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Positive solutions for a nonlocal boundary-value problem with vector-valued response *

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Abstract

Using variational methods, we study the existence of positive solutions for a nonlocal boundary-value problem with vector-valued response. We develop duality and variational principles for this problem and present a numerical version which enables the approximation of solutions and gives a measure of a duality gap between primal and dual functional for approximate solutions for this problem.

1 Introduction

The aim of this paper is to establish the conditions for which the differential equation

$$(k(t)|x'(t)|^{q-2}x'(t))' + V_x(t, x(t)) = 0, \quad \text{a.e. in } [0,T] \tag{1.1}$$

possesses a positive solution $x : [0,T] \to \mathbb{R}^n$ such that $x(0) = 0$, and satisfies the non-local boundary condition

$$|x'(T)|^{q-2}x'(T) = \int_{t_0}^{T} |x'(s)|^{q-2}x'(s)dg(s), \tag{1.2}$$

where $|z| = \sqrt{\sum_{i=1}^{n} z_i^2}$ and the integral is understood in the Riemann-Stieltjes sense. The general assumptions for this article are as follows:

(H) The number q is even and greater than zero, T is an arbitrary positive number, t_0 is a real number in the open interval $(0,T)$, $g = (g_1, \dots, g_n) : [0,T] \to \mathbb{R}^n$ where g_i increases and $g(t_0) = 0$, $V : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is Gateaux differentiable in the second variable and measurable in t, $k : [0,T] \to \mathbb{R}^+$, and $k(T) = 1$.

Of course, when k is an absolutely continuous, the solution of (1.1) belongs to $C^{1,+}([0,T], \mathbb{R}^n)$, the space of continuously differentiable functions whose first

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derivative is absolutely continuous. Note that we do not assume that V_x is superlinear or sublinear.

In this paper we shall apply some variational methods and consider (1.1) as the Euler-Lagrange equation to the functional

$$J(x) = \int_0^T (-V(t, x(t)) + \frac{k(t)}{q}|x'(t)|^q)dt \tag{1.3}$$

defined on the space A_0 of absolutely continuous functions $x : [0, T] \to \mathbb{R}^n$, $x(0) = 0$ with x' in $L^q([0, T], \mathbb{R}^n)$ and the norm $\|x\|_{A_0} = \big(\int_0^T |x'(t)|^q dt\big)^{1/q}$. We shall denote by A_{0b} the subset of A_0 consisting of functions satisfying (1.2) .

This problem appears in mathematical models of physical phenomena and it is associated with the principle of minimal action, which holds true universally in nature. Problems like (1.1), (1.2) have been studied by many authors, mainly in one-dimensional case ($n = 1$) with $q = 2$. This problem is discussed also in [7], where a topological approach is presented and the methods are based on the fixed point theorem in cones by Krasnosielski [9]. In [7], it is assumed that $g(t_0+) > 0$, that V has the special form $V_x(t, x) = q(t)f(x)$ for some continuous functions $q : [0, 1] \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, f is nonnegative for $x > 0$, and V quiet at infinity, and

$$\sup_{x\in[0,v]} f(x) \le \theta v \tag{1.4}$$

for some $v > 0$ and $\theta > 0$. It appears that weaker assumptions ($n \ge 1$, t and x not separated in the left-hand side of the equation, $V_x(\cdot, x)$ only measurable, and $V_x(t, \cdot)$ not necessarily quiet at infinity) are still sufficient to conclude the existence of solutions for (1.1). We consider the general case when V satisfies hypothesis (**H**) which are not as strong as in [7]. We are also able to omit the condition $g(t_0+) > 0$.

In this paper we study (1.1), (1.2) by duality methods analogous to the methods developed for (1.1) in sublinear cases [12, 13]. Functional (1.3) is, in general, unbounded in A_0 (especially in the superlinear case), so that we must look for critical points of (1.3) of "minmax" type, or find subsets X and X^d on which the action functional J or its dual J_D is bounded. We shall apply the second approach; i.e., choose the special sets over which we calculate minimum of J and J_D and then link this value with critical points of J. Of course, we have the Morse theory and its generalization, the saddle points theorems, and the mountain pass theorems [12, 14, 15]. However, because of the boundary condition (1.2) they cannot be applied directly to find critical points of J. Moreover, our assumptions are not strong enough to use, for example, the Mountain Pass Theorem: V is not sufficiently smooth, V_x and V do not have growth conditions. In consequence, J is not necessary of C^1class and it does not satisfy, in general, the PS-condition. We shall develop duality, and because of this theory we are able to omit in our proof of the existence of critical points, the deformation lemmas, the Ekeland variational principle, and the PS type conditions. Our approach also enables the numerical characterization of solutions for this problem. It seems to us that this is the first publication that applies variational methods to problem (1.1), (1.2).

Let the positive cone in $\mathbb{R}^n$ be denoted by

$$P = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0,\ i = 1, \dots, n\}$$

and let $\bar{P} = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \dots, n\}$. We say that $x \geq y$ for $x, y \in \mathbb{R}^n$ if $x - y \in \bar{P}$. If $b, c \in \mathbb{R}^n$ by bc we always mean a vector $[b_i c_i]_{i=1,\dots,n}$ and by $\sqrt[q-1]{b} = b^{\frac{1}{q-1}}$ a vector $[\sqrt[q-1]{b_i}]_{i=1,\dots,n} = [b_i^{\frac{1}{q-1}}]_{i=1,\dots,n}$. We denote $\mathbf{1} = (1, \dots 1)$ the vector in $\mathbb{R}^n$.

Let us investigate the operator $\widetilde{\mathbf{A}}$, for $x \in A_{0b}$, $x(t) \in P$, $t \in [0, T]$.

$$\widetilde{\mathbf{A}}x(t) = \alpha \frac{1}{k(t)} \int_{t_0}^{T} \frac{1}{k(r)} \int_{r}^{T} V_x(s, x(s)) ds dg(r) + \frac{1}{k(t)} \int_{t}^{T} V_x(s, x(s)) ds.$$

where $\alpha := [\alpha_i]_{i=1,\dots,n} = [(1 - a_i)^{-1}]_{i=1,\dots,n}$ and

$$a := \int_{t_0}^{T} \frac{1}{k(t)} dg(t) = \left[\int_{t_0}^{T} \frac{1}{k(t)} dg_i(t) \right]_{i=1,\dots,n}.$$

For the functions that appear in this paper we assume the following:

(H1) The function k is continuous and positive, $V(t, \cdot)$ is convex in $\overline{P}$, $V_x(t, \cdot)$ is continuous and nonnegative in P, $t \in [0, T]$, $|\int_0^T V(t, 0) dt| < \infty$, and $\alpha > 0$.

(H2) There exist a function $u \in L^\infty([0, T], \mathbb{R}^n)$, $u(t) \in P$, for a.e. $t \in (0, T)$, and constants c, $e \in P$, such that for

$$b(t) := \frac{(\alpha a + \mathbf{1})c + e}{k(t)}, \quad v(t) = \int_0^t |b(s)|^{-\frac{q-2}{q-1}} u(s) ds,$$
$$z(t) = \int_0^t |u(s)|^{-\frac{q-2}{q-1}} b(s) ds$$

we have $v(t) < z(t)$, $t \in (0, T)$ and

$$\int_{t_0}^{T} V_x(t, z(t)) dt \leq c, \quad \widetilde{\mathbf{A}}v(t) \geq u(t),\ t \in (0, T).$$

We see that our hypotheses on V concern only convexity of $V(t, \cdot)$ in $\bar{P}$ and that this function is rather of general type.

To construct the set X first we put

$$\overline{X} = \Big\{ x \in A_{0b} : p(t) = k(t)|x'(t)|^{q-2} x'(t),\ t \in [0, T] \quad \text{belongs to } A^{q'},$$
$$x(t) \geq v(t),\ t \in (0, T) \text{ and } \int_{t_0}^{T} V_x(t, x(t)) dt \leq c \Big\}$$

where $A^{q'}$ is the set of absolutely continuous functions $x : [0, T] \to \mathbb{R}^n$ with x' in $L^{q'}([0, T], \mathbb{R}^n)$ and t_0 is given above.

Remark 1.1 Assumptions **(H1)** and **(H2)** imply that $\overline{X} \neq \emptyset$. Indeed; each element $x \in A_{0b}$ such that $v(t) = \int_0^t |b(s)|^{-\frac{q-2}{q-1}} u(s)ds < x(t)$ for all $t \in (0,T)$ and $x(t) < z(t)$ for all $t \in [t_0, T]$ belongs to $\overline{X}$.

We reduce the space $\overline{X}$ to the set

$$X = \{x \in \overline{X} : x \in \bar{P},\ x'(t) \geq 0,\ t \in [0,T]\}.$$

It is clear that X depends strongly on the type of nonlinearity of V. We easily see that X is not in general a closed set in A. Assume additionally that

(H3) For $e \in P$ given in (**H2**)

$$\int_0^{t_0} V_x(t, x(t))dt \leq e \text{ for all } x \in X$$

and there exists $d \in \mathbb{R}$ such that

$$\Big| \int_0^{t_0} V(t, x(t))dt \Big| \leq d \text{ for all } x \in X.$$

Let an operator $\mathbf{A} : X \to \bar{P}$ be given by

$$\mathbf{A}x(t) := \int_0^t \widetilde{\mathbf{A}}x(s) |\widetilde{\mathbf{A}}x(s)|^{-\frac{q-2}{q-1}} ds.$$

Since $\widetilde{\mathbf{A}}x(t) > 0$ for all $t \in [0,T]$, so the operator $\mathbf{A}$ is well-defined. We easily see that solving the boundary problem (1.1)-(1.2) is equivalent to solve the operator equation $x = \mathbf{A}x$ in A_{0b}.

Now we give some auxiliary results

Lemma 1.2 *The set X satisfies $\mathbf{A}X \subset X$; i.e. for each $x \in X$ there exists $w \in X$ such that $w = \mathbf{A}x$.*

Proof. It is clear that if $x \in X$ then $\mathbf{A}x(t) \geq 0$ and

$$(\mathbf{A}x(t))' = \widetilde{\mathbf{A}}x(t) |\widetilde{\mathbf{A}}x(t)|^{-\frac{q-2}{q-1}} \geq 0.$$

First we will prove that $\widetilde{\mathbf{A}}x(t) \leq b(t)$, $t \in (0,T]$ for $x \in X$. Indeed; fix $x \in X$, then using definition of X we have for all $t \in (0,T]$

$$\begin{aligned}
\widetilde{\mathbf{A}}x(t) =& \alpha \frac{1}{k(t)} \int_{t_0}^T \frac{1}{k(r)} \int_r^T V_x(s, x(s))dsdg(r) + \frac{1}{k(t)} \int_t^T V_x(s, x(s))ds \\
\leq& \alpha \frac{1}{k(t)} \int_{t_0}^T \frac{1}{k(r)} dg(r) \int_{t_0}^T V_x(s, x(s))ds + \frac{1}{k(t)} \int_0^T V_x(s, x(s))ds \\
\leq& \frac{(\alpha a + \mathbf{1})c + e}{k(t)} = b(t)
\end{aligned}$$

and

$$\big|\widetilde{\mathbf{A}}x(t)\big|^{-\frac{q-2}{q-1}} \le \big|\widetilde{\mathbf{A}}v(t)\big|^{-\frac{q-2}{q-1}} \le |u(t)|^{-\frac{q-2}{q-1}}.$$

Combining these results we obtain

$$\mathbf{A}x(t) = \int_0^t \widetilde{\mathbf{A}}x(s)|\widetilde{\mathbf{A}}x(s)|^{-\frac{q-2}{q-1}}ds \le \int_0^t b(s)|u(s)|^{-\frac{q-2}{q-1}}ds = z(t).$$

Thus, assumptions **(H1)** and **(H2)** imply

$$\int_{t_0}^T V_x(s,\mathbf{A}x(s))ds \le \int_{t_0}^T V_x(s,z(s))ds \le c.$$

On the other hand we have for all $t \in (0,T)$

$$\begin{aligned}\mathbf{A}x(t) &= \int_0^t \widetilde{\mathbf{A}}x(s)|\widetilde{\mathbf{A}}x(s)|^{-\frac{q-2}{q-1}}ds\\ &\ge \int_0^t |b(s)|^{-\frac{q-2}{q-1}}\widetilde{\mathbf{A}}x(s)ds \ge \int_0^t |b(s)|^{-\frac{q-2}{q-1}}\widetilde{\mathbf{A}}v(s)ds\\ &\ge \int_0^t |b(s)|^{-\frac{q-2}{q-1}}u(s)ds = v(t).\end{aligned}$$

Summarizing: $\mathbf{A}x \in X$. As the dual set to X we shall consider the set

$$X^d = \Big\{p \in A^{q'} : \text{ there exists } x \in X \text{ such that}\\ p(t) = k(t)|x'(t)|^{q-2}x'(t),\ t \in [0,T]\Big\}.$$

Remark 1.3 From the definition of $\mathbf{A}$ and X, we derive that for each $x \in X$ there exists $p \in X^d$ such that $p'(\cdot) = -V_x(\cdot, x(\cdot))$ and therefore

$$\int_0^T \langle -p'(t), x(t)\rangle dt - \int_0^T V^*(t,-p'(t))dt = \int_0^T V(t,x(t))dt.$$

Indeed; fixing $x \in X$, Lemma 1.2 leads to the existence of $\widetilde{x} \in X$ such that $\widetilde{x} = \mathbf{A}x$. So that we have

$$\begin{aligned}|\widetilde{x}'(t)|^{q-2}\widetilde{x}'(t) &=[|\widetilde{\mathbf{A}}x(t)||\widetilde{\mathbf{A}}x(t)|^{-\frac{q-2}{q-1}}]^{q-2}\widetilde{\mathbf{A}}x(t)|\widetilde{\mathbf{A}}x(t)|^{-\frac{q-2}{q-1}}\\ &=[|\widetilde{\mathbf{A}}x(t)|^{\frac{1}{q-1}}]^{q-2}\widetilde{\mathbf{A}}x(t)|\widetilde{\mathbf{A}}x(t)|^{-\frac{q-2}{q-1}} = \widetilde{\mathbf{A}}x(t).\end{aligned}$$

Set $p(t) = k(t)|\widetilde{x}'(t)|^{q-2}\widetilde{x}'(t)$, $t \in [0,T]$. Since $\widetilde{x} \in X$, $p \in X^d$ we get what follows

$$\begin{aligned}(p(t))' &=(k(t)|\widetilde{x}'(t)|^{q-2}\widetilde{x}'(t))'\\ &=(k(t)\widetilde{\mathbf{A}}x(t))'\\ &=\Big(\alpha\int_{t_0}^T \frac{1}{k(r)}\int_r^T V_x(s,x(s))dsdg(r) + \int_t^T V_x(s,x(s))ds\Big)'\end{aligned}$$

$$= - V_x(t, x(t)).$$

Taking into account the properties of the subdifferential, we can infer that the required relation is satisfied.

Now we study the action functional

$$J(x) = \int_0^T (-V(t, x(t)) + \frac{1}{q} k(t)|x'(t)|^q)dt - \langle x(T), |x'(T)|^{q-2}x'(T)\rangle$$

on the set X. To show that $\bar{x} \in X$ realizing "min" is a critical point of J we develop a duality theory between J and dual to it J_D.

2 Duality results

In this section we shall develop the duality, which describes the relationship between the critical value of J and the infimum of the dual functional $J_D : X^d \to \mathbf{R}$,

$$J_D(p) = -\int_0^T \frac{1}{q'}[k(t)]^{1-q'}|p(t)|^{q'} + \int_0^T V^*(t, -p'(t))dt, \tag{2.1}$$

where $q' := q/(q-1)$.

Now we need a kind of perturbation of J and the convexity of a function considered on a whole space. Therefore, for each $x \in X$ the perturbation $J_x : L^q([0,T], \mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}$ for the functional J is defined as

$$\begin{aligned} J_x(y, a) &= \int_0^T (\breve{V}(t, x(t) + y(t)) - \frac{k(t)}{q}|x'(t)|^q)dt + \langle x(T) - a, |x'(T)|^{q-2}x'(T)\rangle \\ &= \int_0^T (\breve{V}(t, x(t) + y(t)) - \frac{k(t)}{q}|x'(t)|^q)dt \\ &\quad + \langle x(T), |x'(T)|^{q-2}x'(T)\rangle - \langle a, |x'(T)|^{q-2}x'(T)\rangle, \end{aligned}$$

with

$$\breve{V}(t, x) = \begin{cases} V(t, x) & \text{if } x \in \bar{P},\ t \in [0, T] \\ \infty & \text{if } x \notin \bar{P},\ t \in [0, T]. \end{cases}$$

We use this notation only for the purpose of duality and we will not change a notation for the functional J containing V or $\breve{V}$. It is associated with the fact that our all investigation reduce to the set X on which $\breve{V}(t, x) = V(t, x)$ for all $t \in [0, T]$. Let $J_x^\# : X^d \to \mathbb{R}$, where $x \in X$, be defined as a type of conjugate of

J_x:

$$\begin{aligned} J_x^{\#}(p) &= \sup_{y\in L^q([0,T],\mathbb{R}^n),a\in\mathbb{R}^n}\Big\{\int_0^T \langle y(t),p'(t)\rangle dt + \langle p(T),a\rangle - J_x(y,a)\Big\} \\ &= \sup_{y\in L^q([0,T],\mathbb{R}^n)}\Big\{\int_0^T \langle y(t),p'(t)\rangle dt - \int_0^T \breve{V}(t,x(t)+y(t))dt\Big\} \\ &\quad + \sup_{a\in\mathbb{R}^n}\big\{\langle a,p(T)\rangle + \langle a,|x'(T)|^{q-2}x'(T)\rangle\big\} \\ &\quad + \int_0^T \frac{k(t)}{q}|x'(t)|^q dt - \langle x(T),|x'(T)|^{q-2}x'(T)\rangle. \end{aligned} \tag{2.2}$$

Put $l:\mathbb{R}^n\to\{0,+\infty\}$, $l(b)=\begin{cases}0, & \text{for } b=0\\ +\infty, & \text{for } b\neq 0.\end{cases}$ We see at once that

$$\begin{aligned} J_x^{\#}(p) &= -\int_0^T \langle x(t),p'(t)\rangle dt + \frac{1}{q}\int_0^T k(t)|x'(t)|^q dt + \int_0^T V^*(t,p'(t))dt \\ &\quad - \langle x(T),|x'(T)|^{q-2}x'(T)\rangle + l(|x'(T)|^{q-2}x'(T)+p(T)) \\ &= \int_0^T \langle x'(t),p(t)\rangle dt - \langle x(T),p(T)\rangle \\ &\quad + \frac{1}{q}\int_0^T k(t)|x'(t)|^q dt + \int_0^T V^*(t,p'(t))dt \\ &\quad - \langle x(T),|x'(T)|^{q-2}x'(T)\rangle + l(|x'(T)|^{q-2}x'(T)+p(T)). \end{aligned} \tag{2.3}$$

Theorem 2.1 *For functionals $J$and J_D, we have the duality relation*

$$\inf_{x\in X} J(x) \le \inf_{p\in X^d} J_D(p).$$

Proof. Our proof starts with the observation that for all $p\in X^d$

$$\inf_{x\in X} J_x^{\#}(-p) = J_D(p) \tag{2.4}$$

and for all $x\in X$

$$\inf_{p\in X^d} J_x^{\#}(-p) \ge J(x). \tag{2.5}$$

Because X is not a linear space we need some trick to avoid calculation of the conjugate with respect to a nonlinear space. To this effect we use the special structure of the sets X^d and X. Indeed; fix $p\in X^d$. The definition of X^d implies that there exists $x_p\in X$ satisfying the equality

$$p(t) = k(t)|x_p'(t)|^{q-2}x_p'(t) \quad \text{for all } t\in[0,T]$$

and, in consequence

$$\int_0^T \langle x_p'(t),p(t)\rangle dt - \int_0^T \frac{k(t)}{q}|x_p'(t)|^q dt = \int_0^T \frac{1}{q'}[k(t)]^{1-q'}|p(t)|^{q'}dt.$$

An easy calculation yields

$$\begin{aligned}
&\int_0^T \langle x_p'(t), p(t)\rangle dt - \int_0^T \frac{k(t)}{q}|x_p'(t)|^q dt\\
&\le \sup_{x\in\{z\in X, p(T)=|z'(T)|^{q-2}z'(T)\}} \Big\{\int_0^T \langle x'(t), p(t)\rangle dt - \int_0^T \frac{k(t)}{q}|x(t)'|^q dt\Big\}\\
&\le \sup_{x\in X} \Big\{\int_0^T \langle x'(t), p(t)\rangle dt - \int_0^T \frac{k(t)}{q}|x(t)'|^q dt\Big\}\\
&\le \sup_{x'\in L^q([0,T],\mathbb{R}^n)} \Big\{\int_0^T \langle x'(t), p(t)\rangle dt - \int_0^T \frac{k(t)}{q}|x'(t)|^q dt\Big\}\\
&= \int_0^T \frac{1}{q'}[k(t)]^{1-q'}|p(t)|^{q'} dt\,.
\end{aligned}$$

Actually all inequalities above are equalities. Finally for $p \in X^d$, we obtain

$$\begin{aligned}
-\inf_{x\in X} J_x^{\#}(-p) &= \sup_{x\in X}(-J_x^{\#}(-p))\\
&= \sup_{x\in X}\Big\{\int_0^T \langle x'(t), p(t)\rangle dt - \frac{1}{q}\int_0^T k(t)|x'(t)|^q dt\\
&\quad - \int_0^T V^*(t, -p'(t))dt - \langle x(T), p(T)\rangle + \langle x(T), |x'(T)|^{q-2}x'(T)\rangle\\
&\quad - l(|x'(T)|^{q-2}x'(T) - p(T))\Big\}\\
&= -\int_0^T V^*(t, -p'(t))dt + \int_0^T \frac{1}{q'}[k(t)]^{1-q'}|p(t)|^{q'} dt = -J_D(p),
\end{aligned}$$

which proves our claim. To show the other assertion, fix $x \in X$. Remark 1.3 leads to the existence of $\bar p \in X^d$ such that

$$\int_0^T \langle -\bar p'(t), x(t)\rangle dt - \int_0^T V^*(t, -\bar p'(t))dt = \int_0^T \check V(t, x(t))dt.$$

Moreover,

$$\begin{aligned}
&\int_0^T \langle -\bar p'(t), x(t)\rangle dt - \int_0^T V^*(t, -\bar p'(t))dt\\
&\quad\le \sup_{p\in X^d}\{\int_0^T \langle -p'(t), x(t)\rangle dt - \int_0^T V^*(t, -p'(t))dt\}\\
&\quad\le \sup_{p'\in L^{q'}([0,T],\mathbb{R}^n)}\{\int_0^T \langle -p'(t), x(t)\rangle dt - \int_0^T V^*(t, -p'(t))dt\}\\
&\quad= \int_0^T \check V(t, x(t))dt.
\end{aligned}$$

Combining both results we infer

$$\begin{aligned}
\sup_{p\in X^d}(-J_x^{\#}(-p)) &= \sup_{p\in X^d}\Big\{\int_0^T \langle x(t),-p'(t)\rangle dt-\int_0^T V^*(t,-p'(t))dt \\
&\quad - l(|x'(T)|^{q-2}x'(T)-p(T))\Big\} \\
&\quad + \langle x(T),|x'(T)|^{q-2}x'(T)\rangle-\frac{1}{q}\int_0^T k(t)|x'(t)|^q dt \\
&\le \sup_{p\in X^d}\big\{-l(|x'(T)|^{q-2}x'(T)-p(T))\big\} \\
&\quad + \sup_{p\in X^d}\Big\{\int_0^T \langle x(t),-p'(t)\rangle dt-\int_0^T V^*(t,-p'(t))dt\Big\} \\
&\quad + \langle x(T),|x'(T)|^{q-2}x'(T)\rangle-\frac{1}{q}\int_0^T k(t)|x'(t)|^q dt \\
&= -\int_0^T(-\check{V}(t,x(t))+\frac{k(t)}{q}|x'(t)|^q)dt+\langle x(T),|x'(T)|^{q-2}x'(T)\rangle \\
&= -J(x);
\end{aligned}$$

so that $\inf_{p\in X^d} J_x^{\#}(-p)\ge J(x)$. Combining (2.4) and (2.5) we obtain the chain of relations

$$\inf_{x\in X}J(x)\le\inf_{x\in X}\inf_{p\in X^d}J_x^{\#}(-p)=\inf_{p\in X^d}\inf_{x\in X}J_x^{\#}(-p)=\inf_{p\in X^d}J_D(p).$$

Denote by $\partial J_x(y)$ the subdifferential of J_x. Calculating ∂J_x at 0 we obtain

$$\begin{aligned}
\partial J_x(0)=\Big\{p'\in L^{q'}([0,T],\mathbb{R}^n):\int_0^T V^*(t,p'(t))dt \\
+\int_0^T\check{V}(t,x(t))dt=\int_0^T\langle p'(t),x(t)\rangle dt\Big\}
\end{aligned}\tag{2.6}$$

Our task is now to prove a variational principle for "min" arguments.

Theorem 2.2 *Let $\bar{x}\in X$ be a minimizer of $J:X\to R$, $J(\bar{x})=\inf_{x\in X}J(x)$. Then there exists $\bar{p}\in X^d$ with $-\bar{p}'\in\partial J_{\bar{x}}(0)$, such that $\bar{p}$ satisfies*

$$J_D(\bar{p})=\inf_{p\in X^d}J_D(p).$$

Furthermore,

$$J_{\bar{x}}(0)+J_{\bar{x}}^{\#}(-\bar{p})=0 \tag{2.7}$$

$$J_D(\bar{p})-J_{\bar{x}}^{\#}(-\bar{p})=0. \tag{2.8}$$

Proof. From Theorem 2.1, $J(\bar{x}) \leq \inf_{p\in X^d} J_D(p)$, so to prove the first assertion we need to show only the existence of $\bar{p} \in X^d$ such that $J(\bar{x}) \geq J_D(\bar{p})$. To this effect we use Remark 1.3 which implies the existence of $\bar{p} \in X^d$ such that

$$\bar{p}'(t) = -V_x(t, \overline{x}(t)) \text{ a.e. on } [0, T]$$

and further

$$\int_0^T \langle -\bar{p}'(t), \overline{x}(t)\rangle dt - \int_0^T V^*(t, -\bar{p}'(t))dt = \int_0^T \check{V}(t, \overline{x}(t))dt. \tag{2.9}$$

Combining (2.9) and (2.6) we get the inclusion $-\bar{p}' \in \partial J_{\bar{x}}(0)$. On the other hand an easy computation gives the equality $J^*_{\bar{x}}(-\bar{p}', -\overline{p}(T)) = J^{\#}_{\bar{x}}(-\bar{p})$ (where $J^*_{\bar{x}}(-\bar{p}', -\overline{p}(T))$ denotes the Fenchel transform of $J_{\bar{x}}$ at $(-\bar{p}', -\overline{p}(T))$). Indeed; from the definitions of $J^*_{\bar{x}}$ and of $J^{\#}_{\bar{x}}$ we have

$$\begin{aligned}
&J^*_{\bar{x}}(-\bar{p}', -\overline{p}(T)) \\
&= \sup_{y\in L^q([0,T],\mathbb{R}^n), a\in\mathbb{R}^n} \Big\{ \int_0^T \langle y(t), -\overline{p}'(t)\rangle dt + \langle a, -\overline{p}(T)\rangle - J_{\bar{x}}(y, a)\Big\} \\
&= \sup_{y\in L^q([0,T],\mathbb{R}^n)} \Big\{ \int_0^T \langle y(t), -\overline{p}'(t)\rangle dt - \int_0^T \check{V}(t, \bar{x}(t) + y(t))dt\Big\} \\
&\quad + \sup_{a\in\mathbb{R}^n} \big\{\langle a, -\overline{p}(T)\rangle + \langle a, |\bar{x}'(T)|^{q-2}\bar{x}'(T)\rangle\big\} \\
&\quad + \int_0^T \frac{k(t)}{q} |\bar{x}'(t)|^q dt - \langle \bar{x}(T), |\bar{x}'(T)|^{q-2}\bar{x}'(T)\rangle \\
&= -\int_0^T \langle \bar{x}(t), -\overline{p}'(t)\rangle dt + \frac{1}{q}\int_0^T k(t)|\bar{x}'(t)|^q dt \\
&\quad + \int_0^T V^*(t, -\overline{p}'(t))dt - \langle \bar{x}(T), |\bar{x}'(T)|^{q-2}\bar{x}'(T)\rangle + l(|\bar{x}'(T)|^{q-2}\bar{x}'(T) - \overline{p}(T)) \\
&= J^{\#}_{\bar{x}}(-\overline{p}).
\end{aligned}$$

Therefore, we obtain (2.7). Finally

$$J(\bar{x}) = J^{\#}_{\bar{x}}(-\bar{p}) \geq \inf_{x\in X} J^{\#}_{x}(-\bar{p}) = J_D(\bar{p}),$$

where the last equality is due to (2.4). Hence $J(\bar{x}) \geq J_D(\bar{p})$. Now Theorem 2.1 leads to $J(\bar{x}) = J_D(\bar{p}) = \inf_{p\in X^d} J_D(p)$. (2.8) follows from (2.7) and the chain of equalities $J_{\bar{x}}(0) = -J(\bar{x}) = -J_D(\overline{p})$.

Corollary 2.3 *Let $\bar{x} \in X$ be such that $J(\bar{x}) = \inf_{x\in X} J(x)$. Then there exists $\bar{p} \in X^d$ such that the pair $(\bar{x}, \bar{p})$ satisfies the relations*

$$J_D(\bar{p}) = \inf_{p\in X^d} J_D(p) = \inf_{x\in X} J(x) = J(\bar{x}) \tag{2.10}$$

and

$$-(k(t)|\bar{x}'(t)|^{q-2}\bar{x}'(t))' = V_x(t, \bar{x}(t)), \tag{2.11}$$

Proof. From Theorems 2.1 and 2.2, we obtain immediately (2.10). To show (2.11) we use (2.7) and (2.8) obtaining the two equalities:

$$\int_0^T V(t,\bar{x}(t))dt + \int_0^T V^*(t,-\bar{p}'(t))dt - \int_0^T \langle \bar{x}(t), -\bar{p}'(t)\rangle dt = 0,$$

$$\int_0^T \frac{1}{q'}[k(t)]^{1-q'}|\bar{p}(t)|^{q'}dt + \int_0^T \frac{k(t)}{q}|\bar{x}'(t)|^q dt - \int_0^T \langle \bar{x}'(t), \bar{p}(t)\rangle dt = 0$$

and further

$$-\bar{p}'(t) = V_x(t,\bar{x}(t)) \text{ and } \bar{p}(t) = k(t)|\bar{x}'(t)|^{q-2}\bar{x}'(t)$$

which gives (2.11).

3 Variational principles and a duality gap for minimizing sequences

In this section we prove that a statement analogous to Theorem 2.2 is true for a minimizing sequence of J. It is worth noting that as a consequence of our duality we obtain for the first time in the superlinear case a measure of a duality gap between primal and dual functional for approximate solutions to (1.1) (for the sublinear case see [13]).

Theorem 3.1 *Let $\{x_j\}$, $x_j \in X$, $j = 1, 2, \ldots$, be a minimizing sequence for J. Then there exist $p_j \in X^d$ with $-p_j' \in \partial J_{x_j}(0)$ such that $\{p_j\}$ is a minimizing sequence for J_D i.e.,*

$$\inf_{x\in X} J(x) = \inf_{j\in N} J(x_j) = \inf_{j\in N} J_D(p_j) = \inf_{p\in X^d} J_D(p). \tag{3.1}$$

Furthermore,

$$J_{x_j}(0) + J^{\#}_{x_j}(-p_j) = 0,$$

and for all $\varepsilon > 0$ there exists $j_0 \in N$ such that for all $j \geq j_0$,

$$J_D(p_j) - J^{\#}_{x_j}(-p_j) \leq \varepsilon, \tag{3.2}$$

$$0 \leq J(x_j) - J_D(p_j) \leq \varepsilon. \tag{3.3}$$

Proof. We first show the boundedness of J on X. By the assumptions made on V and the definition of X we get for all $x \in X$, $0 \leq \int_0^T V_x(r,x(r))dr \leq c+e$ and, in consequence,

$$\|x\| \int_0^T V_x(r,x(r))dr \leq \|x\|(c+e),$$

where $\|x\| = \max\{|x(t)| : t \in [0,T]\}$. Therefore,

$$\int_0^T V_x(r,x(r))x(r)dr \leq \|x\|(c+e).$$

Hence, by the property of the subdifferential

$$\int_0^T V(r,0)dr - \int_0^T V(r,x(r))dr \geq \int_0^T V_x(r,x(r))[0-x(r)]dr \geq -\|x\|(c+e)$$

and further

$$\begin{aligned} J(x) \geq& \int_0^T \frac{k(t)}{q}|x'(t)|^q dt - \int_0^T V(t,0)dt \\ &- \|x\|(c+e) - \langle x(T), |x'(T)|^{q-2}x'(T)\rangle \\ \geq& \int_0^T \frac{k(t)}{q}|x'(t)|^q dt - \int_0^T V(t,0)dt \\ &- T\|x'\|_{L^q([0,T],\mathbb{R}^n)}(c+e) - |x(T)||x'(T)|^{q-1}. \end{aligned} \tag{3.4}$$

It is clear that the definition of J and (3.4) give the estimate

$$\infty > \inf_{j\in N} J(x_j) = a > -\infty.$$

Fix $\varepsilon > 0$. From the above there exists j_0 such that $J(x_j) - a < \varepsilon$, for all $j \geq j_0$. We can now proceed analogously to the proof of Theorem 2.2. First we observe that for each $j \in N$ there exists $p_j \in X^d$ such that $p_j'(t) = -V_x(t,x_j(t))$ a.e. on $[0,T]$. This implies

$$-p_j' \in \partial J_{x_j}(0) \quad \text{for all } j \in \mathbb{N},$$

which gives $J_{x_j}(0) + J^{\#}_{x_j}(-p_j) = 0$ for all $j \in \mathbb{N}$. As in the proof of Theorem 2.2, taking into account (2.4), this assertion yields

$$J(x_j) = J^{\#}_{x_j}(-p_j) \geq \inf_{x\in X} J^{\#}_x(-p_j) = J_D(p_j).$$

Thus, by Theorem 2.1, we deduce that (3.1) holds. Then (3.2) and (3.3) follow from (3.1) and two facts:

$$\begin{gathered} J_D(p_j) + \varepsilon \geq J(x_j) \quad \text{for} \quad j \geq j_0, \\ -J(x_j) = J_{x_j}(0) = -J^{\#}_{x_j}(-p_j) \text{ for each } j \in N. \end{gathered}$$

Theorem 3.1 gives the below corollary

Corollary 3.2 *Let $\{x_j\}$, $x_j \in X$, $j = 1,2,\dots,$ be a minimizing sequence for J. If $-p_j'(t) = V_x(t,x_j(t))$, then*

$$p_j(t) = p_j(T) - \int_t^T p_j'(s)ds, p_j(T) = |x_j'(T)|^{q-2}x_j'(T)$$

which belongs to X^d and $\{p_j\}$ is a minimizing sequence for J_D; i.e.,

$$\inf_{x\in X} J(x) = \inf_{j\in N} J(x_j) = \inf_{j\in N} J_D(p_j) = \inf_{p\in X^d} J_D(p).$$

Furthermore for a given $\varepsilon > 0$ and sufficiently large j,

$$J_D(p_j) - J^{\#}_{x_j}(-p_j) \leq \varepsilon,\ 0 \leq J(x_j) - J_D(p_j) \leq \varepsilon.$$

4 The existence result

In this section we prove the existence of a minimizer of $\bar{x} \in X$ of J on X; i.e., $J(\bar{x}) = \min_{x \in X} J(x)$ being a solution of (1.1) with the non-local boundary condition (1.2).

Theorem 4.1 *Under hypotheses (**H**) and (**H1**)-(**H3**) there exists $\bar{x} \in X$ such that $J(\bar{x}) = \min_{x \in X} J(x)$ and $\bar{x}$ satisfies (1.1)-(1.2).*

Proof. Let $S_z = \{x \in X, J(x) \le z\}$, where $z \in \mathbb{R}$. It follows from (3.4), the assumptions made on V, and the definition of X that S_z is nonempty for sufficiently large z and bounded with respect to the norm $\|x'\|_{L^q([0,T],\mathbb{R}^n)}$. This statement implies that S_z, $z \in \mathbb{R}$, are relatively weakly compact in A_{0b}. It is a well known fact that the functional J is weakly lower semicontinuous in A_0 and thus also in X. Therefore, there exists a sequence $\{x_n\}$, $x_n \in X$, such that $x_n \rightharpoonup \bar{x}$ weakly in A_{0b} with $\bar{x} \in A_{0b}$ (we use the fact that $\{x_n\}$ is uniformly convergent to $\bar{x}$) and $\liminf_{n\to\infty} J(x_n) \ge J(\bar{x})$. Moreover, the uniform convergence of $\{x_n\}$ to $\bar{x}$, implies that $\bar{x} \in \overline{X}$.

Our task is now to show that $\bar{x} \in X$. For this purpose, we recall from Corollary 3.2 that for

$$p_n'(t) = -V_x(t, x_n(t)), \quad t \in [0,T] \tag{4.1}$$

$p_n(t) = p_n(T) - \int_t^T p_n'(s)ds$, where $p_n(T) = |x_n'(T)|^{q-2} x_n'(T)$, belongs to X^d. Then $\{p_n\}$ is a minimizing sequence for J_D. We easily check that $\{p_n(T)\}$ is a bounded sequence and therefore we may assume (up to a subsequence) that it is convergent. From (4.1) we infer that $\{p_n'\}$ is a bounded sequence in L^1 norm and that it is pointwise convergent to

$$\bar{p}'(t) = -V_x(t, \bar{x}(t)). \tag{4.2}$$

Thus $\{p_n\}$ is uniformly convergent to $\bar{p}$ where $\bar{p}(t) = \bar{p}(T) - \int_t^T \bar{p}'(s)ds$. By Corollary 3.1 we also have that for $\varepsilon_n \to 0$ $(n \to \infty)$

$$0 \le \int_0^T \Big(\frac{1}{q'}[k(t)]^{1-q'}|p_n(t)|^{q'} + \frac{k(t)}{q}|x_n'(t)|^q\Big)dt - \int_0^T \langle x_n'(t), p_n(t)\rangle dt \quad \le \varepsilon_n$$

where $1/q + 1/q' = 1$, and so, taking the limit

$$0 = \int_0^T \frac{1}{q'}[k(t)]^{1-q'}|\bar{p}(t)|^{q'}dt + \lim_{n\to\infty}\int_0^T \frac{k(t)}{q}|x_n'(t)|^q dt - \int_0^T \langle \bar{x}'(t), \bar{p}(t)\rangle dt.$$

Therefore, in view of the property of Fenchel inequality,

$$0 = \int_0^T \frac{1}{q'}[k(t)]^{1-q'}|\bar{p}(t)|^{q'}dt + \int_0^T \frac{k(t)}{q}|\bar{x}'(t)|^q dt - \int_0^T \langle \bar{x}'(t), \bar{p}(t)\rangle dt$$

and further

$$\bar{p}(t) = k(t)|\bar{x}'(t)|^{q-2}\bar{x}'(t). \tag{4.3}$$

Thus $\bar{x} \in X$. Substituting (4.3) into (4.2) we can assert that the minimizer $\bar{x} \in X$ is the solution of problem (1.1)-(1.2).

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