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LYAPUNOV FUNCTIONS FOR DICHOTOMIES IN MEAN

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Abstract. We consider the notion of an exponential dichotomy in mean for a cocycle, in which the exponential behavior in the classical notion of an exponential dichotomy is replaced by an average with respect to an invariant probability measure. We give a complete characterization of the exponential behavior in mean in terms of Lyapunov functions, both for cocycles over maps and flows.

1. Introduction

The notion of an exponential dichotomy is central in several parts of the theory of differential equations and dynamical systems. In particular, the existence of an exponential dichotomy causes that there are stable and unstable invariant manifolds under sufficiently small nonlinear perturbations. Moreover, the local instability of the trajectories caused by an exponential dichotomy is one of the main mechanisms for the occurrence of stochastic behavior in the presence of a finite invariant measure. We refer the reader to the books [1, 7, 11] for details and further references.

On the other hand, the existence of an exponential dichotomy is a strong requirement and in view of their important role it is of interest to look for more general types of hyperbolic behavior. Here we consider the more general notion of an *exponential dichotomy in mean* in which the exponential behavior in the classical notion is replaced by an average with respect to an invariant probability measure.

Now we describe briefly the notion of an exponential dichotomy in mean in the particular case when there is only contraction. Let ϕ be a flow preserving a probability measure μ on a measure space Ω, and let Φ be a cocycle over ϕ (with values in the set of bounded linear operators acting on a given Banach space X). We say that Φ has an *exponential contraction in mean* if there exist $K, a > 0$ such that

$$\int_\Omega \|\Phi(t,\omega)x(\omega)\| \, d\mu(\omega) \le K e^{-at} \int_\Omega \|x(\omega)\| \, d\mu(\omega)$$

for all $t \ge 0$ and all measurable functions $x\colon \Omega \to X$ such that

$$\int_\Omega \|x(\omega)\| \, d\mu(\omega) < \infty.$$

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The notion of an exponential dichotomy in mean corresponds to require that there exist both contraction and expansion in mean (see Section 3). We refer the reader to [2] for examples of exponential behavior in mean.

We note that the existence of an exponential dichotomy in mean (even though weaker than the classical notion of an exponential dichotomy) is still robust under sufficiently small linear perturbations. More precisely, given an essentially bounded measurable function $B\colon \Omega \to L(X)$, assume that there exists a unique cocycle Ψ satisfying

$$\Psi(t,\omega) = \Phi(t,\omega) + \int_0^t \Phi(t,\omega)\Phi(\tau,\omega)^{-1}B(\phi_\tau(\omega))\Psi(\omega,\tau)\,d\tau.$$

It is shown in [2] that if Φ admits an exponential dichotomy in mean (respectively, an exponential contraction in mean) and B is sufficiently small, then Ψ also admits an exponential dichotomy in mean (respectively, an exponential contraction in mean). This provides further examples of exponential behavior in mean.

In this paper we study in detail the relation between the notions of an exponential dichotomy in mean and of a *strict Lyapunov function*. In particular, we show that a cocycle over a flow has an exponential dichotomy in mean if and only if it has a strict Lyapunov function (see Theorems 3.1 and 4.1). This includes constructing explicitly a strict Lyapunov function for any exponential dichotomy in mean, which in the particular case of an exponential contraction in mean is given by

$$V(x) = -\sup\Big\{\int_\Omega \|\Phi(\tau,\omega)x(\omega)\|e^{a\tau}\,d\mu(\omega) : \tau \ge 0\Big\}$$

for some appropriate constant $a > 0$. We also consider cocycles over a measurable map (see Section 5).

Our work can be partly seen as a development of related approaches of Dalec'kiĭ and Kreĭn [5] and Massera and Schäffer [10], that go back to Lyapunov. According to Coppel [4], the relation between Lyapunov functions and exponential dichotomies was first considered by Maĭzel' in [9]. Among the first accounts of the theory are the books by LaSalle and Lefschetz [8], Hahn [6] and Bhatia and Szegö [3].

2. Lyapunov functions and exponential contractions

We first introduce some basic notions. Let $\Omega = (\Omega, \mu)$ be a probability space. A measurable map $\phi\colon \mathbb{R}\times\Omega \to \Omega$ is said to be a *flow* on Ω if:

(1) $\phi(0,\omega) = \omega$ for $\omega \in \Omega$;
(2) $\phi(t+s,\omega) = \phi(t,\phi(s,\omega))$ for $t,s \in \mathbb{R}$ and $\omega \in \Omega$.

We also consider the measurable maps $\phi^t = \phi(t,\cdot)$. Now let X be a Banach space and let $L(X)$ be the set of all bounded linear operators acting on X. A measurable map $\Phi\colon \mathbb{R}\times\Omega \to L(X)$ is said to be a *cocycle* over a flow ϕ if:

(1) $\Phi(0,\omega) = \mathrm{Id}$ for $\omega \in \Omega$;
(2) $\Phi(s+t,\omega) = \Phi(s,\phi^t(\omega))\Phi(t,\omega)$ for $s,t \in \mathbb{R}$ and $\omega \in \Omega$.

Taking $s = -t$ we find that $\Phi(t,\omega)$ is invertible for each $t \in \mathbb{R}$ and $\omega \in \Omega$, with $\Phi(t,\omega)^{-1} = \Phi(-t,\phi^t(\omega))$. One can easily verify that Φ is a cocycle over ϕ if and only if one can define a flow on $\Omega\times X$ by

$$(t,\omega,x) \mapsto (\phi^t(\omega), \Phi(t,\omega)x).$$

Now assume that μ is ϕ-invariant (that is, $\mu \circ \phi^t = \mu$ for every $t \in \mathbb{R}$) and consider the space $\mathscr{F}$ of all Bochner measurable functions $x\colon \Omega \to X$ such that

$$\int_\Omega \|x(\omega)\|\, d\mu(\omega) < \infty$$

identified if they are equal μ-almost everywhere. We say that a cocycle Φ has an *exponential contraction in mean* if there exist $K, a > 0$ such that

$$\int_\Omega \|\Phi(t,\omega)x(\omega)\|\, d\mu(\omega) \le Ke^{-at} \int_\Omega \|x(\omega)\|\, d\mu(\omega) \tag{2.1}$$

for $t \ge 0$ and $x \in \mathscr{F}$.

Remark 2.1. In [2] we introduced the notion of a contraction in mean (for a measure μ that need not be invariant) by requiring that

$$\int_\Omega \|\Phi(t,\omega)\Phi(s,\omega)^{-1} z(\omega)\|\, d\mu(\omega) \le Ke^{-a(t-s)} \int_\Omega \|z(\omega)\|\, d\mu(\omega)$$

for $t \ge s$ and $z \in \mathscr{F}$. However, when μ is ϕ-invariant, we have $z = x \circ \phi^s \in \mathscr{F}$ for all $x \in \mathscr{F}$ and so,

$$\begin{aligned}
\int_\Omega \|\Phi(t,\omega)\Phi(s,\omega)^{-1} z(\omega)\|\, d\mu(\omega) &= \int_\Omega \|\Phi(t-s,\phi^s(\omega))x(\phi^s(\omega))\|\, d\mu(\omega) \\
&= \int_\Omega \|\Phi(t-s,\omega)x(\omega)\|\, d\mu(\omega)
\end{aligned}$$

and

$$\int_\Omega \|z(\omega)\|\, d\mu(\omega) = \int_\Omega \|x(\phi^s(\omega))\|\, d\mu(\omega) = \int_\Omega \|x(\omega)\|\, d\mu(\omega).$$

This shows that for a ϕ-invariant measure the notion introduced in [2] can be written as in (2.1).

Now we introduce the notion of a strict Lyapunov function in mean for a cocycle Φ. We first define a flow Φ_t^* on $\mathscr{F}$ by

$$(\Phi_t^* x)(\omega) = \Phi(t, \phi^{-t}(\omega))x(\phi^{-t}(\omega))$$

for $t \in \mathbb{R}$ and $x \in \mathscr{F}$. We say that a function $V\colon \mathscr{F} \to (-\infty, 0]$ is a *strict Lyapunov function in mean for* Φ if:

(1) there exists $C > 0$ such that

$$\frac{1}{C}\int_\Omega \|x(\omega)\|\, d\mu(\omega) \le |V(x)| \le C \int_\Omega \|x(\omega)\|\, d\mu(\omega) \quad \text{for } x \in \mathscr{F}; \tag{2.2}$$

(2) there exists $\theta \in (0,1)$ such that

$$|V(\Phi_t^* x)| \le \theta^t |V(x)| \quad \text{for } t \ge 0 \text{ and } x \in \mathscr{F}.$$

The following result characterizes an exponential contraction in mean in terms of the existence of a strict Lyapunov function in mean.

Theorem 2.2. *The following properties are equivalent.*

(1) *the cocycle Φ has an exponential contraction in mean;*
(2) *there exists a strict Lyapunov function in mean for Φ.*

Proof. Assume that there exists a strict Lyapunov function in mean V for Φ. Then for $t \geq 0$ and $x \in \mathscr{F}$ we have

$$\begin{aligned}\int_\Omega \|\Phi(t,\omega)x(\omega)\|\,d\mu(\omega) &= \int_\Omega \|\Phi(t,\phi^{-t}(\omega))x(\phi^{-t}(\omega))\|\,d\mu(\omega)\\ &\leq C|V(\Phi_t^* x)| \leq C\theta^t|V(x)|\\ &\leq C^2\theta^t\int_\Omega \|x(\omega)\|\,d\mu(\omega).\end{aligned}$$

This shows that Φ has an exponential contraction in mean with $a = -\log\theta$.

Now assume that Φ has an exponential contraction in mean. For $x \in \mathscr{F}$ we define

$$V(x) = -\sup\Big\{\int_\Omega \|\Phi(\tau,\omega)x(\omega)\|e^{a\tau}\,d\mu(\omega) : \tau \geq 0\Big\}.$$

By (2.1) we have

$$|V(x)| \leq K\int_\Omega \|x(\omega)\|\,d\mu(\omega).$$

On the other hand, setting $\tau = 0$ we obtain

$$|V(x)| \geq \int_\Omega \|\Phi(0,\omega)x(\omega)\|\,d\mu(\omega) = \int_\Omega \|x(\omega)\|\,d\mu(\omega),$$

which establishes (2.2). Moreover, since μ is ϕ-invariant, for $t \geq 0$ we have

$$\begin{aligned}|V(\Phi_t^* x)| &= \sup\Big\{\int_\Omega \|\Phi(\tau,\omega)\Phi(t,\phi^{-t}(\omega))x(\phi^{-t}(\omega))\|e^{a\tau}\,d\mu(\omega) : \tau \geq 0\Big\}\\ &= e^{-at}\sup\Big\{\int_\Omega \|\Phi(t+\tau,\phi^{-t}(\omega))x(\phi^{-t}(\omega))\|e^{a(\tau+t)}\,d\mu(\omega) : \tau \geq 0\Big\}\\ &= e^{-at}\sup\Big\{\int_\Omega \|\Phi(t+\tau,\omega)x(\omega)\|e^{a(\tau+t)}\,d\mu(\omega) : \tau \geq 0\Big\}\\ &\leq e^{-at}\sup\Big\{\int_\Omega \|\Phi(r,\omega)x(\omega)\|e^{ar}\,d\mu(\omega) : r \geq 0\Big\}\\ &= e^{-at}|V(x)|\end{aligned}$$

and so V is a strict Lyapunov function in mean with $\theta = e^{-a}$. □

3. Lyapunov functions and exponential behavior

In this section we develop the theory further having in mind the general case when a cocycle Φ exhibits both contraction and expansion. We say that a cocycle Φ has an *exponential dichotomy in mean* if:

(1) there exist projections $P(\omega) \in L(X)$ for $\omega \in \Omega$ satisfying

$$\Phi(t,\omega)P(\omega) = P(\phi^t(\omega))\Phi(t,\omega) \tag{3.1}$$

for $t \in \mathbb{R}$ and μ-almost every $\omega \in \Omega$;

(2) there exist $K, a > 0$ such that

$$\int_\Omega \|\Phi(t,\omega)P(\omega)x(\omega)\|\,d\mu(\omega) \leq Ke^{-at}\int_\Omega \|x(\omega)\|\,d\mu(\omega), \tag{3.2}$$

$$\int_\Omega \|\Phi(-t,\omega)Q(\omega)x(\omega)\|\,d\mu(\omega) \leq Ke^{-at}\int_\Omega \|x(\omega)\|\,d\mu(\omega), \tag{3.3}$$

for $t \geq 0$ and $x \in \mathscr{F}$, where $Q(\omega) = \mathrm{Id} - P(\omega)$.

We define corresponding *stable* and *unstable subspaces* by

$$F^s(\omega) = \operatorname{Im} P(\omega) \quad \text{and} \quad F^u(\omega) = \operatorname{Im} Q(\omega). \tag{3.4}$$

Now we introduce the notion of a strict Lyapunov function in mean. Given a function $V\colon \mathscr{F} \to \mathbb{R}$, we consider the *cones*

$$C^u(V) = \{0\} \cup V^{-1}(0, +\infty) \quad \text{and} \quad C^s(V) = \{0\} \cup V^{-1}(-\infty, 0). \tag{3.5}$$

Moreover, given a cocycle Φ, we define

$$E^u = \bigcap_{t\geq 0} \Phi_t^* \overline{C^u(V)} \quad \text{and} \quad E^s = \bigcap_{t\geq 0} \Phi_{-t}^* \overline{C^s(V)}.$$

Clearly,

$$\begin{gathered} E^u = \big\{x \in \mathscr{F} : V(\Phi_{-t}^* x) \geq 0 \text{ for } t \geq 0\big\}, \\ E^s = \big\{x \in \mathscr{F} : V(\Phi_t^* x) \leq 0 \text{ for } t \geq 0\big\}. \end{gathered}$$

We say that V is a *Lyapunov function in mean* for Φ if:

(1) there exist closed subspaces $D^u \subset E^u$ and $D^s \subset E^s$ such that

$$D^u \oplus D^s = \mathscr{F} \tag{3.6}$$

with continuous projections associated to this splitting;

(2) there exists $C > 0$ such that

$$|V(x)| \leq C \int_\Omega \|x(\omega)\| \, d\mu(\omega) \quad \text{for } x \in \mathscr{F}; \tag{3.7}$$

(3) $V(\Phi_t^* x) \geq V(x)$ for $x \in \mathscr{F}$ and $t \geq 0$.

Moreover, a Lyapunov function in mean V is said to be *strict* if:

(1) there exists $D > 0$ such that

$$|V(x)| \geq \frac{1}{D} \int_\Omega \|x(\omega)\| \, d\mu(\omega) \quad \text{for } x \in E^u \cup E^s; \tag{3.8}$$

(2) there exists $\theta \in (0, 1)$ such that for $t \geq 0$ we have

$$V(\Phi_t^* x) \geq \theta^{-t} V(x) \quad \text{for } x \in E^u, \tag{3.9}$$

$$|V(\Phi_t^* x)| \leq \theta^t |V(x)| \quad \text{for } x \in E^s. \tag{3.10}$$

We note that in the particular case when V takes values only in $(-\infty, 0]$ (and so $E^s = \mathscr{F}$), this notion of a strict Lyapunov function in mean coincides with the corresponding notion introduced in Section 2.

Theorem 3.1. *If V is a strict Lyapunov function in mean for the cocycle Φ, then:*

(1) *$D^u = E^u$, $D^s = E^s$, and for each $t \in \mathbb{R}$ we have*

$$\Phi_t^* E^u = E^u \quad \textit{and} \quad \Phi_t^* E^s = E^s; \tag{3.11}$$

(2) *Φ has an exponential dichotomy in mean with projections $P(\omega)$ determined pointwise by the direct sum $E^s \oplus E^u = \mathscr{F}$, that is, with $\operatorname{Im} P(\omega) = E^s(\omega)$ and $\operatorname{Im} Q(\omega) = E^u(\omega)$ for μ-almost every $\omega \in \Omega$.*

Proof. By definition,

$$E^u \subset \overline{C^u(V)} \quad \text{and} \quad E^s \subset \overline{C^s(V)}.$$

Moreover, by (3.8), the function V is positive on $E^u \setminus \{0\}$ and negative on $E^s \setminus \{0\}$. For each $x \in E^s \setminus \{0\}$ and $t \geq 0$, it follows from (3.8) and (3.10) that

$$\frac{1}{D} \int_\Omega \|\Phi(t, \phi^{-t}(\omega)) x(\phi^{-t}(\omega))\| t \, d\mu(\omega) \leq |V(\Phi_t^* x)| \leq \theta^t |V(x)| \tag{3.12}$$

and so

$$\limsup_{t \to +\infty} \frac{1}{t} \log \int_\Omega \|\Phi(t, \phi^{-t}(\omega)) x(\phi^{-t}(\omega))\| \, d\mu(\omega) \leq \log \theta < 0. \tag{3.13}$$

Similarly, for each $x \in E^u \setminus \{0\}$ and $t \geq 0$, it follows from (3.7) and (3.9) that

$$C \int_\Omega \|\Phi(t, \phi^{-t}(\omega)) x(\phi^{-t}(\omega))\| \, d\mu(\omega) \geq V(\Phi_t^* x) \geq \theta^{-t} V(x) \tag{3.14}$$

and so

$$\limsup_{t \to +\infty} \frac{1}{t} \log \int_\Omega \|\Phi(t, \phi^{-t}(\omega)) x(\phi^{-t}(\omega))\| \, d\mu(\omega) \geq -\log \theta > 0. \tag{3.15}$$

Now let $D^u \subset E^u$ and $D^s \subset E^s$ be closed subspaces as in (3.6). If $D^s \neq E^s$, then there exists $x \in E^s \setminus D^s$ and one can write $x = y + z$, with $y \in D^s$ and $z \in D^u \setminus \{0\}$. By (3.13) and (3.15), since $z \neq 0$ we have

$$\begin{aligned}
&\limsup_{t \to +\infty} \frac{1}{t} \log \int_\Omega \|\Phi(t, \phi^{-t}(\omega)) x(\phi^{-t}(\omega))\| \, d\mu(\omega) \\
&= \max \Big\{ \limsup_{t \to +\infty} \frac{1}{t} \log \int_\Omega \|\Phi(t, \phi^{-t}(\omega)) y(\phi^{-t}(\omega))\| \, d\mu(\omega), \\
&\quad \limsup_{t \to +\infty} \frac{1}{t} \log \int_\Omega \|\Phi(t, \phi^{-t}(\omega)) z(\phi^{-t}(\omega))\| \, d\mu(\omega) \Big\} \\
&= \limsup_{t \to +\infty} \frac{1}{t} \log \int_\Omega \|\Phi(t, \phi^{-t}(\omega)) z(\phi^{-t}(\omega))\| \, d\mu(\omega) > 0
\end{aligned}$$

(with the convention that $\log 0 = -\infty$), which contradicts to (3.13). Therefore, $D^s = E^s$. One can show in a similar manner that $D^u = E^u$.

Moreover, by condition 3 in the notion of a Lyapunov function in mean, for $t \geq 0$ we have

$$\Phi_t^* \overline{C^u(V)} \subset \overline{C^u(V)} \quad \text{and} \quad \Phi_{-t}^* \overline{C^s(V)} \subset \overline{C^s(V)}.$$

This implies that

$$E^u = \bigcap_{t \geq r} \Phi_t^* \overline{C^u(V)} \quad \text{and} \quad E^s = \bigcap_{t \geq r} \Phi_{-t}^* \overline{C^s(V)}$$

for each $r > 0$. Hence,

$$\Phi_{-r}^* E^u = \bigcap_{t \geq r} \Phi_{-r}^* \Phi_t^* \overline{C^u(V)} = \bigcap_{t \geq r} \Phi_{t-r}^* \overline{C^u(V)} = E^u$$

and, similarly, $\Phi_r^* E^s = E^s$. Since Φ_t^* is invertible for each $t \in \mathbb{R}$ and r is arbitrary, this establishes (3.11).

Now we establish the second statement in the theorem. By (3.7) and (3.12), for $x \in E^s$ and $t \geq 0$ we have

$$\int_\Omega \|\Phi(t, \phi^{-t}(\omega)) x(\phi^{-t}(\omega))\| \, d\mu(\omega) \leq D\theta^t |V(x)| \leq CD\theta^t \int_\Omega \|x(\omega)\| \, d\mu(\omega). \tag{3.16}$$

Moreover, it follows from (3.8) and (3.14) that

$$\int_\Omega \|\Phi(t,\phi^{-t}(\omega))x(\phi^{-t}(\omega))\|\,d\mu(\omega) \ge \frac{1}{C}\theta^{-t}V(x) \ge \frac{1}{CD}\theta^{-t}\int_\Omega \|x(\omega)\|\,d\mu(\omega)$$

for $x \in E^u$ and $t \ge 0$. In view of (3.11), this happens if and only if

$$\int_\Omega \|\Phi(-t,\phi^{t}(\omega))y(\phi^{t}(\omega))\|\,d\mu(\omega) \le CD\theta^t\int_\Omega \|y(\omega)\|\,d\mu(\omega) \tag{3.17}$$

for $y \in E^s$ and $t \ge 0$. Indeed, letting

$$y(\omega) = \Phi(t,\phi^{-t}(\omega))x(\phi^{-t}(\omega)) = (\Phi_t^* x)(\omega),$$

we obtain

$$x(\omega) = (\Phi_{-t}^* y)(\omega) = \Phi(-t,\phi^{t}(\omega))y(\phi^{t}(\omega)).$$

By (3.16) and (3.17), for $t \ge 0$ we have

$$\|\Phi_t^*|E^s\| \le CD\theta^t \quad\text{and}\quad \|\Phi_{-t}^*|E^u\| \le CD\theta^t.$$

Since the projections associated to the direct sum $D^u \oplus D^s = \mathscr{F}$ are continuous, this implies that Φ has an exponential dichotomy in mean and the proof of the theorem is complete. □

4. Construction of Lyapunov functions

In this section we construct a strict Lyapunov function in mean for any exponential dichotomy in mean.

Theorem 4.1. *If the cocycle Φ has an exponential dichotomy in mean, then there exists a strict Lyapunov function in mean for Φ.*

Proof. Given $x \in \mathscr{F}$, we write $x = y + z$, with $y(\omega) \in F^s(\omega)$ and $z(\omega) \in F^u(\omega)$ for each $\omega \in \Omega$ (see (3.4)). For $x \in \mathscr{F}$, let

$$V(x) = -V^s(y) + V^u(z),$$

where

$$\begin{gathered} V^s(y) = \sup\Big\{\int_\Omega \|\Phi(r,\omega)y(\omega)\|e^{ar}\,d\mu(\omega) : r \ge 0\Big\}, \\ V^u(z) = \sup\Big\{\int_\Omega \|\Phi(r,\omega)z(\omega)\|e^{-ar}\,d\mu(\omega) : r \le 0\Big\}. \end{gathered}$$

Clearly,

$$V^s(y) \le K\int_\Omega \|y(\omega)\|\,d\mu(\omega) \quad\text{and}\quad V^u(z) \le K\int_\Omega \|z(\omega)\|\,d\mu(\omega).$$

Therefore,

$$|V(x)| \le K\Big(\int_\Omega \|y(\omega)\|\,d\mu(\omega) + \int_\Omega \|z(\omega)\|\,d\mu(\omega)\Big) \le 2K^2\int_\Omega \|x(\omega)\|\,d\mu(\omega).$$

which shows that property (3.7) holds.

Now observe that

$$\begin{aligned} \int_\Omega \|\Phi(r,\omega)y(\omega)\|\,d\mu(\omega) &= \int_\Omega \|\Phi(r,\phi^{-r}(\omega))y(\phi^{-r}(\omega))\|\,d\mu(\omega) \\ &= \int_\Omega \|(\Phi_r^* y)(\omega)\|\,d\mu(\omega) \end{aligned}$$

and similarly,

$$\int_\Omega \|\Phi(r,\omega)z(\omega)\|\,d\mu(\omega) = \int_\Omega \|(\Phi_r^* z)(\omega)\|\,d\mu(\omega).$$

Hence,

$$\begin{gathered}V^s(y) = \sup\Big\{\int_\Omega \|(\Phi_r^* y)(\omega)\| e^{ar}\,d\mu(\omega) : r \ge 0\Big\},\\ V^u(z) = \sup\Big\{\int_\Omega \|(\Phi_r^* z)(\omega)\| e^{-ar}\,d\mu(\omega) : r \le 0\Big\}.\end{gathered}$$

Therefore,

$$V^s(\Phi_t^* y) = \sup\Big\{\int_\Omega \|(\Phi_{t+r}^* y)(\omega)\| e^{ar}\,d\mu(\omega) : r \ge 0\Big\}, \tag{4.1}$$

$$V^u(\Phi_t^* z) = \sup\Big\{\int_\Omega \|(\Phi_{t+r}^* z)(\omega)\| e^{-ar}\,d\mu(\omega) : r \le 0\Big\} \tag{4.2}$$

for $t \in \mathbb{R}$. Now we define projections $P\colon \mathscr{F} \to F^s$ and $Q\colon \mathscr{F} \to F^u$ by

$$(Px)(\omega) = P(\omega)x(\omega) \quad \text{for } \omega \in \Omega$$

and $Q = \mathrm{Id} - P$. Property (3.1) is equivalent to

$$P\Phi_t^* = \Phi_t^* P \quad \text{for } t \in \mathbb{R}. \tag{4.3}$$

Together with (4.3), equations (4.1) and (4.2) imply that for $t \ge 0$ (and in fact for $t \in \mathbb{R}$) we have

$$\begin{gathered}V(\Phi_t^* x) \le 0 \quad \text{for } x \in F^s,\\ V(\Phi_{-t}^* x) \ge 0 \quad \text{for } x \in F^u.\end{gathered}$$

Therefore, $F^s \subset E^u$ and $F^u \subset E^u$. On the other hand, it follows from (3.2) and (3.3) that $\|P\| \le K$ and $\|Q\| \le K$. Hence, the projections P and Q are continuous, and condition 1 in the notion of a Lyapunov function in mean holds taking $D^s = F^s$ and $D^u = F^u$.

Furthermore, for each $t \ge 0$ it follows from (4.1) that

$$\begin{aligned}V^s(\Phi_t^* y) &= e^{-at}\sup\Big\{\int_\Omega \|(\Phi_{t+r}^* y)(\omega)\| e^{a(t+r)}\,d\mu(\omega) : r \ge 0\Big\}\\ &\le e^{-at}V^s(y) \le V^s(y)\end{aligned} \tag{4.4}$$

and it follows from (4.2) that

$$\begin{aligned}V^u(\Phi_t^* z) &= e^{at}\sup\Big\{\int_\Omega \|(\Phi_{t+r}^* z)(\omega)\| e^{-a(t+r)}\,d\mu(\omega) : r \le 0\Big\}\\ &\ge e^{at}V^u(z) \ge V^u(z).\end{aligned} \tag{4.5}$$

Therefore,

$$\begin{aligned}V(\Phi_t^* x) &= -V^s(\Phi_t^* y) + V^u(\Phi_t^* z)\\ &\ge -V^s(y) + V^u(z) = V(x).\end{aligned}$$

This establishes condition 3 in the notion of Lyapunov function in mean.

Now we show that V is in fact strict. For $x \in E^s$ and $t \ge 0$ we have $V(\Phi_t^* x) \le 0$ and so it follows from (4.4) and (4.5) that

$$|V(\Phi_t^* x)| = V^s(\Phi_t^* y) - V^u(\Phi_t^* z)$$

$$\begin{aligned}&\le e^{-at}V^s(y)-e^{at}V^u(z)\\&\le e^{-at}(V^s(y)-V^u(z))=e^{-at}|V(x)|.\end{aligned}$$

Hence, property (3.10) holds with $\theta=e^{-a}$. On the other hand, for $x\in E^u$ and $t\ge 0$ we have $V(\Phi_t^*x)\ge 0$ and so it follows again from (4.4) and (4.5) that

$$\begin{aligned}V(\Phi_t^*x)&=-V^s(\Phi_t^*y)+V^u(\Phi_t^*z)\\&\ge -e^{-at}V^s(y)+e^{at}V^u(z)\\&\ge e^{at}(-V^s(y)+V^u(z))=e^{at}V(x).\end{aligned}$$

Hence, property (3.9) also holds, with the same constant θ.

It remains to establish property (3.8). For $x\in E^s$ we have

$$\begin{aligned}|V(x)|&\ge |V(x)|-|V(\Phi_1^*x)|\\&=V^s(y)-V^s(\Phi_1^*y)-V^u(z)+V^u(\Phi_1^*z).\end{aligned}\tag{4.6}$$

Moreover, by (4.4) and (4.5) we have

$$\begin{aligned}V^s(y)-V^s(\Phi_1^*y)&\ge V^s(y)-e^{-a}V^s(y)=(1-e^{-a})V^s(y)\\&\ge (1-e^{-a})\int_\Omega \|y(\omega)\|\,d\mu(\omega)\end{aligned}$$

and

$$\begin{aligned}-V^u(z)+V^u(\Phi_1^*z)&\ge -V^u(z)+e^aV^u(z)=(e^a-1)V^u(z)\\&\ge (e^a-1)\int_\Omega \|z(\omega)\|\,d\mu(\omega).\end{aligned}$$

Setting $\eta=\min\{1-e^{-a},e^a-1\}$, it follows from (4.6) that

$$\begin{aligned}|V(x)|&\ge \eta\Big(\int_\Omega \|y(\omega)\|\,d\mu(\omega)+\int_\Omega \|z(\omega)\|\,d\mu(\omega)\Big)\\&\ge \eta\int_\Omega \|x(\omega)\|\,d\mu(\omega).\end{aligned}\tag{4.7}$$

Now assume that $x\in E^u$. Then

$$\begin{aligned}V(x)&\ge V(x)-V(\Phi_{-1}^*x)\\&=-V^s(y)+V^s(\Phi_{-1}^*y)+V^u(z)-V^u(\Phi_{-1}^*z).\end{aligned}$$

By (4.4) with y replaced by Φ_{-1}^*y and (4.5) with z replaced by Φ_{-1}^*z we get

$$\begin{aligned}-V^s(y)+V^s(\Phi_{-1}^*y)&\ge -V^s(y)+e^aV^s(y)=(e^a-1)V^s(y)\\&\ge (e^a-1)\int_\Omega \|y(\omega)\|\,d\mu(\omega)\end{aligned}$$

and

$$\begin{aligned}V^u(z)-V^u(\Phi_{-1}^*z)&\ge V^u(z)-e^{-a}V^u(z)=(1-e^{-a})V^u(z)\\&\ge (1-e^{-a})\int_\Omega \|z(\omega)\|\,d\mu(\omega).\end{aligned}$$

So, we obtain again (4.7). Thus, property (3.8) holds with $D=1/\eta$. □

5. The case of discrete time

In this section we consider the case of discrete time and we present corresponding notions and results to those for continuous time.

5.1. **Lyapunov functions and exponential contractions.** Let (Ω,μ) be a probability space and let $f\colon \Omega\to\Omega$ be an invertible measurable map with measurable inverse. A measurable map $\mathscr{A}\colon \mathbb{Z}\times\Omega\to L(X)$ is said to be a *cocycle* over f if $\mathscr{A}(0,\omega)=\mathrm{Id}$ and

$$\mathscr{A}(n+m,\omega)=\mathscr{A}(n,f^m(\omega))\mathscr{A}(m,\omega)$$

for $n,m\in\mathbb{Z}$ and $\omega\in\Omega$. We assume that μ is f-invariant (that is, $\mu(f^{-1}A)=\mu(A)$ for any measurable set $A\subset\Omega$). We say that a cocycle $\mathscr{A}$ has an *exponential contraction in mean* if there exist $K,a>0$ such that

$$\int_\Omega \|\mathscr{A}(m,\omega)x(\omega)\|\,d\mu(\omega)\le Ke^{-am}\int_\Omega \|x(\omega)\|\,d\mu(\omega)$$

for $m\in\mathbb{N}_0$ and $x\in\mathscr{F}$ (where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$).

Now we introduce the notion of a strict Lyapunov function in mean for a cocycle $\mathscr{A}$. We first define a map $T\colon\mathscr{F}\to\mathscr{F}$ by

$$(Tx)(\omega)=\mathscr{A}(1,f^{-1}(\omega))x(f^{-1}(\omega))$$

for $x\in\mathscr{F}$. One can easily verify that T is invertible. We say that a function $V\colon\mathscr{F}\to(-\infty,0]$ is a *strict Lyapunov function in mean for $\mathscr{A}$* if:

(1) property (2.2) holds for some $C>0$;
(2) there exists $\theta\in(0,1)$ such that

$$|V(Tx)|\le\theta|V(x)|\quad\text{for } x\in\mathscr{F}.$$

The following result is an analog of Theorem 2.2 for discrete time. We do not give the proof since it is essentially the same as the proof of Theorem 2.2.

Theorem 5.1. *The following properties are equivalent.*

(1) *the cocycle $\mathscr{A}$ has an exponential contraction in mean;*
(2) *there exists a strict Lyapunov function in mean for $\mathscr{A}$.*

5.2. **Lyapunov functions and exponential behavior.** In this section we consider the general case when a cocycle $\mathscr{A}$ admits both contraction and expansion. We say that a cocycle $\mathscr{A}$ has an *exponential dichotomy in mean* if:

(1) there exist projections $P(\omega)\in L(X)$ for $\omega\in\Omega$ satisfying

$$\mathscr{A}(m,\omega)P(\omega)=P(f^m(\omega))\Phi(m,\omega)$$

for $m\in\mathbb{Z}$ and μ-almost every $\omega\in\Omega$;
(2) there exist $K,a>0$ such that

$$\int_\Omega \|\mathscr{A}(m,\omega)P(\omega)x(\omega)\|\,d\mu(\omega)\le Ke^{-am}\int_\Omega\|x(\omega)\|\,d\mu(\omega),$$
$$\int_\Omega \|\mathscr{A}(-m,\omega)Q(\omega)x(\omega)\|\,d\mu(\omega)\le Ke^{-am}\int_\Omega\|x(\omega)\|\,d\mu(\omega),$$

for $m\in\mathbb{N}_0$ and $x\in\mathscr{F}$, where $Q(\omega)=\mathrm{Id}-P(\omega)$.

Again we define the corresponding *stable* and *unstable subspaces* by (3.4).

Now we introduce the notion of a strict Lyapunov function in mean. Given a function $V\colon\mathscr{F}\to\mathbb{R}$ and a cocycle $\mathscr{A}$, we consider the cones in (3.5) and we define

$$E^u=\bigcap_{m\in\mathbb{N}_0}T^m\overline{C^u(V)}\quad\text{and}\quad E^s=\bigcap_{m\in\mathbb{N}_0}T^{-m}\overline{C^s(V)}.$$

We say that V is a *Lyapunov function in mean* for $\mathscr{A}$ if:

(1) there exist closed subspaces $D^u \subset E^u$ and $D^s \subset E^s$ satisfying (3.6) with continuous projections associated to this splitting;
(2) property (3.7) holds for some $C > 0$;
(3) $V(Tx) \geq V(x)$ for $x \in \mathscr{F}$.

Moreover, a Lyapunov function in mean V is said to be *strict* if:
(1) property (3.8) holds for some $D > 0$;
(2) there exists $\theta \in (0,1)$ such that

$$\begin{gathered} V(Tx) \geq \theta V(x) \quad \text{for } x \in E^u, \\ |V(Tx)| \leq \theta |V(x)| \quad \text{for } x \in E^s. \end{gathered}$$

Again, in the particular case when $E^s = \mathscr{F}$, the last notion coincides with the corresponding notion introduced in Section 5.1. The following result is an analog of Theorem 3.1 for discrete time. The proof is essentially the same as the proof of Theorem 3.1.

Theorem 5.2. *If V is a strict Lyapunov function in mean for the cocycle $\mathscr{A}$, then:*
(1) *$D^u = E^u$, $D^s = E^s$,*

$$TE^u = E^u \quad \text{and} \quad TE^s = E^s;$$

(2) *$\mathscr{A}$ has an exponential dichotomy in mean with projections $P(\omega)$ determined pointwise by the direct sum $E^s \oplus E^u = \mathscr{F}$.*

Moreover, there exists a strict Lyapunov function in mean for any exponential dichotomy in mean.

Theorem 5.3. *If the cocycle $\mathscr{A}$ has an exponential dichotomy in mean, then there exists a strict Lyapunov function in mean for $\mathscr{A}$.*

Proof. Given $x \in \mathscr{F}$, we write $x = y + z$, with $y(\omega) \in F^s(\omega)$ and $z(\omega) \in F^u(\omega)$ for each $\omega \in \Omega$. For $x \in \mathscr{F}$, let

$$V(x) = -V^s(y) + V^u(z),$$

where

$$\begin{gathered} V^s(y) = \sup\Big\{ \int_\Omega \|\mathscr{A}(\ell,\omega)y(\omega)\| e^{a\ell}\, d\mu(\omega) : \ell \in \mathbb{N}_0 \Big\}, \\ V^u(z) = \sup\Big\{ \int_\Omega \|\mathscr{A}(-\ell,\omega)z(\omega)\| e^{a\ell}\, d\mu(\omega) : \ell \in \mathbb{N}_0 \Big\}. \end{gathered}$$

Proceeding as in the proof of Theorem 4.1 (replacing the flow Φ_t^* by T and $\Phi(r,\omega)$ by $\mathscr{A}(\ell,\omega)$) we readily obtain the desired statement. $\square$

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