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POSITIVE SOLUTIONS OF SINGULAR FOURTH-ORDER BOUNDARY-VALUE PROBLEMS

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Abstract. In this paper, we present necessary and sufficient conditions for the existence of positive $C^3[0,1]\cap C^4(0,1)$ solutions for the singular boundary-value problem

$$x''''(t) = p(t)f(x(t)), \quad t \in (0,1);$$
$$x(0) = x(1) = x'(0) = x'(1) = 0,$$

where $f(x)$ is either superlinear or sublinear, $p : (0,1) \to [0,+\infty)$ may be singular at both ends $t = 0$ and $t = 1$. For this goal, we use fixed-point index results.

1. Introduction

In this paper, we consider the fourth order differential equation

$$x''''(t) = p(t)f(x(t)), \quad t \in (0,1); \tag{1.1}$$
$$x(0) = x(1) = x'(0) = x'(1) = 0. \tag{1.2}$$

where $f(x)$ is either superlinear or sublinear, $p : (0,1) \to [0,+\infty)$ may be singular at both ends $t = 0$ and $t = 1$.

Recently, the existence and multiplicity of positive solutions of (1.1)-(1.2) in the non-singular case has been extensively studied in the literature; see [7, 5, 8] and references therein. However for singular fourth order boundary-value problems, the research has proceeded very slowly. Ma and Tisdell [6] studied the singular sublinear fourth order boundary value problems

$$x''''(t) = p(t)x^{\lambda}(t), \quad t \in (0,1); \tag{1.3}$$
$$x(0) = x(1) = x'(0) = x'(1) = 0. \tag{1.4}$$

where $\lambda \in (0,1)$ is given, and $p : (0,1) \to [0,\infty)$ may be singular at both ends $t = 0$ and $t = 1$. Base upon the method of lower and upper solutions, Ma and Tisdell showed that (1.3)-(1.4) has a positive solution in $C^2[0,1]\cap C^4(0,1)$ if and only if

$$0 < \int_0^1 t^{1+2\lambda}(1-t)^{1+2\lambda}p(t)dt < +\infty.$$

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Moreover, this positive solution is in $C^3[0,1] \cap C^4(0,1)$ if and only if

$$0 < \int_0^1 t^{2\lambda}(1-t)^{2\lambda} p(t)dt < +\infty.$$

But necessary and sufficient conditions for the existence of positive solution of superlinear BVPs (1.3)-(1.4) still remain unknown. In this paper, by using the fixed point index, we give some necessary and sufficient conditions for the existence of $C^3[0,1] \cap C^4(0,1)$ positive solutions to the singular boundary value problem (1.1)-(1.2).

In our discussion, by a $C^k[0,1]$ solution ($k = 2, 3$) of (1.1)-(1.2) we mean a function $y(t) \in C^k[0,1] \cap C^4(0,1)$ which satisfies (1.2) and (1.1) on (0,1). We call a solution $y(t)$ is a positive solution if $y(t) > 0$ for $t \in (0,1)$.

This paper is organized as follows. Section 2 gives some preliminary lemmas corresponding to (1.1)-(1.2). Section 3 is devoted to the the existence of $C^3[0,1] \cap C^4(0,1)$ positive solutions for (1.1)-(1.2). At the end of this section we state some lemmas of the fixed point theory, which will be used in Section 3.

Let E be a Banach space, P a cone in E, Ω a bounded open set in E.

Lemma 1.1 ([3]). *Let $\theta \in \Omega$, $A : \overline{\Omega} \cap P \to P$ be completely continuous. Suppose that there exists $u_0 \in P \backslash \{\theta\}$ such that*

$$u - Au \neq \mu u_0, \ \forall \ u \in \partial\Omega \cap P, \ \mu \geq 0,$$

then the fixed point index $i(A, \ \Omega \cap P, \ P) = 0$.

Lemma 1.2 ([3]). *Let $\theta \in \Omega$, $A : \overline{\Omega} \cap P \to P$ be completely continuous. Suppose that*

$$Au \neq \mu u, \quad \forall \ u \in \partial\Omega \cap P, \quad \mu \geq 1,$$

then the fixed point index $i(A, \ \Omega \cap P, \ P)$ is equal to 1.

2. Preliminaries

We give some notations, which will be used below. Let $C[0,1]$, $C^k[0,1]$ and $L^1[0,1]$ be the classical Banach spaces with their usual norms $\|\cdot\|$, $\|\cdot\|_{C^k}$ and $\|\cdot\|_{L^1}$, respectively. Let $AC[0,1]$ be the space of all absolutely continuous functions on [0,1]. Let

$$AC^k[0,1] = \{u \in C^k[0,1] : u^{(k)} \in AC[0,1]\}.$$

Clearly $AC^0[0,1] = AC[0,1]$. Let I be an interval of R. We denote by $L^1_{\text{loc}}I$ the spaces of functions defined by

$$L^1_{\text{loc}}I = \{u : I \to R : u|_{[c,d]} \in L^1[c,d] \ \text{ for every compact interval } [c,d] \subset I\}.$$

For $n, m \in N$, we denote by $X[n,m]$ the Banach space

$$X[n,m] = \{\varphi \in L^1_{\text{loc}}(0,1) \Big| \int_0^1 t^n(1-t)^m |\varphi(t)| dt < +\infty\},$$

equipped with the norm

$$\|\varphi\|_{X[n,m]} = \int_0^1 t^n(1-t)^m |\varphi(t)| dt.$$

Now let $G(t,s)$ be the Green's function of the linear problem

$$x''''(t) = 0, \quad t \in (0,1);$$
$$x(0) = x(1) = x'(0) = x'(1) = 0,$$

which can be explicitly given by

$$G(t,s)=\frac{1}{6}\begin{cases} t^2(1-s)^2[(s-t)+2(1-t)s], & 0\le t\le s\le 1,\\ s^2(1-t)^2[(t-s)+2(1-s)t], & 0\le s\le t\le 1.\end{cases}$$

It is clear that for all $t,s\in[0,1]$,

$$\frac{1}{3}t^2(1-t)^2s^2(1-s)^2\le G(t,s)\le \frac{1}{2}t^2(1-t)^2,\ G(t,s)\le \frac{1}{2}s^2(1-s)^2. \tag{2.1}$$

Suppose that $\varphi\in X[2,2]$. We denote

$$T(\varphi)(t)=\int_0^1 G(t,s)\varphi(s)ds,$$

i.e.

$$\begin{aligned} T(\varphi)(t)=\frac{1}{6}\int_0^t s^2(1-t)^2[(t-s)+2(1-s)t]\varphi(s)ds\\ +\frac{1}{6}\int_t^1 t^2(1-s)^2[(s-t)+2(1-t)s]\varphi(s)ds.\end{aligned}$$

Lemma 2.1 ([6])**.** *Let $\varphi\in X[2,2]$. Then $T(\varphi)(t)$, $[T(\varphi)]'(t)$, $[T(\varphi)]''(t)$, $[T(\varphi)]'''(t)$ are $AC_{\rm loc}(0,1)\cap C^1(0,1)$, and*

$$[T(\varphi)]''''(t)=\varphi(t),\quad a.e.\ t\in(0,1).$$

Lemma 2.2 ([6])**.** *Let $\varphi\in X[2,2]$. Then*

$$T(\varphi)(0)=T(\varphi)(1)=T(\varphi)'(0)=T(\varphi)'(1)=0.$$

Lemma 2.3 ([6])**.** *Let $\varphi\in L^1(0,1)$. Then $[T(\varphi)](t)\in AC^3[0,1]$.*

3. Main Result

We shall assume the following conditions:

(H1) $f:[0,\infty)\to[0,\infty)$ is continuous and nondecreasing in x, $f(x)>0$ on $(0,\infty)$, and there exists $\lambda>1$ such that

$$f(cx)\le c^{\lambda}f(x),\quad \forall\ c\ge 1,\ x\in[0,+\infty). \tag{3.1}$$

(H2) $p:(0,1)\to[0,\infty)$ is continuous, $\int_0^1 s^2(1-s)^2p(s)ds<+\infty$, and there exists $\theta\in(0,1/2)$ such that

$$0<\int_\theta^{1-\theta} s^2(1-s)^2p(s)ds.$$

(H3) $0\le\limsup_{x\to0+}\frac{f(x)}{x}<M_1, m_1<\liminf_{x\to+\infty}\frac{f(x)}{x}\le+\infty$, where

$$M_1=(\max_{t\in[0,1]}\int_0^1 G(t,s)p(s)ds)^{-1},$$

$$m_1=(\min_{t\in[\theta,1-\theta]}\int_\theta^{1-\theta} G(t,s)p(s)ds)^{-1}.$$

Theorem 3.1. *Under assumptions* (H1)-(H3)*, a necessary and sufficient condition for* (1.1)-(1.2) *to have a positive solution in $C^3[0,1]\cap C^4(0,1)$ is that*

$$\int_0^1 p(s)f(s^2(1-s)^2)ds<+\infty. \tag{3.2}$$

Remark 3.2. Inequality (3.1) implies

$$f(cx) \geq c^{\lambda} f(x), \quad \forall\ c \in (0,1),\ x \in [0,+\infty). \tag{3.3}$$

Conversely, (3.3) implies (3.1).

Remark 3.3. (H2) is equivalent to

(H2') $p \in C((0,1),[0,+\infty)) \cap X[2,2]$, and there exists $t_0 \in (0,1)$ with $p(t_0) > 0$.

Proof of Theorem 3.1. Necessity. Let $x \in C^2[0,1] \cap C^4(0,1)$ be a positive solution of (1.1) and (1.2). Then by the fact

$$\begin{aligned} x''(t) = \frac{1}{6}\int_0^t &\{2s^2[(t-s)+2(1-s)t] - 4s^2(1-t)[1+2(1-s)]\}p(s)f(x(s))ds \\ &+ \frac{1}{6}\int_t^1 \{2(1-s)^2[(s-t)+2(1-t)s] + 4t(1-s)^2[-1-2s]\}p(s)f(x(s))ds. \end{aligned}$$

we have that

$$\begin{aligned} x''(0) &= \int_0^1 (1-s)^2 s p(s) f(x(s))ds > 0, \\ x''(1) &= \int_0^1 s^2(1-s)p(s)f(x(s))ds > 0. \end{aligned}$$

and accordingly, there exist $I_1, I_2 \in (0,+\infty)$ such that

$$I_1 t^2(1-t)^2 \leq x(t) \leq I_2 t^2(1-t)^2, \quad t \in [0,1].$$

Let $c_1 \geq \max\{1, 1/I_1\}$, then

$$t^2(1-t)^2 \leq c_1 x(t), \quad t \in [0,1].$$

So by (H1),

$$\begin{aligned} \int_0^1 p(s)f(s^2(1-s)^2)ds &\leq \int_0^1 p(s)f(c_1x(s))ds \\ &\leq c_1^{\lambda}\int_0^1 p(s)f(x(s))ds \\ &= c_1^{\lambda}\int_0^1 x''''(s)ds \\ &\leq c_1^{\lambda}[x'''(1) - x'''(0)] < \infty. \end{aligned}$$

On the other hand, if $c_2 \leq \min\{1/2, 1/I_2\}$, then

$$t^2(1-t)^2 \geq c_2 x(t), \quad t \in [0,1].$$

So by (H1) and (3.3),

$$\int_0^1 p(s)f(s^2(1-s)^2)ds \geq \int_0^1 p(s)f(c_2x(s))ds \geq c_2^{\lambda}\int_0^1 p(s)f(x(s))ds \geq 0$$

Notice that $\int_0^1 p(s)f(x(s))ds > 0$, for otherwise $p(s)f(x(s)) \equiv 0$ on (0,1). In this case (1.1)-(1.2) has only trivial solution $x \equiv 0$. This contradicts the assumption that x is a positive solution. Thus (3.2) holds.

Sufficiency. Suppose that (3.2) holds. we define a set $P \subset C[0,1]$ by

$$P = \big\{x \in C[0,1] : \exists c_x > 0,\ 0 \leq x(t) \leq c_x t^2(1-t)^2,$$

$$x(t) \geq \frac{2}{3}t^2(1-t)^2\|x\|,\ t \in [0,1]\big\}.$$

By its definition, it is easy to verify that P is a cone. We define $T : P \to C[0,1]$ by

$$T(x)(t) = \int_0^1 G(t,s)p(s)f(x(s))ds, \quad t \in [0,1],\ x \in P.$$

In the following, we prove that $T : P \to P$ is completely continuous.

1. We first show that $T : P \to P$ is well defined. For $x \in P$, there exist $c_x \geq 1$ such that $0 \leq x(t) \leq c_x t^2(1-t)^2$ and for $t \in [0,1]$, by (2.1), we get

$$(Tx)(t) = \int_0^1 G(t,s)p(s)f(x(s))ds \leq \frac{1}{2}c_x^{\lambda}t^2(1-t)^2 \int_0^1 p(s)f(s^2(1-s)^2)ds.$$

This implies that $p(t)f(x(t)) \in L^1[0,1]$, by Lemma 2.3, we have $Tx \in C[0,1]$. Let $c_{Tx} = \frac{1}{2}c_x^{\lambda}\int_0^1 p(s)f(s^2(1-s)^2)ds$. By (3.2), we know $c_{Tx} > 0$, so

$$(Tx)(t) \leq c_{Tx}t^2(1-t)^2, \quad t \in [0,1].$$

In addition, for $t \in [0,1]$, by (2.1), we get

$$(Tx)(t) = \int_0^1 G(t,s)p(s)f(x(s))ds \geq \frac{1}{3}t^2(1-t)^2 \int_0^1 s^2(1-s)^2p(s)f(x(s))ds, \tag{3.4}$$

and

$$(Tx)(t) = \int_0^1 G(t,s)p(s)f(x(s))ds \leq \frac{1}{2}\int_0^1 s^2(1-s)^2p(s)f(x(s))ds.$$

Hence

$$\|Tx\| \leq \frac{1}{2}\int_0^1 s^2(1-s)^2p(s)f(x(s))ds.$$

Combining the above with (3.4), we have

$$(Tx)(t) \geq \frac{1}{3}t^2(1-t)^2 \int_0^1 s^2(1-s)^2p(s)f(x(s))ds \geq \frac{2}{3}t^2(1-t)^2\|Tx\|,$$

i.e., $T(P) \subset P$.

2. We show that $T : P \to P$ is compact. Let $D \subset P$ be bounded, i.e., $\|x\| \leq M$ for all $x \in D$ and some $M > 0$. It is clear that if $x \in P$ satisfies $x \in D$, by (H2) we have

$$|(Tx)(t)| \leq \frac{1}{2}\int_0^1 s^2(1-s)^2p(s)f(x(s))ds \leq \frac{1}{2}\int_0^1 s^2(1-s)^2p(s)f(M)ds.$$

So $T(D)$ is uniformly bounded.

Next we prove that $\|(Tx)'\| \leq N$ for all $x \in D$ and some $N > 0$. In fact, for $x \in D$. By Lemma 2.3, we know $Tx \in C^2[0,1]$ and

$$\begin{aligned}
&|(Tx)'(t)| \\
&= \Big|\frac{1}{6}\int_0^t \{-2s^2(1-t)[(t-s)+2(1-s)t] + s^2(1-t)^2[1+2(1-s)]\}p(s)f(x(s))ds \\
&\quad + \frac{1}{6}\int_t^1 \{2t(1-s)^2[(s-t)+2(1-t)s] + t^2(1-s)^2[-1-2s]\}p(s)f(x(s))ds\Big| \\
&\leq \frac{1}{6}\int_0^t \{2s^2(1-s)[(1-s)+2(1-s)] + s^2(1-s)^2[1+2(1-s)]\}p(s)f(M)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{6}\int_t^1\{2t(1-s)^2[s+2s]+s^2(1-s)^2[1+2s]\}p(s)f(M)ds\\
&\le\frac{9}{6}\int_0^t s^2(1-s)^2p(s)f(M)ds \quad +\frac{9}{6}\int_t^1 s^2(1-s)^2p(s)f(M)ds\\
&=\frac{3}{2}\int_0^1 s^2(1-s)^2p(s)f(M)ds=N.
\end{aligned}
$$

This means that $T(D)$ is equicontinuous. From the Ascoli-Arzela theorem, $T(D)$ is relatively compact. This completes the proof that T is compact.

3. We prove $T:P\to P$ is continuous. Assume that $x_n,x\in P$ and $x_n\to x$. Then there exists $M>0$ such that $\|x\|\le M,\ \|x_n\|\le M$ for every $n>0$. Since $f(x)$ is continuous, we have

$$|f(x_n(s))-f(x(s))|\to 0,\quad \text{as } n\to\infty,\quad \forall\ s\in[0,1],$$

and

$$|f(x_n(s))-f(x(s))|\le 2f(M),\quad \forall\ t\in[0,1],\ (n=1,2,3\dots).$$

Consequently, for all $t\in[0,1]$,

$$\|(Tx_n)(t)-(Tx)(t)\|\le\int_0^1 s^2(1-s)^2p(s)|f(x_n(s))-f(x(s))|ds\to 0. \tag{3.5}$$

We now show

$$\|Tx_n-Tx\|\to 0\quad \text{as } n\to\infty). \tag{3.6}$$

If (3.6) is not true, then there exist a positive number $\varepsilon>0$ and a sequence $\{x_{n_i}\}\subset\{x_n\}$ such that

$$\|Tx_{n_i}-Tx\|\ge\varepsilon,\quad (i=1,2,3\dots). \tag{3.7}$$

Since $\{x_n\}$ is bounded, $\{Tx_n\}$ is relatively compact and there is a subsequence of $\{Tx_{n_i}\}$ which converges in $C[0,1]$ to some $y\in C[0,1]$. Without loss of generality, we may assume that $\{Tx_{n_i}\}$ itself converges to y:

$$\|Tx_{n_i}-y\|\to 0,\quad \text{as } i\to\infty. \tag{3.8}$$

By virtue of (3.5) and (3.8), we have $y=Tx$, and so, (3.8) contradicts (3.7). Hence, (3.6) holds, and the continuity of T is proved. To sum up, we have proved $T:P\to P$ is completely continuous.

For all $x\in P$, from the above proof, we know $Tx\in P$, By Lemma 2.1 and Lemma 2.2, the fixed point of the equation

$$Tx=x,\quad x\in P.$$

is the solution of (1.1)-(1.2). Next we will look for the fixed point.

By the first part of (H3), there exist $1>r>0$, $\varepsilon>0$ such that $0<u<r$ implies $f(x)/x\le(M_1-\varepsilon)$. Therefore, we have

$$f(x)\le(M_1-\varepsilon)x\le(M_1-\varepsilon)r,\ \ 0<x\le r.$$

Set $B_r=\{x\in C[0,1]:\|x\|<r\}$. For $\forall\ x\in\partial B_r\cap P$, we have

$$
\begin{aligned}
\|Tx\|&=\max_{t\in[0,1]}\int_0^1 G(t,s)p(s)f(x(s))ds\le(M_1-\varepsilon)r\max_{t\in[0,1]}\int_0^1 G(t,s)p(s)ds\\
&\le r-\varepsilon r\max_{t\in[0,1]}\int_0^1 G(t,s)p(s)ds<r.
\end{aligned}
$$

Then for $x \in \partial B_r \cap P$ and $\mu \geq 1$, we have

$$Tx \neq \mu x.$$

In not, there exist $x_0 \in \partial B_r \cap P$ and $\mu_0 \geq 1$ such that $Tx_0 = \mu_0 x_0$, then $\|Tx_0\| \geq \|x_0\|$, which is a contradiction. According to Lemma 1.2, we have

$$i(T,\ B_r \cap P,\ P) = 1. \tag{3.9}$$

By the second part of (H3), $m_1 < \liminf_{x\to+\infty} \frac{f(x)}{x} \leq +\infty$, there exist $R_1 > \max\{\theta r, 1\}$, $\varepsilon_1 > 0$ such that

$$f(x) \geq (m_1 + \varepsilon_1)x, \quad x \geq R_1.$$

Let $R_2 > \frac{3R_1}{2\theta^2(1-\theta)^2}$, and $B_{R_2} = \{x \in C[0,1] : \|x\| < R_2\}$, then

$$\min_{t\in[\theta,1-\theta]} x(t) \geq \min_{t\in[\theta,1-\theta]} \frac{2}{3} t^2(1-t)^2\|x\| \geq R_1, \quad \forall\ x \in \partial B_{R_2} \cap P.$$

We now prove that

$$x - Tx \neq \mu t^2(1-t)^2, \quad \text{for } \mu \geq 0 \text{ and } x \in \partial B_{R_2} \cap P.$$

If not, then there are $\mu_1 \geq 0$ and $x_1 \in \partial B_{R_2} \cap P$ such that $x_1 - Tx_1 = \mu_1 t^2(1-t)^2$. So $\mu_1 > 0$, otherwise there is a fixed point in $\partial B_{R_2} \cap P$ and this would complete the proof. Let $\eta = \min_{t\in[\theta,1-\theta]} x_1(t)$. Then if $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned}
x_1(t) &= \int_0^1 G(t,s)p(s)f(x_1(s))ds + \mu_1 t^2(1-t)^2 \\
&\geq \int_\theta^{1-\theta} G(t,s)p(s)f(x_1(s))ds + \mu_1 t^2(1-t)^2 \\
&\geq (m_1+\varepsilon_1)\int_\theta^{1-\theta} G(t,s)p(s)x_1(s)ds + \mu_1 t^2(1-t)^2 \\
&\geq \eta(m_1+\varepsilon_1)\int_\theta^{1-\theta} G(t,s)p(s)ds + \mu_1 t^2(1-t)^2 \\
&\geq \eta + \eta\varepsilon_1\int_\theta^{1-\theta} G(t,s)p(s)ds + \mu_1 t^2(1-t)^2.
\end{aligned}$$

Therefore,

$$x_1(t) > \eta, \quad t \in [\theta, 1-\theta],$$

which is a contradiction. According to Lemma 1.1, we get

$$i(T, B_{R_2} \cap P,\ P) = 0. \tag{3.10}$$

By (3.9) and (3.10), we have

$$i(T,\ (B_{R_2}\overline{\backslash B_r}) \cap P,\ P) = i(T,\ B_{R_2} \cap P,\ P) - i(T,\ B_r \cap P,\ P) = -1.$$

Then T has at least a fixed point x^* in $(B_{R_2}\overline{\backslash B_r}) \cap P$ satisfying $0 < r \leq \|x^*\| \leq R_2$. Since $x^* \in P$, there exists $r_{x^*} > 1$ such that $x^* \leq r_{x^*} t^2(1-t)^2$, then

$$\begin{aligned}
\int_0^1 p(s)f(x^*(s))ds &\leq \int_0^1 p(s)f(r_{x^*}s^2(1-s)^2)ds \\
&\leq r_{x^*}^\lambda \int_0^1 p(s)f(s^2(1-s)^2)ds < +\infty,
\end{aligned}$$

that is $p(t)f(x^*(t)) \in L^1(0,1)$, then by Lemma 2.3, we have $x^* \in AC^3[0,1]$, so x^* is a $C^3[0,1] \cap C^4(0,1)$ positive solution of (1.1)-(1.2). This completes the proof of sufficiency. □

Corollary 3.4. *Let p be as above, $0 < \int_0^1 s^2(1-s)^2 p(s)ds < +\infty$, and $\lambda > 1$. Then BVP* (1.3)-(1.4) *has at least a positive solution in $C^3[0,1] \cap C^4(0,1)$*

Proof. The hypotheses on the function $p(s)$ implies $0 < \int_0^1 (s(1-s))^{2\lambda} p(s)ds < +\infty$ for $\lambda > 1$. The result now follows from Theorem 3.1. □

Theorem 3.5. *Assume that (H1) and (H2) are satisfied. If*

$$\lim_{x\to 0+} \frac{f(x)}{x} = 0, \quad \lim_{x\to +\infty} \frac{f(x)}{x} + \infty,$$

Then a necessary and sufficient condition for (1.1)-(1.2) *to have a positive solution in $C^3[0,1] \cap C^4(0,1)$ is that*

$$\int_0^1 p(s)f(s^2(1-s)^2)ds < +\infty.$$

Proof. Clearly (H1)-(H3) hold, and result follows from Theorem 3.1. We omit the detail. □

Next, we shall study (1.1)-(1.2) in the sublinear case. We assume:

(H1') $f : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing in x, $f(x) > 0$ on $(0,\infty)$, and there exists $0 < \lambda_1 < 1$ such that

$$f(cx) \geq c^{\lambda_1} f(x), \quad \forall\ c \in (0,1),\ x \in [0,+\infty).$$

(H3') $0 \leq \limsup_{x\to+\infty} \frac{f(x)}{x} < M_1, m_1 < \liminf_{x\to 0+} \frac{f(x)}{x} \leq +\infty$, where

$$M_1 = (\max_{t\in[0,1]} \int_0^1 G(t,s)p(s)ds)^{-1},$$

$$m_1 = (\min_{t\in[\theta,1-\theta]} \int_\theta^{1-\theta} G(t,s)p(s)ds)^{-1}.$$

Theorem 3.6. *Assume* (H1'), (H2), *and* (H3'). *Then a necessary and sufficient condition for* (1.1)-(1.2) *to have a positive solution in $C^3[0,1] \cap C^4(0,1)$ is that*

$$\int_0^1 p(s)f(s^2(1-s)^2)ds < +\infty. \tag{3.11}$$

Proof. By (H1'), we have $f(cx) \leq c^{\lambda_1} f(x)$, $c \geq 1$, $x \in [0,+\infty)$. The proof of necessity is almost the same as that in Theorem 3.1.

We will show the roof of the sufficiency. We base the proof on the argument in Theorem 3.1 and need only show completely continuous operator $T : P \to P$ has a fixed point.

By the first part of (H3'), there are $R_3 > 1$, $\varepsilon_3 > 0$ such that $x \geq R_3$ implies $f(x) \leq (M_1 - \varepsilon_3)x$. Let $M = \max\{f(x) : 0 \leq x \leq R_3\}$, then

$$f(x) \leq (M_1 - \varepsilon_3)x + M, \quad x \in [0,+\infty).$$

Choose $R_4 > \max\{M\varepsilon_3^{-1}, 1\}$. Let $B_{R_4} = \{x \in C[0,1] : \|x\| < R_4\}$. Then for all $x \in \partial B_{R_4} \cap P$, we have

$$\|Tx\| = \max_{t\in[0,1]} \int_0^1 G(t,s)p(s)f(x(s))ds$$

$$
\begin{aligned}
&\le (M+(M_1-\varepsilon_3)\|x\|)\max_{t\in[0,1]}\int_0^1 G(t,s)p(s)ds\\
&\le M_1R_4\max_{t\in[0,1]}\int_0^1 G(t,s)p(s)ds+(M-\varepsilon_3R_4)\frac{1}{2}\int_0^1 s^2(1-s)^2p(s)ds\\
&= R_4+(M-\varepsilon_3R_4)\frac{1}{2}\int_0^1 s^2(1-s)^2p(s)ds\\
&< R_4=\|x\|.
\end{aligned}
$$

So it is easy to know that $Tx\neq\mu x$ for $x\in\partial B_{R_4}\cap P$ and $\mu\ge 1$. According to Lemma 1.2, we have

$$
i(T,\ B_{R_4}\cap P,\ P)=1. \tag{3.12}
$$

By the second part of (H3'), $m_1<\liminf_{x\to+\infty}\frac{f(x)}{x}\le+\infty$, there exist $0<r_1<1$, $\varepsilon_5>0$ such that $0<x<r_1$ implies

$$
\frac{f(x)}{x}\ge(m_1+\varepsilon_5)x.
$$

Let $B_{r_1}=\{x\in C[0,1]:\|x\|<r_1\}$. We now prove that

$$
x-Tx\neq\mu t^2(1-t)^2,\ \text{for } \mu\ge 0 \text{ and } x\in\partial B_{R_1}\cap P.
$$

If not, there are $\mu_2\ge$ and $x_2\in\partial B_{r_1}\cap P$ such that $x_2-Tx_2=\mu_2t^2(1-t)^2$. So $\mu_2>0$, otherwise there is a fixed point in $\partial B_{r_1}\cap P$ and this would complete the proof. Let $\eta=\min_{t\in[\theta,1-\theta]}x_2(t)$. Then if $t\in[\theta,1-\theta]$, we have

$$
\begin{aligned}
x_2(t)&=\int_0^1 G(t,s)p(s)f(x_2(s))ds+\mu_2t^2(1-t)^2\\
&\ge\int_\theta^{1-\theta} G(t,s)p(s)f(x_2(s))ds+\mu_2t^2(1-t)^2\\
&\ge(m_1+\varepsilon_5)\int_\theta^{1-\theta} G(t,s)p(s)x_2(s)ds+\mu_2t^2(1-t)^2\\
&\ge\eta(m_1+\varepsilon_5)\int_\theta^{1-\theta} G(t,s)p(s)ds+\mu_2t^2(1-t)^2\\
&\ge\eta+\eta\varepsilon_5\int_\theta^{1-\theta} G(t,s)p(s)ds+\mu_2t^2(1-t)^2.
\end{aligned}
$$

Therefore,

$$
x_2(t)>\eta,\quad t\in[\theta,1-\theta].
$$

which is a contradiction. According to Lemma 1.1, we get

$$
i(T,\ B_{r_1}\cap P,\ P)=0. \tag{3.13}
$$

By (3.12) and (3.13), we have

$$
i(T,\ (B_{R_4}\overline{\backslash B_{r_1}})\cap P,\ P)=i(T,\ B_{R_4}\cap P,\ P)-i(T,\ B_{r_1}\cap P,\ P)=1.
$$

Then T has at least a fixed point x^* in $(B_{R_4}\overline{\backslash B_{r_1}})\cap P$, satisfying $0<r_1\le\|x^*\|\le R_4$, and x^* is also a $C^3[0,1]\cap C^4(0,1)$ positive solution of (1.1)-(1.2). This completes the proof. $\square$

Corollary 3.7. *Let p be as above, $0 < \int_0^1 s^2(1-s)^2 p(s)ds < +\infty$, and $0 < \lambda < 1$. Then a necessary and sufficient condition for* (1.3)-(1.4) *to have a positive solution in $C^3[0,1] \cap C^4(0,1)$ is that*
$$0 < \int_0^1 (s(1-s))^{2\lambda} p(s)ds < +\infty.$$

Example 3.8. The singular boundary-value problem
$$x''''(t) = t^{-5/2}(1-t)^{-4/3}x^\lambda, \quad t \in (0,1),\ \lambda > 1,$$
$$x(0) = x(1) = x'(0) = x'(1) = 0,$$
has a solution $x \in C^3[0,1] \cap C^4(0,1)$ with $x(t) > 0$ on $(0,1)$. To see this, we will apply Theorem 3.1 with $p(t) = t^{-5/2}(1-t)^{-4/3}$, $f(x) = x^\lambda$ $(\lambda > 1)$. Clearly (H1) holds. Note that
$$\int_0^1 p(s)s^2(1-s)^2 ds = \int_0^1 s^{-1/2}(1-s)^{2/3} ds \le 2.$$
Consequently (H2) holds (with $\theta = 1/4$). Also note that (H3) holds since
$$\lim_{x\to 0+} \frac{f(x)}{x} = 0, \quad \lim_{x\to +\infty} \frac{f(x)}{x} = +\infty.$$
Finally note that $\int_0^1 p(s)f(s^2(1-s)^2)ds = \int_0^1 p(s)(s(1-s))^{2\lambda}ds < +\infty$. The result now follows from Theorem 3.1.

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