SUM INTEGRALS OF INTERVAL FUNCTIONS

THESIS

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P R E F A C E

The purpose of this paper is to present proofs of theorems showing the existence of sum integrals of functions from real number intervals to real numbers. This is a generalization of the Riemann-Stieltjes Integral used in most elementary and advanced calculus books. Extensive work concerning this type of integral has been done by W. D. L. Appling.

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CHAPTER I

DEFINITIONS AND NOTATIONS

Throughout this paper $[a,b]$ will denote a closed interval for which $a < b$ and (a,b) belongs to SXS, where S is the set of real numbers.

<u>Definition 1.1</u>

The statement that $D = \{x_i\}_{i=0}^{n}$ is a subdivision of $[a,b]$ means D is a finite subset of S such that $a = x_0$, $b = x_n$ and $x_{i-1} < x_i$, where $0 < i \leq n$.

<u>Definition 1.2</u>

The statement that D' is a refinement of a subdivision D of $[a,b]$ means D' is a subdivision of $[a,b]$ and D is a subset of D'.

<u>Definition 1.3</u>

The statement that $\int_a^b H$ exists, where (a,b) belongs to SXS and H is a function from SXS to S, means there exists a number J of S such that if ϵ is a positive number, then there exists a subdivision D of [a,b] such that if $D' = \{x_i\}_{i=0}^{n}$ is a refinement of D, then

$$\left| \sum_{\substack{i=1 \\ D'}}^{n} H(x_{i-1},x_i) - J \right| < \epsilon .$$

<u>Definition 1.4</u>

The statement that I = [x,y] is a subinterval of [a,b] means [x,y] is a closed interval, (x,y), belongs to SXS and $a \leq x < y \leq b$.

<u>Definition 1.5</u>

If f is a function from S to S, then:

(1) f is nondecreasing on [a,b] means, if $a \leq p < q \leq b$, then $f(p) \leq f(q)$.

(2) f is increasing on [a,b] means, if $a \leq p < q \leq b$, then $f(p) < f(q)$.

<u>Definition 1.6</u>

The statement that the number set R has a least upper bound means there is a number M such that

(1) if x belongs to R, then $x \leq M$, and

(2) if $p < M$, then there exists an element x of R such that $p < x$.

Definition 1.7

The statement that the number set R has a greatest lower bound means there is a number M such that

(1) if x belongs to R, then $x \geq M$, and

(2) if $p > M$, then there exists an element x of R such that $p > x$.

Notations

If H is a function from SXS to S, f is a function from S to S, (a,b) belongs to SXS, and $D = \{x_i\}_{i=0}^{n}$ is a subdivision of [a,b], then, where no misunderstanding is likely, the following notations will be used:

(1) lower case letters will be used to denote functions from S to S, and capital letters to denote functions from SXS to S, or functions from number intervals to S;

(2) $H_i = H(x_{i-1}, x_i)$;

(3) $\sum_D H_i = \sum_{i=1}^{n} H_i$;

(4) $\int_i H = \int_{x_{i-1}}^{x_i} H$;

(5) if $I = [x,y]$ is a subinterval of $[a,b]$, then

$$\int_I H = \int_x^y H;$$

(6) if $I = [x,y]$ is a subinterval of $[a,b]$, then $H(I) = H(x,y)$;

(7) if $I = [x,y]$ is a subinterval of $[a,b]$, then $\Delta f = f(y) - f(x)$;

(8) $\Delta f_i = f(x_i) - f(x_{i-1})$;

(9) if x, y belong to S, then the function $G(x,y) = f(y) - f(x)$ will be denoted by df.

CHAPTER II

BASIC THEOREMS

Most of the following theorems were proved in Mathematics 5309 (Foundations of Analysis) during the fall of 1967. The theorems will be stated in this chapter and will be used in establishing proofs for theorems in Chapter III.

<u>Theorem 2.1</u>

If (a,b) belongs to SXS and each of D_1 and D_2 is a subdivision of $[a,b]$, then $D_1 \cup D_2$ is a subdivision of $[a,b]$ and a refinement of D_1 and of D_2.

<u>Theorem 2.2</u>

If f is a function from S to S, defined on $[a,b]$, and D is a subdivision of $[a,b]$, then

$$\sum_{D} \Delta f_i = f(b) - f(a).$$

<u>Theorem 2.3</u>

If f is a function from S to S, defined on [a,b], then $\int_a^b df$ exists and is f(b) - f(a).

<u>Theorem 2.4</u>

If H is a function from SXS to S such that H is integrable on [a,b], and, for each subinterval I of [a,b], $H(I) \geq 0$, then $\int_a^b H \geq 0$.

<u>Theorem 2.5</u>

If H and K are functions from SXS to S such that each is integrable on [a,b], then $\int_a^b (H + K)$ exists and is $\int_a^b H + \int_a^b K$.

<u>Theorem 2.6</u>

If H is a function from SXS to S, integrable on [a,b], and c belongs to S, then $\int_a^b cH$ exists and is $c \int_a^b H$.

<u>Theorem 2.7</u>

If H is a function from SXS to S, integrable on [a,b], and

$$D = \{x_i\}_{i=0}^{n}$$

is a subdivision of [a,b], then

$$\int_a^b H = \sum_D \int_i H.$$

<u>Theorem 2.8</u>

If (a,b) belongs to SXS and H is a function from SXS to S, the following statements are equivalent:

(1) $\int_a^b H$ exists.

(2) If ϵ is a positive number there exists a subdivision D of [a,b] such that if D' and D" are refinements of D, then

$$\left| \sum_{D'} H_i - \sum_{D''} H_j \right| < \epsilon .$$

CHAPTER III

THEOREMS CONCERNING INTERVAL FUNCTIONS

The following sequence of theorems was taken from publications of W. D. L. Appling [1] and [2]. However, all of the proofs shown here were done without reference to previous publications.

Theorem 3.1a

Suppose H is a function from SXS to S, (a,b) belongs to SXS, and, for $a \leq x < y < z \leq b$,

$$H(x,z) \leq H(x,y) + H(y,z).$$

If D is a subdivision of [a,b] and D' is a refinement of D, then

$$\sum_{D} H_i \leq \sum_{D'} H_j.$$

Proof (by induction). For each positive integer n, let S(n) be the statement; if D_n is a refinement of D having exactly n elements which do not belong to D, then

$$\sum_{D} H_i \leq \sum_{D_n} H_j.$$

(a) Show S(1) is true.

Let D_1 be a refinement of D having one element, p, which does not belong to D. Denote the interval of D to which p belongs by $[x_{h-1}, x_h]$, then

$$\sum_{D} H_i = \sum_{\substack{D \\ i \neq h}} H_i + H(x_{h-1}, x_h)$$

$$\leq \sum_{\substack{D \\ i \neq h}} H_i + H(x_{h-1}, p) + H(p, x_h)$$

$$= \sum_{D_1} H_j.$$

(b) Assume S(k) is true.

(c) Show S(k + 1) is true.

Let D_{k+1} be a refinement of D having k + 1 elements which do not belong to D. Now, let D_k be a refinement of D obtained by deleting one of the k + 1 elements of D_{k+1} which do not belong to D. Therefore, D_k is a subdivision of [a,b] and D_{k+1} is a refinement of D_k having exactly one element not belonging to D_k. Hence,

$$\sum_{D} H_i \leq \sum_{D_k} H_j \qquad \text{from (b)}$$

$$\leq \sum_{D_{k+1}} H_t. \qquad \text{from (a)}$$

Theorem 3.1b

Suppose H is a function from SXS to S, (a,b) belongs to SXS, and, for $a \leq x < y < z \leq b$,

$$H(x,z) \geq H(x,y) + H(y,z).$$

If D is a subdivision of [a,b] and D' is a refinement of D, then

$$\sum_{D} H_i \geq \sum_{D'} H_j.$$

Proof. A proof similar to that for Theorem 3.1a will prove this Theorem.

Lemma 3.2a

If f and m are functions from S to S such that m is increasing, then

$$(1)\ \frac{[f(z) - f(x)]^2}{m(z) - m(x)} \leq \frac{[f(y) - f(x)]^2}{m(y) - m(x)} + \frac{[f(z) - f(y)]^2}{m(z) - m(y)},$$

provided $x < y < z$.

Proof (indirect). Assume (1) is false, then

$$(2)\quad \frac{[f(z) - f(x)]^2}{m(z) - m(x)} > \frac{[f(y) - f(x)]^2}{m(y) - m(x)} + \frac{[f(z) - f(y)]^2}{m(z) - m(y)}.$$

Let $f(z) - f(x) = A$ and $f(y) - f(x) = B$; then,

$$f(z) - f(y) = A - B.$$

Also, let $m(z) - m(x) = a$ and $m(y) - m(x) = b$; then,

$$m(z) - m(y) = a - b$$

and since m is increasing $a > b > 0$. Substituting into (2) we have

$$\frac{A^2}{a} > \frac{B^2}{b} + \frac{(A - B)^2}{(a - b)};$$

hence,

$$0 > \frac{B^2}{b} + \frac{(A - B)^2}{(a - b)} - \frac{A^2}{a}$$

and, since $ab(a - b) > 0$,

$$0 > a(a - b) B^2 + ab(A - B)^2 - b(a - b) A^2$$

$$= a^2B^2 - 2abAB + b^2A^2$$

$$= (aB - bA)^2$$

$$\geq 0.$$

Therefore assumption is false.

Theorem 3.2

If f and m are functions from the real numbers to the real numbers and m is increasing, the following statements are equivalent:

(1) $\int_a^b \frac{(df)^2}{dm}$ exists.

(2) There exists a real valued nondecreasing function h on [a,b] such that, for each subinterval I of [a,b], $(\Delta f)^2 \leq \Delta h \Delta m$.

Proof.

Part I, $1 \rightarrow 2$. Assume

$$\int_a^b \frac{(df)^2}{dm}$$

exists; hence, by Theorem 2.7,

$$\int_I \frac{(df)^2}{dm}$$

exists for each subinterval I of [a,b]. Let

$$h(x) = \int_a^x \frac{(df)^2}{dm},$$

for x belonging to [a,b], then for each subinterval I of [a,b],

$$\Delta h = \int_I \frac{(df)^2}{dm} \geq 0,$$

by Theorem 2.4; and hence, h is nondecreasing.

To show $(\Delta f)^2 \leq \Delta h \Delta m$ an indirect argument is used. Suppose $(\Delta f)^2 > \Delta h \Delta m$ for some interval I of [a,b]. Then,

$$\frac{(\Delta f)^2}{\Delta m} > \Delta h = \int_I \frac{(df)^2}{dm}.$$

Since

$$\int_I \frac{(df)^2}{dm}$$

exists and

$$\frac{(\Delta f)^2}{\Delta m} - \int_I \frac{(df)^2}{dm} > 0$$

there is a subdivision D of I such that if D' is a refinement of D, then

$$(1) \quad \left| \sum_{D'} \frac{(\Delta f_i)^2}{\Delta m_i} - \int_I \frac{(df)^2}{dm} \right| < \frac{(\Delta f)^2}{\Delta m} - \int_I \frac{(df)^2}{dm}.$$

Let D' be a refinement of D, then

$$\begin{aligned}
\frac{(\Delta f)^2}{\Delta m} &\leq \sum_{D'} \frac{(\Delta f_i)^2}{\Delta m_i} && \text{(Lemma 3.2a, Th. 3.1a)} \\
&\leq \left| \sum_{D'} \frac{(\Delta f_i)^2}{\Delta m_i} - \int_I \frac{(df)^2}{dm} \right| + \int_I \frac{(df)^2}{dm} \\
&< \frac{(\Delta f)^2}{\Delta m} - \int_I \frac{(df)^2}{dm} + \int_I \frac{(df)^2}{dm} && \text{(Eq. 1)} \\
&= \frac{(\Delta f)^2}{\Delta m}.
\end{aligned}$$

This is a contradiction; hence $(\Delta f)^2 \leq \Delta h \Delta m$ and $1 \to 2$.

Part II, $2 \to 1$. Assume (2) and define R as the set of numbers such that p belongs to R iff there is a subdivision D of [a,b] such that

$$p = \sum_D \frac{(\Delta f_i)^2}{\Delta m_i}.$$

For each subdivision D of [a,b],

$$\sum_D \frac{(\Delta f_i)^2}{\Delta m_i} \leq \sum_D \Delta h_i$$

$$= h(b) - h(a).$$

Hence R is bounded above, by $h(b) - h(a)$, and since R is nonempty, R has a least upper bound, J.

For each $\epsilon > 0$, $J - \epsilon < J$ and there exists a number q belonging to R for which $J - \epsilon < q \leq J$. Since q belongs to R there is a subdivision D of [a,b] such that

$$q = \sum_D \frac{(\Delta f_i)^2}{\Delta m_i}.$$

If D' is any refinement of D, then

$$\begin{aligned} J - \epsilon &< q \\ &= \sum_D \frac{(\Delta f_i)^2}{\Delta m_i} \\ &\leq \sum_{D'} \frac{(\Delta f_j)^2}{\Delta m_j} \quad \text{(Lemma 3.2a, Th. 3.1a)} \end{aligned}$$

$$\leq J \qquad\qquad \text{(Def. of J)}$$

$$< J + \epsilon.$$

This is equivalent to

$$\left| \sum_{D'} \frac{(\Delta f_i)^2}{\Delta m_i} - J \right| < \epsilon.$$

Therefore,

$$\int_a^b \frac{(df)^2}{dm}$$

exists and $2 \longrightarrow 1$.

Theorem 3.3

If H and L are real valued functions of subintervals of [a,b] such that H is bounded and $\int_a^b L$ exists, then $\int_a^b |H|\ |L - \int L|$ exists and is zero.

Proof. Let $J = 0$ and $\epsilon > 0$. Since H is bounded on [a,b], there exists a number M such that $|H(I)| < M$ for each subinterval I of [a,b]. Since $\int_a^b L$ exists and

$$\frac{\epsilon}{2M} > 0,$$

then there exists a subdivision D of [a,b] such that if D' and D" are refinements of D, then

$$(1)\quad \left|\sum_{D'} L_i - \sum_{D''} L_j\right| < \frac{\epsilon}{2M}. \qquad \text{(Th. 2.8)}$$

Let $E = \{x_i\}_{i=0}^{n}$ be a refinement of D. Since

$$\frac{\epsilon}{2nM} > 0,$$

where n is defined by E, and $\int_a^b L$ exists, then for each integer i, i = 1, 2, 3, ..., n, $\int_{x_{i-1}}^{x_i} L$ exists and there exists a subdivision D_i of $[x_{i-1}, x_i]$ such that

$$(2)\quad \left|\sum_{D_i} L_{j_i} - \int_i L\right| < \frac{\epsilon}{2nM}.$$

Hence,

$$\left|\sum_E |H_i|\ |L_i - \int_i L| - J\right| = \left|\sum_E |H_i|\ |L_i - \int_i L| - 0\right|$$

$$= \sum_E |H_i|\ |L_i - \int_i L|$$

$$< M \sum_E |L_i - \int_i L|$$

(Def. of M)

$$\leq M \sum_{E} \left(\left| L_i - \sum_{D_i} L_{j_i} \right| + \left| \sum_{D_i} L_{j_i} - \int_i L \right| \right)$$

$$< M \sum_{E} \left(\left| L_i - \sum_{D_i} L_{j_i} \right| + \frac{\epsilon}{2nM} \right) \qquad \text{(Eq. 2)}$$

$$= M \sum_{E} \left| L_i - \sum_{D_i} L_{j_i} \right| + \frac{\epsilon}{2}$$

$$= M \left[\sum_{\substack{i=1 \\ Q_i \geq 0}}^{n} \left| L_i - \sum_{D_i} L_{j_i} \right| + \sum_{\substack{i=1 \\ Q_i < 0}}^{n} \left| L_i - \sum_{D_i} L_{j_i} \right| \right] + \frac{\epsilon}{2},$$

where

$$Q_i = L_i - \sum_{D_i} L_{j_i}$$

$$= M \left[\sum_{\substack{i=1 \\ Q_i \geq 0}}^{n} \left(L_i - \sum_{D_i} L_{j_i} \right) + \sum_{\substack{i=1 \\ Q_i < 0}}^{n} \left(\sum_{D_i} L_{j_i} - L_i \right) \right] + \frac{\epsilon}{2}$$

$$= M \left[\sum_{\substack{i=1 \\ Q_i \geq 0}}^{n} L_i + \sum_{\substack{i=1 \\ Q_i < 0}}^{n} \left(\sum_{D_i} L_{j_i} \right) - \sum_{\substack{i=1 \\ Q_i < 0}}^{n} L_i - \sum_{\substack{i=1 \\ Q_i \geq 0}}^{n} \left(\sum_{D_i} L_{j_i} \right) \right] + \frac{\epsilon}{2}$$

$$= M \left| \sum_{D'} L_k - \sum_{D''} L_h \right| + \frac{\epsilon}{2},$$

where

$$D' = \left(\bigcup_{\substack{i=1 \\ Q_i<0}}^{n} D_i \right) \cup E$$

and

$$D'' = \left(\bigcup_{\substack{i=1 \\ Q_i \geq 0}}^{n} D_i \right) \cup E$$

$$< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2} \qquad \text{(Eq. 1)}$$

$$= \epsilon.$$

Theorem 3.4

If J is a number and H and L are real valued functions of subintervals of [a,b] such that H is bounded and $\int_a^b L$ exists, the following statements are equivalent:

(1) $\int_a^b H(\int L) = J$.

(2) $\int_a^b HL = J$.

Proof.

Part I, $1 \rightarrow 2$. Assume (1), $\int_a^b H(\int L) = J$, and let $\epsilon > 0$. Then, there exists a subdivision D_1 of [a,b] such that

(1) $\left| \sum_{D'} H_i \int_i L - J \right| < \frac{\epsilon}{2}$,

for each refinement D' of D_1. Also, from Theorem 3.3, $\int_a^b |H|\ |L - \int L| = 0$; hence, there exists a subdivision D_2 of [a,b] such that if D' is a refinement of D_2, then

(2) $\left| \sum_{D'} |H_i|\ |L_i - \int_i L| - 0 \right| < \frac{\epsilon}{2}$.

Let $D = D_1 \cup D_2$ and let D' be a refinement of D; then, D' is a refinement of D_1 and of D_2, and

$$\begin{aligned}
\left| \sum_{D'} H_i L_i - J \right| &\leq \left| \sum_{D'} H_i L_i - \sum_{D'} H_i \int_i L \right| + \left| \sum_{D'} H_i \int_i L - J \right| \\
&< \left| \sum_{D'} H_i (L_i - \int_i L) \right| + \frac{\epsilon}{2} \qquad \text{(Eq. 1)} \\
&\leq \sum_{D'} |H_i|\ |L_i - \int_i L| + \frac{\epsilon}{2} \\
&= \left| \sum_{D'} |H_i|\ |L_i - \int_i L| - 0 \right| + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \text{(Eq. 2)} \\
&= \epsilon .
\end{aligned}$$

Hence, $1 \rightarrow 2$.

Part II, 2$\longrightarrow$1. A proof similar to the proof of Part I can be used to show 2$\longrightarrow$1.

Theorem 3.5

If H is a real nonnegative valued function of subintervals of [a,b] and m is an increasing function such that $\int_a^b (Hdm)^{\frac{1}{2}}$ exists and there is a number $R > 1$ such that $H(I) \leq R\Delta m$, for each subinterval I of [a,b], then:

(1) $\displaystyle\int_a^b \frac{[\int(Hdm)^{\frac{1}{2}}]^2}{dm}$ exists.

(2) $\int_a^b H$ exists.

(3) $\displaystyle\int_a^b \frac{[\int(Hdm)^{\frac{1}{2}}]^2}{dm} = \int_a^b H.$

Proof.

Part (1). Let f and h be functions such that, for each x belonging to [a,b], $f(x) = \int_a^b (Hdm)^{\frac{1}{2}}$ and $h(x) = Rm(x)$. Then, since m is increasing, h is increasing.

To show that

(1) $(\Delta f)^2 \leq \Delta h \Delta m,$

for each subinterval I of [a,b], an indirect argument is used. Assume (1) is false. Then, there is a subinterval

I of [a,b] such that

$$(\Delta f)^2 > \Delta h \Delta m$$

$$= R(\Delta m)^2; \qquad \text{(Def. of h)}$$

hence,

$$\Delta f > R^{\frac{1}{2}}\Delta m.$$

Also,

$$\Delta f = f(y) - f(x), \qquad \text{where } I = [x,y]$$

$$= \int_a^y (Hdm)^{\frac{1}{2}} - \int_a^x (Hdm)^{\frac{1}{2}} \qquad \text{(Def. of f)}$$

$$= \int_I (Hdm)^{\frac{1}{2}}. \qquad \text{(Th. 2.7)}$$

Since $\int_I (Hdm)^{\frac{1}{2}} = \Delta f$ and $\Delta f - R^{\frac{1}{2}}\Delta m > 0$, there exists a subdivision D of I such that if D' is a refinement of D, then

$$(2) \quad \left| \sum_{D'} (H_i \Delta m_i)^{\frac{1}{2}} - \Delta f \right| < \Delta f - R^{\frac{1}{2}}\Delta m.$$

For each refinement D' of D,

$$\Delta f \le \left| \Delta f - \sum_{D'} (H_i \Delta m_i)^{\frac{1}{2}} \right| + \sum_{D'} (H_i \Delta m_i)^{\frac{1}{2}}$$

$$< \Delta f - R^{\frac{1}{2}}\Delta m + \sum_{D'} \left(\frac{H_i}{\Delta m_i} \right)^{\frac{1}{2}} \Delta m \qquad \text{(Eq. 2)}$$

$$\le \Delta f - R^{\frac{1}{2}}\Delta m + R^{\frac{1}{2}} \sum_{D'} \Delta m_i$$

$$= \Delta f - R^{\frac{1}{2}}\Delta m + R^{\frac{1}{2}}\Delta m$$

$$= \Delta f.$$

Hence assumption was false and (1) is true; therefore, by Theorem 3.2,

$$\int_a^b \frac{(df)^2}{dm}$$

exists and

$$\int_a^b \frac{(df)^2}{dm} = \int_a^b \frac{[\int (Hdm)^{\frac{1}{2}}]^2}{dm}. \qquad \text{(Def. of f)}$$

<u>Part (2) and (3)</u>. For each subinterval I of [a,b], let

$$H_1(I) = \frac{\int_I (Hdm)^{\frac{1}{2}}}{\Delta m}$$

and

$$L_1(I) = [H(I)\ \Delta m]^{\frac{1}{2}};$$

then, $\int_a^b L_1$ exists and H_1 is real nonnegative valued.

To show H_1 is bounded on $[a,b]$, let $M = R + 1$. For each subinterval I of $[a,b]$, $\int_I (Hdm)^{\frac{1}{2}}$ exists and there exists a subdivision D of I such that

$$(3)\quad \left| \sum_{D'} (H_i \Delta m_i)^{\frac{1}{2}} - \int_I (Hdm)^{\frac{1}{2}} \right| < \Delta m,$$

for each refinement D' of D. Let D' be a refinement of D, then

$$H_1(I) = \frac{\int_I (Hdm)^{\frac{1}{2}}}{\Delta m}$$

$$\leq \frac{\left| \int_I (Hdm)^{\frac{1}{2}} - \sum_{D'} (H_i \Delta m_i)^{\frac{1}{2}} \right| + \sum_{D'} (H_i \Delta m_i)^{\frac{1}{2}}}{\Delta m}$$

$$< \frac{\Delta m + \sum_{D'} \left(\frac{H_i}{\Delta m_i} \right)^{\frac{1}{2}} \Delta m_i}{\Delta m} \qquad \text{(Eq. 3)}$$

$$< 1 + \frac{R}{\Delta m} \sum_{D'} \Delta m_i$$

$$= 1 + R$$

$$= M;$$

therefore, H_1 is bounded on [a,b]. Now, since

$$\int_a^b \frac{[\int(Hdm)^{\frac{1}{2}}]^2}{dm}$$

exists, from Part (1) of this Theorem, and

$$(4)\quad \int_a^b \frac{[\int(Hdm)^{\frac{1}{2}}]^2}{dm} = \int_a^b H_1(\int L_1); \quad \text{(Def. of } H_1 \text{ and } L_1\text{)}$$

then, $\int_a^b H_1(\int L_1)$ exists, and, by Theorem 3.4,

$$(5)\quad \int_a^b H_1(\int L_1) = \int_a^b H_1L_1.$$

Let H_2 be the function such that

$$H_2(I) = \left[\frac{H(I)}{\Delta m}\right]^{\frac{1}{2}},$$

for each subinterval I of [a,b]. Then H_2 is real nonnegative valued and bounded above by R. From (5), $\int_a^b H_1L_1$ exists and since

$$(6)\quad \int_a^b H_1L_1 = \int_a^b \frac{\int(Hdm)^{\frac{1}{2}}}{dm} \cdot (Hdm)^{\frac{1}{2}}$$

$$= \int_a^b \left(\frac{H}{dm}\right)^{\frac{1}{2}} \int(Hdm)^{\frac{1}{2}}$$

$$= \int_a^b H_2(\int L_1),$$

by Theorem 3.4, $\int_a^b H_2 L_1$ exists and

$$(7)\ \int_a^b H_2(\int L_1) = \int_a^b H_2 L_1.$$

Therefore,

$$\begin{aligned}
\int_a^b \frac{[\int (H\,dm)^{\frac{1}{2}}]^2}{dm} &= \int_a^b H_1(\int L_1) && \text{(Eq. 4)} \\
&= \int_a^b H_1 L_1 && \text{(Eq. 5)} \\
&= \int_a^b H_2(\int L_1) && \text{(Eq. 6)} \\
&= \int_a^b H_2 L_1 && \text{(Eq. 7)} \\
&= \int_a^b \left(\frac{H}{dm}\right)^{\frac{1}{2}} (H\,dm)^{\frac{1}{2}} && \text{(Def. of } H_2 \text{ and } L_1\text{)} \\
&= \int_a^b H.
\end{aligned}$$

Hence, $\int_a^b H$ exists and is

$$\int_a^b \frac{[\int (H\,dm)^{\frac{1}{2}}]^2}{dm}.$$

Theorem 3.6

If P is a real nonnegative valued bounded function of subintervals of [a,b] and m is an increasing function such that $\int_a^b$ Pdm exists, then $\int_a^b P^2 dm$ exists.

Proof. For each subinterval I of [a,b], let $H(I) = [P(I)]^2 \Delta m$; then, H is nonnegative valued. Since P is bounded there is a number M such that $P(I) < M$, for each subinterval I of [a,b], and, since each of P(I) and M is nonnegative, $[P(I)]^2 < M^2$; hence, $H(I) < M^2 \Delta m$. Since $\int_a^b$ Pdm exists and

$$\int_a^b Pdm = \int_a^b (P^2 dm^2)^{\frac{1}{2}}$$
$$= \int_a^b (Hdm)^{\frac{1}{2}},$$

then $\int_a^b (Hdm)^{\frac{1}{2}}$ exists and, by Theorem 3.5, $\int_a^b H$ exists; therefore, $\int_a^b P^2 dm$ exists, since

$$\int_a^b H = \int_a^b P^2 dm. \qquad \text{(Def. of H)}$$

Theorem 3.7

If each of H and K is a real nonnegative valued function of subintervals of [a,b], m is an increasing function and each of $\int_a^b$ Hdm and $\int_a^b$ Kdm exists, then $\int_a^b$ HKdm exists.

Proof. Since $\int_a^b$ Hdm and $\int_a^b$ Kdm exist, then $\int_a^b (H + K)$ dm exists and since $(H + K)$ is nonnegative $\int_a^b (H + K)^2$ dm exists (Th. 3.6). Also, $\int_a^b H^2 dm$ and $\int_a^b K^2 dm$ exist (Th. 3.6). Therefore,

$$\int_a^b (H + K)^2 \, dm - \int_a^b H^2 dm - \int_a^b K^2 dm$$

$$= \int_a^b [(H + K)^2 - H^2 - K^2] \, dm$$

$$= 2 \int_a^b HK dm$$

exists.

Theorem 3.8

If K is a real valued function of subintervals of [a,b] such that $\int_a^b K$ exists, and M is a function of subintervals of [a,b] such that, for each subinterval I of [a,b], M(I) is either K(I) or $\int_I K$, then if $\epsilon > 0$ there exists a subdivision D of [a,b] such that

$$\sum_{D'} |M_i - \int_i K| < \epsilon,$$

for each refinement D' of D.

Proof. Since $\int_a^b K$ exists, from Theorem 3.3, $\int_a^b |K - \int K| = 0$. For each positive number ϵ, there is a subdivision D of [a,b] such that

$$\sum_{D'} |K_i - \int_i K| < \epsilon,$$

for each refinement D' of D. Let $D' = \{x_i\}_{i=0}^n$ be a refinement of D, then

$$\sum_{D'} |M_i - \int_i K| = \sum_{D'} a_i |M_i - \int_i K| + \sum_{D'} b_i |M_i - \int_i K|,$$

where $a_i = 1$ and $b_i = 0$, if $M_i = K_i$, and $b_i = 1$ and $a_i = 0$, if $M_i = \int_i K$

$$= \sum_{D'} a_i |M_i - \int_i K|$$

$$\leq \sum_{D'} |K_i - \int_i K|$$

$$< \epsilon.$$

Theorem 3.9

If K is a real nonnegative valued function of subintervals of [a,b], integrable on [a,b], then $\int_a^b [K\int K]^{\frac{1}{2}}$ exists and is $\int_a^b K$.

Proof. From Theorem 3.3, $\int_a^b |K - \int K| = 0$; hence, for each positive number ϵ, there is a subdivision D of [a,b] such that if D' is a refinement of D, then

(1) $\sum_{D'} |K_i - \int_i K| < \epsilon$.

Therefore, for each refinement D' of D,

$$\left|\sum_{D'} [K_i \int_i K]^{\frac{1}{2}} - \int_a^b K\right| = \left|\sum_{D'} [K_i \int_i K]^{\frac{1}{2}} - \sum_{D'} \int_i K\right| \quad \text{(Th. 2.6)}$$

$$\leq \sum_{D'} |[K_i \int_i K]^{\frac{1}{2}} - \int_i K|$$

$$= \sum_{\substack{i=1 \\ Q_i \geq 0}}^{n} |[K_i \int_i K]^{\frac{1}{2}} - \int_i K| + \sum_{\substack{i=1 \\ Q_i < 0}}^{n} |[K_i \int_i K]^{\frac{1}{2}} - \int_i K|,$$

where $Q_i = [K_i \int_i K]^{\frac{1}{2}} - \int_i K$

$$= \sum_{\substack{i=1 \\ Q_i \geq 0}}^{n} ([K_i \int_i K]^{\frac{1}{2}} - \int_i K) + \sum_{\substack{i=1 \\ Q_i < 0}}^{n} (\int_i K - [K_i \int_i K]^{\frac{1}{2}})$$

$$\leq \sum_{\substack{i=1 \\ Q_i \geq 0}}^{n} (K_i - \int_i K) + \sum_{\substack{i=1 \\ Q_i < 0}}^{n} (\int_i K - K_i)$$

$$= \sum_{D'} |K_i - \int_i K|$$

$$< \epsilon. \qquad \text{(Eq. 1)}$$

Lemma 3.10a

If a, b, c, and d are real nonnegative numbers, then

$$(ac)^{\frac{1}{2}} + (bd)^{\frac{1}{2}} \leq [(a + b)(c + d)]^{\frac{1}{2}}.$$

Proof (indirect). Assume the conclusion is false, then

$$(ac)^{\frac{1}{2}} + (bd)^{\frac{1}{2}} > [(a + b)(c + d)]^{\frac{1}{2}},$$

and

$$2(abcd)^{\frac{1}{2}} > ad + bc;$$

hence,

$$0 > ad - 2(abcd)^{\frac{1}{2}} + bc$$

$$= [(ad)^{\frac{1}{2}} - (bc)^{\frac{1}{2}}]^2$$

$$\geq 0.$$

Therefore assumption is false.

Lemma 3.10b

If for each positive integer n, a_n and b_n are real nonnegative numbers, then

$$\sum_{i=1}^{n} (a_i b_i)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} a_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i\right)^{\frac{1}{2}}.$$

Proof. This Lemma follows directly by using an induction proof and Lemma 3.10a.

Theorem 3.10

If each of H and K are real nonnegative valued functions of subintervals of [a,b], integrable on [a,b], then the following integrals exist and

$$\int_a^b (HK)^{\frac{1}{2}} = \int_a^b [(\int H)(\int K)]^{\frac{1}{2}}$$

Proof. Let R be the set of numbers such that p belongs to R iff there exists a subdivision D of [a,b] such that

$$p = \sum_D [(\int_i H)(\int_i K)]^{\frac{1}{2}}.$$

The set R is bounded below by zero, and is nonempty; hence, R has a greatest lower bound, J. For each positive number

ϵ, $J < J + \epsilon$ and there exists a number q belonging to R such that $J \leq q < J + \epsilon$. Since q belongs to R, there exists a subdivision D of [a,b] such that

$$q = \sum_D [(\int_i H)(\int_i K)]^{\frac{1}{2}}.$$

Hence, for each refinement D' of D,

$$\begin{aligned} J + \epsilon &> q \\ &= \sum_D [(\int_i H)(\int_i K)]^{\frac{1}{2}} \\ &\geq \sum_{D'} [(\int_j H)(\int_j K)]^{\frac{1}{2}} && \text{(Lemma 3.10a, Th. 3.1b)} \\ &\geq J && \text{(Def. of J)} \\ &> J - \epsilon. \end{aligned}$$

Therefore,

(1) $\int_a^b [(\int H)(\int K)]^{\frac{1}{2}} = J.$

To prove $\int_a^b (HK)^{\frac{1}{2}} = J$, we let $\int_a^b H = A$, $\int_a^b K = B$, and ϵ be a positive number. Then, there exists a subdivision D such that if D' is a refinement of D, each of the following statements is true:

(2) $\left|\sum_{D'} H_i - A\right| < 1;$

(3) $\left|\sum_{D'} H_i - A\right| < \frac{1}{18}\left(\frac{\epsilon}{B + 1}\right)^2;$

(4) $\left|\sum_{D'} K_i - B\right| < 1;$

(5) $\left|\sum_{D'} K_i - B\right| < \frac{1}{18}\left(\frac{\epsilon}{A + 1}\right)^2;$

(6) $\left|\sum_{D'} (H_i \int_i H)^{\frac{1}{2}} - A\right| < \frac{1}{36}\left(\frac{\epsilon}{B + 1}\right)^2;$ (Th. 3.9)

(7) $\left|\sum_{D'} (K_i \int_i K)^{\frac{1}{2}} - B\right| < \frac{1}{36}\left(\frac{\epsilon}{A + 1}\right)^2;$ (Th. 3.9)

(8) $\left|\sum_{D'} [(\int_i H)(\int_i K)]^{\frac{1}{2}} - J\right| < \frac{\epsilon}{3}.$ (Eq. 1)

Let D' be a refinement of D, then

$$\left|\sum_{D'} (H_i K_i)^{\frac{1}{2}} - J\right|$$

$$\leq \left|\sum_{D'} (H_i K_i)^{\frac{1}{2}} - \sum_{D'} [(\int_i H)(\int_i K)]^{\frac{1}{2}}\right|$$

$$+ \left|\sum_{D'} [(\int_i H)(\int_i K)]^{\frac{1}{2}} - J\right|$$

$$< \left|\sum_{D'} (H_i K_i)^{\frac{1}{2}} - \sum_{D'} [H_i(\int_i K)]^{\frac{1}{2}} + \sum_{D'} [H_i(\int_i K)]^{\frac{1}{2}}\right.$$

$$\left. - \sum_{D'} [(\int_i H)(\int_i K)]^{\frac{1}{2}}\right| + \frac{\epsilon}{3} \quad \text{(Eq. 8)}$$

$$\leq \sum_{D'} (H_i)^{\frac{1}{2}}\{[(K_i)^{\frac{1}{2}} - (\int_i K)^{\frac{1}{2}}]^2\}^{\frac{1}{2}}$$

$$+ \sum_{D'} (\int_i K)^{\frac{1}{2}}\{[(H_i)^{\frac{1}{2}} - (\int_i H)^{\frac{1}{2}}]^2\}^{\frac{1}{2}} + \frac{\epsilon}{3}$$

$$\leq \left(\sum_{D'} H_i\right)^{\frac{1}{2}}\left\{\sum_{D'} [(K_i)^{\frac{1}{2}} - (\int_i K)^{\frac{1}{2}}]^2\right\}^{\frac{1}{2}}$$

$$+ \left(\sum_{D'} \int_i K\right)^{\frac{1}{2}}\left\{\sum_{D'} [(H_i)^{\frac{1}{2}} - (\int_i H)^{\frac{1}{2}}]^2\right\}^{\frac{1}{2}} + \frac{\epsilon}{3}$$

(Lemma 3.10b)

$$< (A + 1) \left|\sum_{D'} K_i - 2 \sum_{D'} (K_i \int_i K)^{\frac{1}{2}} + \sum_{D'} \int_i K\right|^{\frac{1}{2}}$$

$$+ (B + 1) \left|\sum_{D'} H_i - 2 \sum_{D'} (H_i \int_i H)^{\frac{1}{2}}\right.$$

$$\left. + \sum_{D'} \int_i H\right|^{\frac{1}{2}} + \frac{\epsilon}{3}$$

(Eqs. 2,4)

$$\leq (A + 1) \left[\left|\sum_{D'} K_i - B\right| + 2\left|B - \sum_{D'} (K_i \int_i K)^{\frac{1}{2}}\right|\right]^{\frac{1}{2}}$$

$$+ (B + 1) \left[\left|\sum_{D'} H_i - A\right| + 2\left|A - \sum_{D'} (H_i \int_i H)^{\frac{1}{2}}\right|\right]^{\frac{1}{2}}$$

$$+ \frac{\epsilon}{3}$$

$$< (A + 1)\left[\frac{1}{18}\left(\frac{\epsilon}{A + 1}\right)^2 + \frac{1}{18}\left(\frac{\epsilon}{A + 1}\right)^2\right]^{\frac{1}{2}}$$

$$+ \; (B + 1)\left[\frac{1}{18}\left(\frac{\epsilon}{B + 1}\right)^2 + \frac{1}{18}\left(\frac{\epsilon}{B + 1}\right)^2\right]^{\frac{1}{2}} + \frac{\epsilon}{3}$$

(Eqs. 3,5,6,7)

$$= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

$$= \epsilon .$$

Theorem 3.11

If H is a real nonnegative valued function of subintervals of [a,b], integrable on [a,b], and m is a real valued nondecreasing function on [a,b], then $\int_a^b (Hdm)^{\frac{1}{2}}$ exists.

Proof. This Theorem follows immediately from Theorem 3.10, since $\int_a^b dm$ exists for any real valued function m defined on [a,b].

Theorem 3.12

If H is a real nonnegative valued interval function, m is a real valued nondecreasing function on [a,b], and $\int_a^b Hdm$ exists, then $\int_a^b H^{\frac{1}{2}} dm$ exists.

<u>Proof</u>. Since Hdm and dm are real nonnegative valued interval functions, defined on [a,b], and $\int_a^b Hdm$ and $\int_a^b dm$ exist, then, from Theorem 3.10, $\int_a^b [(Hdm)(dm)]^{\frac{1}{2}}$, which is $\int_a^b H^{\frac{1}{2}} dm$, exists.

B I B L I O G R A P H Y

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