ON DEPTH OF POWERS OF IDEALS

THESIS

Presented to the Graduate Council of
Texas State University-San Marcos
in partial Fulfillment
of the Requirements for the Degree

Master of SCIENCE

By

Heather E. Bruch, B.A.

San Marcos, Texas
May 2011

FAIR USE AND AUTHOR'S PERMISSION STATEMENT

Fair Use

This work is protected by the Copyright Laws of the United States (Public Law 94-553, section 107). Consistent with fair use as defined in the Copyright Laws, brief quotations from this material are allowed with proper acknowledgment. Use of this mterial for financial gain without the author's express written permission is not allowed.

Dublication Permission

As the copyright holder of this work I, Heather Bruch, authorize duplication of this work, in whole or in part, for educational or scholarly purposes only.

ACKNOWLEDGEMENTS

I would like to first thank Dr. Susan Morey for her attention, support, and patience. I would also like to thank the thesis committee members for their time and support, and my friends and family for understanding my demanding workload.

The manuscript was submitted on 11 April 2011.

TABLE OF CONTENTS

ABSTRACT

ON DEPTH OF POWERS OF IDEALS

by

Heather Elaine Bruch, B.A.

Texas State University-San Marcos

April 2011

SUPERVISING PROFESSOR: SUSAN MOREY

The paper “The depth of powers of an ideal,” by Herzog and Hibi is expanded to include background information and proof details. The numerical function $f(k)=\mathrm{depth}(S/I^k)$ is discussed where S is Noetherian local or standard graded and I is a proper ideal of S. A combinatorial example where depth is known for each power is included.

Chapter 1

INTRODUCTION

Throughout this paper, S is assumed to be a commutative ring with identity. Throughout sections 3, 4, and 5, S is assumed to be either a Noetherian local ring with maximal ideal $\mathfrak{m}$, or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$, where K is any field. Also throughout these sections, the ideal I is assumed to be a proper ideal of S. In addition, if S is standard graded, I is assumed to be graded. Also note that the natural numbers will include zero ($\mathbb{N} = \{0, 1, 2, ...\}$).

In this paper, I provide a detailed explanation of two major theorems and one lemma from the paper "Depth of Powers of Ideals," written by Jürgen Herzog and Takayuki Hibi (4). The detail included is sufficient for a second year graduate student reader. In order for a student at such a level to understand this material, an extensive background section is also required.

Herzog and Hibi's "Depth of Powers of Ideals," contains cutting edge results. The depth of objects such as S/I have been studied with results produced by many authors. However, once we pass to powers of I and study the "limit depth of I", $\lim_{k\to\infty}$depth S/I^k, very little is known. The paper "Depth of Powers of Ideals," is one of the first and few commutative algebra papers producing major results involving depths of objects such as S/I^k. In addition to the few papers cited below, Susan Morey's, "Depths of Powers of the Edge Ideal of a Tree," (5) also provides major results pertaining to such objects.

The first theorem (Theorem 3.0.62) will show the existence of the limit of the depths of the components of a graded module and provide a bound. The second theorem (Theorem 3.0.63) utilizes the first to essentially show that $\lim_{k\to\infty}$ depth $(S/I^k) \leq \dim(S) - \ell(I)$ where ℓ denotes analytic spread. These are not new results, but the methods of proof are new and shorter. To be more specific, Markus Brodmann (1) expanded on a result given by Lindsay Burch (2) to show that $\lim_{k\to\infty}$depth $S/I^k \leq \dim(S) - \ell(I)$. Later, David Eisenbud and Craig Huneke (3) showed that if the associated graded ring $\mathrm{gr}_I(S)$ is Cohen-Macaulay, then $\lim_{k\to\infty}$depth $S/I^k = d - \ell(I)$. Herzog and Hibi's proofs involve a great deal of the general commutative algebra background and also make extensive use of the Koszul Complex.

The third major focus of this paper (Lemma 4.0.64), is a lemma in Herzog and Hibi's paper. It is the beginning of new material in their paper to show a strong result. This strong result is that depth (S/I^k) is a non-increasing function of k if I is a graded ideal, and all powers of I have a linear resolution. For Lemma 4.0.64 we will restrict our attention to a polynomial ring over a field. This lemma will make use of Tor and Betti numbers among other various results from the background.

Chapter 2

BACKGROUND

The particular proofs in this paper require various definitions and results from commutative algebra along with an explanation of the Koszul Complex and Tor. Thus the background section is divided into three sections. The first section includes many results that are widely used in Commutative Algebra. The main focus of the second background section is the Koszul Complex. The Koszul complex is defined and some properties are discussed. In the last background section Tor modules and Betti numbers are defined and some properties are included.

2.0.1 Commutative Algebra

The commutative algebra section begins with a famous and frequently cited result known as Nakayama's Lemma.

Lemma 2.0.1. *(10, Lemma VIII.4.5) Nakayama's Lemma:*

If I is an ideal in S, then the following are equivalent.

(i) I is contained in every maximal ideal of S.

(ii) $1_S - x$ is a unit for every $x \in I$.

(iii) If M is a finitely generated S-module such that $IM = M$, then $M = 0$.

(iv) If E is a submodule of a finitely generated S-module M such that $M = IM + E$, then $M = E$.

Notice that if the ring S is local, condition (i) automatically holds simply by the definition of a local ring. Therefore, since we will assume all rings S are local and all ideals I are proper, we will have these four conditions at our disposal throughout much of this paper. We will see later that conditions (iii) and (iv) apply to information about the Koszul Complex and apply to the proof of Theorem 3.0.62.

We will now give our first definition followed by two key properties.

Definition 2.0.2. *(10, Definition VIII.5.2) Let S be an extension ring of R. Let $s \in S$. If there exists a monic polynomial $f(x) \in R[x]$ such that $f(s) = 0$, then s is* ***integral*** *over R. If for each $s \in S$, there exists a monic polynomial $f_s(x) \in R[x]$ such that $f_s(s) = 0$, then S* ***is integral over*** *R or S* ***is an integral extension of*** *R.*

Integral extensions are widely studied in Commutative Algebra. We will only use the following three nice properties, and they will be applied to the proof of Lemma 2.0.28.

Theorem 2.0.3. *(10, Theorem VIII.5.9)* ***Lying Over Theorem***

Let S be an extension ring of R, let S be integral over R, and let P be a prime ideal of R. Then there exists a prime ideal Q of S such that $Q \cap R = P$.

Corollary 2.0.4. *(10, Corollary VIII.5.10)* ***Going Up Theorem***

Let S be an extension ring of R, let S be integral over R, and let P_0 and P_1 be prime ideals of R such that $P_0 \subseteq P_1$. If Q_0 is a prime ideal of S such that $Q_0 \cap R = P_0$, then there exists a prime ideal Q_1 of S such that $Q_0 \subseteq Q_1$ and $Q_1 \cap R = P_1$.

Theorem 2.0.5. *(10, Theorem VIII.5.11) Let S be an integral extension ring of a ring R. Let Q be a prime ideal in R. If P_1 and P_2 are prime ideals in R such that $P_1 \subseteq P_2$, and if P_1 and P_2 both lie over Q, then $P_1 = P_2$.*

Prime ideals of a ring are so useful that we have the following definition.

Definition 2.0.6. *Let S be a ring. The **spectrum** of S is denoted* Spec(S)*, and* Spec$(S) = \{P \mid P$ *is a prime ideal of* $S\}$.

In mathematics we have several notions of "size." Let us define some notions of size in commutative algebra.

Definition 2.0.7. *Let S be a ring. A sequence of prime ideals in S is a **chain** of prime ideals in S if and only if the sequence is finite and strictly increasing, i.e.,* $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$.

Definition 2.0.8. *Let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ be a chain of prime ideals in a ring S. The **length** of the chain is the integer n.*

So length is a notion of size for a chain.

Definition 2.0.9. *Let P be a prime ideal of a ring S. The **height** of P is denoted* $\mathrm{ht}(P)$*, and* $\mathrm{ht}(P) = \sup\{\mathrm{length}(C) \mid C$ *is a chain that ends at* $P\}$.

We may also define the height of any ideal.

Definition 2.0.10. *Let S be a ring and let I be an ideal of S. Then the **height of** I is denoted* $\mathrm{ht}(I)$*, and* $\mathrm{ht}(I) = \min\{\mathrm{ht}(P) \mid P \supseteq I$ *and* $P \in \mathrm{Spec}(S)\}$

Example 2.0.11. *Let S be the ring $\mathbb{R}[x, y, z]$ of three indeterminants over the field of real numbers. Let $I = (x^2, xy, y^2)$. Then $ht(I) = 2$. Note that (x, y) contains I and observe $(0) \subsetneq (x) \subsetneq (x, y)$.*

Thus we have a notion of size for a particular ideal of a ring.

Finally, we have a notion of size for a ring:

Definition 2.0.12. *The **Krull dimension** of a ring S is denoted* $\dim(S)$*, and* $\dim(S) = \sup\{\mathrm{ht}(P) \mid P \in \mathrm{Spec}(S)\}$.

Example 2.0.13. *(11, Example 2.5.1) Consider the ring $\mathbb{Z}$ of integers. The prime ideals in $\mathbb{Z}$ are* $(0), (2), (3), (5), \ldots$. *Each of them are maximal so* dim($\mathbb{Z}$)$= 1$.

It might be useful to note that 0 is prime in S if and only if S is an integral domain. The reader might also take a moment to appreciate the difference between the dimension of a free module F over a ring S (a ring that has the invariant dimension property) and the dimension of a ring S itself (the Krull dimension). The former is also known as rank.

In addition to the dimension of a free module, we have a definition for the dimension of any module. This is actually the Krull dimension, but in commutative algebra, "Krull" is omitted. In order to define the dimension of a module, we need the following definition.

Definition 2.0.14. *Let S be a ring. Let E be an S-module. The* ***annihilator*** *of E is denoted* $\mathrm{ann}_S(E)$ *or simply* $\mathrm{ann}(E)$, *and* $\mathrm{ann}(E) = \{x \in S \mid xE = 0\}$.

Definition 2.0.15. *Let S be a ring and M an S-module. The* ***Krull dimension*** *of M, or rather dimension of M, is denoted* $\dim(M)$, *and is defined as the Krull dimension of* $S/\mathrm{ann}(M)$.

Another commonly used object is the radical of an ideal.

Definition 2.0.16. *Let S be a ring. Let I be an ideal of S. The* ***radical*** *of I is denoted* $\mathrm{rad}(I)$ *or $\sqrt{I}$, and* $\sqrt{I} = \{x \in S \mid x^n \in I \textit{ for some } n \in \mathbb{N}\backslash\{0\}\}$

Example 2.0.17. *Consider the ring $\mathbb{Z}$ and the ideal* (9). *Then* $\sqrt{(9)} = \{x \in \mathbb{Z} \mid x^n \in (9) \textit{ for some } n \textit{ in } \mathbb{N}\backslash\{0\}\} = (3)$.

The next definition and theorem are useful for finding radicals of ideals.

Definition 2.0.18. *(10, Definition VIII.2.8) Let S be a ring. Let Q be an ideal in S. We say Q is* ***primary*** *if given $a, b \in S$, $ab \in Q$ and $a \notin Q$ imples $b^n \in Q$ for some $n \in \mathbb{N} \backslash \{0\}$.*

Theorem 2.0.19. *(10, Theorem VIII.2.9) Let Q be a primary ideal in a ring S. Then $\sqrt{Q}$ is prime.*

Example 2.0.20. *Consider the ring $S = \mathbb{R}[x, y, z]$ and the ideal (x^2, y^3). Then $\sqrt{(x^2, y^3)} = (x, y)$. We also have $\sqrt{(x^2, xy, y^2)} = (x, y)$.*

The next property is so useful, that it is sometimes used as a definition.

Proposition 2.0.21. *(14, Proposition 1.1.10) Let S be a ring and I a proper ideal of S. Then $\sqrt{I} = \cap P$ such that $P \in \{Q \mid Q \supseteq I \text{ and } Q \in \operatorname{Spec}(S)\}$.*

Example 2.0.22. *Consider the ring $\mathbb{Z}$ and the ideal (54). Note that $54 = 3^3 2$. Then $\sqrt{(54)} = (2) \cap (3) = (6)$ and $\sqrt{(27)} = (3)$.*

This concludes our background section on commutative algebra. Commutative algebra is a vast field of mathematics. This background section includes a limited amount of material that specifically applies to the proofs in this paper.

2.0.2 Graded Modules

We now begin a section especially attributed to graded objects. Nearly every object we discuss is graded.

Throughout the Graded Modules section, let $(H, +)$ be an abelian semi-group.

Definition 2.0.23. *An H-**graded ring** is a ring S together with a decomposition*

$S = \bigoplus_{a \in H} S_a$ *(as a* $\mathbb{Z}$*-module or such that for each* $a \in H$*,* S_a *is an abelian group) with the property that for each* a*,* $b \in H$*,* $S_a S_b \subseteq S_{a+b}$*.*

For each $a \in H$, S_a is referred to as a component of S. Given an element $x \in S$, if $x \in S_a$ for some $a \in H$, we say x is homogeneous of degree a and write $\deg(x) = a$.

We will say "graded ring" when the semi-group $H = \mathbb{Z}$. The next definition and two lemmas will be applied to the proof of Theorem 3.0.62.

Example 2.0.24. *Let* S *be the ring* $\mathbb{R}[x]$*. Then* S *has an* $\mathbb{N}$*-grading,* $S = \oplus_{a=0}^{\infty} S_a$ *where* $S_a = \{rx^a \mid r \in \mathbb{R}\}$*. Thus* S_a *is the* $\mathbb{R}$*-vector space generated by* x^a*.*

Definition 2.0.25. *Let* $S = \oplus S_k$ *be a graded Noetherian ring. The* r*th* ***Veronese subring*** *of* S *is denoted* $S^{(r)}$ *and* $S^{(r)} = \oplus S_{rk}$*.*

Example 2.0.26. *The rth Veronese subring of* $\mathbb{R}[x]$ *is* $\mathbb{R}[x]^{(r)}$*, and* $\mathbb{R}[x]^{(r)} = \mathbb{R}[x^r]$*.*

Lemma 2.0.27. *Let* S *be a graded ring. Then* S *is integral over the Veronese subring* $S^{(r)}$*.*

Proof. Let $s \in S$. Say $\deg(s) = k$. Then $\deg(s^r) = kr$. Now $s^r \in S_{kr}$. Thus $s \in S^{(r)}$. Then $f(x) = x^r - s^r \in S^{(r)}[x]$, and $f(s) = 0$. □

Lemma 2.0.28. *Let* S *be a Noetherian graded ring. Then* dim (S) =dim $(S^{(r)})$.

Proof. Let $P^{(r)} \in \operatorname{Spec}(S^{(r)})$. Say $\operatorname{ht}(P^{(r)}) = k$. Then there exists a chain $P^{(r)} = P_0^{(r)} \supsetneq P_1^{(r)} \supsetneq \cdots \supsetneq P_k^{(r)}$. By the Lying Over Theorem, since S is integral over $S^{(r)}$, and for all $i \in \{1, ..., k\}$, $P_i^{(r)}, P_{i-1}^{(r)} \in \operatorname{Spec}(S^{(r)})$, then for all $i \in \{1, ..., k\}$, there exist $P_i, P_{i-1} \in \operatorname{Spec}(S)$ such that $P_i \cap S^{(r)} = P_i^{(r)}$ and $P_{i-1} \cap S^{(r)} = P_{i-1}^{(r)}$. Now since for all $i \in \{1, ..., k\}$, $P_{i-1}^{(r)} \supsetneq P_i^{(r)}$, then for all $i \in \{1, ..., k\}$,

$P_{i-1} \cap S^{(r)} \supsetneq P_i \cap S^{(r)}$. Then for all $i \in \{1, ..., k\}$, $P_{i-1} \supsetneq P_i$. So there exists $P \in \operatorname{Spec}(S)$ such that $P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_k$ where each $P_i \in \operatorname{Spec}(S)$. So there exists $P \in \operatorname{Spec}(S)$ such that $\operatorname{ht}(P) \geq k = \operatorname{ht}(P^{(r)})$. Then $\dim(S) \geq \dim(S^{(r)})$.

We will show $\dim(S) \leq \dim(S^{(r)})$ by contradiction. Suppose $\dim(S) > \dim(S^{(r)})$. Then there exists $P \in \operatorname{Spec}(S)$ such that for all $Q \in \operatorname{Spec}(S^{(r)})$, $\operatorname{ht}(P) > \operatorname{ht}(Q)$. Say $\operatorname{ht}(P) = k$. So there exists a chain $P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_k$. Consider $P \cap S^{(r)} = P_0 \cap S^{(r)} \supseteq P_1 \cap S^{(r)} \supseteq \cdots P_k \cap S^{(r)}$. Since $P \cap S^{(r)} \in \operatorname{Spec}(S^{(r)})$, $\operatorname{ht}(P \cap S^{(r)}) < k$. So there exists $i \in \{0, ..., k-1\}$ such that $P_i \cap S^{(r)} = P_{i+1} \cap S^{(r)}$. Now P_i lies over $P_i \cap S^{(r)}$, and P_{i+1} lies over $P_{i+1} \cap S^{(r)}$. Also, $P_{i+1} \subseteq P_i$. Then by Theorem 2.0.5, $P_i = P_{i+1}$. This is a contradiction. Therefore $\dim(S) \leq \dim(S^{(r)})$

Thus, $\dim(S) = \dim(S^{(r)})$.

□

Now let us move from graded rings to graded modules.

Definition 2.0.29. *Let S be an H-graded ring. An H-**graded** S-**module** M is a module together with a decomposition $M = \bigoplus_{i \in H} M_i$ such that for each i, $j \in H$, $S_i M_j \subseteq M_{i+j}$*

Similar to the case of a homogeneous element of a graded ring we have a homogeneous element of a graded module. An element $m \in M$ is **homogeneous** of degree j if $m \in M_j$. Natural curiosity leads to the idea of a graded submodule.

Definition 2.0.30. *Let M be an H-graded module. A submodule $N \subseteq M$ is an H-graded **submodule** of M if N is generated by homogeneous elements of M.*

Definition 2.0.31. *Let $\phi : M \to N$ be a map between H-graded modules. Then ϕ is called **homogeneous** if for each $a \in H$, $\phi(M_a) \subseteq N_a$.*

Observe that a homogeneous map between graded modules is a degree-preserving map. Also note that if M is an H-graded module and N is an H-graded submodule of M, then for all $a \in H$, $(M/N)_a = M_a/N_a$.

The following definition provides a useful tool for creating maps in complexes. This definition will be applied to the proof of Lemma 4.0.64.

Definition 2.0.32. *(14, Definition 2.54)*

Let $S = \oplus_{i=0}^{\infty} S_i$ be a positively graded ring and $a \in \mathbb{N}$. The graded S-module obtained by a ***shift in the graduation*** *of S is given by $S(-a) = \oplus_{i=0}^{\infty} S(-a)_i$ where the ith graded component of $S(-a)$ is $S(-a)_i = S_{-a+i}$. Note that if $-a+i < 0$, $S_{-a+i} = 0$.*

Sometimes the ith graded component of $S(-a)$ is denoted $S(-a_i)$. Then $S(-a_i) = S_{-a+i}$. This is useful when multiple subscripts are needed as we will see in the proof of Lemma 4.0.64.

Definition 2.0.33. *An H-**graded** S-**algebra** A is an H-graded S-module such that for each i, $j \in H$, $A_iA_j \subseteq A_{i+j}$.*

The following proposition is useful throughout the whole paper and is worth verifying.

Proposition 2.0.34. *(8, Exercise I.4.1)*

Let S be a ring. Let A be an S-algebra. Let E be an A-module. The structure map from S to A makes E an S-module. If A is finite over S, and E is finitely generated as an A-module, then E is finitely generated as an S-module.

Proof. First note that E is an S-module by the map $S \times E \to E$ given by $(s, x) \mapsto ((s1_A)x)$. Since E is finitely generated as an A-module, let $X = \{x_1, ..., x_t\}$

be the generating set of E as an A-module. Since A is finite over S, let $Y = \{y_1, ..., y_m\}$ be the generating set of A as an S-algebra. Now let v be an element of the A-module E. Then $v = \sum_{i=1}^{t} r_i x_i$ where $r_i \in A$. Since $r_i \in A$, $r_i = \sum_{j=1}^{m} s_j y_j$ where $s_j \in S$. Now $v = \sum_{i=1}^{t}(\sum_{j=1}^{m} s_{j_i} y_{j_i})x_i$. Since each $s_{j_i} y_{j_i}$ is an element of A and each x_i is an element of E and E is an A-module, we may distribute the x_i over the $s_{j_i} y_{j_i}$. Then we have a sum of the form $\sum_{k=1}^{tm}(s_k y_k)x_k$. Given elements $s \in S$, $y \in A$, and $x \in E$, $(sy)x = (s(1_A y))x$ since A has an identity. Then $(s(1_A y))x = ((s1_a)y)x$ since A is an S-algebra. Now $((s1_a)y)x = s1_A(yx)$ since E is an A-module. Finally, $s1_A(yx) = s(yx)$ by the map the structure map of the S-module E. Thus $\{y_1x_1, y_2x_1, ..., y_m x_t\}$ is a finite generating set of E as an S-module.

□

Now let us focus our attention to polynomial rings in particular.

Definition 2.0.35. *(14, Definition 1.2.2) Let $S = K[x_1, ..., x_n]$ be a polynomial ring over a field K. The* **standard grading** *of S is an $\mathbb{N}$-grading created by setting $deg(x_i) = 1$ for each i. For $a = (a_1, ..., a_n) \in \mathbb{N}^n$, set $x^a = x_1^{a_1} \cdots x_n^{a_n}$. The induced $\mathbb{N}$-grading is*

$$S = \oplus_{j=0}^{\infty} S_j \text{ where } S_j = \oplus_{deg(x^a)=j} K x^a.$$

To extend to a $\mathbb{Z}$-grading, set $S_j = 0$ for all $i < 0$.

Definition 2.0.36. *Let S be a ring, t an indeterminate of S, and I an ideal of S. Consider $S[t]$ with the standard grading. The* **Rees ring** *of I is denoted $\mathcal{R}(I)$, and $\mathcal{R}(I) = S \oplus_k I^k t^k$.*

Note that $\mathcal{R}(I) \subseteq S[t]$.

Definition 2.0.37. *Let S be a ring and I an ideal of S. The **analytic spread** of I is denoted $\ell(I)$, and $\ell(I)$ =*$\dim(\mathcal{R}(I)/\mathfrak{m}\,\mathcal{R}(I))$.

Definition 2.0.38. *Let S be a ring and I an ideal of S. The **associated graded ring** of S is denoted $gr_I(S)$ and $gr_I(S) = S/I \oplus I/I^2 \oplus \cdots \oplus I^i/I^{i+1} \oplus \cdots$. Let $a + I^i \in I^{i-1}/I^i$ and let $b + I^j \in I^{j-1}/I^j$. The multiplication operation is defined as $(a + I^i)(b + I^j) = ab + I^{i+j-1}$.*

Remark. *$\mathcal{R}(I)/\mathfrak{m}\,\mathcal{R}(I) = gr_I(S)/\mathfrak{m} gr_I(S)$ is a ring called the **fiber cone** or the **fiber ring of I**. Observe $\mathcal{R}(I)/\mathfrak{m}\,\mathcal{R}(I) = \frac{S\oplus_k I^k t^k}{\mathfrak{m}\,S\oplus_k \mathfrak{m}\,I^k t^k} = (S/\mathfrak{m}) \oplus_k (I^k/\mathfrak{m}\,I^k)$ since $\mathfrak{m}$ is a graded sub-module of S, and $\mathfrak{m}\,I^k$ is a graded sub-module of I^k. Also, $gr_I(S)/\mathfrak{m} gr_I(S) = \frac{S/I \oplus I/I^2 \oplus \cdot \oplus I^i/I^{i+1} \oplus}{\mathfrak{m}\,S/I \oplus \mathfrak{m}\,I/I^2 \oplus \ \oplus \mathfrak{m}\,I^i/I^{i+1} \oplus \cdot} = \frac{S/I}{\mathfrak{m}\,S/I} \oplus \frac{I/I^2}{\mathfrak{m}\,I/I^2} \oplus \frac{I^2/I^3}{\mathfrak{m}\,I^2/I^3} \oplus \cdots = (S/\mathfrak{m}) \oplus_k (I^k/\mathfrak{m}\,I^k)$ by the third isomorphism theorem.*

Example 2.0.39. *Let $S = \mathbb{R}[x, y]$, $I = (x^2, xy, y^2) = (x, y)^2$ be an ideal of S, and T_1, T_2, T_3 be indeterminates of S. The Rees ring is $\mathcal{R}(I) \cong \frac{\mathbb{R}[x,y,T_1,T_2,T_3]}{(xT_1, yT_2, xT_2+yT_3, T_1T_3-T_2^2)}$. The fiber cone is $\mathfrak{F}(I) \cong \frac{\mathbb{R}[T_1,T_2,T_3]}{(T_1T_3-T_2^2)}$. The dimension of $\mathfrak{F}(I)$ is 2, so $\ell(I) = 2$. For more information see (5).*

Definition 2.0.40. *A sequence of module homomorphisms $\cdots \xrightarrow{\phi_{i+2}} E_{i+1} \xrightarrow{\phi_{i+1}} E_i \xrightarrow{\phi_i} E_{i-1} \xrightarrow{\phi_{i-1}} \cdots$ is said to be **exact** at E_k if* $\ker(\phi_k) = \mathrm{im}(\phi_{k+1})$. *A sequence of module homomorphisms is said to be an **exact sequence** if for each $i \in \mathbb{Z}$,* $\ker(\phi_i) = \mathrm{im}(\phi_{i+1})$. *A **short exact sequence** is an exact sequence of the form $0 \xrightarrow{f} U \xrightarrow{g} M \xrightarrow{h} N \to 0$*

Sometimes the terminology used is “exact sequence of modules,” versus “exact sequence of module homomorphisms.”

This concludes our section on graded objects, we now move to complexes, which are not so different.

2.0.3 The Koszul Complex

The Koszul Complex is a very interesting object. In our paper, its homology modules help us to measure depths of ideals. We begin with the definitions of a complex and a homology module, without which there would be no homology theory.

Definition 2.0.41. *Let S be a ring. Let $\mathcal{F} : \cdots F_{i+1} \overset{\phi_{i+1}}{\to} F_i \overset{\phi_i}{\to} F_{i-1} \overset{\phi_{i-1}}{\to} \cdots$ be a sequence of S-module homomorphisms. $\mathcal{F}$ is a* ***complex of module homomorphisms*** *if for each i, $\phi_i\phi_{i+1} = 0$.*

Complexes are common in topology and algebraic geometry, and for historical, topological reasons, the homomorphisms in a complex are often called differentials.

Definition 2.0.42. *If $\mathcal{F}$ is a complex of module homomorphisms, the* ***homology module*** *of $\mathcal{F}$ at i is* $\ker(\phi_i)/\mathrm{im}(\phi_{i+1})$.

Note that a complex of module homomorphisms becomes an exact sequence if for each i, $\ker(\phi_i)/\mathrm{im}(\phi_{i+1}) = 0$.

Remark. *Also notice that a graded S-module $M = \oplus M_i$ may be viewed as a complex by letting the differential maps be the zero map.*

The next theorem is a useful fact that relates exact sequences with homology. It will be applied in the proof of Theorem 3.0.63.

Theorem 2.0.43. *(9, Theorem 2.16) Any short exact sequence of complexes, $0 \to U \to M \to N \to 0$, leads to a long exact sequence of homology modules $\cdots \to H_{n-1}(N) \to H_n(U) \to H_n(M) \to H_n(N) \to H_{n+1}(U) \to \cdots$.*

We next define a free module. This is a commonly used object. In our paper, it is necessary for the definition of the Koszul Complex and also of resolutions.

Definition 2.0.44. *(10, Theorem IV.2.1) Let S be a ring. A unitary S-module F is a* ***free S-module*** *if F is an S-module isomorphic to a direct sum of copies of the left S-module S.*

We will now define the Koszul complex of a sequence of elements of a ring. In this definition and when we make use of the Koszul Complex, we use abbreviated notation. The object $\oplus Se_{j_1 \dots j_i}$ is shorter notation for $Se_{j_1} \oplus \dots \oplus Se_{j_n}$. The Koszul Complex is a very interesting object that can be viewed as a graded algebra. The symbol $\wedge$ is the product operation in the algebra (wedge product). Given elements a, b in the algebra,

(i) $a \wedge b = (-1)^{deg(a))(deg(b))} b \wedge a$

(ii) $a \wedge a = 0$ if a is homogeneous with odd deg(a).

We will not need to use the wedge product, however in order to fully understand the definition of the complex, it is necessary to give a description so that we may understand the following two objects. The object $e_{j_1 \dots j_i}$ is shorter notation for $e_{j_1} \wedge \dots \wedge e_{j_i}$, and the object $e_{j_1 \dots \hat{e_{j_r}} \dots j_i}$ is shorter notation for $e_{j_1} \wedge \dots \wedge \hat{e_{j_r}} \wedge \dots \wedge e_{j_i}$ where the hat over e_{j_r} denotes the removal of e_{j_r}.

Definition 2.0.45. *Let S be a ring. Let $\underline{x} = x_1, ..., x_n \in S$. Let $K_0 = S$ and $K_i = 0$ if $i < 0$ or $n < i$. For $1 \leq i \leq n$, let $K_i = \oplus Se_{j_1 \dots j_i}$ be the free S-module of rank $\binom{n}{i}$ with basis $\{e_{j_1 \dots j_i} | 1 \leq j_1 < \dots < j_i \leq n\}$. Define the differential map $d : K_i \to K_{i-1}$ by $e_{j_1 \dots j_i} \mapsto \sum_{r=1}^{i} (-1)^{r-1} x_{j_r} e_{j_1 \dots \hat{j_r} \dots j_i}$. For $i = 1$, set $d(e_j) = x_j$. This complex is the* ***Koszul complex*** *of the sequence $\underline{x}$ and is denoted $K_\bullet(\underline{x})$.*

Example 2.0.46. *Let S be a ring, and let x, y be in S. The Koszul complex of x, y is $K(x, y) : 0 \to S \xrightarrow{\binom{x}{y}} S \oplus S \xrightarrow{(-x,y)} S$. See (8) page 424.*

Remark. *Let S be a Noetherian local ring, let I be an ideal of S, and let $\underline{x} = x_1, ..., x_n$ and $\underline{y} = y_1, ..., y_n$ be minimal systems of generators of I. Then the Koszul complexes $K_\bullet(\underline{x})$ and $K_\bullet(\underline{y})$ are isomorphic. See (7).*

We now define the tensor product of two modules. This definition is needed in order to extend the definition of the Koszul complex of a sequence of elements in a ring to the Koszul complex of a module.

Definition 2.0.47. *(10, Definition IV.5.1)*

Let S be a ring. Let M be a right S-module and let N be a left S-module. Let F be the free abelian group on the set $M \times N$. Let K be the subgroup of F generated by elements of the following forms where $m_0, m_1 \in M, n_0, n_1 \in N$, and $s \in S$.

(i) $(m_0 + m_1, n_0) - (m_0, n_0) - (m_1, n_0)$;

(ii) $(m_0, n_0 + n_1) - (m_0, n_0) - (m_0, n_1)$;

(iii) $(m_0 s, n_0) - (m_0, s n_0)$.

*The **tensor product** of the modules M and N is the quotient group F/K. It is denoted $M \otimes_S N$. If $S = \mathbb{Z}$, the tensor product of M and N is denoted $M \otimes N$.*

However, it should be noted that when the ring S is understood and not necessarily $\mathbb{Z}$, $M \otimes_S N$ is often written $M \otimes N$. If (m, n) is an element in F, the coset $(m, n) + K$ in $M \otimes_S N$ is denoted $m \otimes n$. An element in F is of the form $\sum_{i=1}^{r} j_i(m_i, n_i)$ where $j_i \in \mathbb{Z}, m_i \in M, n_i \in N$. So an element in $F/K = M \otimes_S N$ is of the form $\sum_{i=1}^{r} j_i(m_i \otimes m_i)$.

Finally, we may define the Koszul complex of a module.

Definition 2.0.48. ***The Koszul complex of a Module****. Let M be an S-module and let $\underline{x} = x_1, ..., x_n$ be a sequence of elements in S. The Koszul Complex of $\underline{x}$ with coefficients in M is denoted $K_\bullet(\underline{x}, M)$, and $K_\bullet(\underline{x}, M) = K_\bullet(\underline{x}) \otimes_S M$.*

Remark. *The homology modules of a Koszul complex $K_\bullet(\underline{x}, M)$ are denoted $H_i(\underline{x}, M)$. Some useful properties of the Koszul complex are that $H_0(\underline{x}, M) \cong M/\underline{x}M$ ($\underline{x}M = \sum_i x_i M$) and $H_n(\underline{x}, M) \cong \{a \in M \mid x_1 a = \cdots x_n a = 0\}$.*

Remark. *If $\mathcal{F} : \cdots F_{i-1} \to F_i \to F_{i+1} \to \cdots$ is a complex and M is a module, then $\mathcal{F} \otimes M = \cdots F_{i-1} \otimes M \to F_i \otimes M \to F_{i+1} \otimes M \cdots$.*

The following proposition would not be completely obvious to a beginning graduate student. The proposition is provided here so that it shortens the proof of Theorem 3.0.62.

Proposition 2.0.49. *Recall that for each $k \in \mathbb{N}$, E_k is a finitely generated S-module. Note that $H_i(\underline{x}, M)$ denotes the ith homology module of the Koszul Complex of a module M, and k represents the index of the component. The proposition is $H_i(\underline{x}, E)_k = H_i(\underline{x}, E_k)$.*

Proof. Consider $H_i(\underline{x}, E)_k$. Then $H_i(\underline{x}, E)_k$ is the kth component of the ith homology module of the complex $K_\bullet(\underline{x}) \otimes E$.

Then $H_i(\underline{x}, E)_k$ is the kth component of:

$$\left[\frac{\ker(\oplus Se_{j_1\;\;j_i} \otimes E \to \oplus Se_{j_1\;\;j_{i-1}} \otimes E)}{\mathrm{im}(\oplus Se_{j-1\;\;j_{i+1}} \otimes E \to \oplus Se_{j_1\;\;j_i} \otimes E)}\right].$$

Now using (10, Thm I.8.10, Thm IV.5.9, and Thm I.8.11), we have the following equalities.

$$\begin{aligned}
&H_i(\underline{x}, E) \\
&= \left[\frac{\ker(\oplus Se_{j_1\;\;j_i} \otimes \oplus E_m \to \oplus Se_{j_1\;\;j_{i-1}} \otimes \oplus E_m)}{\mathrm{im}(\oplus Se_{j-1\;\;j_{i+1}} \otimes \oplus E_m \to \oplus Se_{j_1\;\;j_i} \otimes \oplus E_m)}\right] \\
&= \left[\frac{(\ker(\oplus(\oplus Se_{j_1\;\;j_i} \otimes E_m) \to \oplus(\oplus Se_{j_1\;\;j_{i-1}} \otimes E_m))}{\mathrm{im}(\oplus(\oplus Se_{j-1\;\;j_{i+1}} \otimes E_m) \to \oplus(\oplus Se_{j_1\;\;j_i} \otimes E_m))}\right] \\
&= \left[\frac{\oplus(\ker(\oplus Se_{j_1\;\;j_i} \otimes E_m \to \oplus Se_{j_1\;\;j_{i-1}} \otimes E_m))}{\oplus(\mathrm{im}(\oplus Se_{j-1\;\;j_{i+1}} \otimes E_m \to \oplus Se_{j_1\;\;j_i} \otimes E_m))}\right]
\end{aligned}$$

Now the kth component of $H_i(x; E)$ is the term of this summation where $m = k$ which is $= \left[\frac{\ker(\oplus Se_{j_1\;\;j_i} \otimes E_k \to \oplus Se_{j_1\;\;j_{i-1}} \otimes E_k)}{\mathrm{im}(\oplus Se_{j-1\;\;j_{i+1}} \otimes E_k \to \oplus Se_{j_1\;\;j_i} \otimes E_k)}\right]$. Thus $H_i(x; E)_k = H_i(\underline{x}; E_k)$

□

The following definition is crucial in understanding Theorem 3.0.62, and it may give the reader an initial idea of how useful the Koszul complex is in finding depth.

Definition 2.0.50. *(7, Definition 9.1.1) Let S be a ring, I an ideal generated by $x_1, ..., x_n$, and E an S-module. If there exists $\imath \in \{1, ..., n\}$ s.t. $H_\imath(x, E) \neq 0$, then*

grade$(I, E) = n - \max\{i : H_\imath(\underline{x}, E) \neq 0\}$; *otherwise* $\text{grade}(I, E) = \infty$. *The* ***depth*** *of E is denoted* depth (E) *and* depth $(E) = \text{grade}(\mathfrak{m}, E)$.

Some readers might be familiar with the alternate definition of grade of a module that makes use of regular sequences. The above definition is consistent with the other definition and more information can be found in (7) on pages 10 and 336.

The next lemma is another famous one. It has many versions. The one given here, is the most useful for this paper.

Lemma 2.0.51. ***The Depth Lemma*** *(7, Proposition 1.2.9)*

Let S be a Noetherian ring, I an ideal of S and $0 \to U \to M \to N \to 0$ an exact sequence of finite S-modules. Then

$$grade(I, M) \geq min\{grade(I, U), grade(I, N)\},$$

$$grade(I, U) \geq \min\{grade(I, M), grade(I, N) + 1\},$$

$$grade(I, N) \geq \min\{grade(I, U) - 1, grade(I, M)\}.$$

This concludes our section on the Koszul complex.

2.0.4 Tor and Betti Numbers

Aside from being used in many areas of mathematics, the information provided in this section will be applied to Lemma 4.0.64.

We will now list several definitions and propositions to make our way to the Definitions 2.0.57 and 2.0.58 which define Tor and Betti numbers, respectively.

Definition 2.0.52. *Let S be a ring. Let A and B be S-modules. Let $f : A \to B$ be a module homomorphism. Then B/imf is the* ***cokernel of*** *f, denoted $coker(f)$.*

Definition 2.0.53. *A* ***free resolution of an S-module*** *M is a complex $\mathcal{F} : \cdots \to F_n \xrightarrow{\phi_n} \cdots F_1 \xrightarrow{\phi_1} F_0$ of free S-modules where coker $\phi_1 = M$ and $\mathcal{F}$ is exact. $\mathcal{F}$ is* ***minimal*** *means for each $i \in \mathbb{N}$, $im\phi_i \subseteq \mathfrak{m} F_{i-1}$.*

Definition 2.0.54. *Let $\mathcal{F} : \cdots \to F_n \xrightarrow{\phi_n} \cdots F_1 \xrightarrow{\phi_1} F_0$ be a free resolution of an S-module M. $\mathcal{F}$ is a* ***graded free resolution*** *if S is a graded ring, if for each i, F_i is a graded free module, and if for each i, ϕ_i is a homogeneous map of degree 0.*

Theorem 2.0.55. *(12, Theorem 4.63) Let $S = K[x_1, ..., x_n]$ be a polynomial ring over K. If K is a principal ideal domain, then all finitely generated projective S-modules are free.*

Proposition 2.0.56. *Let S be a polynomial ring over a field k with the standard grading. Let I be a graded ideal of S. The minimal graded resolution of S/I by free S-modules can be expressed as*

$$0 \to \oplus_{i=1}^{c_g} S^{b_{gi}}(-d_{gi}) \xrightarrow{\phi_g} \cdots \to \oplus_{i=1}^{c_1} S^{b_{1i}}(-d_{1i}) \xrightarrow{\phi_1} S \to S/I \to 0,$$

where all maps are degree-preserving and the d_{ij} are positive integers.

Definition 2.0.57. *Let M and N be S-modules. Let $... \to F_{i+1} \to F_i \to F_{i-1} \to ... \to F_0 \to 0$ be a free resolution of M as an S-module. Then* ***Tor_i^S***(M, N) *is the homology module at $F_i \otimes N$ of the complex*

$F_{i+1} \otimes N \to F_i \otimes N \to F_{i-1} \otimes N$. *So* $Tor_i^S(M,N) = \ker(F_i \otimes N \to F_{i-1} \otimes N)$ / $\mathrm{im}(F_{i+1} \otimes N \to F_i \otimes N)$

Remark. *The following is a well known and useful property of* Tor. *Given any short exact sequence* $0 \to I \to J \to K \to 0$ *of* S*-modules and any* S*-module* M*, there exists a long exact sequence of* Tor*:*

$$\cdots \to Tor_i(M,I) \to Tor_i(M,J) \to Tor_i(M,K) \to Tor_{i-1} \to \cdots$$
$$\cdots \to Tor_0(M,I) \to Tor_0(M,J) \to Tor_0(M,K) \to 0.$$

Definition 2.0.58. *Let* S *be a ring. Let* $\mathcal{F} : \cdots \to F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \to \cdots \xrightarrow{\phi_1} F_0$ *be a free minimal resolution of a module* M. *If rank* $F_i = b_i$*, then* $Tor_i(S/\mathfrak{m}, M) = (S/\mathfrak{m})^{b_i}$. *The* b_i *are the* ***Betti numbers*** *of* M.

We will denote the graded Betti numbers of a graded module M over S by $\beta(M)_{ij}$. The following definition is based on the resolution given in Proposition 2.0.56.

Definition 2.0.59. *Set* $F_j = \oplus_{i=1}^{c_j} S^{b_{ji}}(-d_{ji})$ *where* $\oplus_{i=1}^{c_j} S^{b_{ji}}(-d_{ji})$ *is a component from Proposition 2.0.56. The* ***jth Betti number*** *of* I *is rank* F_j. *The* b_{ji} *are the* ***graded Betti numbers****.*

The notations b_{ji} or $b_{j,i}$ will be useful when reading the proof of Lemma 4.0.64.

Example 2.0.60. *Consider the ring* $S = \mathbb{R}[x,y]$ *and the ideal* $I = (y^2, x^2y, x^4)$. *Then we have the complex* $0 \to S^{b_{2,1}} \oplus S^{b_{2,2}} \xrightarrow{\phi_2} S^{b_{1,1}} \oplus S^{b_{1,2}} \oplus S^{b_{1,3}} \xrightarrow{\phi_1} S \to S/I \to 0$.

Where $\phi_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *and* $\phi_2 = (y^2, x^y, x^4)$. *The Betti numbers are:* $b_{1,1} = 2$, $b_{1,2} = 3$, $b_{1,3} = 4$, $b_{2,1} = 4$, *and* $b_{2,2} = 5$.

We are now finished with the background material and are ready to delve into the first few results of Herzog and Hibi's (4).

Chapter 3

LIMIT BEHAVIOR OF DEPTH S/I^K

We may now begin to discuss the limit depth of an ideal, about which very little is known. This section coincides with the first section of (4). The following lemma will apply to the proof of Theorem 3.0.62.

Lemma 3.0.61. *Let A be a Noetherian local ring, and let M be a finitely generated graded A-module. Then* $\dim M = 0$ *implies* $M_k = 0 \; \forall k >> 0$.

Proof. By the definition of dimension, $\dim M = 0 \implies \dim(A/annM) = 0$. Since A is local, let $\mathfrak{m}$ be the unique maximal ideal of A. Notice that since $annM$ is an ideal of A, $annM \subseteq \mathfrak{m}$. If $annM = A$, then $A \subseteq \mathfrak{m}$ But $\mathfrak{m}$ is proper by definition so $annM \neq A$.

Now since A is commutative with $1 \neq 0$, $\mathfrak{m}$ is prime. Then $\mathfrak{m}/annM$ is prime in $A/annM$.

Claim: $\mathfrak{m}$ is a minimal prime of A containing $annM$:

Suppose not. Then there exists a prime ideal in A such that $annM \subsetneq P \subsetneq \mathfrak{m}$. Then $P/annM$ is a prime ideal in $A/annM$ such that $annM/annM \subsetneq P/annM \subsetneq \mathfrak{m}/annM$. Or $0 \subsetneq P/annM \subsetneq \mathfrak{m}/annM$. This implies $ht(\mathfrak{m}/annM) \neq 0$ which implies $ht(\mathfrak{m}/annM) > 0$. Then $\dim(A/annM) > 0$. But this contradicts $\dim(A/annM) = 0$. Thus $\mathfrak{m}$ is a minimal prime of A containing $annM$, and this concludes the claim.

Now by Proposition 2.0.21, $\bigcap_{P \supseteq annM} P = \sqrt{annM}$ where each P is a prime ideal. We know $\mathfrak{m}$ is prime, and we know $\mathfrak{m} \supseteq annM$ since $\mathfrak{m}$ is he maximal ideal of

A. By the claim above, $\mathfrak{m}$ is also a minimal prime containing $annM$. So $\bigcap_{P \supseteq annM} P = \mathfrak{m}$.

Now $\mathfrak{m} = \sqrt{annM} = \{x \in A | x^n \in annM$ for some $n \in \mathbb{N}\}$. Since A is Noetherian, $\mathfrak{m}$ is finitely generated. Then there exists $t \in \mathbb{N}$ such that $\mathfrak{m}^t \subseteq annM$.

We know M is finitely generated. Let $Y \subseteq M$ such that $Y = y_1, ..., y_m$, and $M = (y_1, ..., y_m)$. Let c be the greatest degree of any generator y_i. A is commutative with identity implies $M = AY = \{\sum_{i=1}^{s} a_i y_i | a_i \in A; y_i \in Y; s \in \mathbb{N}\}$. Now consider the graded component M_{c+t}. Let $b \in M_{c+t}$. Then deg$b = c + t$, and b is a homogeneous element. So $b = \sum_{i=1}^{s} a_i y_i$. Now since c is the highest degree of any y_i and b is homogeneous implies each $a_j y_i$ has degree $c + t$ then each a_i has degree at least t. Then each $a_i \in \mathfrak{m} \subseteq$ ann$M$. Then each $a_i y_i = 0$. Then $b = 0$. Then $M_{c+t} = 0$.

□

We are now ready to provide a more detailed proof of the first major result in "The Depth of Powers of Ideals." The results of Theorem 3.0.62 and Theorem 3.0.63 were previously and collectively shown by Burch, Brodmann, Eisenbud, and Huneke.

Theorem 3.0.62. *(4, Theorem 1.1) Let A be a finitely generated, standard graded S-algebra, and let E be a finitely generated graded A-module. Recall Proposition 2.0.34, then each graded component E_k of E is a finitely generated S-module. The depth of E_k is constant for $k >> 0$, and hence $lim_{k\to\infty}$*depth E_k *exists. Moreover, $lim_{k\to\infty}$*depth $E_k \leq$ dim $E -$ dim $E/\mathfrak{m} E$, *and if E is Cohen-Macaulay* (depth$E =$ dim$E$ *or $E = (0)$), equality holds.*

Proof. Let $x_1, ..., x_n$ be a minimal set of generators of $\mathfrak{m}$. Note that by a remark from section 2.3, we may take any minimal set of generators of $\mathfrak{m}$ when discussing the Koszul Complex.

Recall that $H(\underline{x}, E)$ denotes the Koszul homology of the module E with respect to the sequence $\underline{x}$. Then $H_\imath(\underline{x}, E)$ is the $\imath$th graded homology module, with componenets $H_\imath(\underline{x}, E)_k$.

Now recall that by Definition 2.0.50, the grade of a module with respect to an ideal I can be determined by the Koszul homology modules. If the ring S is local, the grade of an S-module with respect to $\mathfrak{m}$ is the **depth** of the module. We have that $\mathfrak{m}$ is an ideal in S generated by $x_1, ..., x_n$, and E_k is an S-module. We must show that there exists i such that $H_\imath(x, E_k) \neq 0$.

Since $\mathfrak{m}$ is the unique maximal ideal of S, $\mathfrak{m}$ is contained in every maximal ideal of S. Then by Nakayama's Lemma (Lemma 2.0.1), $\mathfrak{m}\, E_k = E_k$ implies $E_k = 0$. But we know $E_k \neq 0$, so $\mathfrak{m}\, E_k \neq E_k$. Then $E_k / \mathfrak{m}\, E_k \neq 0$.

By a remark from section 2.3 , $H_0(\underline{x}, E_k) \cong E_k / \mathfrak{m}\, E_k$. So for $i = 0$, $H_\imath(\underline{x}, E_k) \neq 0$. Thus depth $E_k = n - \max\{i : H_\imath(x; E_k) \neq 0\}$

Now let $c = \max\{i | \dim H_\imath(\underline{x}, E) > 0\}$. If $i > c$, $\dim H_\imath(\underline{x}; E) = 0$. Then by Lemma 3.0.61, for all $\imath > c$, and $k >> 0$, $H_\imath(\underline{x}; E_k) = 0$. Also by Lemma 3.0.61 $\dim H_c(\underline{x}; H) > 0 \implies$ for all $t > 0$, there exists $k > t$, such that $H_c(\underline{x}, E_k) \neq 0$.

Then for $k >> 0$, $c = \max\{\imath | H_\imath(\underline{x}, E_k) \neq 0\}$. Then for $k >> 0$, depth $E_k = n - c$. Thus $\lim_{k \to \infty}$depth $E_k = n - c$.

Since E is finitely generated, we may assume that $E_0 = 0$ after a shift in the grading. By Definition 2.0.25, $E^{(r)} = \oplus_\imath E_{ir}$.

By Lemma 2.0.28, dim $E^{(r)}$ =dim E. Since $(\mathfrak{m}\, E)^{(r)} = \mathfrak{m}\, E^{(r)}$, $\dim E / \mathfrak{m}\, E = \dim E^{(r)} / \mathfrak{m}\, E^{(r)}$. Since $E_k^{(r)} = E_{kr}$ by definition, then depth $(E_k^{(r)}) =$ depth (E_{kr}). Note that since $E = E_0 \oplus E_1 \oplus E_2 \oplus \cdots$, $\lim_{k \to \infty}$ depth $E_k =$ $\lim_{k \to \infty}$ depth E_{kr} .

Recall that in general, depth $E \leq$ dim E (7, Proposition 1.2.12). So depth $E \leq$ dim $E-$ dim $E/\mathfrak{m}\, E$.

Then depth $E^{(r)} \leq \dim E^{(r)} - \dim E^{(r)}/\mathfrak{m}\, E^{(r)}$. By the above, depth $E^{(r)} \leq \dim E - \dim E/\mathfrak{m}\, E$.

Claim: There exists $r \in \mathbb{N}$ such that depth $E^{(r)} = lim_{k\to\infty}$depth E_k.

Since $\lim_{k\to\infty}$ depth E_k exists, then there exists $t \in \mathbb{N}$ such that for all $k > t$, depth E_k =depth E_t. Consider $E^{(t)}$.

Since for all $k > t$, depth E_k =depth E_t, and for all $k \in \mathbb{N}$, $kt > t$, then depth E_{kt} = depth E_t. Then for all $k \in \mathbb{N}$, $n-$ $\max\{\imath | H_\imath(x; E_t) \neq 0\} = n-\max\{\imath | H_\imath(x; E_{kt}) \neq 0\}$. So for all $k \in \mathbb{N}$, $\max\{\imath | H_\imath(x; E_t) \neq 0\}$ $=\max\{\imath | H_\imath(x; E_{kt}) \neq 0\}$ $=\max\{\imath | H_\imath(x; E_k^{(t)}) \neq 0\}$. Say $\max\{\imath | H_\imath(x; E_t) \neq 0\} = s$. Then $H_s(x; E_t) \neq 0$, and for all $k \in \mathbb{N}$, $H_s(x; E_{kt}) \neq 0$. Then $\oplus_k H_s(x; E_k^{(t)}) \neq 0$. Then $H_s(x; E^{(t)}) \neq 0$. On the other hand, s $=\max\{\imath | H_\imath(x; E_t) \neq 0\}$. So for all $i > s$, $H_\imath(x; E_t) = 0$, and for all $\imath > s$, $H_\imath(x; E_{kt}) = 0$. We assumed $E_0 = 0$ so for all $i > s$, $H_\imath(x; E^{(t)}) = 0$. So $\max\{\imath | H_\imath(x; E^{(t)}) \neq 0\} = s$. So depth $E^{(t)} = n - s$. Now $\lim_{k\to\infty}$depth $E_{tk} = n-\max\{\imath | H_\imath(x; E_{tk}) \neq 0\}$. Since $k \in \mathbb{N}$, $\max\{\imath | H_\imath(x; E_{tk}) \neq 0\} = s$. So $\lim_{k\to\infty}$depth $E_{tk} = n - s$. Thus, depth $E^{(t)}$ $=\lim_{k\to\infty} E_{tk} = \lim_{k\to\infty} E_k$. Thus there exists $r \in \mathbb{N}$ such that depth $E^{(r)} = lim_{k\to\infty}$depth E_k, completing the claim.

Now depth $E^{(r)}$ $\leq$dim $E-$dim $E/\mathfrak{m}\, E$ gives $\lim_{k\to\infty}$depth E_k $\leq$dim $E-$dim $E/\mathfrak{m}\, E$. If E is Cohen-Macaulay, then by (7, Theorem 2.12) , $\lim_{k\to\infty} E_k$ =dim $E-$dim $E/\mathfrak{m}\, E$.

□

The following theorem is the next major result of (4). Theorem 3.0.62 is an application of Theorem 3.0.63.

Theorem 3.0.63. *(4, Theorem 1.2) Let A be a finitely generated, standard graded S-algebra, and let E be a finitely generated graded A-module. Then each graded component E_k of E is a finitely generated S-module. The limits $lim_{k\to\infty}depth\ I^k$, $lim_{k\to\infty}depth\ S/I^k$, and $lim_{k\to\infty}depth\ I^k/I^{k+1}$ exist, and $lim_{k\to\infty}depth$ $S/I^k \leq lim_{k\to\infty}depth\ I^k - 1 = lim_{k\to\infty}depth\ I^k/I^{k+1} \leq \dim S - \ell(I)$. If S is Cohen-Macaulay and height $I > 0$, then $lim_{k\to\infty}$ depth $S/I^k = lim_{k\to\infty}depth\ I^k - 1$. If S is Cohen -Macaulay and the associated graded ring $gr_1(S)$ is Cohen-Macaulay, then all limits are equal to* $\dim S - \ell(I)$.

Proof. Recall I is a proper ideal of S, and $\mathcal{R}(I)$ is the Rees ring of I. Note $\mathcal{R}(I)$ is a finitely generated graded A-module. Then each $I^k t^k$ is a finitely generated S-module. Then by Theorem 3.0.62, $\lim_{k\to\infty}$depth $I^k t^k$ exists. Thus $\lim_{k\to\infty}$depth I^k exists.

Now consider the associated graded ring $gr_I(S)$. $gr_I(S)$ a finitely generated graded A-module Then each I^k/I^{k+1} is a finitely generated S-module Then by 3.0.62, $\lim_{k\to\infty}$ depth I^k/I^{k+1} exists.

Now we will show that $\lim_{k\to\infty}$ depth $I^k/I^{k+1} \leq$dim $S - \ell(I)$. By definition, $\ell(I)$ is the analytic spread of I, and $\ell(I) =$dim $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. Also, dim $gr_I(S) =$dim(S) (7, Theorem 4.4.6). By a remark from section 2.2, dim $gr_I(S)/\mathfrak{m}\, gr_I(S) =$dim $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. Note that the kth component of $gr_I(S)$ is I^k/I^{k+1}.

Now if we substitute $gr_I(S)$ for E in Theorem 3.0.62, then we have

$\lim_{k\to\infty}$depth $E_k \leq$dim $E-$dim $E/\mathfrak{m}E \implies$

$\lim_{k\to\infty}$depth $[gr_I(S)]_k \leq$dim $gr_I(S)-$dim $gr_I(S)/\mathfrak{m}\, gr_I(S) \implies$

$\lim_{k\to\infty}$depth $I^k/I^{k+1} \leq$dim $gr_I(S)-$dim $gr_I(S)/\mathfrak{m}\, gr_I(S) \implies$

$\lim_{k\to\infty}$depth $I^k/I^{k+1} \leq$dim$(S)-$dim $gr_I(S)/\mathfrak{m}\, gr_I(S) \implies$

$\lim_{k\to\infty}$depth $I^k/I^{k+1} \leq$dim$(S)-$dim $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) \implies$

$\lim_{k\to\infty}$depth $I^k/I^{k+1} \leq$dim$(S) - \ell(I)$.

Now we will show that $\lim_{k\to\infty}$depth S/I^k exists. We know $\mathfrak{m} \subset S$ is an ideal, and for all $k \in \mathbb{N}$, I^k/I^{k+1} and S/I^k are S-modules. Consider the short exact sequence $0 \to I^k/I^{k+1} \to S/I^{k+1} \to S/I^k \to 0$ of S-modules homomorphisms. Since grade$(\mathfrak{m}, S/I^{k+1}) =$ depth (S/I^{k+1}) and grade$(\mathfrak{m}, I^k/I^{k+1})$ =depth (I^k/I^{k+1}), then by the Depth Lemma,

depth $(S/I^{k+1}) \geq$min{depth (I^k/I^{k+1}),depth (S/I^k)} and

depth $(I^k/I^{k+1}) \geq$min{depth (S/I^{k+1}),depth $(S/I^k) + 1$}.

Now since we know $\lim_{k\to\infty}$depth I^k/I^{k+1} exists, then there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$, we have

(i) depth $(S/I^{k+1}) \geq$min{$\lim_{k\to\infty}$depth (I^k/I^{k+1}),depth (S/I^k)}, and

(ii) $\lim_{k\to\infty}$depth $(I^k/I^{k+1}) \geq$min{depth (S/I^{k+1}),depth $(S/I^k) + 1$}.

We will prove that $\lim_{k\to\infty}$depth (S/I^k) exists using two cases.

Case one: Assume there exists $k_t \geq k_0$ such that depth $(S/I^{k_t}) \geq \lim_{k\to\infty}$depth (I^k/I^{k+1}).

Since depth $(S/I^{k_t}) + 1 \geq$depth $(S/I^{k_t}) \geq \lim_{k\to\infty}$depth (I^k/I^{k+1}), then (ii) gives

min{depth (S/I^{k_t+1}),depth $(S/I^{k_t}) + 1$} =depth (S/I^{k_t+1}).

Then $\lim_{k\to\infty}$depth $(I^k/I^{k+1}) \geq$depth (S/I^{k_t+1}). Since depth $(S/I^{k_t}) \geq \lim_{k\to\infty}$depth (I^k/I^{k+1}), min{$\lim_{k\to\infty}$depth (I^k/I^{k+1}),depth (S/I^{k_t})} =$\lim_{k\to\infty}$depth (I^k/I^{k+1}).

By (i), depth $(S/I^{k_t+1}) \geq \lim_{k\to\infty}$depth (I^k/I^{k+1}).

Then $\lim_{k\to\infty}$depth $(I^k/I^{k+1}) \geq$ depth (S/I^{k_t+1}) and $\lim_{k\to\infty}$depth $(I^k/I^{k+1}) \leq$ depth (S/I^{k_t+1}) imply $\lim_{k\to\infty}$depth (I^k/I^{k+1}) =depth (S/I^{k_t+1}).

Now we have shown depth $(S/I^{k_t}) \geq \lim_{k\to\infty}$depth (I^k/I^{k+1}) implies

$\lim_{k\to\infty}$depth (I^k/I^{k+1}) = depth (S/I^{k_t+1}). Also, we have depth $(S/I^{k_t}) \geq \lim_{k\to\infty}$depth (I^k/I^{k+1}).

Thus $\lim_{k\to\infty}$depth $(I^k/I^{k+1}) \geq$depth (S/I^{k_t+1}). Then $\lim_{k\to\infty}$depth $(I^k/I^{k+1}) \geq$ depth (S/I^{k_t+1}) implies $\lim_{k\to\infty}$depth (I^k/I^{k+1}) =depth (S/I^{k_t+2}).

By induction, for all $k \geq k_t$, $\lim_{k\to\infty}$depth (I^k/I^{k+1}) =depth (S/I^{k_t}).

Case 2: Assume for all $k \geq k_0$, depth $(S/I^k) \leq \lim_{k\to\infty}$depth (I^k/I^{k+1}). Then for all $k \geq k_0$, min$\{\lim_{k\to\infty}$depth (I^k/I^{k+1}),depth $(S/I^k)\}$ =depth (S/I^k). Then by (i), for all $k \geq k_0$, depth $(S/I^{k+1}) \geq$depth (S/I^k). Then $\{$depth $(S/I^k)\}_k$ is a non-decreasing sequence for all $k \geq k_0$.

Since for all $k \geq k_0$, depth $(S/I^k) \leq \lim_{k\to\infty}$depth (I^k/I^{k+1}) (as we assumed), $\{$depth $(S/I^k)\}_k$ is bounded above by $\lim_{k\to\infty}$depth (I^k/I^{k+1}) for all $k \geq k_0$.

Then the fact that $\{$depth $(S/I^k)\}_k$ is a non-decreasing sequence for all $k \geq k_0$ together with the fact that $\{$depth $(S/I^k)\}_k$ is bounded above by $\lim_{k\to\infty}$depth (I^k/I^{k+1}) for all $k \geq k_0$ implies $\lim_{k\to\infty}$depth (S/I^k) exists and $\lim_{k\to\infty}$depth $(S/I^k) \leq \lim_{k\to\infty}$depth (I^k/I^{k+1}).

Thus in both cases, $\lim_{k\to\infty}$ depth (S/I^k) exists and $\lim_{k\to\infty}$ depth $(S/I^k) \leq$ $lim_{k\to\infty}$ depth (I^k/I^{k+1}).

Now we will show $\lim_{k\to\infty}$depth $I^k - 1 = \lim_{k\to\infty}$depth I^k/I^{k+1}. For all $k \in \mathbb{N}$, $0 \to I^{k+1} \to I^k \to I^k/I^{k+1} \to 0$ is a short exact sequence. Then for all $k \in \mathbb{N}$, the Depth Lemma gives depth $(I^k/I^{k+1}) \geq$ min$\{$depth $(I^{k+1}) - 1$,depth $(I^k)\}$.

Since we know $\lim_{k\to\infty}$depth (I^k) and $\lim_{k\to\infty}$depth (I^k/I^{k+1}) exist, then we have $\lim_{k\to\infty}$ depth $(I^k/I^{k+1}) \geq$ min$\{$ $\lim_{k\to\infty}$ depth $(I^{k+1}) - 1$, $\lim_{k\to\infty}$ depth $(I^k)\}$. Now of course $\lim_{k\to\infty}$ depth $(I^{k+1}) = \lim_{k\to\infty}$ depth (I^k). So $\lim_{k\to\infty}$ depth $(I^k/I^{k+1}) \geq \lim_{k\to\infty}$ depth $(I^{k+1}) - 1$. Then $\lim_{k\to\infty}$depth $(I^k/I^{k+1}) \geq \lim_{k\to\infty}$ depth $(I^k) - 1$.

We will now adopt a new notation in order to identify particular homology modules. The ability to identify these particular homology modules will help us show that

$lim_{k\to\infty}$depth $(I^k/I^{k+1}) \leq \lim_{k\to\infty}$ depth $(I^k) - 1$. Let n be the minimal number of generators of m. Let $g = \lim_{k\to\infty}$depth I^k.

Suppose $\lim_{k\to\infty}$depth $(I^k/I^{k+1}) > \lim_{k\to\infty}$depth $(I^k) - 1$.

Then

$\lim_{k\to\infty}(n - \min\{i|H_i(\underline{x}; I^k/I^{k+1}) \neq 0\}) > \lim_{k\to\infty}(n - \min\{i|H_i(\underline{x}; I^k) \neq 0\}) - 1$.

Then

$n - \lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k/I^{k+1}) \neq 0\}) > n - \lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k) \neq 0\}) - 1$.

Then $-n + \lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k/I^{k+1}) \neq 0\}) < -n + \lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k) \neq 0\}) + 1$. Then

$\lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k/I^{k+1}) \neq 0\}) < \lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k) \neq 0\}) + 1$.

Now observe $n - g = n - \lim_{k\to\infty}$depth $I^k = n - \lim_{k\to\infty} n - \min\{i|H_i(\underline{x}; I^k) \neq 0\}$

$= lim_{k\to\infty}\min\{i|H_i(\underline{x}; I^k) \neq 0\}$. So there exists $k_1 \in \mathbb{N}$ such that for all $k > k_1$,

$H_{n-g}(\underline{x}; I^k) \neq 0$ and $H_{n-g+1}(\underline{x}; I^k) = 0$.

Also observe $n - g + 1 = \lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k) \neq 0\}) + 1$. From above

$\lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k/I^{k+1}) \neq 0\}) < \lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k) \neq 0\}) + 1$ which can be written as $\lim_{k\to\infty}(\min\{i|H_i(\underline{x}; I^k/I^{k+1}) \neq 0\}) < n - g + 1$.

Then there exists $k_0 \geq k_1$ such that for all $k \geq k_0$, $H_{n-g+1}(\underline{x}; I^k/I^{k+1}) = 0$ and

$H_{n-g}(\underline{x}; I^k) \neq 0$.

We again consider the short exact sequence $0 \to I^{k+1} \to I^k \to I^k/I^{k+1} \to 0$.

Recall that each of the modules in this sequence is graded, that is $I^k = \oplus_n I^k_n$.

Thus we have a short exact sequence of complexes. (See the next page.)

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & (I^{k+1})_{n-g+2} \arrow[r] \arrow[d] & (I^{k+1})_{n-g+1} \arrow[r] \arrow[d] & (I^{k+1})_{n-g} \arrow[r] \arrow[d] & (I^{k+1})_{n-g-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & (I^{k})_{n-g+2} \arrow[r] \arrow[d] & (I^{k})_{n-g+1} \arrow[r] \arrow[d] & (I^{k})_{n-g} \arrow[r] \arrow[d] & (I^{k})_{n-g-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & (I^k/I^{k+1})_{n-g+2} \arrow[r] \arrow[d] & (I^k/I^{k+1})_{n-g+1} \arrow[r] \arrow[d] & (I^{k+1}/I^k)_{n-g} \arrow[r] \arrow[d] & (I^{k+1}/I^k)_{n-g-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

Observe the sequence

$\cdots \to H_{n-g+1}(x; I^k/I^{k+1}) \to H_{n-g}(x; I^{k+1}) \to H_{n-g}(x; I^k) \to$

$H_{n-g}(x; I^{k+1}/I^k) \to$

$\mathrm{H}_{n-g-1}(x; I^{k+1}) \cdots$

By Theorem 2.0.43, the sequence

$\cdots \to H_{n-g+1}(x; I^k/I^{k+1}) \to H_{n-g}(x; I^{k+1}) \to H_{n-g}(x; I^k) \to \cdots$ is exact for all k.

Now consider the natural map $H_{n-g}(x; I^{k+1}) \to H_{n-g}(x; I^k)$. Since the sequence

$\cdots \to H_{n-g+1}(x; I^k/I^{k+1}) \to H_{n-g}(x; I^{k+1}) \to H_{n-g}(x; I^k) \to \cdots$ is exact for all k, and since $H_{n-g+1}(x; I^k/I^{k+1}) = 0$ for all $k \geq k_0$, then $H_{n-g}(x; I^{k+1}) \to H_{n-g}(x; I^k)$ is injective for all $k \geq k_0$. Since the composition of injective maps is injective, and since for all $l > k \geq k_0$, $H_{n-g+1}(x; I^l/I^{l+1}) = 0$ then for each $l > k$, $H_{n-g}(x; I^l) \to H_{n-g}(x; I^k)$ is injective for $k \geq k_0$. But the Artin-Rees lemma (14, Theorem 3.4.8) implies that if M is a finitely generated S-module, then the natural homomorphism $H_{n-g}(x; I^l M) \to H_{n-g}(x; M)$ is the zero map for $l >> 0$. This implies that $H_{n-g}(x; I^l) = 0$ for $l >> 0$, or equivalently $H_{n-g}(x; I^k) = 0$ for $k >> 0$. However, $n - g =\lim_{k\to\infty}\min\{\imath | H_\imath(\underline{x}; I^k) \neq 0\}$. Thus we have a contradiction. Therefore, $\lim_{k\to\infty}$depth $(I^k) - 1 \geq \lim_{k\to\infty}$ depth I^k/I^{k+1}. Finally,

$\lim_{k\to\infty}$depth $(I^k) - 1 = \lim_{k\to\infty}$ depth I^k/I^{k+1}. Then we have

$\lim_{k\to\infty}$depth $S/I^k \leq \lim_{k\to\infty}$depth $I^k - 1 = \lim_{k\to\infty}$depth $I^k/I^{k+1} \leq$dim $S - \ell(I)$.

Now assume S is Cohen-Macaulay. Consider the short exact sequence

$0 \to I^k \to S \to S/I^k \to 0$. Since S is Cohen Macaulay, depth $S \geq$ depth $I^k - 1$, and depth $S \geq$ depth $S/I^k + 1$. Then by the Depth Lemma, depth $S/I^k \geq$ depth $I^k - 1$ and depth $S/I^k \leq$ depth $I^k - 1$. So depth $S/I^k =$ depth $I^k - 1$.

Therefore, $\lim_{k\to\infty}$ depth $S/I^k = \lim_{k\to\infty}$ depth $I^k - 1$.

Now if we assume $\mathrm{gr}_I(S)$ is Cohen-Macaulay and substitute $\mathrm{gr}_I(S)$ for E as we did in the beginning of this proof, then by Theorem 3.0.62, $\lim_{k\to\infty}$ depth $[\mathrm{gr}_I(S)]_k = \dim \mathrm{gr}_I(S) - \dim \mathrm{gr}_I(S)/\mathfrak{m}\mathrm{gr}_I(S)$. As before, this implies $\lim_{k\to\infty}$depth $I^k/I^{k+1} = \dim(S) - \ell(I)$.

Finally, if we have that S is CM and $\mathrm{gr}_I(S)$ is CM then $\lim_{k\to\infty}$depth S/I^k $=\lim_{k\to\infty}$depth $I^k - 1$ $=\lim_{k\to\infty}$depth I^k/I^{k+1} $=\dim S - \ell(I)$.

$\square$

Chapter 4

THE INITIAL BEHAVIOR OF DEPTH S/I^K

This section corresponds with the second section of (4). The following theorem is the first proven result of their second section.

The least degree of the homogeneous generators of a module M will be referred to as the $initial\ degree\ of\ M$.

Lemma 4.0.64. *(4, Lemma 2.2) Let $S = K[x_1, ..., x_n]$ be a polynomial ring over the field K such that for each $i \in \{1, ..., n\}$, $degx_i = 1$. Let $J \subseteq I$ be graded ideals of S, and let d be the initial degree of I. Then $\beta_{i,i+d}(J) \leq \beta_{i,i+d}(I)\ \forall i$.*

Proof. By a Remark from section 2.4, the short exact sequence
$0 \to J \to I \to I/J \to 0$ yields the long exact sequence
$\cdots \to Tor_{i+1}(K, I/J) \to Tor_i(K, J) \to Tor_i(K, I) \to \cdots$.

Recall Proposition 2.0.56. Since S is a polynomial ring over a field k with the usual grading and J is a graded ideal of I, the minimal graded resolution of I/J by free S-modules can be expressed as

$$0 \to \oplus_{m=1}^{c_g} S^{b_{gm}}(-a_{gm}) \xrightarrow{\phi_g} \cdots \to \oplus_{m=1}^{c_1} S^{b_{1m}}(-a_{1m}) \xrightarrow{\phi_1} \oplus_{m=1}^{c_0} S^{b_{0m}} \to I/J,$$

where g is the projective dimension of I/J. Now $Tor_{i+1}(K, I/J)$ is the homology module at $i+1$ of the sequence

$$\cdots \to \oplus_{m=1}^{c_i+2} S^{b_{i+2,m}}(-a_{i+2,m}) \otimes K \xrightarrow{\phi_{i+2}} \oplus_{m=1}^{c_i+1} S^{b_{i+1,m}}(-a_{i+1,m}) \otimes K \xrightarrow{\phi_{i+1}}$$
$$\oplus_{m=1}^{c_i} S^{b_{i,m}}(-a_{i,m}) \otimes K \to \cdots.$$

So $Tor_{i+1}(K, I/J) = \mathrm{ker}\phi_{i+1}/\mathrm{im}\phi_{i+2}$.

Now recall Definitions 2.0.57 and 2.0.58. Let $\mathcal{F} : \cdots \to F_i \xrightarrow{\psi_i} \cdots \to F_1 \xrightarrow{\psi_1} F_0$ be a free minimal resolution of the S-module J. Then $Tor_i(K, J)$ is the ith homology module of the sequence $\cdots \to F_{i+1} \otimes K \xrightarrow{\psi_{i+1}} F_i \otimes K \xrightarrow{\psi_i} F_{i-1} \otimes K \to \cdots$. So $Tor_i(K, J) = \ker\psi_i / \mathrm{im}\psi_{i+1}$.

Similarly, let $\mathcal{G} : \cdots \to G_i \xrightarrow{\sigma_i} \cdots \to G_1 \xrightarrow{\sigma_1} G_0$ be a free minimal resolution of the S-module I. Then $Tor_i(K, I)$ is the ith homology module of the sequence $\cdots \to G_{i+1} \otimes K \xrightarrow{\sigma_{i+1}} G_i \otimes K \xrightarrow{\sigma_i} G_{i-1} \otimes K \to \cdots$. So $Tor_i(K, I) = \ker\sigma_i / \mathrm{im}\sigma_{i+1}$.

Since ϕ_{i+1}, ψ_i, and σ_i are homogeneous maps, we have the long exact sequence $\cdots \to Tor_{i+1}(K, I/J)_{i+d} \to Tor_i(K, J)_{i+d} \to Tor_i(K, I)_{i+d} \to \cdots$. Or we can write $\cdots \to Tor_{i+1}(K, I/J)_{i+1+(d-1)} \to Tor_i(K, J)_{i+d} \to Tor_i(K, I)_{i+d} \to \cdots$.

Now let d be the initial degree of I, and let q be the initial degree of I/J. If $\{d_1, d_2, ..., d_n\}$ is the generating set of homogeneous generators of I, then $\{d_1 + J, d_2 + J, ..., d_n + J\}$ is sufficient for the set of homogeneous generators of I/J. Thus $q \geq d$.

Note that

$Tor_{i+1}(K, I/J) = \ker\phi_{i+1} / \mathrm{im}\phi_{i+2}$

$\subseteq \oplus_{m=1}^{c_{i+1}} S^{b_{i+1,m}}(-a_{i+1,m}) \otimes K /$

$\mathrm{im}(\oplus_{m=1}^{c_{i+2}} S^{b_{i+2,m}}(-a_{i+2,m}) \otimes K \to \oplus_{m=1}^{c_{i+1}} S^{b_{i+1,m}}(-a_{i+1,m}) \otimes K)$.

Now we are interested in the component $m = 1 + i + (d-1)$. So we are interested in

$S^{b_{i+1,i+1+(d-1)}}(-a_{i+1,i+1+(d-1)}) \otimes K$. Since we have a free minimal resolution, we have $a_{i+1} \geq a_1 + i \geq i + 1$. So

$a_{i+1} \geq i+1 \implies -a_{i+1} \leq i+1 \implies -a_{i+1}+i+1+d-1 \leq i+1+i+1+d-1 = d-1$.

By the definition of a shift in the graduation,

$S^{b_{i+1,i+1+d-1}}(-a_{i+1,i+1+d-1}) = S^{b_{i+1,i+1+d-1}}_{-a_{i+1}+i+1+d-1} = S^{b_{i+1,i+1+d-1}}_t$ where $t \leq d-1$. Since

d is the initial degree of I and $q \geq d$, t is strictly less than q. So t is strictly less than the least degree of homogeneous generators of I/J. Thus $Tor_{i+1}(K, I/J)_{i+d} = 0$ implying $Tor_i(K, J)_{i+d} \to Tor_i(K, I)_{i+d}$ is injective.

Then $Tor_i(K, J)_{i+d} \to Tor_i(K, I)_{i+d}$ is injective implies $\text{Rank}(F_i)_{i+d} \leq \text{Rank}(G_i)_{i+d}$. Then by the definition of Betti number, $\beta(J)_{i,i+d} \leq \beta(I)_{i,i+d}$.

$\square$

Chapter 5

A CLASS OF EXAMPLES ARISING IN COMBINATORICS

This section of the paper corresponds to section 3 of Herzog and Hibi's paper. They give several classes of examples, whereas we will focus on just one. This example is one of the only known cases where the exact depth of S/I^k can be computed.

Example 5.0.65. *(4, Corollary 3.4) Let d and n be integers such that $2 \leq d < n$. Let $I = I_{n,d}$ be the square-free Veronese ideals of degree d in the variables $x_1, ..., x_n$. That is, the ideal of S generated by all square-free monomials in $x_1, ..., x_n$ of degree d. Then depth S/I^k =max$\{0,\ n - k(n-d) - 1\}$.*

Example 5.0.66. *Set $d = 2$ (this gives us a complete graph). We must have $n > 2$. Let us first consider what happens when $k = 1$. Then depth*
S/I =max$\{0,\ n - (n-2) - 1\}$ =max$\{0,\ 1\} = 1$.
Now consider $k = 2$. Then depth
S/I^2 =max$\{0,\ n - 2(n-2) - 1\}$ =max$\{0,\ -n+3\}$. Since we must have $n > 2$, then in this case ($d = 2$), depth $S/I^2 = 0$ for all n.

Let's consider an example where k varies while d and n remain fixed.

Example 5.0.67. *Fix $d = 2$ and $n = 3$. We already know depth S/I from Example 5.0.66. Let $k = 2$. Then depth*
S/I^2 =max$\{0,\ 3 - 2(3-2) - 1\}$ =max$\{0,\ 3 - 2(3-2) - 1\} = 0$.
Observe that we have depth S/I^k =max$\{0,\ 3 - k(3-2) - 1\}$ =max$\{0,\ 2-k\}$. Thus in this case, depth $S/I^k = 0$ if $k > 1$.

BIBLIOGRAPHY

[1] M. Atiyah, I. MacDonald. 1969. *Introduction to Commutative Algebra.* Massachusetts: Addison Wesley Publishing Company, Inc.

[2] M. Brodmann. 1979. The Asymptotic Nature of the Analytic Spread. *Math. Proc. Cambridge Philos. Soc.* 86: 35-39.

[3] W. Bruns, J. Herzog. 1993. *Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, vol. 39.* Cambridge: Cambridge University Press.

[4] L. Burch. 1972. Codimension and Analytic Spread. *Proc. Cambridge Philos. Soc.* 72: 369-373.

[5] D. Eisenbud. 1996. *Commutative Algebra with a View Toward Algebraic Geometry, Graduate Texts in Mathematics, vol. 150.* New York: Springer.

[6] D. Eisenbud, C. Huneke. 1983. Rees Algebras and Their Specialization. *J. Algebra* 81: 202-224.

[7] A. Hatcher. 2001. *Algebraic Topology*. Cambridge: Cambridge University Press.

[8] J. Herzog, T. Hibi. 2005. The depth of powers of an ideal. *Journal of Algebra* 291: 534-550.

[9] T. Hungerford. 2000. *Algebra, Graduate Texts in Mathematics, vol. 73*. New York: Springer.

[10] H. Matsumura. 1992. *Commutative Ring Theory, Cambridge Studies in Advanced Mathematics, vol. 8*. Cambridge: Cambridge University Press.

[11] S. Morey. 2010. Depths of Powers of the Edge Ideal of a Tree. *Communications in Algebra* 38:11, 4042-4055.

[12] J. Rotman. 1979. *An Introduction to Homological Algebra, Pure and Applied Mathematics, vol. 85*. New York: Academic Press.

[13] W. Vasconcelos. 1994. *London Mathematical Society Lecture Note Series, 195.* Cambridge: Cambridge University Press.

[14] R. Villareal. 2001. *Monomial Algebras, Pure and Applied Mathematics*. New York: Marcel Dekker Inc.

VITA

Heather Elaine Bruch was born in Austin, Texas, on 10 November 1983, the daughter of Debra Bruch and Mark Bruch. After completing her work at Lockhart High School, Lockhart, Texas, in 2002, she entered Saint Edward's University in Austin, Texas. During the summer of 2006, she participated in a research project at the University of Iowa in Iowa City. She received the degree of Bachelor of Arts from St. Edward's in 2007. She taught as a high school math teacher for the Austin Independent School District in Texas. In August 2009, she entered the Graduate College of Texas State.

Permanent Email Address: heatherbruch@gmail.com

This thesis was typed by Heather E. Bruch.