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THE MAXIMUM PRINCIPLE FOR EQUATIONS WITH COMPOSITE COEFFICIENTS

GARY M. LIEBERMAN

Abstract. It is well-known that the maximum of the solution of a linear elliptic equation can be estimated in terms of the boundary data provided the coefficient of the gradient term is either integrable to an appropriate power or blows up like a small negative power of distance to the boundary. Apushkinskaya and Nazarov showed that a similar estimate holds if this term is a sum of such functions provided the boundary of the domain is sufficiently smooth and a Dirichlet condition is prescribed. We relax the smoothness of the boundary and also consider non-Dirichlet boundary conditions using a variant of the method of Apushkinskaya and Nazarov. In addition, we prove a Holder estimate for solutions of oblique derivative problems for nonlinear equations satisfying similar conditions.

Introduction

We are concerned here with various estimates for solutions of the linear elliptic equation

$$a^{ij}D_{ij}u + b^iD_iu + cu = f$$

in some domain $\Omega \subset \mathbb{R}^n$ under weak hypotheses on the coefficients a^{ij}, b^i, and c. We always assume that $[a^{ij}]$ is a positive-definite matrix-valued function with minimum eigenvalue λ and determinant $\mathcal{D}$, and that $c \le 0$. It was shown in [1] that, if $u = 0$ on $\partial\Omega$, then u satisfies an estimate of the form

$$\sup_\Omega u \le C(n, \|b/\lambda\|_n)(\operatorname{diam}\Omega)\left\|\frac{f}{\mathcal{D}^{1/n}}\right\|_n .$$

On the other hand, if $\partial\Omega$ is sufficiently smooth and the vector b grows like a (sufficiently small, negative) power of the distance to $\partial\Omega$, then a similar estimate holds via the maximum principle. To state this estimate, let us use d to denote distance to $\partial\Omega$. If there are positive constants $\alpha < 1$, B_0, and μ such that $\partial\Omega \in C^{1,\alpha}$, $|b| \le B_0\lambda d^{\alpha-1}$, and $|a^{ij}| \le \mu\lambda$, then

$$\sup_\Omega u \le C(n, \alpha, B_0, \mu, \Omega) \sup_\Omega |fd^{1-\alpha}/\lambda|.$$

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(Although this precise form of the maximum principle does not seem to be stated anywhere, we point out that its proof is contained in arguments that go back as least as far as [3]; see also [6].) In [2], the authors showed that a composite condition on b leads to an analogous maximum principle. Specifically, assume that there are positive constants α, B_0, and μ such that $|a^{ij}| \le \mu\lambda$, $b = b_1 + b_2$ with

$$\left\|\frac{b_1}{\lambda}\right\|_n + \sup_{x\in\Omega}\frac{|b_2(x)|}{\lambda(x)}(x^n)^{1-\alpha} \le B_0$$

and $|x'| < R$ and $x^n > 0$ in Ω for some $R \in (0,1)$. Then [2, Theorem 2.1′] states that

$$\sup_\Omega u \le C(B_0, n, \alpha, \mu)R\left\|\frac{f}{\lambda}\right\|_n.$$

(Actually, the proof of [2, Theorem 2.1′] seems to need a smallness condition on B_0 for the reasons discussed in Remark 3.3 of [5].) These estimates were used in [2] to infer the regularity of solutions to the Dirichlet problem for nonlinear elliptic equations with coefficients satisfying similar composite conditions; parabolic problems are also considered in [2].

Our goal here is to prove a maximum principle for problems with composite coefficients under Dirichlet and non-Dirichlet boundary conditions. In addition, we obtain various consequences of the maximum principle not used in [2]. We shall apply these estimates, such as the Hölder estimates (Corollaries 4.3 and 4.4 below), to studying the smoothness of solutions to such problems in [11]. On the other hand, we shall not discuss analogs of the boundary gradient estimates from [2]. Our approach is based on that in [2], but there are some technical differences which we indicate below. In principle, our estimates could be proved by modifying the method in [2], but our techniques are of independent interest.

We begin in Section 1 with a review of the relevant notation. Next, we construct some supersolutions for the linear operator $L_2 = a^{ij}D_{ij} + b_2^iD_i$ in Section 2. Similar supersolutions for the full operator $L = a^{ij}D_{ij} + b^iD_i$ were used in [2]. These supersolutions are used to derive our maximum principles for subsolutions of elliptic equations with composite coefficients in Section 3. Harnack and Hölder inequalities for solutions of such equations are stated in Section 4; we include them for completeness, but their proofs from the maximum principles in the previous section contain no new ingredients. Results for parabolic problems are given in Section 5. We close with some remarks, mostly about comparing our results with those in [5], in Section 6. Our investigation of this problem was heavily influenced by that paper although the methods used here are quite different.

1. Notation

Although our notation is generally quite standard, we list here some elements that may not be immediately apparent. The reader is also directed to [4] for further information in the elliptic case and to [9] in the parabolic case.

First, we write Ω for a bounded domain in $\mathbb{R}^n$ and we write d for the function defined by $d(x) = \inf\{|x-y| : y \in \partial\Omega\}$. Points in $\mathbb{R}^n$ are written as $x = (x^1, \dots, x^n)$ and we sometimes abbreviate $x' = (x^1, \dots, x^{n-1})$. For $x_0 \in \mathbb{R}^n$ and $R > 0$, we define

$$\Omega[x_0, R] = B(x_0, R) \cap \Omega,\ \Sigma[x_0, R] = B(x_0, R) \cap \partial\Omega,$$

and we suppress x_0 from the notation when we assume that x_0 is the origin. Note that, even when we refer to $\Omega[R]$ rather than Ω, d denotes the distance to $\partial\Omega$.

We say that a vector β is *inward pointing* at $x_0 \in \partial\Omega$ if there is a positive constant ε such that $x_0 + t\beta \in \overline{\Omega}$ for $t \in [0,\varepsilon]$. For $x_0 \in \partial\Omega$, a vector β which is inward pointing at x_0, a constant k, and a function $u \in C^0(\overline{\Omega})$, we say that $\beta \cdot Du(x_0) \geq k$ if

$$\liminf_{t\to 0^+} \frac{u(x_0+t\beta)-u(x_0)}{t} \geq k.$$

Similar definitions apply to the conditions $\beta \cdot Du(x_0) \leq k$ and $\beta \cdot Du(x_0) = k$. Moreover, if β is a vector field defined on (a portion of) $\partial\Omega$, we say that $\beta \cdot Du \geq g$ for some function g if $\beta(x_0) \cdot Du(x_0) \geq g(x_0)$ for each x_0 in (the portion of) $\partial\Omega$ with similar definitions for $\beta \cdot Du \leq g$ and $\beta \cdot Du = g$.

We recall that a continuous, increasing function ζ, defined on $[0,1]$ is called *Dini* if the function $\bar\zeta$ defined by $\bar\zeta(s) = \zeta(s)/s$ is in $L^1(0,1)$ and it is 1-*decreasing* if

$$\frac{\zeta(s)}{s} \geq \frac{\zeta(\sigma)}{\sigma}$$

for all $s \leq \sigma$ in $(0,1)$. If ζ is Dini, we define

$$I(\zeta)(s) = \int_0^s \frac{\zeta(\sigma)}{\sigma}\,d\sigma,\ \ J(\zeta)(s) = \int_0^s \frac{1}{\sigma^2}\int_0^\sigma \zeta(t)\,dt\,d\sigma.$$

It follows from [13, Section 5] that J is continuous, increasing, and 1-decreasing and that $J(\zeta) \leq I(\zeta) \leq 2J(\zeta)$. We shall say that ζ is a D_1 function if ζ is Dini and 1-decreasing with $\zeta(1) = 1$.

2. Construction of supersolutions

The major step is to show that, for any D_1 function ζ and any $\mu \geq 1$, there is a nonnegative function w such that $a^{ij}D_{ij}w \leq -\lambda\zeta(d/R)/d$ for any $[a^{ij}]$ satisfying

$$|a^{ij}| \leq \mu\lambda. \tag{2.1}$$

It will be convenient to construct such a function locally first.

Lemma 2.1. *Suppose that there are constants $\omega_0 \geq 0$ and $R > 0$ along with a function ω, defined for $|x'| < R$, such that*

$$\Omega[R] = \{x : x^n > \omega(x'),\ |x| < R\},\ \ |\omega(x') - \omega(y')| \leq \omega_0|x'-y'| \tag{2.2}$$

for all x' and y' with $|x'|, |y'| < R$, let ζ be a D_1 function, and set $\kappa = (1+\omega_0^2)^{1/2}$. Then, for any $\rho \in (0, R/(2\kappa))$ and any $\mu \in (0,1]$, there is a nonnegative function $w \in C^2(\Omega[\rho])$ such that

$$a^{ij}D_{ij}w \leq -\lambda\frac{\zeta(d/R)}{d} \text{ in } \Omega[\rho] \tag{2.3}$$

for any a^{ij} satisfying (2.1). In addition, there is a constant $C(n,\mu,\omega_0)$ such that

$$|w| \leq CI(\zeta)(\rho/R)\rho,\ \ |Dw| \leq CI(\zeta)(\rho/R) \text{ in } \Omega[\rho]. \tag{2.4}$$

Proof. We note that [12, Theorem 3.7] shows that there are positive constants $\alpha \in (0,1)$ and C (both determined only by n, μ, and ω_0) along with a function v such that

$$a^{ij}D_{ij}v \leq -\lambda d^{\alpha-2},\ d^\alpha \leq v \leq Cd^\alpha,\ |Dv| \leq Cd^{\alpha-1}$$

in $\Omega[R]$. Next, we define $v_1 = J(\zeta)(R^{-1}\rho^{1-\alpha}v)$ and abbreviate $\bar v = R^{-1}\rho^{1-\alpha}v$. Then

$$\begin{aligned}\frac{1}{\lambda}a^{ij}D_{ij}v_1 =& J(\zeta)'(\bar v)R^{-1}\rho^{1-\alpha}a^{ij}D_{ij}v \\ &+ J(\zeta)''(\bar v)R^{-2}\rho^{2-2\alpha}a^{ij}D_ivD_jv \\ \le& -J(\zeta)'(\bar v)R^{-1}\rho^{1-\alpha}d^{\alpha-2}.\end{aligned}$$

From the explicit form of $J(\zeta)'$, we see that $\zeta(s)/s \ge J(\zeta)'(s) \ge \zeta(s)/(2s)$ for any $s \in (0,1)$. Since $R^{-1}\rho^{1-\alpha}d^\alpha \le \bar v \le CR^{-1}\rho^{1-\alpha}d^\alpha$, it follows that

$$a^{ij}D_{ij}v_1 \le -\lambda\frac{\zeta(R^{-1}\rho^{1-\alpha}d^\alpha)}{d^2},$$
$$|Dv_1| \le C\zeta(R^{-1}\rho^{1-\alpha}d^\alpha)/d,\ 0 \le v_1 \le CI(\zeta)(R^{-1}\rho^{1-\alpha}d^\alpha)$$

in $\Omega[R]$. Since $(x', s) \in \Omega[R]$ for $\omega(x') < s \le 2\kappa\rho$ and $|x'| \le \rho$, we can define W by

$$W(x) = \int_{x^n}^{2\kappa\rho} v_1(x', s)\,ds.$$

By construction W is nonnegative and $W \in C^2(\Omega[\rho])$.

We now prove an upper bound for $a^{ij}D_{ij}W$. We compute

$$D_iW(x) = -\delta_{in}v_1(x', 2\kappa\rho) - \int_{x^n}^{2\kappa\rho} D_iv_1(x', s)\,ds.$$

A similar expression can be obtained for $D_{ij}W$, and hence (noting that $\rho \le d(x', 2\kappa\rho) \le (2\kappa+1)\rho$ for $|x'| \le \rho$)

$$\begin{aligned}a^{ij}D_{ij}W &= \int_{x^n}^{2\kappa\rho} a^{ij}D_{ij}v_1(x', s)\,ds - \sum_{i=1}^n a^{in}D_iv_1(x', 2\kappa\rho) \\ &\le -\lambda\int_{x^n}^{2\kappa\rho}\frac{\zeta(R^{-1}\rho^{1-\alpha}d(x', s)^\alpha)}{d(x', s)^2}\,ds + C\lambda\frac{\zeta(\rho/R)}{\rho}.\end{aligned}$$

To estimate this integral, we first observe that

$$\frac{1}{\kappa}(s - \omega(x')) \le d(x', s) \le s - \omega(x')$$

for any s. It follows that $d(x', s) \ge d(x)/\kappa$. In addition, d is Lipschitz with Lipschitz constant 1 (see, for example, [4, Section 14.6]) and therefore $d(x', s) \le d(x)+s-x^n$. Because $\kappa \ge 1$, we also have $2\kappa\rho \ge x^n + d(x)$, so

$$\frac{\zeta(R^{-1}\rho^{1-\alpha}d(x', s)^\alpha)}{d(x', s)^2} \ge \frac{\zeta(R^{-1}\rho^{1-\alpha}(d(x)/\kappa)^\alpha)}{(d(x)+s-x^n)^2} \ge \varepsilon\frac{\zeta(R^{-1}\rho^{1-\alpha}d(x)^\alpha)}{d(x)^2}$$

for $s \in [x^n, x^n + d(x)]$ and $\varepsilon = 1/(4\kappa^\alpha)$. Therefore

$$\begin{aligned}\int_{x^n}^{2\kappa\rho}\frac{\zeta(R^{-1}\rho^{1-\alpha}d(x', s)^\alpha)}{d(x', s)^2}\,ds &\ge \int_{x^n}^{x^n+d(x)}\frac{\zeta(R^{-1}\rho^{1-\alpha}d(x', s)^\alpha)}{d(x', s)^2}\,ds \\ &\ge \varepsilon\frac{\zeta(R^{-1}\rho^{1-\alpha}d(x)^\alpha)}{d(x)}.\end{aligned}$$

It follows that

$$a^{ij}D_{ij}W \le -\lambda\varepsilon\frac{\zeta(d(x)/R)}{d(x)} + C\lambda\frac{\zeta(\rho/R)}{\rho}$$

in $\Omega[\rho]$ becasue $\rho^{1-\alpha}d(x)^\alpha \ge d(x)$ there. In addition,

$$\begin{aligned}|D'W| &\le C\int_{x^n}^{2\kappa\rho}\frac{\zeta(R^{-1}\rho^{1-\alpha}d(x',s))}{d(x',s)}\,ds\\ &\le C\int_{\omega(x')}^{2\kappa\rho}\frac{\zeta(R^{-1}\rho^{1-\alpha}(s-\omega(x'))^\alpha)}{(s-\omega(x'))}\,ds\\ &= CI(\zeta)\left(\frac{\rho}{R}\left(\frac{2\kappa\rho-\omega(x')}{\rho}\right)^\alpha\right) \le CI(\zeta)(\frac{\rho}{R}),\end{aligned}$$

and

$$|D_nW| = v_1 \le CI(\zeta)(R^{-1}\rho^{1-\alpha}d(x)^\alpha) \le CI(\zeta)(\rho/R),$$

The proof is completed by taking $w = W/\varepsilon + AI(\zeta)(\rho/R)(\rho^2-|x|^2)/\rho$ for a suitable constant A. □

Let us note that we can prove a similar result more easily if we use the geometric situation in [2]. Specifically, suppose $0 < x^n < R$ for all $x \in \Omega$ and a^{ij} only satisfies the lower bound $a^{ij}\xi_i\xi_j \ge \lambda|\xi|^2$. If we take $v = (x^n)^\alpha$ (with $\alpha \in (0,1)$ arbitrary) and define v_1 as in the proof of Lemma 2.1, then w given by

$$w(x) = 2\int_{x^n}^{\rho} v_1(x',s)\,ds.$$

satisfies $a^{ij}D_{ij}w \le -\lambda\zeta(x^n/R)/x^n$ in $\Omega[\rho]$ as well as estimate (2.4).

For our local estimates including lower order terms, we can use a simple improvement of the preceding result.

Lemma 2.2. *Suppose that there are constants $\omega_0 \ge 0$ and $R > 0$ along with a function ω, defined for $|x'| < R$, such that (2.2) holds for all x' and y' with $|x'|, |y'| < R$, and let ζ be a D_1 function. Then for any $\mu \ge 1$ and $B_0 \ge 0$, there is a constant $\rho_0(n,\mu,\omega_0,B_0,\zeta) \in (0,1)$ and a function $\bar w$ such that $\rho \le \rho_0 R$ implies*

$$a^{ij}D_{ij}\bar w + b^iD_i\bar w \le -\lambda\frac{\zeta(d/R)}{d} \tag{2.5}$$

in $\Omega[\rho]$ for all $[a^{ij}]$ satisfying (2.1) and all b such that $|b| \le B_0\zeta(d/R)/d$.

Proof. Take $\bar w = 2w$, where w is the function from Lemma 2.1, and note that

$$a^{ij}D_{ij}\bar w + b^iD_i\bar w \le \frac{\lambda}{\zeta(d/R)}d\left(-2 + CB_0I(\zeta)(\rho/R)\right).$$

The proof is completed by taking ρ_0 so small that $CB_0I(\zeta)(\rho_0) \le 1$. □

For global estimates, we use a more careful argument. We first quantify the assumption that Ω is a Lipschitz domain by noting that there are constants N, R and ω_0 along with N points $x_1,\dots,x_N$ on $\partial\Omega$ such that, after a translation and rotation taking x_j to 0, $\Omega[x_j,R]$ can be written in the form (2.2) and such that $\partial\Omega$ can be covered by the N balls $B(x_j,R/(3\kappa))$, where κ was defined in Lemma 2.1. For simplicity, we call N, R, and ω_0 the *Lipschitz constants* of Ω.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be Lipschitz with Lipschitz constants N, R, and ω_0, and let ζ be a D_1 function. Then, for any $\mu \ge 1$ and $B_2 \ge 0$, there is a nonnegative function $w_1 \in C^2(\overline{\Omega})$ such that*

$$a^{ij}D_{ij}w_1 + b^iD_iw_1 \le -\lambda\frac{\zeta(d/\operatorname{diam}\Omega)}{d}\ \text{in}\ \Omega \tag{2.6}$$

for any matrix-valued function $[a^{ij}]$ *satisfying* (2.1) *and any vector-valued function* b *such that*

$$|b| \le B_2\lambda \frac{\zeta(d/\operatorname{diam}\Omega)}{d}. \tag{2.7}$$

In addition, there is a constant C *determined only by* N*,* ζ*,* B_2*,* μ*,* ω_0*, and* $R/\operatorname{diam}\Omega$ *such that*

$$|w_1| \le C\operatorname{diam}\Omega,\ |Dw_1| \le C. \tag{2.8}$$

Proof. Let ζ_1 be a D_1 function to be further specified. From Lemma 2.1, there are functions $W_1, \dots, W_N$ such that

$$a^{ij}D_{ij}W_k \le -\lambda\frac{\zeta_1(d/\operatorname{diam}\Omega)}{d}$$

in $\Omega[x_k, R]$. In addition, the function $W_0 = (\operatorname{diam}\Omega)^2 - |x-x_1|^2$ satisfies $a^{ij}D_{ij}W_0 \le -2\lambda$ in Ω. Now set

$$\Omega' = \bigcup_{k=1}^{N} \Omega[x_k, \frac{2R}{5\kappa}],$$

let (η_k) be a partition of unity on Ω' subordinate to the covering $(\Omega[x_k, R/(2\kappa)])$, and set

$$w_0 = AW_0 + \sum_{k=1}^{N}(\eta_k W_k)$$

with A a positive constant to be chosen. It follows that there is a constant C^*, determined only by N, μ, ω_0, and $R/\operatorname{diam}\Omega$, such that

$$\frac{1}{\lambda}a^{ij}D_{ij}w_0 \le -\frac{\zeta_1(d/\operatorname{diam}\Omega)}{d} - 2A + \frac{C^*}{R}\,I(\zeta_1)(1)$$

in Ω' and

$$\frac{1}{\lambda}a^{ij}D_{ij}w_0 \le -2A + \frac{C^*}{R}\,I(\zeta_1)(1)$$

in $\Omega\setminus\Omega'$. We now write d^* for the infimum of d over $\Omega\setminus\Omega'$ and note that d^*/R is bounded above and below by positive constants determined by n and ω_0. By choosing $A = \zeta_1(d^*/R)/d^* + C^*I(\zeta_1)(1)/R$, we infer that

$$a^{ij}D_{ij}w_0 \le -\lambda\frac{\zeta_1(d/\operatorname{diam}\Omega)}{d},\ |Dw_0| \le C_0 I(\zeta_1)(1)$$

in all of Ω for some constant C_0 determined only by n, μ, $R/\operatorname{diam}\Omega$, and ω_0.

Let us set $K = C_0B_2$. Then there is a positive constant $\varepsilon_1(n,\mu,\omega_0,B_2)$ such that $KI(\zeta)(s) \le 1/2$ for all $s \le \varepsilon_1$. Because ζ is Dini, it follows that

$$\liminf_{s\to 0^+} \zeta(s)|\ln s| = 0,$$

and hence there is a constant $\varepsilon_0 \in (0, \min\{\varepsilon_1, 1/3\})$ such that

$$K\zeta(\varepsilon_0)|\ln\varepsilon_0| \le 1/8.$$

We now choose ζ_1 as follows:

$$\zeta_1(s) = \begin{cases} 6K\zeta(s) & \text{if } 0 \le s \le \varepsilon_0 \\ \dfrac{(1-6K\zeta(\varepsilon_0))s + 6K\zeta(\varepsilon_0) - \varepsilon_0}{1-\varepsilon_0} & \text{if } \varepsilon_0 < s \le 1. \end{cases}$$

It's easy to check that ζ_1 is a D_1 function. A direct calculation shows that

$$I(\zeta_1)(1) = 6KI(\zeta)(\varepsilon_0) + 1 - 6K\zeta(\varepsilon_0) + \frac{6K\zeta(\varepsilon_0)|\ln \varepsilon_0| - \varepsilon_0}{1-\varepsilon_0} \le 5$$

and therefore

$$KI(\zeta_1)(1)\zeta(s) \le \frac{5}{6}\zeta_1(s)$$

for $s \le \varepsilon_0$, and hence

$$B_2|Dw_0(x)|\zeta(d(x)/\operatorname{diam}\Omega) \le (5/6)\zeta_1(d(x)/\operatorname{diam}\Omega)$$

if $d(x) \le \varepsilon_0 \operatorname{diam}\Omega$.

We now let $g \in C^2([0, \sup w_0])$ be an increasing, concave function to be further specified and we define $w_1 = g(w_0)$. Then

$$a^{ij}D_{ij}w_1 + b^iD_iw_1 = g'(w_0)[a^{ij}D_{ij}w_0 + b^iD_iw_0] + g''(w_0)a^{ij}D_iw_0D_jw_0,$$

so

$$\begin{aligned} a^{ij}D_{ij}w_1 + b^iD_iw_1 &\le \lambda g'(w_0)[-\frac{\zeta_1(d/\operatorname{diam}\Omega)}{d} + B_2|Dw_0|\frac{\zeta(d/\operatorname{diam}\Omega)}{d}] \\ &\le -\lambda g'(w)\frac{\zeta_1(d/\operatorname{diam}\Omega)}{6d} \end{aligned}$$

wherever $d \le \varepsilon_0 \operatorname{diam}\Omega$. In addition,

$$\begin{aligned} a^{ij}D_{ij}w_1 + b^iD_iw_1 \le &- \lambda g'\frac{\zeta_1(d/\operatorname{diam}\Omega)}{2d} \\ &+ \lambda g'[-\frac{\zeta_1(\varepsilon_0)}{2\operatorname{diam}\Omega} + B_2|Dw_0|\frac{\zeta(\varepsilon_0)}{\varepsilon_0 R}] + \lambda g''|Dw_0|^2 \end{aligned}$$

wherever $d > \varepsilon_0 \operatorname{diam}\Omega$. But

$$B_2|Dw_0|\frac{\zeta(\varepsilon_0)}{\varepsilon_0 R} = \frac{\zeta_1(\varepsilon_0)}{6C_0\varepsilon_0 R}|Dw_0| \le \frac{\zeta_1(\varepsilon_0)}{2\operatorname{diam}\Omega} + \frac{\bar B}{R}|Dw_0|^2,$$

where $\bar B = \operatorname{diam}\Omega/(72C_0^2\varepsilon_0 R)$ since ζ_1 is 1-decreasing. It follows that

$$a^{ij}D_{ij}w_1 + b^iD_iw_1 \le -\lambda g'\frac{\zeta_1(d/\operatorname{diam}\Omega)}{2d} + \lambda\left[\frac{\bar B g'}{R} + g''\right]|Dw_0|^2$$

wherever $d > \varepsilon_0 \operatorname{diam}\Omega$. We now note that there is a positive constant K_0 such that $K_0\zeta_1 \ge 6\zeta$ on the whole interval $[0,1]$, and we choose

$$g(s) = K_0\frac{R}{\bar B}\exp(\bar B \sup w_0/R)[1 - \exp(-\bar B s/R)].$$

Then straightforward calculations show that (2.6) and (2.8) hold. □

3. The elliptic composite maximum principle

In order to prove our maximum principle for elliptic equations with composite coefficients, we first prove an intermediate result which is a variant of the Aleksandroff maximum principle. Instead of the usual upper contact set (see [4, (9.5)]), for a function $u \in C^0(\overline{\Omega})$ and a constant $\varepsilon \in (0,1)$, we introduce $\Gamma_\varepsilon(u)$, the set of all $y \in \Omega$ such that $u(y) \ge 0$ and there is a vector p with $|p| \le \varepsilon \sup u/(\operatorname{diam}\Omega + \beta_0)$ and $u(x) \le u(y) + p\cdot(x-y)$ for all $x \in \Omega$. We also have the normal mapping χ defined by

$$\chi(y) = \{p \in \mathbb{R}^n : u(x) \le u(y) + p\cdot(x-y) \text{ for all } x \in \Omega\}.$$

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and define the operators L and $\mathcal{M}$ by*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu, \tag{3.1a}$$

$$\mathcal{M}u = -u + \beta \cdot Du, \tag{3.1b}$$

with $[a^{ij}]$ a positive-definite matrix-value function and $c \le 0$. Suppose there are constants B_1 and β_0 such that

$$\left\| \frac{b}{\mathcal{D}^{1/n}} \right\|_{n,\Omega} \le B_1, \tag{3.2a}$$

$$|\beta| \le \beta_0. \tag{3.2b}$$

Let $u \in W^{2,n}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ and suppose there is a nonpositive function f with $f/\mathcal{D}^{1/n} \in L^n(\Omega)$ such that $Lu \ge f$ in Ω and $\mathcal{M}u \ge 0$ on $\partial\Omega$. Then, there is a constant $C(n, B_1)$ such that, for any $\varepsilon \in (0,1)$,

$$\sup u \le C \frac{\operatorname{diam}\Omega + \beta_0}{\varepsilon} \left\| \frac{f}{\mathcal{D}^{1/n}} \right\|_{n,\Gamma_\varepsilon(u)}. \tag{3.3}$$

Proof. As in the proof of [4, Lemma 9.4] (see also [10, Proposition 2.1]), there is a constant R_0 with

$$R_0 \le C(B_1, n) \left\| \frac{f}{\mathcal{D}^{1/n}} \right\|_{n,\Gamma_\varepsilon(u)}. \tag{3.4}$$

such that, for any $\delta > 0$, there is $p_0 \in \mathbb{R}^n \setminus \chi(\Gamma_\varepsilon(u))$ with $|p_0| \le R_0 + \delta$. If $|p_0| \le \varepsilon \sup u/(\operatorname{diam}\Omega + \beta_0)$, we proceed as in [7, Lemma 1.1] to see that

$$\sup_\Omega u \le (\operatorname{diam}\Omega + \beta_0)|p_0|,$$

which implies that $\sup u = 0$. On the other hand, if $|p_0| > \varepsilon \sup u/(\operatorname{diam}\Omega + \beta_0)$, then

$$\sup u \le \frac{1}{\varepsilon}(R_0 + \delta)$$

for any $\delta > 0$. Combining these two cases and using (3.4) yields the desired estimate. □

We are now ready to state and prove our main maximum estimate.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be Lipschitz with Lipschitz constants N, R, and ω_0, and define the operators L and $\mathcal{M}$ by (3.1) with $[a^{ij}]$ a positive-definite matrix-value function and $c \le 0$. Suppose there is a constant β_0 such that condition (3.2b) holds. Suppose also that $\partial\Omega \in C^{0,1}$ and that there is a constant μ such that condition (2.1) holds. Suppose finally that there are constants B_1 and B_2, vector-valued functions b_1 and b_2, and a D_1 function ζ such that $b = b_1 + b_2$ and*

$$\left\| \frac{b_1}{\mathcal{D}^{1/n}} \right\| \le B_1, \tag{3.5a}$$

$$|b_2| \le B_2 \lambda \frac{\zeta(d/\operatorname{diam}\Omega)}{d} \tag{3.5b}$$

Let $u \in W^{2,n}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ and suppose there are nonpositive functions f_1 and f_2 with $f_1/\mathcal{D}^{1/n} \in L^n(\Omega)$ and $f_2 d/(\lambda\zeta(d/\operatorname{diam}\Omega)) \in L^\infty(\Omega)$ and a nonpositive constant g such that $Lu \ge f_1 + f_2$ in Ω and $\mathcal{M}u \ge g$ on $\partial\Omega$. Then, there are constants C and

B^*, determined only by n, μ, B_2, N, $R/\operatorname{diam}\Omega$, and ω_0 such that $B_1 \le B^*$ implies that

$$\sup u \le |g| + C(\operatorname{diam}\Omega + \beta_0)\left[\left\|\frac{f_1}{\mathcal{D}^{1/n}}\right\|_{n,\Gamma^*(u)} + \left\|\frac{f_2 d}{\lambda\zeta(d/\operatorname{diam}\Omega)}\right\|_\infty\right], \tag{3.6}$$

where

$$\Gamma^* = \{x \in \Omega : u(x) > 0,\ |Du| \le \frac{\sup u}{\operatorname{diam}\Omega + \beta_0} + C\left\|\frac{f_2 d}{\lambda\zeta(d/\operatorname{diam}\Omega)}\right\|_\infty\}. \tag{3.7}$$

Proof. We define the operators L_k for $k = 1,2$ by $L_k u = a^{ij}D_{ij}u + b^i_k D_i u + cu$, and, for $A \ge 0$ and $\varepsilon \in (0,1/2)$ constants to be determined, we set $\bar u = u + g$, $\bar w_1 = w_1 + \beta_0 \sup|Dw_1|$, and $v = \bar u - A\bar w_1$. Next, we assume that $B^* \le 1$ and we apply Lemma 3.1 to v using the operator L_1 in place of L. It follows that

$$\sup v \le \frac{CR_0}{\varepsilon}\left\|\frac{(f^*)^-}{\mathcal{D}^{1/n}}\right\|_{n,\Gamma}, \tag{3.8}$$

where we have used the abbreviations

$$f^* = f_1 + f_2 - b^i_2 D_i v - AL_2 w_1 - Ab^i_1 D_i w_1,$$

$\Gamma = \Gamma_\varepsilon(v)$, and $R_0 = \operatorname{diam}\Omega + \beta_0$. To proceed, we obtain a lower bound for f^* using the abbreviation $z = \zeta(d/\operatorname{diam}\Omega)$.

First, we note that $\sup v \le \sup \bar u$ and hence

$$-b^i_2 D_i v \ge -\lambda\frac{B_2}{2}\sup \bar u \frac{z}{R_0 d}$$

on Γ. Then we set

$$F_1 = \left\|\frac{f}{\mathcal{D}^{1/n}}\right\|_{n,\Gamma},\ F_2 = \left\|\frac{f_2 d}{\lambda\zeta(d/\operatorname{diam}\Omega)}\right\|_\infty,$$

and note that

$$f_2 - b^i_2 D_i v - AL_2 w_1 \ge \lambda(-F_2 - \frac{B_2}{R_0}\varepsilon \sup \bar u + A)\frac{z}{d}.$$

In addition, $-Ab^i_1 D_i w_1 \ge -CA|b_1|$. Taking $A = F_2 + \varepsilon B_2 \sup \bar u / R_0$ then yields

$$f^* \ge f_1 - CF_2|b_1| - C\frac{B_2}{2R_0}|b_1| \sup \bar u$$

Now we use this inequality and our choice of A along with (3.8) to see that

$$\sup v \le C\left(1 + \frac{1}{\varepsilon}\right)F_2 R_0 + \frac{C}{\varepsilon}F_1 R_0 + CB_1 B_0 \sup \bar u.$$

On the other hand,

$$\sup v \ge \sup \bar u - CF_2 \operatorname{diam}\Omega - CB_2\varepsilon \sup \bar u.$$

By choosing $B^* = \varepsilon = 1/(4 + 4CB_2)$, we find that

$$\sup \bar u \le C(F_1 + F_2)R_0,$$

and that

$$|Du| \le |Dv| + A|Dw_1| \le \frac{(1+B_2)\varepsilon}{R_0}\sup \bar u + CF_2$$

on Γ. Combining these two inequalities and recalling our choice of ε easily implies (3.6) since $\bar u \le u$. □

Note that the smallness condition on B_1 can be modified. By paying more attention to the values of the constants generically denoted by C in this proof, we see that (3.6) holds provided B_1 and B_2 satisfy the joint condition

$$K_1(B_1)K_2(B_2)B_1B_2 < 1,$$

where $K_1(B_1)$ is the constant from Lemma 3.1 and $K_2(B_2)$ is the constant from Lemma 2.3.

4. Local estimates for elliptic problems

Next, we discuss various local estimates for elliptic oblique derivative problems. Our main concern is with a Hölder estimate for u, which will be useful in applications, so we just sketch the major ideas. First, for a positive-definite matrix-valued function $\mathcal{A} = [a^{ij}]$ and a continuous increasing function ζ, we say that a measurable function f is an $(n, \zeta, \mathcal{A})$-composite function if there is a decomposition $f = f_1 + f_2$ along with constants F_1 and F_2 such that

$$\|\frac{f_1}{\mathcal{D}^{1/n}}\|_{n,\Omega[R]} \le F_1, \ |f_2/\lambda| \le F_2 \frac{\zeta(d/R)}{d}$$

in $\Omega[R]$. We call (F_1, F_2) the composite norm of f. In general, there will be more than one such decomposition and hence this norm is not unique, so we shall choose any convenient choice. In particular, if f is nonnegative or nonpositive, then we shall assume that f_1 and f_2 are both nonnegative or nonpositive, respectively. We also write

$$F_1(\rho) = \|f_1/\mathcal{D}^{1/n}\|_{n,\Omega[\rho]}$$

for $\rho \in (0, R)$. We use similar notation for the coefficients b and c.

Now suppose that there are positive constants $\varepsilon < 1$, R, and ω_0 such that

$$\{x \in \mathbb{R}^n : x^n > \omega_0|x'|, |x| < R\} \subset \Omega, \tag{4.1a}$$

$$|\beta'| \le \frac{1-\varepsilon}{\omega_0}\beta^n \text{ on } \Sigma[R]. \tag{4.1b}$$

In addition, we suppose that there is a constant $\theta_0 \in (0, \pi/2]$ such that, for each point $x_0 \in \Sigma[R]$, there is a cone with height R, semi-vertex angle θ_0, and vertex x_0 which does not intersect Ω. It is easy to check that the basic estimate Lemma 3.1 from [10] continues to hold provided we replace each nonnegative L^n function (that is, $(Lu)^+$, $|b|$, and $|c|$) by an $(n, \zeta, \mathcal{A})$-composite function. Specifically, we define the operator M by

$$Mu = \beta \cdot Du + \beta_0 u, \tag{4.2}$$

we set $A = 2^{1+2\varepsilon}\varepsilon^{-4\varepsilon}$ and for $x_1 \in \Omega[R]$ and $\rho \in (0, R)$ and $\alpha \in (0, 1)$, we define

$$E(x_1, \rho) = \{x \in \Omega : \left(\frac{|x' - x_1'|^2}{\rho^2} + \alpha\right)^{(1+\varepsilon)/2} + \frac{|x^n - x_1^n|^2}{(A\omega_0\rho)^2} < 1\}. \tag{4.3}$$

Finally, we use $F_1^*(\rho)$ to denote

$$\left\|\frac{f_1}{\mathcal{D}^{1/n}}\right\|_{n,E(x_1,\rho)}$$

and similarly for $B_1^*(\rho)$ and $C_1^*(\rho)$.

Lemma 4.1. *Let u be a nonnegative $W^{2,n}_{loc}(\Omega[R]) \cap C^0(\overline{\Omega})$ function, let ζ be a D_1 function, define L by* (3.1a) *and suppose that b and c are $(n, \zeta, \mathcal{A})$-composite functions. Suppose there are a nonnegative $(n, \zeta, \mathcal{A})$-composite function f and a nonnegative constant g such that*

$$Lu \leq f \text{ in } \Omega[R], \ Mu \leq g\beta^n \text{ on } \Sigma[R], \tag{4.4}$$

let $\rho \in (0, R)$, $x_1 = (0, x_1^n) \in \Omega[R]$ and $\alpha \in (0, 1)$, and set

$$\bar{u} = u + \rho[F_1^*(\rho) + F_2 I(\zeta)(\rho/R) + |g|]. \tag{4.5}$$

In addition to conditions (4.1) *and* (2.1)*, suppose that $c \leq 0$ in $\Omega[R]$ and that there is a positive constant μ_2 such that $0 \geq \beta^0 \geq -\mu_2\beta^n$ on $\Sigma[R]$. Then there is a positive constant $\alpha_1(\varepsilon)$ such that if*

$$x_1^n \geq (A - \alpha_1)\omega_0\rho, \tag{4.6}$$

and $E(x_1, \rho) \subset \Omega[R]$, then for any positive constants δ and δ_1 in $(0, 1)$, there are positive constants K_1 and ζ_1 determined only by n, α, ρC_2, B_2, ε, μ, $\mu_2\rho$, δ, δ_1, θ_0, and ω_0 such that if $\rho C_1^*(\rho) + B_1^*(\rho) \leq \zeta_1$ and

$$|\{x \in E(x_1, \delta\rho) : \bar{u}(x) < h\}| \leq \zeta_1^n \rho^n \tag{4.7}$$

for some $h \geq 0$, then $h \leq K_1\bar{u}$ in $E(x_1, \delta_1\rho)$.

From this lemma, the argument of [10, Section 4] leads to the following weak Harnack inequality.

Theorem 4.2. *Let $0 \in \partial\Omega$ and suppose conditions* (4.1) *and* (2.1) *hold. Let ζ be a D_1 function, let $\rho \in (0, R/4)$ and suppose $u \in C^0(\overline{\Omega[4\rho]}) \cap W^{2,n}_{loc}(\Omega[4\rho])$ satisfy the inequalities*

$$a^{ij}D_{ij}u \leq \lambda\nu_1|Du|^2 + b|Du| + cu + f, \ u \geq 0 \text{ in } \Omega[4\rho], \tag{4.8a}$$

$$\beta \cdot Du \leq \beta^n[\mu_1 u + g] \text{ on } \partial\Omega[4\rho] \tag{4.8b}$$

for nonnegative constants ν_1, g, and μ_1 and nonnegative $(n, \zeta, \mathcal{A})$-composite functions b, c, and f. Then there are constants K_2, ε_2, and κ (determined only by n, $\nu_1 \sup u$, B_2, $\rho C_1(\rho)$, ρC_2, ε, μ, $\rho\mu_1$, θ_0, and ω_0) such that $B_1(\rho) \leq \varepsilon_2$ implies

$$\left(\rho^{-n}\int_{\Omega[\rho]} u^\kappa \, dx\right)^{1/\kappa} \leq K_2\left(\inf_{\Omega[\rho]} u + \rho(F_1(\rho) + F_2 I(\zeta)(\rho/R) + g)\right). \tag{4.9}$$

From this weak Harnack inequality, a Hölder estimate follows by standard methods.

Corollary 4.3. *Suppose condition* (2.2) *holds, let $[a^{ij}]$ satisfy* (2.1)*, and let β satisfy* (4.1b)*. Let $\rho \in (0, R/4)$, let ζ be a D_1 function, let $u \in C^0(\overline{\Omega[\rho]}) \cap W^{2,n}_{loc}(\Omega[\rho])$, and suppose that there are nonnegative functions b, c, and f satisfying the hypotheses in Theorem 4.2 and a nonnegative constant ν_1 such that*

$$|a^{ij}D_{ij}u| \leq \lambda\nu_1|Du|^2 + b|Du| + c|u| + f \tag{4.10}$$

in $\Omega[\rho]$. Suppose also that there are nonnegative constants μ_1 and g such that

$$|\beta \cdot Du| \leq \beta^n[\mu_1|u| + g] \tag{4.11}$$

on $\Sigma[\rho]$. Then there are constants C, θ, and ε_3 determined only by n, $\nu_1 \sup |u|$, ω_0, ε, μ, $\mu_1\rho$, B_2, and $\rho(C_1(\rho)+C_2)$ such that if $B_1(\rho) \le \varepsilon_3$, then u satisfies the estimate

$$\operatorname*{osc}_{\Omega[\tau\rho]} u \le C\tau^\theta \left(\operatorname*{osc}_{\Omega[\rho]} u + \rho[g + F_1(\rho) + F_2 I(\zeta)(\rho/R)] \right) \tag{4.12}$$

In fact, we can relax the condition (2.2) to just (4.1a) by invoking the obvious analog of the so-called "displaced" weak Harnack inequality [10, Theorem 3.4]; for our intended applications, this improvement will not be important. On the other hand, the use of the $L^n(\Omega[\rho])$ norm for f_1 in this Hölder estimate will be important.

A similar argument along with the proof of [10, Corollary 8.4] shows that an analogous Hölder estimate is valid for mixed problems which we state in terms of the sets

$$O[y,\rho] = \{x \in \Omega : |x-y| < \rho\},\ O^+[y,\rho] = \{x \in O[y,\rho] : |x| = |y|\}$$

for a point $y \in \partial\Omega[R]$ and $\rho \le R$.

Corollary 4.4. *Suppose condition (2.2) holds, let $[a^{ij}]$ satisfy (2.1), and let β satisfy (4.1b). Let $\rho \in (0, R/4)$, let ζ be a D_1 function, let $y \in \Sigma[R]$ with $|y| = \rho$, let $u \in C^0(\overline{O[y,\rho]}) \cap W^{2,n}_{loc}(O[y,\rho])$ and suppose that there are $(n,\zeta,\mathcal{A})$-composite nonnegative functions b, c, and f and a nonnegative constant ν_1 such that (4.10) holds in $O[y,\rho]$. Suppose also that there are nonnegative constants μ_1 and g such that (4.11) holds on $\Sigma[\rho] \cap \overline{O[y,\rho]}$. Then, there are constants C, θ, and ε_0 determined only by $\nu_1 \sup|u|$, ω_0, ε, Λ/λ, B_2, and $\rho(C_1(\rho)+C_2)$ such that if $B_1(\rho) \le \varepsilon_0$, then u satisfies the estimate*

$$\begin{aligned} \operatorname*{osc}_{O[y,\tau\rho]} u \le C &\operatorname*{osc}_{O^+[y,2\tau\rho]} u \\ &+ C\tau^\theta \left(\operatorname*{osc}_{O[y,\rho]} u + \rho[g + \|f_1/\mathcal{D}^{1/n}\|_{n,O[y,\rho]} + F_2 I(\zeta_1)(\rho/R)] \right). \end{aligned} \tag{4.13}$$

We next point out that all the elliptic results in [10] have their analogs when the coefficients b and c (and f when it appears) are composite as in this paper. The only result that requires some specific comment is the Harnack inequality. It is not difficult to see that its proof applies if we can write $b = b_1 + b_2$ with $b_1 \in L^q$ for some $q > n$ and $|b_2| \le B_2 d^{\alpha-1}$ for some $\alpha \in (0,1)$.

We conclude this section with a maximum principle for mixed boundary value problems in $\Omega[\rho]$. This maximum principle will be useful in studying gradient estimates for oblique derivative problems; see [11].

Lemma 4.5. *Let $R > 0$, and suppose conditions (2.1) and (2.2) are satisfied. Let $\rho \in (0,R)$, let ζ be a D_1 function, let b and c be $(n,\zeta,\mathcal{A})$-composite function with $c \le 0$, and define the operator L by (3.1a). Suppose β is an inward pointing direction field defined on $\Sigma[\rho]$ with $|\beta| \le \mu_1\beta^n$ for some constant μ_1. Let $u \in W^{2,n}_{loc}(\Omega[\rho]) \cap C(\overline{\Omega[\rho]})$ and suppose there are nonnegative constants μ_0 and g along with a nonpositive $(n,\zeta,\mathcal{A})$-composite function f such that $Lu \ge f$ in $\Omega[\rho]$ and $\beta \cdot Du \ge -g\beta^n$ on $\Sigma[\rho]$. Then there are positive constants ε_1 and ρ_0, determined only by n, μ, and μ_1 such that $B_1 \le \varepsilon_1$ and $\rho \le \rho_0 R$ imply that*

$$\sup_{\Omega[\rho]} u \le \sup_{E^+(\rho)} u^+ + C(n, B_2, \mu, \mu_1)\rho[g + F_1(\rho) + F_2 I(\zeta_1)(\rho/R)], \tag{4.14}$$

where $E^+(\rho) = \{x \in \Omega : |x| = \rho\}$.

Proof. We define v by

$$v(x) = \exp(\varepsilon x^n/\rho)(u(x) - \sup_{E^+(\rho)} u^+),$$

and we define $\bar{L}$ by

$$\bar{L}u = a^{ij} D_{ij} u + \left(b^i - \frac{2\varepsilon}{\rho} a^{in}\right) D_i u + cu.$$

It's easy to see that

$$\bar{L}v \geq \exp(\varepsilon x^n/\rho) f - \left(\varepsilon \frac{|b|}{\rho} + \frac{\varepsilon^2 a^{nn}}{\rho^2}\right) v$$

in $\Omega[\rho]$. Now we set $\bar{\beta} = (\rho\beta)/(\varepsilon\beta^n(0))$ and extend $\bar{\beta}$ to be zero on $E^+(\rho)$. Then $\bar{\beta} \cdot Dv - v \geq -(\rho g/\varepsilon)$ on $\partial(\Omega[\rho])$.

We now assume that $\varepsilon_1 \leq B^*$, the constant from Theorem 3.2, and we apply that theorem to v with $\bar{L}$ and $\bar{\beta}$ replacing L and β, respectively. In this way, we obtain

$$\begin{aligned}\sup_{\Omega[\rho]} v \leq{}& C(1 + \frac{\mu_1}{\varepsilon})\rho[g + F_1(\rho)] + F_2 I(\zeta_1)(\rho/R) \\ &+ C(\varepsilon + \mu_1)(\varepsilon_1 + \varepsilon + B_2 I(\rho/R)) \sup_{\Omega[\rho]} v.\end{aligned}$$

The proof is completed by choosing ε, ε_1, and ρ_0 sufficiently small and rewriting the resulting inequality in terms of u. □

5. The parabolic composite maximum principle

For parabolic problems, we modify our notation slightly. Let Ω be a bounded domain in $\mathbb{R}^{n+1}$ with parabolic boundary $\mathcal{P}\Omega$ and suppose R_0 is so large that $|x| \leq R_0$ for all $X = (x, t) \in \Omega$. Let u be a continuous function defined on $\overline{\Omega} \setminus \mathcal{P}\Omega$. For constants $\beta_0 \geq 0$ and $\varepsilon > 0$, we define $E_\varepsilon(u, \beta_0)$ to be the set of all $X \in \overline{\Omega} \setminus \mathcal{P}\Omega$ such that there is $\xi \in \mathbb{R}^n$ with $u(Y) \leq u(X) + \xi \cdot (x - y)$ for all $Y \in \Omega$ with $s \leq t$, $u(X) > 0$, and

$$\frac{R_0 + \beta_0}{\varepsilon} |\xi| \leq u(X) - \xi \cdot x < \frac{1}{2} \sup_\Omega u.$$

The rest of the notation from Section 1 is then modified in the obvious way.

Before presenting our main maximum principle, we begin with a simple variant of an intermediate result.

Lemma 5.1. *Suppose β is an inward pointing direction field on $\mathcal{P}\Omega$ with $\beta^{n+1} \equiv 0$ on $\mathcal{P}\Omega$ and $\beta \equiv 0$ on $B\Omega$. Suppose also that there is a constant β_0 such that $|\beta| \leq \beta_0$ on $\mathcal{P}\Omega$. If $u \in W^{2,1}_{n+1;loc}(\Omega) \cap C^0(\overline{\Omega})$ and $\beta \cdot Du \geq u$ on $\mathcal{P}\Omega$, then*

$$\sup_\Omega u \leq C(n) \left(\frac{R_0 + \beta_0}{\varepsilon}\right)^{n/(n+1)} \left(\int_{E_\varepsilon(u,\beta_0)} |u_t \det D^2 u| \, dX\right)^{1/(n+1)}. \tag{5.1}$$

Proof. The proof is virtually identical to that of [9, Lemma 7.2] (which is modeled, in turn, on that in [14]), so we only give a sketch. First, we assume that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and we define the function $\Phi\colon \Omega \to \mathbb{R}^{n+1}_0$ by $\Phi(X) = (Du(X), u(X)-$

$x \cdot Du(X))$. Since the Jacobian determinant of this function is just $u_t \det D^2 u$, it follows that

$$\int_E |u_t \det D^2 u| \, dX \ge |\Phi(E)|,$$

where we use E to abbreviate $E_\varepsilon(u, \beta_0)$. Next, we set $M = \sup_\Omega u$ and we define

$$\Sigma = \{\Xi = (\xi, h) \in \mathbb{R}_0^{n+1} : \frac{R_0 + \beta_0}{\varepsilon} |\xi| < h < \frac{M}{2}\}.$$

The discussion on p. 107 of [9] shows that $\Sigma \subset \Phi(E)$, so

$$\int_E |u_t \det D^2 u| \, dX \ge |\Sigma| = C(n) \left(\frac{\varepsilon}{R_0 + \beta_0} \right)^n M^{n+1},$$

and the desired result (for smooth u) follows from this one by simple algebra. The hypothesis $u \in C^2$ is relaxed to $u \in W^{2,1}_{n+1;\mathrm{loc}}$ as in [9, Proposition 7.3]. □

In analogy to the elliptic definition for composite functions, for a positive-definite matrix-valued function $\mathcal{A} = [a^{ij}]$ and a continuous increasing function ζ, we say that a measurable function f is an $(n+1, \zeta, \mathcal{A})$-composite function if there is a decomposition $f = f_1 + f_2$ along with constants F_1 and F_2 such that $\|f_1/\mathcal{D}^{1/(n+1)}\|_{n+1,\Omega[R]} \le F_1$ and

$$|f_2/\lambda| \le F_2 \frac{\zeta(d/R)}{d}$$

in $\Omega[R]$. We call (F_1, F_2) the composite norm of f.

Several different measures of regularity for $\mathcal{P}\Omega$ will be used to quantify the dependence of the estimates on the domain. First, we refer to p. 76 of [9] for the definition of $\mathcal{P}\Omega \in H_1$ although we shall rewrite the definition to emphasize the connection to β. If $\mathcal{P}\Omega \in H_1$, then there are positive constants N, R, T_0, and ω_0 along with points $X_1, \dots, X_N$ in $S\Omega$ such that, after a translation and rotation (in the x-variables only) which takes X_i to the origin, we have

$$\Omega[R] = \{X \in \mathbb{R}^n : |X| < R, \ x^n > \omega(X'), \ t > -T_0\} \tag{5.2}$$

for some function ω (which will generally be different for each X_i) satisfying

$$|\omega(x', t) - \omega(y', s)| \le \omega_0 |X' - Y'|. \tag{5.3}$$

In addition, $S\Omega$ is covered by the cylinders $Q(X_i, R/(3\kappa))$ with $\kappa = (1 + 2\omega_0^2)^{1/2}$. Next, a *tusk* is a set of the form

$$\{X : -T < t < 0, \ |x - (-t)^{1/2} x_0| < R(-t)^{1/2}\}$$

for some point $x_0 \in \mathbb{R}^n$ and positive constants R and T. We then say (compare with [8, p. 26]) that Ω satisfies an exterior θ_0-tusk condition at $X_1 \in S\Omega$ (for $\omega_0 \in (0, \pi/2)$) if $T = \infty$, and

$$(t_1 - t)^{1/2} < \tan\theta_0 \left| x - x_1 - \frac{|X - X_1|}{2^{1/2}|x_0|} x_0 \right|$$

for $X \in \Omega$. Note that θ_0 can be determined explicitly in terms of R and $|x_0|$.

We then have the following maximum principle for parabolic operators with composite coefficients.

Theorem 5.2. *Let $\mathcal{P}\Omega \in H_1$ with H_1 constants N, R, T_0, and ω_0. Define the operators L by*

$$Lu = -u_t + a^{ij} D_{ij} u + b^i D_i u + cu \tag{5.4}$$

and $\mathcal{M}$ by (3.1b) *with $[a^{ij}]$ positive definite, $c \le 0$, and β satisfying* (3.2b). *Suppose there are positive constants λ_0 and Λ_0 so that $[a^{ij}]$ satisfies*

$$\lambda_0 |\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda_0 |\xi|^2 \tag{5.5}$$

in Ω, and suppose $\beta \equiv 0$ on $B\Omega$ and $\beta^{n+1} \equiv 0$ on $S\Omega$. Let ζ be a D_1 function and suppose that b and c are $(n+1, \zeta, \mathcal{A})$-composite functions. Let $u \in W^{2,1}_{n+1,loc} \cap C^0(\overline{\Omega})$ and suppose there are a nonpositive, $(n+1, \zeta, \mathcal{A})$-composite function f and a nonpositive constant g such that $Lu \ge f$ in Ω, $\mathcal{M}u \ge g$ on $\mathcal{P}\Omega$. If $|x| < R_0$ in Ω, then there is a constant C, determined only by B_2, n, N, R/R_0, T_0, λ_0, Λ_0, and ω_0 such that

$$\sup_\Omega u \le |g| + C(B_1^{n+1} + R_0 + \beta_0)^{n/(n+1)} \left[\left\| \frac{f_1}{\mathcal{D}^{1/(n+1)}} \right\|_{n+1, \Gamma^*(u)} + F_2 \right], \tag{5.6}$$

with Γ^* given by

$$\Gamma^* = \{x \in \Omega : u(x) \ge 0,\ |Du| \le \frac{\sup u}{R_0 + \beta_0} + CF_2\}. \tag{5.7}$$

Proof. We first note that the proof of Lemma 2.3 can be modified to the parabolic case. The only significant differences are that we use the remarks following Lemma 13.1 of [8] in place of [12, Theorem 3.7] and we replace $\operatorname{diam}\Omega$ by R_0. We denote the resulting function also by w_1.

Next, we use the matrix inequality $(\det A \det B)^{n+1} \le (\operatorname{tr} AB)/(n+1)$, true for any $(n+1) \times (n+1)$, positive semidefinite matrices A and B, and we set $v = u + g - A(w_1 - \beta_0 \sup |Dw_1|)$ to se that

$$\sup_\Omega v \le C(n) \left(\frac{R_0 + \beta_0}{\varepsilon} \right)^{n/(n+1)} \left\| \frac{f^*}{\mathcal{D}^{1/(n+1)}} \right\|_{n+1, E},$$

where $f^* = -v_t + a^{ij} D_{ij} v$ and $E = E_\varepsilon(v, \beta_0)$. Some straightforward calculation shows that

$$f^* \ge f_1 - C \left[F_2 + \frac{\varepsilon \sup_\Omega v}{R_0 + \beta_0} \right] |b_1|$$

on E if $A = F_2 + (\varepsilon B_2 \sup_\Omega v)/(R_0 + \beta_0)$, so

$$\sup_\Omega v \le C \left(\frac{R_0 + \beta_0}{\varepsilon} \right)^{n/(n+1)} \left(F_1 + F_2 + \frac{B_1 \varepsilon}{R_0 + \beta_0} \sup_\Omega v \right),$$

and the proof is completed by taking ε sufficiently small. □

Note that the form of the estimate (5.6) agrees with that in [14] and it improves the form stated in [9, Theorem 7.1] (although the choice

$$\mu = (R + \|b/\mathcal{D}^*\|^{n+1} + \beta_0)^{-1/(n+1)} \|f^-/D^*\|$$

in the proof of that theorem, on p. 159 of [9], does give this form). Of course, if we replace the assumption $c \le 0$ by $c \le K$ for some nonpositive constant K, then we can apply this theorem to $u \exp(-Kt)$ to obtain an analogous estimate for u.

We leave the statements of the local estimates for parabolic equations to the reader, mentioning [10, Section 7] as a source for the descriptions. In particular, we point out that the appropriate hypothesis for b_1 is that $b_1/\mathcal{D}^{1/(n+1)}$ should be in the Morrey space $M^{n+1,1}$ and that β is assumed to satisfy condition (4.1b) under the assumption that (5.3) is modified to

$$|\omega(x',t)-\omega(y',s)| \le \omega_0|x'-y'| + \omega_1|t-s||^{1/2}$$

for some ω_1.

6. Additional remarks

Our method gives an alternative approach for some of the results in [5] which were used in [2]. To illustrate this point, we consider the following result, which is approximately the elliptic analog of [5, Lemma 1.2]. (See also Lemma 3.3 from that paper.)

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^n$ and define the operator L by* (3.1a) *with $c \le 0$. If there is a nonnegative function w such that $Lw \le -|b|$ and if $u \in W^{2,n}_{loc} \cap C^0(\overline{\Omega})$ with $u \le 0$ on $\partial\Omega$, then*

$$\sup_\Omega u \le C(n)(\sup w + \operatorname{diam}\Omega) \left\| \frac{(Lu)^-}{\mathcal{D}^{1/n}} \right\|_{n,\Omega^+}, \tag{6.1}$$

where Ω^+ is the subset of Ω on which $u \ge 0$.

Proof. Set $M = \sup_\Omega u$ and, with $\varepsilon \in (0,1)$ to be determined, set $v = u - \varepsilon M/R$ and $f = Lu^-$. Then $a^{ij}D_{ij}v \ge f$ in $\Gamma_\varepsilon(v)$ and $v \le 0$ on $\partial\Omega$, so Lemma 3.1 with $B_0 = \beta_0 = 0$ implies that

$$\sup_\Omega v \le C(n)\frac{\operatorname{diam}\Omega}{\varepsilon} \left\| \frac{f}{\mathcal{D}^{1/n}} \right\|_{n,\Omega^+},$$

and hence

$$M(1 - \frac{\varepsilon}{\operatorname{diam}\Omega}\sup w) \le C(n)\frac{\operatorname{diam}\Omega}{\varepsilon} \left\| \frac{f}{\mathcal{D}^{1/n}} \right\|_{n,\Omega^+}.$$

The proof is completed by taking $\varepsilon = \operatorname{diam}\Omega/(2(\sup w + \operatorname{diam}\Omega))$. □

This result is weaker than Krylov's in that he proves the pointwise inequality

$$u \le C(n)(w + \operatorname{diam}\Omega) \left\| \frac{(Lu)^-}{\mathcal{D}^{1/n}} \right\|_{n,\Omega^+}.$$

On the other hand, our method considers situations in which we only have a supersolution to part of the operator; that is, we only need a function w (like w_1 in Section 3) such that $a^{ij}D_{ij}w + b_2^iD_iw \le -|b_2|$ with $b = b_1 + b_2$.

Via similar considerations, we can prove essentially all the results in [5] for solutions of elliptic and parabolic equations. The main differences are that we only obtain global estimates for u and we always assume that $p = n$. (Here, Krylov's d is the same as our n.) In a future work, we shall examine the case $p > n$.

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Gary M. Lieberman
Department of Mathematics, Iowa State University, Ames, Iowa 50011
E-mail address: lieb@iastate.edu