| & \leq |\{Q_r(z) : \mathcal{M}_{Q_{3r/2}(z)}(|\nabla u - \nabla v|^2) > C_0\}| \\ & \leq \frac{C(n)|Q_{3r/2}(z)|}{C_0} \int_{Q_{3r/2}(z)} |\nabla u - \nabla v|^2 dx dt \leq C' \eta |Q_r(z)|, \end{aligned}$$

where $C' > 0$ depends only on Λ, M_0, s, α , and n . Then, from Lemma 2.3, there is $\beta = \beta([\omega]_{A_q}, n) > 0$ such that

$$\omega(E) \leq C([\omega]_{A_q}, n) \left(\frac{|E|}{|Q_r(z)|} \right)^\beta \omega(Q_r(z)) \leq C_* \eta^\beta \omega(Q_r(z)),$$

where $C_* > 0$ is a constant depending only on $\Lambda, M_0, s, \alpha, [\omega]_{A_q}$ and n . By choosing $\eta = (\varepsilon/C_*)^{1/\beta}$, we obtain the desired result. \square

Lemma 5.2. *Let $\Lambda > 0, M_0 > 0$ be fixed and $\omega \in A_q$ with some $1 < q < \infty$. Let s, λ, α be as in (1.7). There exists a constant $N > 1$ depending only on Λ, M_0, s, λ and n such that for any $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon, \Lambda, M_0, s, \lambda, [\omega]_{A_q}, n) > 0$ such that if (1.5) holds for the matrix a , $\operatorname{div}(b) = 0$, and $|F|, |b| \in L^2(Q_2)$.*

$$\sup_{0 < \rho \leq 1} \sup_{(y,s) \in Q_1} \int_{Q_\rho(y,s)} |a(x,t) - \bar{a}_{B_\rho(y)}(t)|^2 dx dt \leq \delta, \quad \|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} \leq M_0,$$

and a weak solution u of (4.1) satisfying

$$\omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N\}) \leq \varepsilon \omega(Q_1(z)), \quad \forall z \in Q_1.$$

Then

$$\begin{aligned} \omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N\}) &\leq \varepsilon_1 \left\{ \omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > 1\}) \right. \\ &\quad \left. + \omega(\{Q_1 : \mathcal{M}_{Q_2}(|F|^2) + \mathcal{M}_{\alpha, Q_2}(B)^{2/s} > \delta\}) \right\}, \end{aligned}$$

where ε_1 is defined in Lemma 2.7.

Proof. In view of Lemma 5.1, we can apply Lemma 2.7, for

$$E = \{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N\},$$

$$K = \{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > 1\} \cup \{Q_1 : \mathcal{M}_{Q_2}(|F|^2) + \mathcal{M}_{\alpha, Q_2}(B)^{2/s} > \delta\}$$

to obtain the desired estimate. \square

Lemma 5.3. *Let $\Lambda > 0, M_0 > 0$ be fixed and $\omega \in A_q$ with some $1 < q < \infty$. Let s, λ, α be as in (1.7). There exists a constant $N > 1$ depending only on Λ, M_0, s, λ and n such that for any $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon, \Lambda, M_0, s, \lambda, [\omega]_{A_q}, n) > 0$ such that if (1.5) holds for the matrix a , $\operatorname{div}(b) = 0$, and $|F|, |b| \in L^2(Q_2)$.*

$$\sup_{0 < \rho \leq 1} \sup_{(y,s) \in Q_1} \int_{Q_\rho(y,s)} |a(x,t) - \bar{a}_{B_\rho(y)}(t)|^2 dx dt \leq \delta, \quad \|b\|_{L^\infty(\Gamma_2, \mathcal{V}^{1,2}(B_2))} \leq M_0,$$

and a weak solution u of (4.1) satisfying

$$\omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N\}) \leq \varepsilon \omega(Q_1(z)), \quad \forall z \in Q_1.$$

Then for every $k = 1, 2, \dots$,

$$\begin{aligned} &\omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N^k\}) \\ &\leq \varepsilon_1^k \omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > 1\}) \\ &\quad + \sum_{i=1}^k \varepsilon_1^i \omega(\{Q_1 : \mathcal{M}_{Q_2}(|F|^2) + \mathcal{M}_{\alpha, Q_2}(B)^{2/s} > \delta N^{k-i}\}), \end{aligned} \tag{5.6}$$

where ε_1 is defined in Lemma 2.7.

Proof. Let N, δ be defined as in Lemma 5.2 and we prove (5.6) holds with these N, δ by using induction on k . If $k = 1$, then (5.6) holds because of Lemma 5.2. Assume now that (5.6) holds with some $m \in \mathbb{N}$, and we prove that it holds with for $k = m + 1$. For given u, b satisfying the assumptions of the lemma, we define

$$u' = u/\sqrt{N}, \quad F' = F/\sqrt{N}, \quad B' = [u']_{s', \lambda, Q_1} |b|^s.$$

Observe that u' is a weak solution of

$$\partial_t u' - \operatorname{div}[a(x,t)\nabla u'] - b(x,t)\nabla u' = \operatorname{div}[F], \quad \text{in } Q_2.$$

Moreover, for every $z \in Q_1$, it is easy to see that

$$\begin{aligned} \omega(Q_1 : \mathcal{M}_{Q_2}(|\nabla u'|^2) > N) &= \omega(Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N^2) \\ &\leq \omega(Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N) \leq \varepsilon|Q_1(z)|. \end{aligned}$$

Therefore, we can apply the induction hypothesis for u', F', B' to obtain

$$\begin{aligned} &\omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u'|^2) > N^m\}) \\ &\leq \varepsilon_1^k \omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u'|^2) > 1\}) \\ &\quad + \sum_{i=1}^m \varepsilon_1^i \omega(\{Q_1 : \mathcal{M}_{Q_2}(|F'|^2) + \mathcal{M}_{\alpha, Q_2}(B')^{2/s} > \delta N^{k-i}\}). \end{aligned}$$

By changing back to u, F, B and using the Lemma 5.2 again, we see that (5.6) holds with $k = m + 1$. The proof is complete. \square

5.2. Proof of Theorem 1.3. The proof is quite standard once Lemma 5.3 is established; we give it here for completeness. Let $N = N(\Lambda, M_0, s, \lambda, n) > 1$ be as in Lemma 5.3, and let $q = p/2$, and $\varepsilon_1 = (20)^{nq} [\omega]_{A_q}^2 \varepsilon$. Choose ε sufficiently small depending only on $\Lambda, M_0, s, \lambda, p, n$ and $[\omega]_{A_q}$ such that

$$N^p \varepsilon_1 < 1/2.$$

With this choice of ε , let δ be as in Lemma 5.3 depending on $\Lambda, M_0, s, \lambda, p, n$ and $[\omega]_{A_q}$. We first prove Theorem 1.3 with the additional assumption that

$$\omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N\}) \leq \varepsilon \omega(Q_1(z)), \quad \forall z \in Q_1. \tag{5.7}$$

Assume that (5.7) holds, and let us denote

$$S = \sum_{k=1}^{\infty} N^{kq} \omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N\}).$$

By Lemma 5.3, we see that

$$\begin{aligned} S &\leq \sum_{k=1}^{\infty} N^{kq} \left[\sum_{i=1}^k \varepsilon_1^i \omega(\{Q_1 : \mathcal{M}_{Q_2}(|F|^2) + \mathcal{M}_{\alpha, Q_2}(B)^{2/s} > \delta N^{k-i}\}) \right] \\ &\quad + \sum_{k=1}^{\infty} N^{kq} \varepsilon_1^k \omega(\{Q_1 : \mathcal{M}_{Q_1}(|\nabla u|^2) \geq 1\}). \end{aligned}$$

Then, applying the Fubini's theorem, and Lemma 2.6, we obtain

$$\begin{aligned} S &\leq \sum_{j=1}^{\infty} (N^q \varepsilon_1)^j \left[\sum_{k=j}^{\infty} N^{q(k-j)} \omega(\{Q_1 : \mathcal{M}_{Q_2}(|F|^2) + \mathcal{M}_{\alpha, Q_2}(B)^{2/s} > \delta N^{k-j}\}) \right] \\ &\quad + \sum_{k=1}^{\infty} N^{kq} \varepsilon_1^k \omega(\{Q_1 : \mathcal{M}_{Q_1}(|\nabla u|^2) \geq 1\}) \\ &\leq C \left(\|\mathcal{M}_{Q_2}(|F|^2)\|_{L^q(Q_1, \omega)}^q + \|\mathcal{M}_{\alpha, Q_2}(B)^{2/s}\|_{L^q(Q_1, \omega)}^q + \omega(Q_1) \right) \sum_{k=1}^{\infty} (N^q \varepsilon_1)^k. \end{aligned}$$

Then, by our choice of ε , and Lemma 2.5, we obtain

$$S \leq C \left[\|F\|_{L^p(Q_2, \omega)}^p + \|\mathcal{M}_{\alpha, Q_2}(B)^{2/s}\|_{L^{p/2}(Q_2, \omega)}^{p/2} + \omega(Q_1) \right].$$

By Lemma 2.6, it follows that

$$\begin{aligned} \|\nabla u\|_{L^p(Q_1, \omega)}^p &\leq C(S + \omega(Q_1)) \\ &\leq C\left[\|F\|_{L^p(Q_2, \omega)}^p + \|\mathcal{M}_{\alpha, Q_2}(B)^{2/s}\|_{L^{p/2}(Q_2, \omega)}^{p/2} + \omega(Q_1)\right]. \end{aligned} \quad (5.8)$$

Hence, we have proved (5.8) under the additional assumption (5.7). To remove (5.7), let us define

$$u' = u/\kappa, \quad F' = F/\kappa, \quad B' = ([u']_{s', \lambda, Q_1} |b|)^s,$$

for some constant $\kappa > 0$ to be determined. Observe that u' is a weak solution of

$$\partial_t u' - \operatorname{div}[a \nabla u'] - b \cdot \nabla u' = \operatorname{div}(F'), \quad \text{in } Q_2.$$

Let us define

$$E := \{Q_1 : \mathcal{M}_{Q_2}(|\nabla u'|^2) > N\} \subset Q_2.$$

Then, from Lemma 2.3 it follows that for every $z \in Q_1$,

$$\frac{\omega(E)}{\omega(Q_1(z))} = \frac{\omega(E)}{\omega(Q_2)} \frac{\omega(Q_2)}{\omega(Q_1(z))} \leq [\omega]_{A_q} \frac{\omega(E)}{\omega(Q_2)} \left(\frac{|Q_2|}{|Q_1(z)|}\right)^q = C([\omega]_{A_q}, n, q) \frac{\omega(E)}{\omega(Q_2)}.$$

Then, using Lemma 2.3 again, we can find $\beta = \beta([\omega]_{A_q}, n) > 0$ such that

$$\frac{\omega(E)}{\omega(Q_1(z))} \leq C([\omega]_{A_q}, q, n) \left(\frac{|E|}{|Q_2|}\right)^\beta.$$

On the other hand, by the weak type (1,1) estimate, Lemma 2.5, we see that

$$|E| = |\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u|^2) > N\kappa^2\}| \leq \frac{C(n)}{N\kappa^2} \int_{Q_2} |\nabla u|^2 dx dt = \frac{C|Q_2|}{\kappa^2} \|\nabla u\|_{L^2(Q_2)}^2.$$

Hence, combining the last two estimates, we can see

$$\frac{\omega(E)}{\omega(Q_1(z))} \leq C_*([\omega]_q, q, n) \left(\frac{\|\nabla u\|_{L^2(Q_2)}}{\kappa}\right)^{2\beta}, \quad \forall z \in Q_1.$$

Then, by taking

$$\kappa = \|\nabla u\|_{L^2(Q_2)} (C_*/\varepsilon)^{1/(2\beta)}, \quad (5.9)$$

we obtain

$$\omega(E) = \omega(\{Q_1 : \mathcal{M}_{Q_2}(|\nabla u'|^2) > N\}) \leq \varepsilon \omega(Q_1(z)), \quad \forall z \in Q_1.$$

This means that (5.7) holds for u' . Hence, it follows from (5.8) that

$$\|\nabla u'\|_{L^p(Q_1, \omega)} \leq C[\|F'\|_{L^p(Q_2, \omega)} + \|\mathcal{M}_{\alpha, Q_2}(B')^{2/s}\|_{L^{p/2}(Q_2, \omega)}^{1/2} + \omega(Q_1)^{1/p}].$$

This and (5.9) imply

$$\begin{aligned} &\|\nabla u\|_{L^p(Q_1, \omega)} \\ &\leq C[\|F\|_{L^p(Q_2, \omega)} + \|\mathcal{M}_{\alpha, Q_2}(B)^{2/s}\|_{L^{p/2}(Q_1, \omega)}^{1/2} + \omega(Q_1)^{1/p} \|\nabla u\|_{L^2(Q_2)}] \\ &= C[\|F\|_{L^p(Q_2, \omega)} + [u]_{s', \lambda, Q_1} \|\mathcal{M}_{\alpha, Q_2}(|b|^s)^{2/s}\|_{L^{p/2}(Q_2, \omega)}^{1/2} + \omega(Q_1)^{1/p} \|\nabla u\|_{L^2(Q_2)}]. \end{aligned}$$

The proof is complete.

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