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EXISTENCE OF POSITIVE SOLUTIONS FOR HIGHER ORDER SINGULAR SUBLINEAR ELLIPTIC EQUATIONS

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Abstract. We present existence result for the polyharmonic nonlinear problem

$$\begin{gathered}(-\Delta)^{pm}u=\varphi(.,u)+\psi(.,u),\quad \text{in } B\\ u>0,\quad \text{in } B\\ \lim_{|x|\to 1}\frac{(-\Delta)^{jm}u(x)}{(1-|x|)^{m-1}}=0,\quad 0\le j\le p-1,\end{gathered}$$

in the sense of distributions. Here m,p are positive integers, B is the unit ball in $\mathbb{R}^n (n\ge 2)$ and the nonlinearity is a sum of a singular and sublinear terms satisfying some appropriate conditions related to a polyharmonic Kato class of functions $\mathcal{J}^{(p)}_{m,n}$.

1. Introduction

In this paper, we investigate the existence and the asymptotic behavior of positive solutions for the following iterated polyharmonic problem involving a singular and sublinear terms:

$$\begin{gathered}(-\Delta)^{pm}u=\varphi(.,u)+\psi(.,u),\quad \text{in } B\\ u>0\quad \text{in } B\\ \lim_{|x|\to 1}\frac{(-\Delta)^{jm}u(x)}{(1-|x|)^{m-1}}=0,\quad \text{for } 0\le j\le p-1,\end{gathered}\tag{1.1}$$

in the sense of distributions. Here B is the unit ball of $\mathbb{R}^n$ $(n\ge 2)$ and m,p are positive integers. This research is a follow up to the work done by Shi and Yao [14], who considered the problem

$$\begin{gathered}\Delta u+k(x)u^{-\gamma}+\lambda u^{\alpha}=0,\quad \text{in } D,\\ u>0,\quad \text{in } D\end{gathered}\tag{1.2}$$

where D is a bounded $C^{1,1}$ domain in $\mathbb{R}^n(n\ge 2)$, γ,α are two constants in $(0,1),\lambda$ is a real parameter and k is a Hölder continuous function in $\overline{\Omega}$. They proved the existence of positive solutions. Choi, Lazer and Mckenna in [8] and [11] have studied a variety of singular boundary value problems of the type $\Delta u+p(x)u^{-\gamma}$, in a regular

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domain D, $u = 0$ on ∂D, where $\gamma > 0$ and p is a nonnegative function. They proved the existence of positive solutions. This has been extended by Mâagli and Zribi [13] to the problem $\Delta u = -f(.,u)$ in D, $u = 0$ on ∂D, where $f(x,.)$ is nonnegative and nonincreasing on $(0,\infty)$.

On the other hand, problem (1.1) with a sublinear term $\psi(.,u)$ and a singular term $\varphi(.,u) = 0$, has been studied by Mâagli, Toumi and Zribi in [12] for $p = 1$ and by Bachar [2] for $p \geq 1$.

Thus a natural question to ask, is for more general singular and sublinear terms combined in the nonlinearity, whether or not the problem (1.1) has a solution, which we aim to study in this paper.

Our tools are based essentially on some inequalities satisfied by the Green function $\Gamma^{(p)}_{m,n}$ (see (2.1) below) of the polyharmonic operator $u \mapsto (-\Delta)^{pm}u$, on the unit ball B of $\mathbb{R}^n$ $(n \geq 2)$ with boundary conditions $(\frac{\partial}{\partial\nu})^j(-\Delta)^{im}u\big|_{\partial B} = 0$, for $0 \leq i \leq p-1$ and $0 \leq j \leq m-1$, where $\frac{\partial}{\partial\nu}$ is the outward normal derivative. Also, we use some properties of functions belonging to the polyharmonic Kato class $\mathcal{J}^{(p)}_{m,n}$ which is defined as follows.

Definition 1.1 ([2])**.** A Borel measurable function q in B belongs to the class $\mathcal{J}^{(p)}_{m,n}$ if q satisfies the condition

$$\lim_{\alpha\to 0}\Big(\sup_{x\in B}\int_{B\cap B(x,\alpha)}(\frac{\delta(y)}{\delta(x)})^m\Gamma^{(p)}_{m,n}(x,y)|q(y)|dy\Big) = 0, \tag{1.3}$$

where $\delta(x) = 1 - |x|$, denotes the Euclidean distance between x and ∂B.

Typical examples of elements in the class $\mathcal{J}^{(p)}_{m,n}$ are functions in $L^s(B)$, with

$$s > \frac{n}{2pm} \quad \text{if } n > 2pm$$

or with

$$s > \frac{n}{2(p-1)m}, \quad \text{if } 2(p-1)m < n < 2pm$$

or with

$$s \in (1,\infty] \quad \text{if } n \leq 2(p-1)m$$

or with $n = 2pm$; see [2]. Furthermore, if $q(x) = (\delta(x))^{-\lambda}$, then $q \in \mathcal{J}^{(p)}_{m,n}$ if and only if

$$\lambda < 2m, \quad \text{if } p = 1 \quad \text{(see [4]) or}$$
$$\lambda < 2m+1, \quad \text{if } p \geq 2 \quad \text{(see [2])}.$$

For the rest of this paper, we refer to the potential of a nonnegative measurable function f, defined in B by

$$V_p(f)(x) = \int_B \Gamma^{(p)}_{m,n}(x,y)f(y)dy.$$

The plan for this paper is as follows. In section 2, we collect some estimates for the Green function $\Gamma^{(p)}_{m,n}$ and some properties of functions belonging to the class $\mathcal{J}^{(p)}_{m,n}$. In section 3, we will fix $r > n$ and we assume that the functions φ and ψ satisfy the following hypotheses:

- (H1) φ is a nonnegative Borel measurable function on $B \times (0,\infty)$, continuous and nonincreasing with respect to the second variable.
- (H2) For each $c > 0$, the function $x \mapsto \varphi(x, c(\delta(x))^m)/(\delta(x))^m$ is in $\mathcal{J}^{(1)}_{m,n}$.

(H3) For each $c > 0$, the function $x \mapsto \varphi(x, c(\delta(x))^m)$ is in $L^r(B)$.

(H4) ψ is a nonnegative Borel measurable function on $B \times [0,\infty)$, continuous with respect to the second variable such that there exist a nontrivial nonnegative function $h \in L^1_{\rm loc}(B)$ and a nontrivial nonnegative function $k \in \mathcal{J}^{(1)}_{m,n}$ such that
$$h(x)f(t) \le \psi(x,t) \le (\delta(x))^m k(x) g(t), \quad \text{for } (x,t) \in B \times (0,\infty), \tag{1.4}$$
where $f : [0,\infty) \to [0,\infty)$ is a measurable nondecreasing function satisfying
$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty, \tag{1.5}$$
and g is a nonnegative measurable function locally bounded on $[0,\infty)$ satisfying
$$\limsup_{t\to\infty} \frac{g(t)}{t} < \|V_p((\delta(.))^m k)\|_\infty. \tag{1.6}$$

(H5) The function $x \mapsto (\delta(x))^m k(x)$ is in $L^r(B)$.

Using a fixed point argument, we shall prove the following existence result.

Theorem 1.2. *Assume (H1)–(H5). Then* (1.1) *has at least one positive solution* $u \in C^{2pm-1}(B)$*, such that*
$$a_j(\delta(x))^m \le (-\Delta)^{jm} u(x) \le V_{p-j}(\varphi(., a_j(\delta(.))^m))(x) + b_j V_{p-j}((\delta(.))^m k)(x),$$
for $j \in \{0,\dots,p-1\}$*. In particular,*
$$a_j(\delta(x))^m \le (-\Delta)^{jm} u(x) \le c_j(\delta(x))^m,$$
where a_j, b_j, c_j *are positive constants.*

Typical examples of nonlinearities satisfying (H1)–(H5) are:
$$\varphi(x,t) = k(x)(\delta(x))^{m\gamma+m} t^{-\gamma},$$
for $\gamma \ge 0$, and
$$\psi(x,t) = k(x)(\delta(x))^m t^\alpha Log(1+t^\beta),$$
for $\alpha, \beta \ge 0$ such that $\alpha + \beta < 1$, where k is a nontrivial nonnegative functions in $L^r(B)$.

Recently Ben Othman [5] considered (1.1) when $p = 1$ and the functions φ, ψ satisfy hypotheses similar to the ones stated above. Then she proved that (1.1) has a positive continuous solutions u satisfying
$$a_0(\delta(x))^m \le u(x) \le V_1(\varphi(., a_0(\delta(.))^m))(x) + b_0 V_1((\delta(.))^{m-1} k)(x).$$
Here we prove an existence result for the more general problem (1.1) and obtain estimates both on the solution u and their derivatives $(-\Delta)^{jm} u$, for all $j \in \{1,\dots,p-1\}$.

To simplify our statements, we define some convenient notations:

(i) Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ and let $\overline{B} = \{x \in \mathbb{R}^n : |x| \le 1\}$, for $n \ge 2$.

(ii) $\mathcal{B}(B)$ denotes the set of Borel measurable functions in B, and $\mathcal{B}^+(B)$ the set of nonnegative ones.

(iii) $C(\overline{B})$ is the set of continuous functions in $\overline{B}$.

(iv) $C^j(B)$ is the set of functions having derivatives of order $\le j$, continuous in B ($j \in \mathbb{N}$).

(v) For $x, y \in B$, $[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$.

(vi) Let f and g be two positive functions on a set S. We call $f \preceq g$, if there is $c > 0$ such that $f(x) \leq cg(x)$, for all $x \in S$.
We call $f \sim g$, if there is $c > 0$ such that $\frac{1}{c}g(x) \leq f(x) \leq cg(x)$, for all $x \in S$.

(vii) For any $q \in \mathcal{B}(B)$, we put

$$\|q\|_{m,n,p} := \sup_{x\in B} \int_B (\frac{\delta(y)}{\delta(x)})^m \Gamma^{(p)}_{m,n}(x,y)|q(y)|dy.$$

2. Properties of the iterated Green function and the Kato class

Let $m \geq 1$, $p \geq 1$ be a positive integer and $\Gamma^{(p)}_{m,n}$ be the iterated Green function of the polyharmonic operator $u \mapsto (-\Delta)^{pm}u$, on the unit ball B of $\mathbb{R}^n$ $(n \geq 2)$ with boundary conditions $(\frac{\partial}{\partial\nu})^j(-\Delta)^{im}u\big|_{\partial B} = 0$, for $0 \leq i \leq p-1$ and $0 \leq j \leq m-1$, where $\frac{\partial}{\partial\nu}$ is the outward normal derivative.

Then for $p \geq 2$ and $x, y \in B$,

$$\Gamma^{(p)}_{m,n}(x,y) = \int_B \dots \int_B G_{m,n}(x,z_1)G_{m,n}(z_1,z_2)\dots G_{m,n}(z_{p-1},y)dz_1\dots dz_{p-1}, \tag{2.1}$$

where $G_{m,n}$ is the Green function of the polyharmonic operator $u \mapsto (-\Delta)^m u$, on B with Dirichlet boundary conditions $(\frac{\partial}{\partial\nu})^j u = 0$, $0 \leq j \leq m-1$.

Recall that Boggio in [6] gave an explicit expression for $G_{m,n}$: For each x, y in B,

$$G_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n}\int_1^{\frac{[x,y]}{|x-y|}} \frac{(v^2-1)^{m-1}}{v^{n-1}}dv,$$

where $k_{m,n}$ is a positive constant.

In this section we state some properties of $\Gamma^{(p)}_{m,n}$ and of functions belonging to the Kato class $\mathcal{J}^{(p)}_{m,n}$. These properties are useful for the statements of our existence result, and their proofs can be found in [2].

Proposition 2.1. *On B^2, the following estimates hold*

$$\Gamma^{(p)}_{m,n}(x,y) \sim \begin{cases} \frac{(\delta(x)\delta(y))^m}{|x-y|^{n-2pm}[x,y]^{2m}}, & \text{for } n > 2pm, \\ \frac{(\delta(x)\delta(y))^m}{[x,y]^{2m}}\log(1+\frac{[x,y]^2}{|x-y|^2}), & \text{for } n = 2pm \\ \frac{(\delta(x)\delta(y))^m}{[x,y]^{n-2(p-1)m}}, & \text{for } 2(p-1)m < n < 2pm. \end{cases} \tag{2.2}$$

Proposition 2.2. *With the above notation,*

$$(\delta(x)\delta(y))^m \preceq \Gamma^{(p)}_{m,n}(x,y),$$
$$\Gamma^{(p)}_{m,n}(x,y) \preceq \Gamma^{(p-1)}_{m,n}(x,y), \quad \text{for } p \geq 2.$$
$$\Gamma^{(p)}_{m,n}(x,y) \preceq \delta(x)\delta(y)\Gamma^{(p)}_{m-1,n}(x,y), \quad \text{for } m \geq 2.$$

In particular,

$$\mathcal{J}^{(1)}_{m,n} \subset \mathcal{J}^{(2)}_{m,n} \dots \subset \mathcal{J}^{(p)}_{m,n}\ ,\ \mathcal{J}^{(p)}_{1,n} \subset \mathcal{J}^{(p)}_{2,n} \subset \dots \subset \mathcal{J}^{(p)}_{m,n}. \tag{2.3}$$

Proposition 2.3. *Let q be a function in $\mathcal{J}^{(p)}_{m,n}$. Then*

$$\text{The function } x \mapsto (\delta(x))^{2m}q(x) \text{ is in } L^1(B). \tag{2.4}$$

$$\|q\|_{m,n,p} < \infty. \tag{2.5}$$

3. Existence result

We are concerned with the existence of positive solutions for the iterated polyharmonic nonlinear problems (1.1). For the proof, we need the next Lemma. For a given nonnegative function q in $\mathcal{J}^{(p)}_{m,n}$, we define

$$\mathcal{M}_q = \{\theta \in \mathcal{B}(B), |\theta| \leq q\}.$$

Lemma 3.1. *For any nonnegative function $q \in \mathcal{J}^{(p)}_{m,n}$, the family of functions*

$$\Big\{ \int_B \big(\frac{\delta(y)}{\delta(x)}\big)^m \Gamma^{(p)}_{m,n}(x,y)|\theta(y)|dy : \theta \in \mathcal{M}_q \Big\} \tag{3.1}$$

is uniformly bounded and equicontinuous in $\overline{B}$ and consequently it is relatively compact in $C(\overline{B})$.

Proof. Let q be a nonnegative function $q \in \mathcal{J}^{(p)}_{m,n}$ and L be the operator defined on $\mathcal{M}_q$ by

$$L\theta(x) = \int_B \big(\frac{\delta(y)}{\delta(x)}\big)^m \Gamma^{(p)}_{m,n}(x,y)|\theta(y)|dy.$$

By (2.5), for each $\theta \in \mathcal{M}_q$, we have

$$\sup_{x\in B} \int_B \big(\frac{\delta(y)}{\delta(x)}\big)^m \Gamma^{(p)}_{m,n}(x,y)|\theta(y)|dy \leq \|q\|_{m,n,p} < \infty.$$

Then the family $L(\mathcal{M}_q)$ is uniformly bounded. Next, we prove the equicontinuity of functions in $L(\mathcal{M}_q)$ on $\overline{B}$. Indeed, let $x_0 \in \overline{B}$ and $\varepsilon > 0$. By (1.3), there exists $\alpha > 0$ such that for each $x, x' \in B(x_0,\alpha) \cap B$, we have

$$\begin{aligned}
&|L\theta(x) - L\theta(x')| \\
&\leq \int_B \Big|\frac{\Gamma^{(p)}_{m,n}(x,y)}{(\delta(x))^m} - \frac{\Gamma^{(p)}_{m,n}(x',y)}{(\delta(x'))^m}\Big|(\delta(y))^m|q(y)|\,dy \\
&\leq \varepsilon + \int_{B\cap B(x_0,2\alpha)\cap B^c(x,2\alpha)} \Big|\frac{\Gamma^{(p)}_{m,n}(x,y)}{(\delta(x))^m} - \frac{\Gamma^{(p)}_{m,n}(x',y)}{(\delta(x'))^m}\Big|(\delta(y))^m|q(y)|\,dy \\
&\quad + \int_{B\cap B^c(x_0,2\alpha)\cap B^c(x,2\alpha)} \Big|\frac{\Gamma^{(p)}_{m,n}(x,y)}{(\delta(x))^m} - \frac{\Gamma^{(p)}_{m,n}(x',y)}{(\delta(x'))^m}\Big|(\delta(y))^m|q(y)|\,dy
\end{aligned}$$

Now since for $y \in B^c(x,2\alpha)\cap B$, from Proposition 2.1, we have

$$\Gamma^{(p)}_{m,n}(x,y) \preceq (\delta(x)\delta(y))^m.$$

We deduce that

$$\begin{aligned}
&\int_{B\cap B(x_0,2\alpha)\cap B^c(x,2\alpha)} \Big|\frac{\Gamma^{(p)}_{m,n}(x,y)}{(\delta(x))^m} - \frac{\Gamma^{(p)}_{m,n}(x',y)}{(\delta(x'))^m}\Big|(\delta(y))^m|q(y)|\,dy \\
&\preceq \int_{B\cap B(x_0,2\alpha)} (\delta(y))^{2m}|q(y)|\,dy,
\end{aligned}$$

which tends by (2.4) to zero as $\alpha \to 0$.

Since for $y \in B^c(x_0,2\alpha)\cap B$, the function $x \mapsto (\frac{\delta(y)}{\delta(x)})^m \Gamma^{(p)}_{m,n}(x,y)$ is continuous on $B(x_0,\alpha)\cap B$, by (2.4) and by the dominated convergence theorem, we have

$$\int_{B\cap B^c(x_0,2\alpha)\cap B^c(x,2\alpha)} \Big|\frac{\Gamma^{(p)}_{m,n}(x,y)}{(\delta(x))^m} - \frac{\Gamma^{(p)}_{m,n}(x',y)}{(\delta(x'))^m}\Big|(\delta(y))^m|q(y)|\,dy \to 0$$

as $|x-x'| \to 0$. This proves that the family $L(\mathcal{M}_q)$ is equicontinuous in $\overline{B}$. It follows by Ascoli's theorem, that $L(\mathcal{M}_q)$ is relatively compact in $C(\overline{B})$. □

The next remark will be used to obtain regularity of the solution.

Remark 3.2. Let $r > n$ and f be a nonnegative measurable function in $L^r(B)$. Then $V_p f \in C^{2pm-1}(B)$.

Indeed, by using the regularity theory of [1] (see also [10, Theorem 5.1], and [7, Theorem IX.32]), we obtain that $V_p f \in W^{2pm,r}(B)$. Furthermore, since $r > n$, then one finds that $V_p f \in C^{2pm-1}(B)$ (see [9, Chap. 7, p.158], or [7, Corollary IX.15]).

Proof of Theorem 1.2. Let K be compact in B such that $\gamma := \int_K h(y)dy > 0$ and define $r_0 := \min_{y\in K}(\delta(y))^m > 0$.

By (2.2) there exists a constant $c > 0$ such that for each $x, y \in B$,

$$c(\delta(x)\delta(y))^m \le \Gamma^{(p)}_{m,n}(x,y). \tag{3.2}$$

By (1.5) we can find $a > 0$ such that $cr_0\gamma f(ar_0) \ge a$.

By (H4) and (2.3), the function $k \in \mathcal{J}^{(1)}_{m,n} \subset \mathcal{J}^{(p)}_{m,n}$; then it follows from (2.5) that

$$\delta := \|V_p((\delta(.))^m k)\|_\infty \le \|k\|_{m,n,p} < \infty.$$

Let $0 < \alpha < \frac{1}{\delta}$, then using (1.6) we can find $\eta > 0$ such that for each $t \ge \eta$, $g(t) \le \alpha t$. Put $\beta := \sup_{0\le t\le \eta} g(t)$. Then we have

$$0 \le g(t) \le \alpha t + \beta, \text{ for } t \ge 0. \tag{3.3}$$

On the other hand, using (3.2) and (2.4), there exists a constant $c_0 > 0$ such that

$$V_p((\delta(.))^m k)(x) \ge c_0(\delta(x))^m. \tag{3.4}$$

From (H2), (2.5) and (2.3) we derive that

$$\nu := \|V_p(\varphi(., a(\delta(.))^m)\|_\infty < \infty.$$

Put $b = \max\{\frac{a}{c_0}, \frac{\alpha\nu+\beta}{1-\alpha\delta}\}$ and let Λ be the convex set given by

$$\Lambda = \left\{u \in C(\overline{B}) : a(\delta(x))^m \le u(x) \le V_p(\varphi(., a(\delta(.))^m)(x) + bV_p((\delta(.))^m k)(x)\right\}.$$

and T be the operator defined on Λ by

$$Tu(x) = \int_B \Gamma^{(p)}_{m,n}(x,y)[\varphi(y,u(y)) + \psi(y,u(y))]dy.$$

¿From (3.4), $\Lambda \ne \emptyset$. We will prove that T has a fixed point in Λ. Indeed, for $u \in \Lambda$, we have by (1.4), (3.2) and the monotonicity of f that

$$\begin{aligned}
Tu(x) &\ge \int_B \Gamma^{(p)}_{m,n}(x,y)\psi(y,u(y))dy \\
&\ge c(\delta(x))^m \int_B (\delta(y))^m h(y) f(u(y))dy \\
&\ge c(\delta(x))^m f(ar_0) r_0 \int_K h(y)dy \\
&\ge a(\delta(x))^m.
\end{aligned}$$

On the other hand, using (H1), (1.4) and (3.3), we deduce that

$$\begin{aligned} Tu(x) &\leq V_p(\varphi(., a(\delta(.))^m)(x) + \int_B \Gamma^{(p)}_{m,n}(x,y)(\delta(y))^m k(y) g(u(y)) dy \\ &\leq V_p(\varphi(., a(\delta(.))^m)(x) + \int_B \Gamma^{(p)}_{m,n}(x,y)(\delta(y))^m k(y)(\alpha u(y) + \beta) dy \\ &\leq V_p(\varphi(., a(\delta(.))^m)(x) + (\alpha(\nu + b\delta) + \beta) V_p((\delta(.))^m k)(x) \\ &\leq V_p(\varphi(., a(\delta(.))^m)(x) + b V_p((\delta(.))^m k)(x). \end{aligned}$$

Let $v(x) = \varphi(x, a(\delta(x))^m/(\delta(x))^m$. Then using similar arguments as above, we deduce that for each $u \in \Lambda$

$$\begin{gathered} \varphi(., u) \leq \varphi(., a(\delta(.))^m) = (\delta(.))^m v, \\ \psi(., u) \leq g(u)(\delta(.))^m k \leq b(\delta(.))^m k. \end{gathered} \tag{3.5}$$

That is, $\varphi(., u) + \psi(., u) \in \mathcal{M}_{(v+bk)(\delta(.))^m}$. Now since by (H2) and (H4), the function $(v + bk)(\delta(.))^m \in \mathcal{J}^{(1)}_{m,n} \subset \mathcal{J}^{(p)}_{m,n}$, we deduce from Lemma 3.1, that $T(\Lambda)$ is relatively compact in $C(\overline{B})$. In particular, for all $u \in \Lambda$, $Tu \in C(\overline{B})$ and so $T(\Lambda) \subset \Lambda$. Next, let us prove the continuity of T in Λ. We consider a sequence $(u_j)_{j\in\mathbb{N}}$ in Λ which converges uniformly to a function $u \in \Lambda$. Then we have

$$|Tu_j(x) - Tu(x)| \leq V_p[|\varphi(., u_j(.) - \varphi(., u(.))| + |\psi(., u_j(.)) - \psi(., u(.)|].$$

Now, by (3.5), we have

$$|\varphi(., u_j(.) - \varphi(., u(.))| + |\psi(., u_j(.)) - \psi(., u(.)| \leq 2(1 + b)(\delta(.))^m (v + k)$$

and since φ, ψ are continuous with respect on the second variable, we deduce by (2.5) and the dominated convergence theorem that

$$\forall x \in B, Tu_j(x) \to Tu(x) \quad \text{as } j \to \infty$$

Since $T\Lambda$ is relatively compact in $C(\overline{B})$, we have the uniform convergence, namely

$$\|Tu_j - Tu\|_\infty \to 0 \quad \text{as } j \to \infty.$$

Thus we have proved that T is a compact mapping from Λ to itself. Hence by the Schauder fixed point theorem, there exists $u \in \Lambda$ such that

$$u(x) = \int_B \Gamma^{(p)}_{m,n}(x,y)[\varphi(y, u(y)) + \psi(y, u(y))] dy. \tag{3.6}$$

Using (3.5), (H3) and (H5), for each $y \in B$,

$$\varphi(y, u(y)) + \psi(y, u(y)) \leq \varphi(y, a(\delta(y))^m) + b(\delta(y))^m k(y) \in L^r(B). \tag{3.7}$$

So it is clear that u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^{pm} u = \varphi(., u) + \psi(., u), \quad \text{in } B.$$

Furthermore, by (3.6), (3.7) and Remark 3.2, we deduce that $u \in C^{2pm-1}(B)$. Therefore, using again (3.6) and (2.1) we obtain for $j \in \{0, \dots, p-1\}$,

$$(-\Delta)^{jm} u(x) = \int_B \Gamma^{(p-j)}_{m,n}(x,y)[\varphi(y, u(y)) + \psi(y, u(y))] dy. \tag{3.8}$$

Using similar arguments as above, we obtain for all $j \in \{0, \dots, p-1\}$,

$$a_j(\delta(x))^m \leq (-\Delta)^{jm} u(x) \leq V_{p-j}(\varphi(., a_j(\delta(.))^m))(x) + b_j V_{p-j}((\delta(.))^m k)(x), \tag{3.9}$$

where a_j, b_j are positive constants. Finally, for $j \in \{0, \dots, p-1\}$, from (3.9), (2.3) and (2.5), we have

$$\begin{aligned} a_j(\delta(x))^m &\leq (-\Delta)^{jm} u(x) \\ &\leq (\delta(x))^m (\| \frac{\varphi(., a_j(\delta(.))^m)}{(\delta(.))^m} \|_{m,n,p-j} + b_j \|k\|_{m,n,p-j}) \\ &\preceq (\delta(x))^m. \end{aligned}$$

So u is the required solution. □

Example 3.3. *Let $r > n$, $\lambda < m + \frac{1}{r}$, $\gamma \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Let ρ_1, ρ_2 be a nontrivial nonnegative Borel measurable functions on B satisfying $\rho_1(x) \leq (\delta(x))^{m(1+\gamma)-\lambda}$ and $\rho_2(x) \leq (\delta(x))^{m-\lambda}$. Then the problem*

$$\begin{gathered} (-\Delta)^{pm} u = \rho_1(x) u^{-\gamma} + \rho_2(x) u^{\alpha} \log(1 + u^{\beta}), \quad \text{in } B \\ u > 0 \quad \text{in } B \\ \lim_{|x| \to 1} \frac{(-\Delta)^{jm} u(x)}{(1 - |x|)^{m-1}} = 0, \quad \text{for } 0 \leq j \leq p-1, \end{gathered}$$

has at least one positive solution, $u \in C^{2pm-1}(B)$, satisfying

$$(-\Delta)^{jm} u(x) \sim (\delta(x))^m, \quad \forall j \in \{0, \dots, p-1\}.$$

Remark 3.4. If $m = 1$ and $p \geq 1$, one can obtain similar existence result for (1.1) on a bounded domain $D \subset \mathbb{R}^n$ $(n \geq 2)$ of class $C^{2p,\alpha}$ with $\alpha \in (0, 1]$.

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