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Cauchy problem for derivors in finite dimension *

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Abstract

In this paper we study the uniqueness of solutions to ordinary differential equations which fail to satisfy both accretivity condition and the uniqueness condition of Nagumo, Osgood and Kamke. The evolution systems considered here are governed by a continuous operators A defined on $\mathbb{R}^N$ such that A is a derivor; i.e., $-A$ is quasi-monotone with respect to $(\mathbb{R}^+)^N$.

1 Introduction

For $T > 0$, we study the Cauchy Problem (CP)

$$\begin{gathered} \dot{u}(t) + Au(t) = f(t), \quad t \in [0,T] \\ u(0) = u_0, \end{gathered} \tag{1.1}$$

where A is a continuous operator on $\mathbb{R}^N$ and f belongs to $L^1([0,T] : \mathbb{R}^N)$. We require in addition that A be a derivor on $\mathbb{R}^N$ (or equivalently that $-A$ be quasi-monotone with respect to the cone $(\mathbb{R}^+)^N$) and has an additional order property (see Assumption H1T in Section 2). The existence of local solutions of (1.1) is proved by standard arguments (see [17] and Lemma 4.2). For instance, in the continuous case, this local existence comes from the Peano's Theorem. So the problem is essentially to prove the uniqueness of a local solution and the existence of global solutions. An important remark is that the identity operator minus the limit of infinitesimal generators of increasing semigroups is a derivor on the domains of the operators (see remark 2.1.d). The aim of this paper consists of giving a special converse of this previous property. General studies of evolution problems governed by derivors can be found in [2, 8, 9, 17] (for existence of extremal solutions of differential inclusions in $\mathbb{R}^N$) and in [13] for the behavior of the flow (stability, etc.) in the regular case: A is C^1. This work establishes uniqueness for the Cauchy Problem and complements previous studies.

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Let us point out that derivors often occur in the theory of production processes in Economics (for cooperative systems, see [10, 14]), in Chemistry [12], and in Biology [12]. Our uniqueness result given in the sequel applies to these situations. Notice also that the additional order property; namely, existence of uniform ascents, (see Definition 2.2) has obvious interpretations in applications and may be considered as a special extension of the submarkovian property (see remark 2.2.3) and [1]). Nevertheless the notion of uniform ascents is a new concept built from the concept of progressions in [10]. This ascent notion which extends the usual submarkovian property seems to lead naturally to the maximum principle worked in [10]. Finally, we emphasize that the ascent notion is the key to obtain a suitable increasing resolvent (see Proposition 2.4 and Theorem 3.2).

In this paper, the operator A does not satisfy either uniqueness conditions such as those given by Nagumo, Osgood and Kamke [4, 15, 16] nor accretivity conditions, even in a generalized sense as in [4, 6, 16]. We will exhibit in Section 5 a simple example of operator on $\mathbb{R}^2$ which satisfies all our conditions and none of the uniqueness conditions quoted above. Consequently our framework is not included in the submarkovian case, since a continuous submarkovian derivor is accretive in $(\mathbb{R}^N, \|\cdot\|_\infty)$. Moreover based in our analysis, it appear that a simple natural-order property can replace a classical Lipschitz condition about uniqueness in the Cauchy Problem.

Uniqueness and order-preserving dependence with respect to the initial value u_0 are stated in Theorem 3.1. In the case $f = 0$, Theorem 3.2 guarantees the existence of a global solution and a special form of the Crandall-Ligget exponential formula [7, p. 319] involving suitable selections of the multi-valued operators $(I + \lambda A)^{-1}$ (while in [8] $(I + \lambda A)^{-1}$ is single valued and Lipschitz).

This paper is organized as follows. Section 2 is devoted to general definitions and preliminaries. The main results are stated in Section 3, while the proofs are given in the next section. Section 5 gives an example in $\mathbb{R}^2$ which demonstrates the need for Theorems 3.1 and 3.2. Some remarks about the asymptotic behavior follow in Section 6.

2 Generalities

We supply $\mathbb{R}^N$ with the usual partial order relation $u \le v$ if $u^i \le v^i$ for all $i = 1, \dots, N$, where u^i is the i-th component. The vector in $\mathbb{R}^N$ whose components are $C, \dots, C$ is denoted by C. The symbol $\|\cdot\|$ stands for any norm in $\mathbb{R}^N$. The symbol $\mathbb{N}^*$ denotes the set of integers greater than zero.

Definition 2.1 We say that the map A is a **derivor** on $\mathbb{R}^N$ if it satisfies the condition

(i) For each $(u, v) \in (\mathbb{R}^N)^2$ and each $i \in \{1, \dots, N\}$

$$(u \le v \text{ and } u^i = v^i) \quad \text{implies that} \quad A^i u \ge A^i v \tag{2.1}$$

We say that the map A is a **moderate derivor** (resp. a **strong derivor**) if, in addition to (i), it satisfies

(ii) For each $u \in \mathbb{R}^N$, there exist $u_1, u_2 \in \mathbb{R}^N$ (resp. two sequences $(u_k)_k \to +\infty$, $(v_k)_k \to -\infty$) such that $u_1 \leq u \leq u_2$ and $Au_1 \leq 0 \leq Au_2$ (see [4]). (resp. $\lim_{k\to+\infty} Au_k = +\infty$ and $\lim_{k\to+\infty} Av_k = -\infty$).

The previous notation $\lim_{k\to+\infty} w_k = +\infty$ in $\mathbb{R}^N$ may be interpreted to mean that $\lim_{k\to+\infty} w_k^j = +\infty$ in $\mathbb{R}^N$ for each $j \in \{1, \dots, N\}$.

The derivor notion coincides with the notion of quasimonotone operator on $\mathbb{R}^N$, except the sign (see [2], [8, p. 91], [15]). In these references, $-A$ is quasimonotone with respect to $(\mathbb{R}^+)^N$ if

$$(u \leq v \text{ and } x^*(u) = x^*(v)) \quad \text{implies that} \quad x^*(-Au) \leq x^*(-Av) \tag{2.2}$$

for any linear positive form x^* on $\mathbb{R}^N$. Hence, A is a derivor, because if x^* is a linear positive form on $\mathbb{R}^N$, x^* is a linear combination with positive coefficients of coordinate forms on $\mathbb{R}^N$.

Remark 2.1 a) Condition (i) in the definition of derivor is automatically fulfilled for any operator A from $\mathbb{R}$ to $\mathbb{R}$, but it is not in the case of Condition (ii). A special case where (ii) holds for an operator A from $\mathbb{R}$ to $\mathbb{R}$ is the case where A is a non-decreasing operator such that there is $v \in \mathbb{R}$ satisfying $Av = 0$.
b) When A is a linear derivor, the reader can check that Condition (ii) is equivalent to: there is $u \geq 1$ satisfying $Au \geq 0$.
c) An equivalent form of definition 2.1.(i) is: A^i is decreasing with respect to x^j for each $i \neq j$ with $i, j \in \{1, \dots, N\}$ (see [14]).
d) If P is an increasing operator on $\mathbb{R}^N$, then $A = I - P$ is a derivor. Therefore, when (P_t) is an increasing semi-group on $\mathbb{R}^N$, then $A_t = \frac{I-P_t}{t}$ with $t > 0$ is a derivor and so is A_0, defined by $A_0 u = \lim_{t\downarrow 0} A_t u$ (on the domain where this limit exists).

Ascents

We denote by $V_K(u_0)$ the set of compact neighborhoods of u_0.

Definition 2.2 We say that a derivor A has a (strict) **uniform ascent** at u_0 if there are $V \in V_K(u_0)$ and a sequence (v_k) in $\mathbb{R}^N$ convergent to 0 such that $(v_k^i)_{k\in\mathbb{N}^*}$ is strictly decreasing for all $i = 1, \dots, N$ and

$$\min_{i\in\{1,\dots,N\}} (A^i(u + v_k) - A^i u) > 0 \tag{2.3}$$

for each $k \in \mathbb{N}^*$ and each $u \in V$.

Remark 2.2 1) In terms of production operator ($A^i u$ is the production of the i-th input of the product u), the uniform ascent property at u_0 means that in a neighborhood of u_0 it is possible to increase the level of production by means of small uniform augmentations around u_0.
2) The notion of uniform ascent plays a crucial part in this work. In our opinion, this concept is new, but it was inspired from the progression notion carried out in [10].
3) The uniform ascent property may be connected to the submarkovian property namely,

$$A(u + C) - Au \geq 0 \tag{2.4}$$

for all $u \in \mathbb{R}^N$ and all $C \in \mathbb{R}^+$. Notice that a submarkovian derivor in $\mathbb{R}^N$ is accretive in $(\mathbb{R}^N, \|\cdot\|_\infty)$; the verification of this claim is left to the reader.
4) The following dual notion of uniform ascent at u_0 provides again the results of Section 3: There are $V \in V_K(u_0)$ and a strictly increasing sequence (v_k) in $\mathbb{R}^N$ convergent to 0 such that $(v_k^i)_{k\in\mathbb{N}^*}$ is strictly increasing for all $i = 1, \dots, N$ and

$$\sup_{i\in\{1,\dots,N\}} (A^i(u + v_k) - A^i u) < 0$$

for each $k \in \mathbb{N}^*$ and each $u \in V$.
5) In the case $A = I - P$ with an increasing operator P, (see Remark 2.1.d)), Definition 2.2 means that the required sequence (v_k) satisfies

$$P^i(u + v_k) - P^i(u) < v_k^i$$

for all $i = 1, \dots, N$, for each $k \in \mathbb{N}^*$ and all $u \in V$.

Assumptions

In the sequel, by hypothesis **H1T** stands for the following three conditions:

- A is a continuous derivor on $\mathbb{R}^N$
- A is a moderate derivor with uniform ascent at each u_0
- $f \in L^1([0, T]; \mathbb{R}^N)$.

When necessary, we will make precise the arguments involved for the Cauchy Problem (1.1) as follows: $CP(A, f, u_0)$ or $CP(A, f, u_0, T)$ for the domain $[0, T]$, and $CP(A, f, u_0, +\infty)$ for $[0, +\infty[$.

The hypothesis **H2T** stand for the following condition (cf Section 6).

- For each $u_0 \in \mathbb{R}^N$, each local solution of $CP(A, f, u_0)$ can be extended to a solution on $[0, T]$.

It is well-known that sublinearity at infinity ($\|Au\| \leq a(\|u\| + 1)$) guarantees H2T, [13]. Moreover we will see in Theorem 3.2 that in the autonomous case $f = 0$, H1T implies H2T.

Resolvents for a moderate continuous derivor

In this section, we assume that A is a continuous moderate derivor. When B is a continuous derivor, we have the following Theorem [10].

Theorem 2.4 *Let u and w fixed in $\mathbb{R}^N$, if the system*

$$\begin{aligned} v &\geq u \\ Bv &\geq w \end{aligned} \tag{2.5}$$

with v as unknown quantity has a solution then it has a smallest solution. Analogously, the system

$$\begin{aligned} v &\leq u \\ Bv &\leq w \end{aligned} \tag{2.6}$$

has a largest solution v whenever it has a solution. In addition, in these two systems the constraints are optimal, i.e. $\forall i \in \{1, \dots, N\}$, $v^i = u^i$ or $Bv^i = w^i$.

In the case where A is accretive, the resolvent operators $(I + \lambda A)^{-1}$, $\lambda > 0$ are single-valued contractions. But in our case $(I + \lambda A)^{-1}$ is a priori multi valued. Nevertheless it is possible to define suitable selectors J_λ of $(I + \lambda A)^{-1}$ as claimed in the following lemma.

Lemma 2.4 *Let A be a moderate continuous derivor on $\mathbb{R}^N$. Let u in $\mathbb{R}^N$, $\lambda \in \mathbb{R}^+$ and v a solution of*

$$\begin{aligned} v &\leq u \\ Av &\leq 0. \end{aligned} \tag{2.7}$$

Then the system

$$\begin{aligned} w &\geq v \\ (I + \lambda A)w &\geq u \end{aligned} \tag{2.8}$$

has a smallest solution denoted by $J_{\lambda,v}u$. Moreover we have

$$J_{\lambda,v}u \in (I + \lambda A)^{-1}(u). \tag{2.9}$$

Proof. According to (ii) in definition 2.1, Systems (2.7) and (2.8) have solutions. Let v be a solution of (2.7). Since $B = I + \lambda A$ is a continuous derivor, the existence of the smallest solution $J_{\lambda,v}u$ of (2.8) is guaranteed by the Theorem 2.4.

It remains to prove

$$(I + \lambda A)J_{\lambda,v}u = u. \tag{2.10}$$

Since the constraints are optimal in (2.8), we have for each $i \in \{1, \dots, N\}$, $((I+\lambda A)J_{\lambda,v}u)^i = u^i$ or $(J_{\lambda,v}u)^i = v^i$. Thus we have to prove $((I+\lambda A)J_{\lambda,v}u)^i =$

u^i when $(J_{\lambda,v}u)^i = v^i$. So assume $(J_{\lambda,v}u)^i = v^i$ for some $i \in \{1,\dots,N\}$, Relation (2.1) and $J_{\lambda,v}u \geq v$ yield

$$(AJ_{\lambda,v}u)^i \leq (Av)^i. \tag{2.11}$$

Now (2.11) and (2.7) provide

$$(J_{\lambda,v}u)^i + \lambda(AJ_{\lambda,v}u)^i = v^i + \lambda(AJ_{\lambda,v}u)^i \leq u^i + \lambda(AJ_{\lambda,v}u)^i \leq u^i.$$

Therefore, $((I+\lambda A)J_{\lambda,v}u)^i \leq u^i$. But from (2.8) we have $((I+\lambda A)J_{\lambda,v}u)^i \geq u^i$. Finally (2.10) is proved. □

In the same way, let v be a solution of

$$\begin{gathered} v \geq u \\ Av \geq 0. \end{gathered} \tag{2.12}$$

Then the system

$$\begin{gathered} w \leq v \\ (I+\lambda A)w \leq u \end{gathered} \tag{2.13}$$

has a largest solution $w = \tilde{J}_{\lambda,v}u$. Moreover $\tilde{J}_{\lambda,v}u$ satisfies again (2.9).

Set $J_\lambda u = J_{\lambda,v}u$ (resp. $J_\lambda u = \tilde{J}_{\lambda,v}u$) for an arbitrary v satisfying (2.7) (resp. (2.12)). Let us notice that J_λ is defined on $D_v = \{u \in \mathbb{R}^N, u \geq v\}$ (resp. $D_v = \{u \in \mathbb{R}^N, u \leq v\}$). The family of selectors $(J_\lambda)_{\lambda\geq 0}$ of $(I+\lambda A)^{-1}$ is said to be the **resolvent** of A.

Definition 2.4 For u given, the notation $\overline{u}$ (resp. $\hat{u}$) stands for the largest solution of (2.7) (resp. the smallest solution of (2.12)).

Thanks to Theorem 2.4, such extremal elements $\overline{u}$ and $\hat{u}$ exist. Furthermore we have clearly

$$u \leq v \Longrightarrow (\overline{u} \leq \overline{v} \quad \text{and} \quad \hat{u} \leq \hat{v}) \tag{2.14}$$

and

$$\overline{\overline{u}} = \overline{u} \quad \text{and} \quad \hat{\hat{u}} = \hat{u}. \tag{2.15}$$

The resolvent operators satisfy the following properties.

Proposition 2.4 *For a given $u \in \mathbb{R}^N$, let $v, v' \in \mathbb{R}^N$ satisfying (2.7) and $w, w' \in \mathbb{R}^N$ satisfying (2.12). Then*

(a) *The map J_λ is single-valued and increasing on D_v.*

(b) *We have*

$$v \leq J_\lambda u \leq w \tag{2.16}$$

In particular

$$\overline{u} \leq J_\lambda u \leq \hat{u} \tag{2.17}$$

(c) *If* $Au \geq 0$ *(resp.* $Au \leq 0$*), then* $AJ_\lambda u \geq 0$ *(resp.* $AJ_\lambda u \leq 0$*) for each* $\lambda \geq 0$.

(d) *If* $Au \geq 0$*, then* $\lambda \to J_\lambda u$ *is decreasing on* $\mathbb{R}^N$ *(and increasing if* $Au \leq 0$*).*

(e) *We have* $J_{\lambda,v}u \leq \tilde{J}_{\lambda,w}u$*. In particular* $J_{\lambda,\overline{u}}u \leq \tilde{J}_{\lambda,\hat{u}}u$.

(f) $J_{\lambda,\overline{u}}\overline{u} = \overline{u}$ *and* $\tilde{J}_{\lambda,\hat{u}}\hat{u} = \hat{u}$.

(g) *If* $v \leq v'$ *and* $w \leq w'$*, then* $J_{\lambda,v}u \leq J_{\lambda,v'}u$ *and* $\tilde{J}_{\lambda,w}u \leq \tilde{J}_{\lambda,w'}u$.

Proof. We prove only results (a),(b),(c),(d) in the case $J_\lambda = J_{\lambda,v}$.
(a) Let $u \geq w$. Then $J_{\lambda,v}u$ satisfies

$$\begin{gathered} J_{\lambda,v}u \geq v \\ (I+\lambda A)J_{\lambda,v}u \geq u \geq w. \end{gathered}$$

Hence we get (a) from minimality of $J_{\lambda,v}w$ for the previous system.
(b) Inequality $J_{\lambda,v}u \geq v$ is required in the definition of $J_{\lambda,v}$. Since w satisfies

$$\begin{gathered} w \geq v \\ (I+\lambda A)w \geq w \geq u, \end{gathered}$$

we get (b) from minimality of $J_{\lambda,v}u$ in the previous system.
(c) Let $Au \geq 0$ and $\lambda \geq 0$. We have

$$\begin{gathered} (I+\lambda A)u \geq u \\ u \geq v \end{gathered}$$

so $J_{\lambda,v}u \leq u$. Hence $u = J_{\lambda,v}u + \lambda AJ_{\lambda,v}u \leq u + \lambda AJ_{\lambda,v}u$ and so $AJ_\lambda u \geq 0$.
(d) Let $0 \leq \lambda \leq \mu$. Then $u = (I+\lambda A)J_{\lambda,v}u \leq (I+\mu A)J_{\lambda,v}u$. Since we have $J_{\lambda,v}u \geq v$, from minimality of $J_\mu u$ for these two constraints, it comes $J_{\mu,v}u \leq J_{\lambda,v}u$. The proof is similar when $Au \leq 0$.
(e) Since $Av \leq 0$ and $Aw \geq 0$, from (c) it follows $AJ_{\lambda,v}u \geq 0$ and $A\tilde{J}_{\lambda,w}u \leq 0$. Hence (e) results from $J_{\lambda,v}u + \lambda AJ_{\lambda,v}u = u = \tilde{J}_{\lambda,w}u + \lambda A\tilde{J}_{\lambda,w}u$.
Properties (f) and (g) result immediately from the definitions. □

Solution of (1.1)

We recall that a (local) **strong solution** of (1.1) is a continuous function u defined on $[0,\theta) \subset [0,T], \theta > 0$ such that $u(t) = u_0 + \int_0^t (-Au(\tau) + f(\tau))d\tau$ for $t \in [0,\theta)$. In the sequel we only look for (local) strong solutions of (1.1).

A **maximal** (resp. **minimal**) **solution** of (1.1) is the strong solution $u = S^{\max}_{A,f}(t)u_0$ (resp. $u = S^{\min}_{A,f}(t)u_0$) of (1.1) defined as follows:
(i) The interval of definition $[0,\theta)$ of $S^{\max}_{A,f}(.)u_0$ (resp. $S^{\min}_{A,f}(.)u_0$) is maximal on $[0,T]$, i.e. there is no solution $v \neq u$, such that $v = u$ on $[0,\theta]$.
(ii) For each solution v of (1.1) on $[0,T_1) \subset [0,T]$, we have $v(t) \leq S^{\max}_{A,f}(t)u_0$ (resp. $v(t) \geq S^{\min}_{A,f}(t)u_0$) on $[0,\inf(\theta,T_1))$.

3 Main results

For the following results, we assume the hypothesis H1T defined in Section 2.

Theorem 3.1 *The problem $CP(A,f,u_0)$ has a unique local solution denoted by $S_{A,f}(t)u_0$ (or $S_A(t)u_0$ if $f=0$) and defined on a maximal interval $[0,T_{\max})\subset[0,T]$. Moreover if $u_0\le u_1$ in $\mathbb{R}^N$ and if $f\le g$ in $L^1([0,T],\mathbb{R}^N)$ then $S_{A,f}(t)u_0\le S_{A,g}(t)u_1$ on the common domain of existence of these two solutions.*

The next result concerns the autonomous case, for which we have global solutions.

Theorem 3.2 *Assume that $f\equiv 0$. Then $S_A(.)u_0$ is defined on the whole interval $[0,T]$ and*

$$S_A(t)u_0=\lim_{n\to+\infty} J_{t/n,n}(u_0), \tag{3.1}$$

for $t\in[0,T]$, where $J_\lambda=J_{\lambda,\overline{u}_0}$ is as defined in Section 2.

This is an exponential Crandall-Liggett's type formula, but here $(I+\lambda A)^{-1}$ is a priori multi-valued. In the non-autonomous case $f\not\equiv 0$, it is possible to exhibit a formula as (3.1) which gives the solution of (1.1) as a limit of a discrete scheme. But such a formula is more complicated than (3.1) and thus, is not of a particular interest. When $f\in L^\infty([0,T],\mathbb{R}^N)$, from Theorem 3.1 and Theorem 3.2, we can deduce that $CP(A,f)$ has solution on $[0,T]$ if A is a strong continuous derivor (see def. 2.1). Unfortunately, we do not know what happens in the general case $f\in L^1([0,T],\mathbb{R}^N)$ without extra assumptions.

4 Proofs

The proof of Theorem 3.1 follows immediately from the three lemmas below.

Lemma 4.1 *Let A be a continuous derivor. Let V be an element of $V_K(u_0)$. Then the operator B defined by*

$$B(v):=\inf_{w\in V}[A(w+v)-A(w)] \tag{4.1}$$

is a continuous derivor.

Proof. 1.) Let us show that B is a derivor on $\mathbb{R}^N$. If $u\le v$ and $u^i=v^i$ for some $i\in\{1,\dots,N\}$, we have $u+w\le v+w$ and $(u+w)^i=(v+w)^i$ for each $w\in V$. Since A is a derivor, it follows $A^i(u+w)-A^iw\ge A^i(v+w)-A^iw$. Thus

$$\inf_{w\in V}(A^i(u+w)-A^iw)\ge\inf_{w\in V}(A^i(v+w)-A^iw).$$

So $B^iu\ge B^iv$ for $u\le v$ and $u^i=v^i$.
2.) At this stage we will show that B is continuous on $\mathbb{R}^N$. According to (4.1),

B is clearly upper semi-continuous (see [3, pp. 132-137]). So it is enough to prove that for each $i \in \{1, \dots, N\}$, B^i is lower semi-continuous on $\mathbb{R}^N$. Fix $i \in \{1, \dots, N\}$. For each $u \in \mathbb{R}^N$, thanks to the compactness of V, there exists $\chi(u)$ (which depends on i) in V satisfying

$$B^i u = A^i(u + \chi(u)) - A^i(\chi(u)) \tag{4.2}$$

We have to prove now that B^i is lower semi-continuous, that is $(B^i)^{-1}(]-\infty, \alpha])$ is closed for all $\alpha \in \mathbb{R}$. In this goal, consider $\alpha \in \mathbb{R}$ and a sequence $(u_k)_{k \in \mathbb{N}^*}$ of elements of $\mathbb{R}^N$ such that $\lim u_k = u_\infty$ and $B^i(u_k) \leq \alpha$. It suffices to prove $B^i(u_\infty) \leq \alpha$.

By contradiction, let us suppose $B^i(u_\infty) > \alpha$. Without loss of generality, thanks to the compactness of V, we can suppose

$$\lim_{k \to +\infty} \chi(u_k) = v_\infty \in V.$$

Equation (4.1) yields

$$\alpha < B^i(u_\infty) = A^i(\chi(u_\infty) + u_\infty) - A^i(\chi(u_\infty)) \leq A^i(v_\infty + u_\infty) - A^i(v_\infty). \tag{4.3}$$

From the continuity of A^i, it results

$$A^i(v_\infty + u_\infty) - A^i(v_\infty) = \lim_{k \to +\infty} A^i(\chi(u_k) + u_k) - A^i(\chi(u_k)) = B^i(u_k) \leq \alpha. \tag{4.4}$$

Equations (4.3) and (4.4) lead to a contradiction. $\square$

Lemma 4.2 *Let A be a continuous derivor.*
(a) Problem (1.1) has a local unique maximal solution $S^{max}_{A,f}(t)u_0$ defined on its maximal interval of existence $[0, T^1)$ (resp. a unique minimal solution $S^{min}_{A,f}(t)u_0$ on $[0, T^2)$).
(b) If $v_0 \leq u_0$ and if $v(t)$ satisfies $v(0) = v_0$ and $v'(t) \leq -Av(t) + f(t)$ a.e. on $[0, \tilde{T})$ with $\tilde{T} < T^1$, then for $t \in [0, \tilde{T})$ we have $v(t) \leq S^{max}_{A,f}(t)u_0$.
(c) if $v_0 \geq u_0$ and if $v(t)$ satisfies $v(0) = v_0$ and $v'(t) \geq -Av(t) + f(t)$ a.e. on $[0, \tilde{T})$ with $\tilde{T} < T^2$, then for $t \in [0, \tilde{T})$ we have $v(t) \geq S^{min}_{A,f}(t)u_0$.

The previous lemma will be proved by standards arguments in an analogous way as the Kamke's Lemma [15] and the arguments given in [13, 17].

Proof. We shall prove only parts (a) and (b). The proof of part (c) can be obtained in an analogous way. Let v be a solution on $[0, \tilde{T}) \subset [0, T]$ of

$$\begin{gathered} \dot{v}(t) \leq -Av(t) + f(t), \quad t \in [0, \tilde{T}) \\ v(0) = v_0. \end{gathered}$$

For each $n \in \mathbb{N}^*$, Problem $CP(A, f + \frac{1}{n}, u_0)$ has at least a local solution u_n (see [17]) defined on a maximal interval of $[0, T_n]$.

1) Let us show $v \leq u_n$ on $[0, \tilde{T} \wedge T_n)$, where $\tilde{T} \wedge T_n$ means $\min(\tilde{T}, T_n)$. One has

$$\begin{gathered} u_n(t) - v(t) \geq u_n(t_0) - v(t_0) + \int_{t_0}^{t} (\epsilon_n(\tau) + \frac{1}{n}) d\tau \\ \epsilon_n(\tau) = -Au_n(\tau) + Av(\tau) \end{gathered} . \tag{4.5}$$

for all $t_0, t \in [0, \tilde{T} \wedge T_n)$, $t_0 \leq t$. Let

$$E = \{t \in [0, \tilde{T} \wedge T_n), v(\tau) \leq u_n(\tau) \quad \text{for all} \quad \tau \in [0, t]\}$$

First, remark that E is (not empty and) closed on $[0, \tilde{T} \wedge T_n)$. Second, if $t_0 \in E$, $t_0 < \tilde{T} \wedge T_n$ and $(v(t_0))^i = (u_n(t_0))^i$, for some $i \in \{1, \dots, N\}$ then the derivor property of Definition 2.1 (i) yields

$$\epsilon_n^i(t_0) \geq 0 \tag{4.6}$$

Consequently, relations (4.5), (4.6) and the definition of t_0 provide some $\eta > 0$ such that $v^i(\tau) \leq u_n^i(\tau)$ for $\tau \in [t_0, t_0 + \eta] \subset [0, \tilde{T} \wedge T_n)$. Finally E is open in $[0, \tilde{T} \wedge T_n)$ and thus $E = [0, \tilde{T} \wedge T_n)$.
2) We have $u_{n+1} \leq u_n$ on $[0, T_{n+1} \wedge T_n)$. Indeed the proof is the same as 1) if we replace v by u_{n+1} and $-Au_n(\tau) + Av(\tau)$ by $-Au_n(\tau) + Au_{n+1}(\tau)$.
3) We have $\tilde{T} \wedge T_n \geq \tilde{T} \wedge T_1$. Indeed, from parts 1) and 2), for each $n \in \mathbb{N}^*$ we have

$$v \leq u_{n+1} \leq u_n \leq u_1 \tag{4.7}$$

on the common interval of existence of these solutions. Then the extension principle of solutions implies $\tilde{T} \wedge T_{n+1} \geq \tilde{T} \wedge T_n$ since a bounded solution is extendable.
4) The sequence (u_n) converges uniformly to u_∞ on each compact sub-interval of $[0, \tilde{T} \wedge T_1)$ thanks to (4.7) and the Lebesgue's Dominated Convergence Theorem. Furthermore u_∞ is solution of $CP(A, f, u_0, \tilde{T} \wedge T_1)$ on $[0, \tilde{T} \wedge T_1)$. Moreover, clearly u_∞ is the maximal solution of $CP(A, f, u_0, \tilde{T} \wedge T_1)$ (see Section 2).

Let F be the set of $S \in [0, T]$ such that u_∞ is extendable into a continuous function on $[0, S)$ which is the maximal solution of $CP(A, f, u_0, S)$. One has $\tilde{T} \wedge T_1 \in F$. By considering $S_\infty = \sup F$, we obtain a maximal extension of u_∞ as a local solution of $CP(A, f, u_0, T)$ which is by construction the maximal solution of $CP(A, f, u_0, T)$. □

The next lemma makes use of the ascent assumption.

Lemma 4.3 *With the notation in Lemma 4.2, if H1T holds, we have*

$$S_{A,f}^{min}(t)u_0 = S_{A,f}^{max}(t)u_0$$

on $[0, T^1 \wedge T^2) = [0, T^1)$.

Proof. Thanks to Lemma 4.2(a), (1.1) has a maximal solution $S^{\max}_{A,f}(t)u_0$ defined on a sub-interval $[0,T^1)$ of $[0,T]$ and a minimal solution $u(t)=S^{\min}_{A,f}(t)u_0$ defined on a sub-interval $[0,T^2)$ of $[0,T]$. Set $T_3=T^1\wedge T^2$ and

$$w(t):=S^{\max}_{A,f}(t)u_0-S^{\min}_{A,f}(t)u_0 \tag{4.8}$$

for $t\in[0,T_3)$. We have to prove $w=0$ on $[0,T_3)$, that is $E=[0,T_3)$ where $E=\{t\in[0,T_3), w(\tau)=0, \forall\tau\in[0,t]\}$. Since $E=w^{-1}(0)$ is closed in $[0,T_3)$ (w being continuous), it just remains to show that E is open to the right. Let $t_0\in E, t_0<T_3$. We have to prove that there exists $h>0$ such that $w=0$ on $[t_0,t_0+h]$. Eventually, by changing w into $w(t_0+.)$ and f into $f(t_0+.)$, we will suppose $t_0=0$.

Let $V\in V_K(u_0)$ and B as in (4.1), in view of the continuity of u at 0, there exists $T_4\in]0,T_3[$ such that, for each $t\in[0,T_4]$, $u(t)\in V$, hence w satisfies a.e.:

$$\begin{gathered} w'(t)=-(A(u(t)+w(t))-Au(t))\le -Bw(t)\\ w(0)=0, \end{gathered} \tag{4.9}$$

a.e. $t\in[0,T_4]$. By using Lemma 4.2 (b) with B instead of A, we have

$$w(t)\le S^{\max}_B(t)(0) \tag{4.10}$$

for each $t\in[0,T_4\wedge T_5]$, where $[0,T_5]$ is the maximal interval of existence of $S^{\max}_B(t)(0)$. The function $x(t)=S^{\max}_B(t)(0)$ satisfies

$$\begin{gathered} x'(t)=-Bx(t)\\ x(0)=0. \end{gathered} \tag{4.11}$$

Let $(v_k)_{k\in\mathbb{N}^*}$ be a sequence which defines a uniform ascent at the point u_0 for the operator A on the set$V$ (see section 2).

$$B^i(v_k)=A^i(v_k+\hat v_k(i))-A^i(\hat v_k(i))>0 \tag{4.12}$$

for $k\in\mathbb{N}^*$ and $i\in\{1,\dots,N\}$ where $\hat v_k(i)$ is a vector minimizing $v\to A^i(v_k+v)-A^i(v)$ on V.

Let $k\in\mathbb{N}$ be fixed, then due to Lemma 4.2(b) there exists $s_k>0$ such that $s_k\le T_4\wedge T_5$ and

$$S^{\max}_B(t)(0)\le S^{\max}_B(t)(v_k) \tag{4.13}$$

for each $t\in[0,s_k]$.

Equation (4.12) and the continuity of B give the existence of $t_k>0$ and $t_k\le s_k$ such that:

$$B(S^{\max}_B(t)(v_k))\ge 0$$

for $t\in[0,t_k]$. Thus $t\to S^{\max}_B(t)(v_k)$ is decreasing on $[0,t_k]$. Consequently, from (4.10) and (4.13), it results

$$w(t)\le S^{\max}_B(t)(v_k)\le S^{\max}_B(0)(v_k)=v_k \tag{4.14}$$

for each $t \in [0, t_k]$.

In particular, we have $w(t_k) \le v_k$. If we put $y(t) = w(t_k + t)$, we get

$$\begin{gathered} y'(t) \le -By(t) \\ y(0) = w(t_k) \le v_k \end{gathered} \tag{4.15}$$

for a.e. $t \in [0, t_k]$. Hence, according to (4.14) and (4.15), one has

$$w(t_k + t) \le S_B^{\max}(t)(v_k) \le v_k$$

for $t \in [0, t_k]$. So $w(t) \le v_k$ for $t \in [0, 2t_k \wedge T_4]$. Whence by induction, we get

$$0 \le w(t) \le v_k \tag{4.16}$$

for $t \in [0, T_4]$. Since (4.16) is valid for each $k \in \mathbb{N}^*$ and $\lim v_k = 0$, it follows $w(t) = 0$ for each $t \in [0, T_4]$. Hence for $h = T_4 > 0$, we have $[0, h] \subset E$ which completes the proof. □

Proof of Theorem 3.2

In this subsection, we assume that A satisfies H1T, and $f \equiv 0$ on $[0, T]$. First, let us recall some basic facts about the discretization (1.1) in the Theory of Nonlinear Semigroups. It is known [7] that a strong solution of (1.1) is a mild solution, i.e. a continuous function which is a uniform limit of Euler's implicit discrete schemes. Such discrete schemes are defined as follows.

Let $\epsilon > 0$ be fixed. Then an ϵ-discretization on $[0, T]$ of $\dot{u} + Au = 0$ on $[0, T]$ consists of a partition $0 = t_0 \le t_1 \le \cdots \le t_n$ of the interval $[0, t_n]$ and a finite sequence $(f_1, f_2, \dots, f_n)$ in $\mathbb{R}^N$ such that
(a) $t_i - t_{i-1} < \epsilon$ for $i = 1, \dots, n$ and $T - \epsilon < t_n \le T$.
(b) $\Sigma_{i=1}^n (t_i - t_{i-1})\|f_i\| \le \epsilon$.

We will indicate these data by writing $D_A(0 = t_0, t_1, \dots, t_n : f_1, \dots, f_n)$.

A solution of a discretization $D_A(0 = t_0, t_1, \dots, t_n : f_1, \dots, f_n)$ is a piecewise constant function $v : [0, t_n] \to \mathbb{R}^N$ whose values v_i on $(t_{i-1}, t_i]$ satisfy

$$\begin{gathered} \frac{v_i - v_{i-1}}{t_i - t_{i-1}} + Av_i = f_i \\ v_0 = u_0 \end{gathered} \tag{4.17}$$

for $i \in \{1, \dots, n\}$. An ϵ-approximate solution of $CP(A, 0, u_0)$ is a solution v of an ϵ-discretization $D_A(0 = t_0, t_1, \dots, t_n : f_1, \dots, f_n)$.

A **mild solution** of $CP(A, 0, u_0)$ on $[0, T]$ is a continuous function u on $[0, T]$ with the property that for each $\epsilon > 0$ there is an ϵ-approximate solution v of $CP(A, 0, u_0)$ on $[0, T]$ such that $\|v(t) - u(t)\| \le \epsilon$ for t in the domain of v.

Now, for $n \in \mathbb{N}^*$, let $J = J_{T/n, \overline{u}_0}$, and define the function u_n by $u_n(0) = u_0$ and $u_n(t) = J^i(u_0)$ for $(i-1)T/n < t \le iT/n$ where J^i is the i^{th} power of J. Then, thanks to (2.9), (4.17) holds with $v_i = u_n(iT/n)$, $t_i = iT/n$ and $f_i = 0$ (Lemma 4.4 below guarantees $\overline{u}_0 \le v_{i-1}$ for all $i \ge 1$). In other words u_n is a T/n-approximate solution of $CP(A, 0, u_0)$.

Then Theorem 3.2 results immediately from the following two lemmas.

Lemma 4.4 *With the previous notations, for each $t \in [0,T]$, we have*

$$\overline{u}_0 \leq u_n(t) \leq \hat{u}_0. \tag{4.18}$$

Proof. We set v_i for $u_n(iT/n)$. By Proposition 2.4 (b), we have

$$\overline{u}_0 \leq v_1 = J(u_0) \leq \hat{u}_0 \tag{4.19}$$

Then (4.18) results by induction from (4.19) and Proposition 2.4 parts (a) and (f). □

The following lemma studies the continuous and discrete approach and gives an exponential formula such as the Crandall-Liggett's formula (for the accretive autonomous case in [7]).

Lemma 4.5 *The sequence of approximate solutions (u_n) defined in Lemma 4.4 converges uniformly on $[0,T]$ to $S_A(.)u_0$. Moreover, for all $t \in [0,T]$,*

$$\overline{u}_0 \leq S_A(t)u_0 \leq \hat{u}_0 \tag{4.20}$$

and

$$S_A(t)u_0 = \lim_n J^n_{t/n}(u_0)$$

where $J_{t/n} = J_{t/n,\overline{u}_0}$.

Proof. The approximate solutions u_n satisfy an Ascoli-Arzel's type condition $\mathcal{A}$ on $[0,T]$ [11, p. 260-268], namely: for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ and $\eta_\epsilon > 0$ such that ($n \geq N_\epsilon$ and $|t-s| \leq \eta_\epsilon$) implies $\|u_n(t)-u_n(s)\|_\infty \leq \epsilon$. Indeed, relations (4.17) lead to

$$u_n(t^n_j) - u_n(t^n_i) = -\int_{t^n_i}^{t^n_j} Au_n(t)dt. \tag{4.21}$$

Using (4.18), Relation (4.21) yields

$$\|u_n(t) - u_n(s)\| \leq M(|t-s| + 2\frac{T}{n}), \tag{4.22}$$

where $M = \sup_{\overline{u}_0 \leq w \leq \hat{u}_0} \|Aw\|$. Consequently (see [11, p. 260]) the sequence (u_n) is relatively compact in the Banach space $\mathcal{B}([0,T], \mathbb{R}^N, \| \ \|_\infty)$ of bounded functions on $[0,T]$ with values in $\mathbb{R}^N$. So there exists a subsequence (u_{n_k}) converging to a continuous function u_∞ which is a mild solution of $CP(A,0,u_0)$. Then, passing to the limit in (4.21) (or from [7, p. 314]), we see that u_∞ is a strong (even a classical) solution of $CP(A,0,u_0)$ on $[0,T]$.

From Theorem 3.1, it results

$$u_\infty = S_A(.)u_0 \tag{4.23}$$

on $[0,T]$. Thus (4.20) follows from (4.23) and (4.18) on $[0,T]$. Then, taking $T=t$, (4.23) yields

$$S_A(t)u_0 = \lim_{n\to+\infty} J^n_{t/n}(u_0),$$

where $J_{t/n} = J_{t/n,\overline{u}_0}$. The proof is complete.

5 An example in $\mathbb{R}^2$

Let A_0 be the operator defined on $\mathbb{R}^2$ by

$$A_0\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}x+(x-2y)^{1/3}\\y+(2y-x)^{1/5}\end{pmatrix} \tag{5.1}$$

Lemma 5.1 *The operator A_0 satisfies H1T and H2T for all $T>0$.*

The proof is left to the reader. In particular, the relation

$$A_0\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}2t\\t\end{pmatrix} = A_0\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}2t\\t\end{pmatrix} \tag{5.2}$$

for $t\in\mathbb{R}^+$, provides uniform ascents at each point. The sublinearity at infinity implies H2T. Therefore we can apply the results of Section 3 to the operator A_0 for any $T>0$. Hence $CP(A_0,f,u_0,+\infty)$ has a unique global solution, on $[0,+\infty[$. Now, our task is to prove that no condition of Nagumo-Osgood-Kamke and no accretivity condition (even in a generalized sense) can be applied to obtain the uniqueness of solutions of $CP(A_0,f,u_0)$.

Generalized accretivity conditions

Let $\|\cdot\|_p$, $p\in[1,+\infty]$, be the classical l_p-norm in $\mathbb{R}^2$. As usual (see [7, 8]), we set

$$[u,v] = \lim_{\lambda\downarrow 0}\frac{\|u+\lambda v\|-\|u\|}{\lambda} \tag{5.3}$$

for $u,v\in\mathbb{R}^2$. For $p\in[1,+\infty]$, the notation $[u,v]_p, p\in[1,+\infty]$ means $[u,v]$, with $\|\cdot\|_p$ instead of $\|\cdot\|$ in (5.3).

In the sequel, ϕ stands for a continuous function $\phi:\mathbb{R}\to\mathbb{R}^+$ satisfying the following condition $\mathcal{U}$: For each T_0, the function $x\equiv 0$ is the unique positive solution on $[0,T_0]$ of

$$\begin{gathered}\dot{x}(t) = \phi(x(t))\\ x(0)=0.\end{gathered}$$

Definition 5.2 We will say that an operator B defined on $\mathbb{R}^2$ is ϕ**-accretive** in $(\mathbb{R}^2, \|\cdot\|)$ if

$$-[u-v, Bu-Bv] \leq \phi(\|u-v\|) \tag{5.4}$$

for all $u, v \in \mathbb{R}^2$.

We will say that B satisfies a ϕ**-Osgood condition** if

$$\|Bu - Bv\| \leq \phi(\|u-v\|)$$

for all $u, v \in \mathbb{R}^2$.

Remark a) The condition $B+\omega I$ is accretive ($\omega \geq 0$) means B is ϕ-accretive with $\phi(x) = \omega x$. General studies of ϕ-accretive conditions can be found in [6, 16].
b) A ϕ**-Osgood condition** is a particular case of ϕ-accretivity.

Lemma 5.2 *Let $p \in [1, +\infty]$. Then, there is no ϕ, such that A_0 is ϕ-accretive in $(\mathbb{R}^2, \|\cdot\|_p)$. Moreover, there are no ϕ and no norm $\|\cdot\|$ such that A_0 satisfies a ϕ-Osgood condition in $(\mathbb{R}^2, \|\cdot\|)$.*

Proof. a) Suppose first $p = +\infty$. By contradiction, suppose that A_0 is ϕ-accretive in $(\mathbb{R}^2, \|\cdot\|_\infty)$ for some ϕ. Let $x \in [0,1[$. A direct computation yields $A_0 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and

$$[\begin{pmatrix} x \\ x - \frac{1}{2}x^2 \end{pmatrix}, A \begin{pmatrix} x \\ x - \frac{1}{2}x^2 \end{pmatrix}]_\infty = x^{1/3}(x^{2/3} + (-1+x)^{1/3}). \tag{5.5}$$

So, thanks to the ϕ-accretivity, (5.5) implies

$$\frac{1}{2}x^{1/3} \leq \phi(x), \tag{5.6}$$

for $x \geq 0$ sufficiently small. Set

$$z(t) = H^{-1}(t)\,, \quad H(\sigma) = \int_0^\sigma \frac{d\xi}{\phi(\xi)}. \tag{5.7}$$

From (5.6), H is defined for $\sigma \geq 0$ sufficiently small and

$$z(t) > 0 \tag{5.8}$$

on some interval $]0, T_0]$ with $T_0 > 0$. By using (5.7), a straightforward computation gives $z'(t) = \phi(z(t))$ and $z(0) = 0$. Then $\mathcal{U}$ provides

$$z \equiv 0 \quad \text{on } [0, T_0]. \tag{5.9}$$

Hence there is a contradiction between (5.8) and (5.9).

b) Suppose now $p \in [1, +\infty[$. By contradiction again, suppose that A_0 is ϕ-accretive in $(\mathbb{R}^2, \|\cdot\|_p)$. In this case, for $x \in [0,1]$, by setting

$$u = \begin{pmatrix} x \\ \frac{1}{2}x - \frac{1}{2}x^2 \end{pmatrix}$$

a direct computation gives

$$[u, A_0 u]_p = \left(1 + \frac{x^{p-\frac{1}{3}} - (\frac{x-x^2}{2})^{p-1} x^{2/5}}{\|u\|_p^p}\right) \|u\|_p. \tag{5.10}$$

According to (5.10), the reader can check that the ϕ-accretivity property implies $\phi(\|u\|_p) \geq -[u, A_0 u]_p \geq \frac{1}{2^p+1} x^{2/5}$ for $x \in [0,1]$ sufficiently small. Then we can deduce that for some $x_0 \in]0,1]$, there is $C > 0$ (for instance $C = \frac{e^{-1/5}}{2(2^p+1)}$), such that

$$C\|u\|_p^{2/5} \leq \phi(\|u\|_p)$$

for all $x \in [0, x_0]$. Finally, there exists $\xi_0 > 0$ such that $\phi(\xi) \geq C\xi^{2/5}$ for $\xi \in [0, \xi_0]$. Now, as in step a), using the function H defined in (5.7), we can easily derive a contradiction.

c) Let $\|\cdot\|$ be a norm in $\mathbb{R}^2$ and suppose that A_0 satisfies a ϕ-Osgood condition in $(\mathbb{R}^2, \|\cdot\|)$. Then, by taking $u = \begin{pmatrix} 0 \\ x \end{pmatrix}$, in the ϕ-Osgood property we obtain $\phi(\xi) \geq c\xi^{1/5}$ for a constant $c > 0$, $\xi_1 > 0$ and all $\xi \in [0, \xi_1]$. So we can conclude as before and the lemma is proved. $\square$

6 Asymptotic behavior

Figure 5.1 motivates the following remarks about asymptotic behavior of solutions of (1.1). Hypothesis **H3** stands for following three conditions

- $f \equiv 0$
- The assumption H2T holds for all $T > 0$
- A is a continuous derivor on $\mathbb{R}^N$.

We do not assume the uniqueness of solutions of $CP(A, 0, u_0, +\infty)$. We set $A^+ = \{u; Au \geq 0\}$ and $A^- = \{u; Au \leq 0\}$.

Definition 6.1 A derivor A is *absorbent* if $u_0 \in A^+$ (resp. $u_0 \in A^-$) implies $u(t) \in A^+$ (resp. $u(t) \in A^-$) for all $t \geq 0$ and each solution $u(.)$ of the autonomous problem $CP(A, 0, u_0, +\infty)$. We say that A is u_∞-absorbent if B defined by $Bu = Au - Au_\infty$ is absorbent.

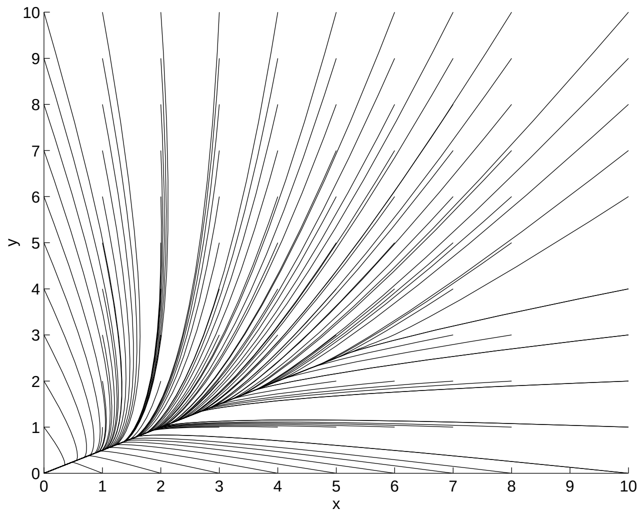

Figure 1: Flow relative to A_0

Proposition 6.2 *Assume H3. Let u_0, v_0, w_0 be in $\mathbb{R}^N$ such that $Av_0 \le 0$, $Aw_0 \ge 0$, $v_0 \le w_0$ and $v_0 \le u_0 \le w_0$. Suppose A is a continuous u_∞-absorbent derivor on $\mathbb{R}^N$ such that the equation $Av = 0$ has a unique solution u_∞ in $[v_0, w_0]$. Then every solution u of $CP(A, 0, u_0, +\infty)$ satisfies*

$$\lim_{t\to+\infty} u(t) = u_\infty$$

Proof. It is sufficient to prove the result for $S_A^{\max}(t)w_0$ and $S_A^{\min}(t)v_0$ since from Lemma 4.2 such extremal solutions exist and satisfy $S_A^{\min}(t)v_0 \le u(t) \le S_A^{\max}(t)w_0$, $t \in [0, +\infty[$. If $w(t) = S_A^{\max}(t)w_0$ we have

$$w(t) - w_0 = -\int_0^t Aw(x)dx. \tag{5.11}$$

Consequently $t \to w(t)$ is decreasing because from the absorbent property $w'(t) = -Aw(t) \le 0$. In an analogous way $v(t) = S_A^{\min}(t)u$ is increasing because $v'(t) = -Av(t) \le 0$ for each $t \in [0, +\infty[$. Hence we get

$$v_0 \le v(t) \le w(t) \le w_0.$$

Then $l_1 = \lim_{t\to+\infty} w(t)$ and $l_2 = \lim_{t\to+\infty} v(t)$ exist in $\mathbb{R}^N$. Hence, according to (5.11), $\int_0^{+\infty} Av(\tau)d\tau$ and $\int_0^{+\infty} Aw(\tau)d\tau$ converge. Since $\lim_{t\to+\infty} Aw(t) = Al_1$ and $\lim_{t\to+\infty} Av(t) = Al_2$, we have necessarily $Al_1 = Al_2 = 0$. So by hypothesis $\lim_{t\to+\infty} w(t) = \lim_{t\to+\infty} v(t) = u_\infty$.

Corollary 6.3 *For the operator A_0 introduced in (5.1), we have*

$$\lim_{t\to\infty} S_{A_0}(t)(u_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Proof. We can show that $A_0 u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ holds if and only if $u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Moreover A_0 is absorbent. Indeed, with the notation of Lemma 4.4, let $u_0 \in A_0^+$ (resp. A_0^-) and $u_n(t) = J^i_{T/n}(u_0)$ for $(i-1)T/n < t \le iT/n$. Then, owing to Proposition 2.4.(c), $u_n(t) \in A_0^+$ (resp. $u_n(t) \in A_0^-$). Consequently, Lemma 4.5 yields $S_{A_0}(t)u_0 \in A_0^+$ (resp. $S_{A_0}(t)u_0 \in A_0^-$) for all $t \ge 0$. So Corollary 6.3 is a direct consequence of Proposition 6.2.

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