Electronic Journal of Differential Equations, Vol. **2000**(2000), No. 65, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

POSITIVE SOLUTIONS TO A SECOND ORDER MULTI-POINT BOUNDARY-VALUE PROBLEM

Daomin Cao & Ruyun Ma

Abstract. We prove the existence of positive solutions to the boundary-value problem

$$u'' + \lambda a(t) f(u, u') = 0$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)\,,$$

where a is a continuous function that may change sign on $[0,1]$, f is a continuous function with $f(0,0) > 0$, and λ is a samll positive constant. For finding solutions we use the Leray-Schauder fixed point theorem.

1. Introduction

The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [8, 9]. Motivated by the study of Il'in and Moiseev [8, 9], Gupta [4] studied certain three point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors using the Leray-Schauder Continuation Theorem, Nonlinear Alternative of Leray-Schauder, coincidence degree theory or fixed point theorem in cones. We refer the reader to [1-3, 5, 10-12] for some existence results of nonlinear multi-point boundary value problems. Recently, the second author[12] proved the existence of positive solutions for the three-point boundary value problem

$$u'' + b(t) g(u) = 0, \quad t \in (0,1) \tag{1.1}$$
$$u(0) = 0, \quad \alpha u(\eta) = u(1)\,, \tag{1.2}$$

where $\eta \in (0,1)$, $0 < \alpha < \frac{1}{\eta}$, $b \geq 0$, and $g \geq 0$ is either superlinear or sublinear by the simple application of a fixed point theorem in cones.

In this paper, we consider the nonlinear eigenvalue m-point boundary value problem

$$u'' + \lambda a(t) f(u, u') = 0 \tag{1.3}$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \tag{1.4}$$

2000 Mathematics Subject Classifications: 34B10.
Key words: Multi-point boundary value problem, positive solution, fixed point theorem.

Submitted September 18, 2000. Published Ocotber 30, 2000.
R.Ma was supported by the Natural Science Foundation of China (grant 19801028)

where λ is a positive parameter.

We make the following assumptions:

(A1) $a_i \geq 0$ for $i = 1, \cdots, m-3$ and $a_{m-2} > 0$; $\xi_i : 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $\sum_{i=1}^{m-2} a_i\xi_i < 1$.

(A2) $f : [0,\infty) \times R \to R$ is continuous and $f(0,0) > 0$;

(A3) $a \in C[0,1]$ and there exist $r_0 \in [0,1]$ and $\theta > 0$ such that $a(r_0) \neq 0$, and the solution of the linear problem

$$u'' + a^+(t) - (1+\theta)a^-(t) = 0, \quad t \in (0,1)$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$$

is nonnegative in $[0,1]$, where a^+ is the positive part of a and a^- is the negative part of a.

(A4) There exist a constant k in $(1,\infty)$ such that

$$P(t) \geq kQ(t) \tag{1.5}$$

where

$$P(t) = \int_0^t a^+(s)ds + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)a^+(s)ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} + \frac{\int_0^1 (1-s)a^+(s)ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i}$$

and

$$Q(t) = \int_0^t a^-(s)ds + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)a^-(s)ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} + \frac{\int_0^1 (1-s)a^-(s)ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i}$$

Our main result is

Theorem 1. *Let (A1), (A2), (A3), and (A4) hold. Then there exists a positive number λ^* such that (1.3)-(1.4) has at least one positive solution for $0 < \lambda < \lambda^*$.*

The proof of this theorem is based upon the Leray-Schauder fixed point theorem and motivated by [7].

2. Preliminary lemmas

In the sequel we shall denote by I the interval $[0,1]$ of the real line. E will stand for the space of functions $u : I \to R$ such that $u(0) = 0$, $u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$ and u' is continuous on I. We furnish the set E with the norm $|u|_E = \max\{|u|_0, |u'|_0\} = |u'|_0$, where $|u|_0 = \max\{u(t) \mid t \in I\}$. Then E is a Banach space.

To prove Theorem 1, we need the following preliminary results.

Lemma 1 [6]. *Let $a_i \geq 0$ for $i = 1, \cdots, m-2$, and $\sum_{i=1}^{m-2} a_i\xi_i \neq 1$, then for $y \in C(I)$, the problem*

$$u'' + y(t) = 0, \quad t \in (0,1) \tag{2.1}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \tag{2.2}$$

has a unique solution,

$$u(t) = -\int_0^t (t-s)y(s)ds - t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)y(s)ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} + t\frac{\int_0^1 (1-s)y(s)ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i}$$

The following two results extend Lemma 2 and Lemma 3 of [12].

Lemma 2. *Let $a_i \geq 0$ for $i = 1, \cdots, m-2$, and $\sum_{i=1}^{m-2} a_i\xi_i < 1$. If $y \in C(I)$ and $y \geq 0$, then the unique solution u of the problem (2.1)-(2.2) satisfies*

$$u(t) \geq 0, \quad \forall t \in I$$

Proof From the fact that $u''(x) = -y(x) \leq 0$, we know that the graph of $u(t)$ is concave down on I. So, if $u(1) \geq 0$, then the concavity of u together with the boundary condition $u(0) = 0$ implies that $u \geq 0$ for all $t \in I$.

If $u(1) < 0$, then from the concavity of u we know that

$$\frac{u(\xi_i)}{\xi_i} \geq \frac{u(1)}{1}, \quad \text{for } i = 1, \cdots, m-2 \tag{2.3}$$

This implies

$$u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq \sum_{i=1}^{m-2} a_i\xi_i u(1) \tag{2.4}$$

This contradicts the fact that $\sum_{i=1}^{m-2} a_i\xi_i < 1$.

Lemma 3. *Let $a_i \geq 0$ for $i = 1, \cdots, m-3$, $a_{m-2} > 0$, and $\sum_{i=1}^{m-2} a_i\xi_i > 1$. If $y \in C(I)$ and $y(t) \geq 0$ for $t \in I$, then (2.1)-(2.2) has no positive solution.*

Proof Assume that (2.1)-(2.2) has a positive solution u, then $u(\xi_i) > 0$ for $i = 1, \cdots, m-2$, and

$$\begin{aligned} u(1) &= \sum_{i=1}^{m-2} a_i u(\xi_i) = \sum_{i=1}^{m-2} a_i\xi_i \frac{u(\xi_i)}{\xi_i} \\ &\geq \sum_{i=1}^{m-2} a_i\xi_i \frac{u(\bar\xi)}{\bar\xi} > \frac{u(\bar\xi)}{\bar\xi} \end{aligned} \tag{2.5}$$

(where $\bar\xi \in \{\xi_1, \cdots, \xi_{m-2}\}$ satisfies $\frac{u(\bar\xi)}{\bar\xi} = \min\{\frac{u(\xi_i)}{\xi_i} | i = 1, \cdots, m-2\}$). This contradicts the concavity of u.

If $u(1) = 0$, then applying $a_{m-2} > 0$ we know that

$$u(\xi_{m-2}) = 0 \tag{2.6}$$

From the concavity of u, it is easy to see that $u(t) \leq 0$ for all t in I.

In the rest of this paper, we assume that $a_i \geq 0$ for $i = 1, \cdots, m-3$, $a_{m-2} > 0$, and $\sum_{i=1}^{m-2} a_i\xi_i < 1$. We also assume that $f(u,p) = f(0,p)$ for $(u,p) \in (-\infty, 0)$.

Lemma 4. *Let (A1) and (A2) hold. Then for every $0 < \delta < 1$, there exists a positive number $\bar\lambda$ such that, for $0 < \lambda < \bar\lambda$, the problem*

$$u'' + \lambda a^+(t) f(u, u') = 0 \tag{2.7}$$

$$u(0) = 0, \ u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \tag{2.8}$$

has a positive solution $\tilde u_\lambda$ with $|\tilde u_\lambda|_E \to 0$ and $|\tilde u'_\lambda|_0 \to 0$ as $\lambda \to 0$, and

$$\tilde u_\lambda \geq \lambda\delta f(0,0) p(t), \quad t \in I \tag{2.9}$$

where

$$p(t)=-\int_0^t (t-s)a^+(s)ds - t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)a^+(s)ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} + t\frac{\int_0^1 (1-s)a^+(s)ds}{1-\sum_{i=1}^{m-2} a_i\xi_i}$$

Proof. By Lemma 2, we know that $p(t)\geq 0$ for $t\in I$. From Lemma 1, (2.7)-(2.8) is equivalent to the integral equation

$$\begin{aligned}u(t)=&\lambda\Big[-\int_0^t (t-s)a^+(s)f(u(s),u'(s))ds\\ &- t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)a^+(s)f(u(s),u'(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i}\\ &+ t\frac{\int_0^1 (1-s)a^+(s)f(u(s),u'(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i}\Big]\\ &\stackrel{\text{def}}{=} Au(t)\end{aligned}$$

where $u\in C^1(I)$. Further, we have that

$$\begin{aligned}(Au)'(t)=&\lambda\Big[-\int_0^t a^+(s)f(u(s),u'(s))ds\\ &- \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)a^+(s)f(u(s),u'(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i}\\ &+ \frac{\int_0^1 (1-s)a^+(s)f(u(s),u'(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i}\Big]\end{aligned}\tag{2.10}$$

Then $A: C^1(I)\to C^1(I)$ is completely continuous and fixed points of A are solutions of (2.7)-(2.8). We shall apply the Leray-Schauder fixed point theorem to prove A has a fixed point for λ small.

Let $\epsilon>0$ be such that

$$f(u,y)\geq \delta f(0,0),\quad \text{for } (u,y)\in[0,\epsilon]\times[-\epsilon,\epsilon]\tag{2.11}$$

Suppose that

$$\lambda<\frac{\epsilon}{2|P|_0\tilde f(\epsilon)}:=\bar\lambda\tag{2.12}$$

where $\tilde f(r)=\max_{(u,y)\in[0,r]\times[-r,r]} f(u,y)$. By (A2) we know that

$$\lim_{r\to 0^+}\frac{\tilde f(r)}{r}=+\infty.\tag{2.13}$$

It follows that there exists $r_\lambda\in(0,\epsilon)$ such that

$$\frac{\tilde f(r_\lambda)}{r_\lambda}=\frac{1}{2\lambda|P|_0}\tag{2.14}$$

We note that (2.14) implies

$$r_\lambda \to 0, \quad \text{as } \lambda \to 0 \tag{2.15}$$

Now, consider the homotopy equations

$$u = \theta A u, \quad \theta \in (0,1) \tag{2.16}$$

Let $u \in C^1(I)$ and $\theta \in (0,1)$ be such that $u = \theta A u$. We claim that $|u|_E \neq r_\lambda$. In fact,

$$\begin{aligned} u'(t) =& \theta\lambda\Big[-\int_0^t a^+(s) f(u(s),u'(s))ds \\ & - \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)a^+(s)f(u(s),u'(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\ & + \frac{\int_0^1 (1-s)a^+(s)f(u(s),u'(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i}\Big] \end{aligned} \tag{2.17}$$

This implies that

$$|u'(t)| \le \lambda \tilde{f}(|u|_E)P(t), \quad t \in [0,1] \tag{2.18}$$

hence

$$|u|_E \le \lambda |P|_0 \tilde{f}(|u|_E) \tag{2.19}$$

or

$$\frac{\tilde{f}(|u|_E)}{|u|_E} \ge \frac{1}{\lambda |P|_0} \tag{2.20}$$

which implies that $|u|_E \neq r_\lambda$. Thus by Leray-Schauder fixed point theorem, A has a fixed point $\tilde{u}_\lambda$ with

$$|\tilde{u}_\lambda|_E \le r_\lambda < \epsilon \tag{2.21}$$

Moreover, combining (2.21) and (2.11) and using (2.10) and Lemma 2, we have that

$$\tilde{u}_\lambda(t) \ge \lambda\delta f(0,0)p(t), \tag{2.20}$$

for $t \in I$, $\lambda \le \bar{\lambda}$.

3. Proof of the main reuslt

Proof of Theorem 1. Let

$$q(t) = -\int_0^t (t-s)a^-(s)ds - t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)a^-(s)ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} + t\frac{\int_0^1 (1-s)a^-(s)ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \tag{3.1}$$

then from Lemma 2, we know that $q(t) \ge 0$. By (A3) and (A4), there exist positive numbers $c, d \in (0,1)$ such that for $t \in I$,

$$\begin{aligned} q(t)\max\{|f(u,y)| \mid 0 \le u \le c, -c \le y \le c\} &\le dp(t)f(0,0), \\ Q(t)\max\{|f(u,y)| \mid 0 \le u \le c, -c \le y \le c\} &\le dP(t)f(0,0)\,. \end{aligned} \tag{3.2}$$

Fix $\delta \in (d,1)$ and let $\lambda^* > 0$ be such that

$$|\tilde{u}_\lambda|_E + \lambda\delta f(0,0)|P|_0 \le c \tag{3.3}$$

for $\lambda < \lambda^*$, where $\tilde{u}_\lambda$ is given by Lemma 4, and

$$|f(u_1,y_1) - f(u_2,y_2)| \le f(0,0)\big(\frac{\delta - d}{2}\big) \tag{3.4}$$

for $(u_1,y_1),(u_2,y_2) \in [0,c]\times[-c,c]$ with

$$\max\{|u_1-u_2|,|y_1-y_2|\} \le \lambda^*\delta f(0,0)|P|_0.$$

Let $\lambda < \lambda^*$. We look for a solution u_λ of the form $\tilde{u}_\lambda + v_\lambda$. Here v_λ solves

$$v'' + \lambda a^+(t)(f(\tilde{u}_\lambda + v, \tilde{u}'_\lambda + v') - f(\tilde{u}_\lambda,\tilde{u}'_\lambda)) - \lambda a^-(t) f(\tilde{u}_\lambda + v, \tilde{u}'_\lambda + v') = 0 \tag{3.5}$$

$$v(0) = 0,\ v(1) = \sum_{i=1}^{m-2} a_i v(\xi_i) \tag{3.6}$$

For each $w \in C^1(I)$, let $v = T(w)$ be the solution of

$$v'' + \lambda a^+(t)(f(\tilde{u}_\lambda + w, \tilde{u}'_\lambda + w') - f(\tilde{u}_\lambda,\tilde{u}'_\lambda)) - \lambda a^-(t) f(\tilde{u}_\lambda + w, \tilde{u}'_\lambda + w') = 0$$

$$v(0) = 0, quad v(1) = \sum_{i=1}^{m-2} a_i v(\xi_i)$$

Then $T : C^1(I) \to C^1(I)$ is completely continuous.

Let $v \in C^1(I)$ and $\theta \in (0,1)$ be such that $v = \theta T v$. Then we have

$$v'' + \theta\lambda a^+(t)(f(\tilde{u}_\lambda + v, \tilde{u}'_\lambda + v') - f(\tilde{u}_\lambda,\tilde{u}'_\lambda)) - \theta\lambda a^-(t)(f(\tilde{u}_\lambda + v, \tilde{u}'_\lambda + v')) = 0 \tag{3.7}$$

$$v(0) = 0,\ v(1) = \sum_{i=1}^{m-2} a_i v(\xi_i) \tag{3.8}$$

We claim that $|v|_E \ne \lambda\delta f(0,0)|P|_0$. Suppose to the contrary that $|v|_E = \lambda\delta f(0,0)|P|_0$. Then by (3.3), we obtain

$$\begin{gathered} |\tilde{u}_\lambda + v|_E \le |\tilde{u}_\lambda|_E + |v|_E \le c, \\ |\tilde{u}_\lambda + v|_0 \le |\tilde{u}_\lambda|_0 + |v|_0 \le c\,. \end{gathered} \tag{3.9}$$

These inequalities and (3.4) imply

$$|f(\tilde{u}_\lambda + v, \tilde{u}'_\lambda + v') - f(\tilde{u}_\lambda,\tilde{u}'_\lambda)|_0 \le f(0,0)\big(\frac{\delta - d}{2}\big)\,. \tag{3.10}$$

Using (3.10)and (3.2) and applying Lemma 1 and Lemma 2, we have that

$$\begin{aligned} |v(t)| &\leq \lambda\frac{\delta-d}{2}f(0,0)p(t)+\lambda\max\{|f(u,y)|\mid 0\leq u\leq c,-c\leq y\leq c\}q(t) \\ &\leq \lambda\frac{\delta-d}{2}f(0,0)p(t)+\lambda df(0,0)p(t) \\ &= \lambda\frac{\delta+d}{2}f(0,0)p(t), \quad t\in I \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} |v'(t)| &\leq \lambda\frac{\delta-d}{2}f(0,0)P(t)+\lambda\max\{|f(u,y)|\mid 0\leq u\leq c,-c\leq y\leq c\}Q(t) \\ &\leq \lambda\frac{\delta-d}{2}f(0,0)P(t)+\lambda df(0,0)P(t) \\ &= \lambda\frac{\delta+d}{2}f(0,0)P(t), \quad t\in I \end{aligned} \tag{3.12}$$

In particular

$$|v|_E \leq \lambda\frac{\delta+d}{2}f(0,0)|P|_0 < \lambda\delta f(0,0)|P|_0 \tag{3.13}$$

a contradiction, and the claim is proved. Thus by Leray-Schauder fixed point theorem, T has a fixed ponit v_λ with

$$|v_\lambda|_E \leq \lambda\delta f(0,0)|P|_0 \tag{3.14}$$

Finally, using (2.9) and (3.11), we obtain

$$\begin{aligned} u_\lambda &\geq \tilde{u}_\lambda - |v_\lambda| \\ &\geq \lambda\delta f(0,0)p(t)-\lambda\frac{\delta+d}{2}f(0,0)p(t) \\ &= \lambda\frac{\delta-d}{2}f(0,0)p(t), \quad t\in I \end{aligned} \tag{3.15}$$

i.e., u_λ is a positive solution of (1.3)-(1.4).

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