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UNIQUENESS FOR A SEMILINEAR ELLIPTIC EQUATION IN NON-CONTRACTIBLE DOMAINS UNDER SUPERCRITICAL GROWTH CONDITIONS

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Abstract. We apply the Pohozaev identity to sub-domains of a tubular neighbourhood of a closed or broken curve in $\mathbb{R}^n$ and establish uniqueness results for the smooth solutions of the Dirichlet problem for $-\Delta u + |u|^{p-1}u = 0$. We require the domain to be in $\mathbb{R}^n$ with $n \geq 4$ and with $p > (n+1)/(n-3)$.

1. Introduction

In this note, we consider the uniqueness of smooth solutions for the Dirichlet problem

$$\begin{aligned} -\Delta u = |u|^{p-1}u \quad &\text{in } \Omega \subset R^n, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{1}$$

in some non-starshaped and non-contractible domains. Since Pohozaev's work [P], there have been many uniqueness results for (1) and its generalizations (see, for example [PS, V, M]). These results are based on Pohozaev's identity [P] and are established on star-shaped domains. Under the critical growth condition $p = (n+2)/(n-2)$, it is known [BC] that (1) has nontrivial solutions when the topology of the domain is nontrivial. For some simply connected domains, there are examples [Da, Di] that (1) can have nontrivial solutions when $p = (n+2)/(n-2)$ is the critical Sobolev exponent.

Recently, possible generalizations have been considered for 'nearly star-shaped' domains [DZ] and for carefully designed non-starshaped rotation domains [CZ] on which (1) does not have nontrivial smooth solutions.

In [CZ] a special class of non-star shaped domains was constructed by rotating a two-dimensional graph designed by using inversions in Euclidean spaces. The first result of the present note is to generalize this result to domains including all rotation domains. Since there is much less restriction on the graph, we have a weaker result, that is, when $n > 3$ and $p \geq (n+1)/(n-3)$, the only smooth solution is $u \equiv 0$. We also show that when $p > (n+1)/(n-3)$ the same result holds for sufficiently small tubular neighbourhood of a given closed, smooth embedded curve in $\mathbb{R}^n$. A simple example of such a non-contractible domain is the solid torus in $\mathbb{R}^4$. In general, our non-contractible domains have the same homotopic type as the unit circle S^1.

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When $p > (n+2)/(n-2)$, there are examples of non-starshaped domains [CZ, DZ] on which (1) has only trivial solutions. However, for domains with nontrivial topology, examples I can find such that the same uniqueness result holds are in $\mathbb{R}^n$ with $n > 3$ and with the growth condition $p > (n+1)/(n-3)$.

The method we use is to apply the Pohozaev identity [P, PS] to certain sub-domains. We carefully divide a tubular neighbourhood of a closed curve into sub-domains by using the normal planes of the central curve, such that each sub-domain is star-shaped. We apply the Pohozaev identity on each of these sub-domains. Then we collect the resulting terms and pass to the limit by using the definition of Riemann integral. In the limit, we obtain quantities which are comparable. By adjusting the thickness of the tubular domain, we can show that, at least for $n > 3$ and $p > (n+1)/(n-3)$, the uniqueness result remains true.

In this note all domains are open, bounded, and connected. Recall that a domain Ω is star-shaped if there is a point $x_0 \in \Omega$ such that any line segment $\overline{x_0 x}$ is contained in Ω when $x \in \Omega$. For convenience, we call x_0 a central point.

We need the following Pohozaev identity [P, PS].

For the Dirichlet problem (1), the equation is the Euler-Lagrange equation for the energy density

$$F(u, Du) = \frac{1}{2}|Du|^2 - \frac{|u|^{p+1}}{p+1}. \tag{2}$$

Let $\Omega \subset \mathbb{R}^n$ be a piecewise smooth domain. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a smooth solution of the Euler-Lagrange equation of the variational integral

$$I(u) = \int_\Omega F(u(x), Du(x))dx, \tag{3}$$

Then the identity

$$\begin{aligned}\int_{\partial\Omega} \Bigg[\left(\frac{1}{2}|Du|^2 - \frac{|u|^{p+1}}{p+1}\right) &\sum_{\alpha=1}^{n} (x - x^0)_\alpha \nu_\alpha \\ - \left(\sum_{\alpha,\,\beta=1}^{n} h_\beta \nu_\alpha \frac{\partial u}{\partial x_\beta}\frac{\partial u}{\partial x_\alpha}\right) &- au \sum_{\alpha=1}^{n} \nu_\alpha \frac{\partial u}{\partial x_\alpha}\Bigg] dS \\ = \int_\Omega \left[\left(\frac{n-2}{2} - a\right)|Du|^2 \right.&+ \left.\left(a - \frac{n}{p+1}\right)|u|^{p+1}\right] dx\end{aligned} \tag{4}$$

holds, where a is any fixed constant and $h(x) = x - x^0$ with $x^0 \in \mathbb{R}^n$ is a fixed vector. We use $\langle\cdot,\cdot\rangle$ to denote the inner product in $\mathbb{R}^n$. Then we can write (4) as

$$\begin{aligned}&\int_{\partial\Omega} [F(u, Du)\langle h, \nu\rangle - \langle Du, h\rangle\langle Du, \nu\rangle - au\langle Du, \nu\rangle]\, dS \\ &= \int_\Omega \left[\left(\frac{n-2}{2} - a\right)|Du|^2 + \left(a - \frac{n}{p+1}\right)|u|^{p+1}\right] dx.\end{aligned} \tag{4'}$$

If we further assume that Ω is star-shaped with $x^0 \in \bar{\Omega}$ a central point, and $u = 0$ on a portion Γ of $\partial\Omega$, then on Γ we have $\frac{\partial u}{\partial x_\alpha} = \frac{\partial u}{\partial \nu}\nu_\alpha$, so that

$$\int_\Gamma [F(u, Du)\langle h, \nu\rangle - \langle Du, h\rangle\langle Du, \nu\rangle - au\langle Du, \nu\rangle]\, dS = -\frac{1}{2}\int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2 \langle h, \nu\rangle dS \le 0, \tag{5}$$

because Ω is star-shaped and $x^0 \in \bar{\Omega}$ is a central point.

The following are the main results of this paper. Theorem 1 deals with general rotation-like domains while Theorem 2 treats tubular neighbourhoods of a closed or broken curve.

Theorem 1. *Suppose $\Omega \subset \mathbb{R}^n$ is a smooth domain with $n \geq 4$, and suppose the orthogonal projection of the closure of the domain onto the first component is an interval $[a, b]$. We assume that there is a $\delta > 0$, such that for all $a \leq t_1 < t_2 \leq b$, $|t_2 - t_1| \leq \delta$, the set*

$$\Omega_{t_1,t_2} = \{x = (x_1, x_2, \dots, x_n) \in \Omega,\ t_1 \leq x_1 \leq t_2\}$$

is star-shaped and there is some $t_0 \in [t_1, t_2]$ such that $x_0 = (t_0, 0, \dots, 0)$ is a central point. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a smooth solution of (1) with $p \geq (n+1)/(n-3)$. Then $u \equiv 0$ in $\bar{\Omega}$.

Remark. A rotation domain is a special case of those treated in Theorem 1. More precisely, suppose $x_2 = f(x_1) > 0$ is a smooth function defined in $[a, b]$. Then the rotation in $\mathbb{R}^{n-1}$ around the x_1-axis of the two-dimensional region bounded by f and the x_1-axis satisfies the hypotheses of Theorem 1. In particular, the domains we treat are much more general than those in [CZ].

Theorem 2 below deals with the uniqueness problem in general tubular neighbourhoods of embedded curves under a technical condition. We assume that there is a smooth orthogonal moving frame along the curve [S, Ch 1]. Suppose that $\gamma : [0, l] \to \mathbb{R}^n$ is a smooth curve parameterized by its arc-length $s \in [0, l]$. Suppose that there is a smooth orthogonal basis $e_2(s), \dots, e_n(s)$ on the normal hyperplane of $\gamma(s)$. Let $\dot{\gamma}(s) = e_1(s)$. Then

$$\begin{aligned}
\dot{e}_1(s) &= -k_1(s)e_2,\\
\dot{e}_j(s) &= k_{j-1}(s)e_{j-1} - k_j(s)e_{j+1}, \quad 2 \leq j \leq n-1,\\
\dot{e}_n(s) &= k_{n-1}e_{n-1}.
\end{aligned}$$

We call $k_1(s) \geq 0$ [S] the first curvature of γ and $E(s) := \{e_1(s),\, e_2(s), \dots, e_n(s)\}$, $0 \leq s \leq l$ a moving orthogonal frame along γ.

Notice that if $\gamma \subset \mathbb{R}^2$ is a planar curve, such a moving frame always exists. Let $\gamma(s) = (x_1(s), x_2(s))$, $\alpha(s) = \dot{\gamma}(s)$, $\beta(s) = (-\dot{x}_2(s), \dot{x}_1(s))$, and let $e_3, \dots e_n$ be the standard Euclidean basis for $\mathbb{R}^{n-2}$. Then $\alpha(s), \beta(s), e_3, \dots, e_n$ form an orthogonal moving frame along γ.

Let $\gamma : [0, l] \to \mathbb{R}^n$ be a simple, smooth and closed curve with bounded curvatures. Then it is easy to see that the r-neighbourhood

$$\Omega_r = \{x \in \mathbb{R}^n,\ \mathrm{dist}(x, \gamma) < r\}$$

is a tubular neighbourhood of γ for $r > 0$ small, with $(n-1)$-dimensional open balls of radius r as its fibres. If γ is a broken curve, Ω_r is the union of a tubular neighbourhood $\cup_{0<s<l} B_s$ and two half-balls at each end of the curve, where B_s is an $(n-1)$-dimensional open ball lying in the normal hyperplane of $\gamma(s)$ and centered at $\gamma(s)$.

We have

Theorem 2. *Let $n \geq 4$, and let γ be an embedded smooth (C^2) curve (closed or broken) in $\mathbb{R}^n$ with an associated smooth moving frame as defined above. Let $p > (n+1)/(n-3)$. Let Ω_r be the r-neighbourhood of γ. Then for sufficiently small $r > 0$, the only smooth solution of (1) on Ω_r is $u \equiv 0$.*

Corollary 1. *Let γ be an embedded smooth (C^2)-planar curve (closed or broken) in $\mathbb{R}^2$. Let Ω_r be its r-neighbourhood in $\mathbb{R}^2 \times \mathbb{R}^{n-2}$ with $n \geq 4$ and $p > (n+1)/(n-3)$. Then for sufficiently small $r > 0$ the only smooth solution of (1) on Ω_r is $u \equiv 0$.*

Proof of Theorem 1. We divide $[a, b]$ evenly as $a = t_0 < t_1 < \cdots < t_N = b$, with $t_{i+1} - t_i = (b - a)/N$, $i = 0, 1, 2, \ldots, N$ such that $(b - a)/N < \delta$. Let

$$\Omega_i = \{x \in \Omega,\ t_i \leq x_1 \leq t_{i+1}\}$$

for $i = 0, 1, \ldots, N - 1$. From the property of Ω, we see that Ω_i is star-shaped and there is some $t'_i \in [t_i, t_{i+1}]$ such that $x^i = (t'_i, 0, \ldots, 0)$ is a central point of Ω_i. We divide the boundary of Ω_i into three parts:

$$\partial\Omega_i = \Gamma_i \cup \Gamma_{i+1} \cup S_i,$$

where $\Gamma_i = \{x \in \bar{\Omega},\ x_1 = t_i\}$, and $S_i = \partial\Omega \cup \bar{\Omega}_i$. Notice that both Γ_0 and Γ_N are contained in $\partial\Omega$.

Now we apply (4') to u over the sub-domain Ω_i for each fixed i with $h^i = x - x^i$ to obtain

$$\begin{aligned}&\int_{\partial\Omega_i} \left[F(u, Du)\langle h^i, \nu\rangle - \langle Du, h^i\rangle\langle Du, \nu\rangle - au\langle Du, \nu\rangle\right] dS \\ &= \int_{\Omega_i} \left[\left(\frac{n-2}{2} - a\right)|Du|^2 + \left(a - \frac{n}{p+1}\right)|u|^{p+1}\right] dx.\end{aligned} \tag{6}$$

Now, let I_i and J_i be the left hand side and right hand side of (6), respectively. If $0 < i < N - 1$, we have $\partial\Omega_i = \Gamma_i \cup \Gamma_{i+1} \cup S_i$, and on S_i, $u = 0$ so that (5) implies

$$\begin{aligned}&\int_{S_i} \left[F(u, Du)\langle h^i, \nu\rangle - \langle Du, h^i\rangle\langle Du, \nu\rangle - au\langle Du, \nu\rangle\right] dS \\ &= -\frac{1}{2}\int_{S_i} \left|\frac{\partial u}{\partial \nu}\right|^2 \langle h^i, \nu\rangle dS \leq 0.\end{aligned}$$

Therefore

$$\begin{aligned}I_i \leq &\int_{\Gamma_{i+1}} \left[F(u, Du)\langle h^i, \nu\rangle - \langle Du, h^i\rangle\langle Du, \nu\rangle - au\langle Du, \nu\rangle\right] dS \\ &- \int_{\Gamma_i} \left[F(u, Du)\langle h^i, \nu\rangle - \langle Du, h^i\rangle\langle Du, \nu\rangle - au\langle Du, \nu\rangle\right] dS,\end{aligned} \tag{7}$$

where we have chosen the normal vector of Γ_i as towards the positive direction of the x_1-axis.

If $i=0$, we have

$$I_0 \le \int_{\Gamma_1} \big[F(u,Du)\langle h^0,\nu\rangle - \langle Du,h^0\rangle\langle Du,\nu\rangle - au\langle Du,\nu\rangle\big]\,dS. \tag{8}$$

This is because that on $\Gamma_0 \cup S_0$, $u=0$. Similarly, When $i=N-1$, we have,

$$I_{N-1} \le -\int_{\Gamma_{N-1}} \big[F(u,Du)\langle h^{N-1},\nu\rangle - \langle Du,h^{N-1}\rangle\langle Du,\nu\rangle - au\langle Du,\nu\rangle\big]\,dS. \tag{9}$$

Now we sum (7), (8) and (9) for $i=0,1,\dots,N-1$ to obtain

$$\sum_{i=0}^{N-1} J_i \le \sum_{i=0}^{N-2}\Big\{\int_{\Gamma_{i+1}} \big(F(u,Du)\langle x^{i+1}-x^i,\nu\rangle - \langle Du,x^{i+1}-x^i\rangle\langle Du,\nu\rangle\big)\,dS\Big\}. \tag{10}$$

Since $x^{i+1}-x^i = (t'_{i+1}-t'_i,0,\dots,0)$ and the normal vector ν on every Γ_i is $\nu=(1,0,\dots,0)$, we have in (10),

$$\begin{aligned}\sum_{i=0}^{N-2}\int_{\Gamma_{i+1}} &\big[F(u,Du)\langle x^{i+1}-x^i,\nu\rangle - \langle Du,x^{i+1}-x^i\rangle\langle Du,\nu\rangle - au\langle Du,\nu\rangle\big]\,dS \\ &= \sum_{i=0}^{N-2}\int_{\Gamma_{i+1}}\left[F(u,Du) - \left|\frac{\partial u}{\partial x_1}\right|^2\right] dS(t'_{i+1}-t'_i).\end{aligned} \tag{11}$$

We also see that

$$\sum_{i=0}^{N-1} J_i = \int_\Omega \left[\left(\frac{n-2}{2}-a\right)|Du|^2 + \left(a - \frac{n}{p+1}\right)|u|^{p+1}\right] dx.$$

Therefore we obtain

$$\begin{aligned}&\int_\Omega \left[\left(\frac{n-2}{2}-a\right)|Du|^2 + \left(a - \frac{n}{p+1}\right)|u|^{p+1}\right] dx \\ &\le \sum_{i=0}^{N-2}\left[\int_{\Gamma_{i+1}}\left(F(u,Du) - \left|\frac{\partial u}{\partial x_1}\right|^2\right) dS\right](t'_{i+1}-t'_i).\end{aligned} \tag{12}$$

Now we let $N\to\infty$ so that $\max_i\{t'_{i+1}-t'_i\}\to 0$ in (12). We have, by the definition of Riemann integral,

$$\begin{aligned}&\int_\Omega \left[\left(\frac{n-2}{2}-a\right)|Du|^2 + \left(a - \frac{n}{p+1}\right)|u|^{p+1}\right] dx \\ &\le \int_\Omega \left[\left(\frac{1}{2}|Du|^2 - \frac{|u|^{p+1}}{p+1}\right) - \left|\frac{\partial u}{\partial x_1}\right|^2\right] dx.\end{aligned} \tag{13}$$

Therefore,

$$\int_\Omega \left[\left(\frac{n-3}{2}-a\right)|Du|^2 + \left(a - \frac{n-1}{p+1}\right)|u|^{p+1}\right] dx \le -\int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^2 dx. \tag{14}$$

If

$$\frac{n-3}{2} > \frac{n-1}{p+1}, \qquad \text{hence} \qquad p > \frac{n+1}{n-3}$$

we may find a constant a such that

$$\frac{n-3}{2} > a > \frac{n-1}{p+1}$$

and conclude from (14) that $u \equiv 0$.

If

$$\frac{n-3}{2} = \frac{n-1}{p+1}, \qquad \text{which implies} \qquad p = \frac{n+1}{n-3},$$

we can only choose $a = (n-3)/2$ and (14) is reduced to

$$\int_\Omega \left| \frac{\partial u}{\partial x_1} \right|^2 dx = 0,$$

which gives that $\frac{\partial u}{\partial x_1} = 0$ in Ω. The zero boundary condition implies that $u \equiv 0$.

□

Proof of Theorem 2. Let $\gamma : [0, l] \to \mathbb{R}^n$ be a C^2 closed embedded curve parameterized by its arc-length, so that $\gamma(0) = \gamma(l)$. Define $k_0 = \max_{0 \le s \le l} k_1(s)$. Let $\bar{\Omega}_r$ be the closed r-neighbourhood in $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$ with $n \ge 4$, where $0 < rk_0 < 1$.

We first choose $r > 0$ small enough so that the periodic mapping (in s with period l)

$$F : (s, x_2, x_3, x_4, \dots, x_n) \to \gamma(s) + x_2 e_2(s) + x_3 e_3(s) + \dots + x_n e_n(s)$$

is one-to-one from $[0, l] \times \bar{B}_r(0)$ to $\bar{\Omega}_r$ except at 0 and l where $F(0, \cdot) = F(l, \cdot)$, with

$$\bar{B}_r(0) = \{(x_2, x_3, \dots, x_n) \in \mathbb{R}^{n-1},\ x_2^2 + x_3^2 + \cdots x_n^2 \le r^2\}$$

the closed ball in $\mathbb{R}^{n-1}$. The Jacobian of this mapping is $\pm(1 + x_2 k_1(s))$, where $k_1(s)$ is the first curvature of γ.

Now we divide $[0, l]$ evenly as

$$0 = s_0 < s_1 < \dots < s_{N-1} < s_N = l, \qquad s_{i+1} - s_i = \frac{l}{N},\ i = 0, 1, \dots N-1$$

and let s_i' be the midpoint of $[s_i, s_{i+1}]$. We let Γ_i be the intersection of the normal hyperplane of γ at $s = s_i$ and Ω_r and define $\bar{\Omega}_i$ to be the closed sub-domain of Ω_r bounded by Γ_i and Γ_{i+1}. Notice that γ is a closed curve so that $\Gamma_N = \Gamma_0$ and $\Omega_N = \Omega_0$.

As in the proof of Theorem 1, we apply (4') to each Ω_i with $h^i(x) = x - \gamma(s_i')$. We have

$$\begin{aligned} &\int_{\partial\Omega_i} \left[F(u, Du)\langle h^i, \nu\rangle - \langle Du, h^i\rangle\langle Du, \nu\rangle - au\langle Du, \nu\rangle \right] dS \\ &= \int_{\Omega_i} \left[\left(\frac{n-2}{2} - a \right) |Du|^2 + \left(a - \frac{n}{p+1} \right) |u|^{p+1} \right] dx. \end{aligned} \tag{15}$$

As in the proof of Theorem 1, we let I_i and J_i be the left and right hand sides of (15), respectively, and let $\partial\Omega_i = \Gamma_i \cup \Gamma_{i+1} \cup S_i$, where $S_i = \partial\Omega_i \cap \partial\Omega_r$.

Let us first consider the surface integral over $S_i \subset \partial\Omega_r$. Notice that $u = 0$ on S_i, so that (5) gives

$$\begin{aligned}&\int_{S_i} \big[F(u, Du)\langle h^i, \nu\rangle - \langle Du, h^i\rangle\langle Du, \nu\rangle - au\langle Du, \nu\rangle\big]\, dS \\ &= -\frac{1}{2}\int_{S_i} \left|\frac{\partial u}{\partial \nu}\right|^2 \langle h^i, \nu\rangle dS.\end{aligned} \tag{16}$$

We claim that for sufficiently large $N > 0$, $\langle h^i, \nu\rangle \geq 0$ on S_i. A general point $x \in S_i$ can be written as

$$x = \gamma(s) + x_2e_2(s) + x_3e_3(s) + \cdots + x_ne_n(s)$$

with $x_2^2 + x_3^2 + \cdots + x_n^2 = r^2$, for some $s \in [s_i, s_{i+1}]$, and the outward normal vector at x is

$$\nu = [x_2e_2(s) + x_3e_3(s) + \cdots + x_ne_n(s)]/r.$$

We have

$$\begin{aligned}&r\langle h^i, \nu\rangle = r\langle x - \gamma(s_i'), \nu\rangle \\ &= \langle \gamma(s) + x_2e_2(s) + x_3e_3(s) + \cdots x_ne_n(s) - \gamma(s_i'), x_2e_2(s) + x_3e_3(s) + \cdots x_ne_n(s)\rangle \\ &= \langle \gamma(s) - \gamma(s_i'), x_2e_2(s) + x_3e_3(s) + \cdots + x_ne_n(s)\rangle + r^2 \\ &\geq r^2 - |\gamma(s) - \gamma(s_i')|r \geq r^2 - r|s - s_i'| > 0,\end{aligned}$$

when $|s - s_i'| \leq l/N$ is sufficiently small.

Now we sum up I_i's as in the proof of Theorem 1 to obtain

$$\begin{aligned}&\sum_{i=0}^{N-1} I_i \leq \\ &\sum_{i=0}^{N-1} \int_{\Gamma_{i+1}} \big[F(u, Du)\langle \gamma(s_{i+1}') - \gamma(s_i'), \nu\rangle - \langle Du, \gamma(s_{i+1}') - \gamma(s_i')\rangle\langle Du, \nu\rangle\big]\, dS \\ &= \sum_{i=0}^{N-1} \left[\int_{\Gamma_{i+1}} \left(\frac{1}{2}|Du|^2 - \frac{|u|^{p+1}}{p+1}\right)\langle \gamma(s_{i+1}') - \gamma(s_i'),\ \nu\rangle dS\right. \\ &\left. - \int_{\Gamma_{i+1}} \langle Du,\ \gamma(s_{i+1}') - \gamma(s_i')\rangle\langle Du, \nu\rangle dS\right] \\ &= A_N.\end{aligned} \tag{17}$$

Notice that $\Gamma_N = \Gamma_0$, $\nu = \dot\gamma(s_{i+1})$,

$$\begin{aligned}&\langle \gamma(s_{i+1}') - \gamma(s_i'), \nu\rangle \\ &= \langle \dot\gamma(s_{i+1})(s_{i+1}' - s_i'), \dot\gamma(s_{i+1})\rangle \\ &+ \langle \frac{1}{2}\ddot\gamma(\xi_{i+1})(s_{i+1}' - s_{i+1})^2 - \frac{1}{2}\ddot\gamma(\eta_{i+1})(s_{i+1} - s_i')^2, \dot\gamma(s_{i+1})\rangle,\end{aligned}$$

where ξ_{i+1} and η_{i+1} are two points in (s_{i+1}, s'_{i+1}) and (s'_i, s_{i+1}) respectively. Now we have

$$\begin{aligned}&\langle\dot\gamma(s_{i+1})(s'_{i+1}-s'_i),\dot\gamma(s_{i+1})\rangle\\&=s'_{i+1}-s'_i.\end{aligned}\tag{18}$$

Since γ is of class C^2, there is a constant $C_0>0$ such that $|\ddot\gamma(s)|\le C_0$ for all $s\in[0.l]$. Therefore we also have

$$\begin{aligned}&\Big|\langle\frac{1}{2}\ddot\gamma(\xi_{i+1})(s'_{i+1}-s_{i+1})^2-\frac{1}{2}\ddot\gamma(\eta_{i+1})(s_{i+1}-s'_i)^2,\,\dot\gamma(s_{i+1})\rangle\Big|\\&\le\frac{1}{2}C_0\left[(s'_{i+1}-s_{i+1})^2+(s_{i+1}-s'_i)^2\right]\\&\le C_0(s'_{i+1}-s'_i)^2.\end{aligned}\tag{19}$$

Similarly, we have

$$\begin{aligned}&\langle\gamma(s'_{i+1})-\gamma(s'_i),Du\rangle\\&=\langle\dot\gamma(s_{i+1}),Du\rangle(s'_{i+1}-s'_i)\\&+\langle\frac{1}{2}\ddot\gamma(\xi'_{i+1})(s'_{i+1}-s_{i+1})^2-\frac{1}{2}\ddot\gamma(\eta'_{i+1})(s_{i+1}-s'_i)^2,Du\rangle,\end{aligned}\tag{20}$$

with

$$\begin{aligned}&\Big|\langle\frac{1}{2}\ddot\gamma(\xi'_{i+1})(s'_{i+1}-s_{i+1})^2-\frac{1}{2}\ddot\gamma_r(\eta'_{i+1})(s_{i+1}-s'_i)^2,Du\rangle\Big|\\&\le C_0|Du|(s'_{i+1}-s'_i)^2.\end{aligned}\tag{21}$$

Now we can estimate the sum A_N in (17):

$$\begin{aligned}A_N&\le\sum_{i=0}^{N-1}\int_{\Gamma_{i+1}}\left[F(u,Du)-\langle Du,\dot\gamma(s_{i+1})\rangle^2\right]dS(s'_{i+1}-s'_i)\\&+C_0\sum_{i=0}^{N-1}\frac{l}{N}\left[\int_{\Gamma_{i+1}}\left|\frac{1}{2}|Du|^2-\frac{|u|^{p+1}}{p+1}\right|+|Du|^2dS\right](s'_{i+1}-s'_i)\\&=B_1(N)+B_2(N),\end{aligned}$$

where

$$\begin{aligned}B_1(N)&=\sum_{i=0}^{N-1}\int_{\Gamma_{i+1}}\left[F(u,Du)-\langle Du,\dot\gamma(s_{i+1})\rangle^2\right]dS(s'_{i+1}-s'_i)\\&\to\int_0^l\int_{\Gamma_s}\left[\left(\frac{1}{2}|Du|^2-\frac{|u|^{p+1}}{p+1}\right)-\langle Du,\dot\gamma(s)\rangle^2\right]dSds,\end{aligned}$$

as $N\to\infty$, where

$$\Gamma_s=\{\gamma(s)+x_2e_2(s)+x_3e_3(s)+\cdots+x_ne_n(s),\ x_2^2+x_3^2+\cdots+x_n^2\le r^2\}.\tag{22}$$

We also have

$$B_2=C_0\sum_{i=0}^{N-1}\frac{l}{N}\int_{\Gamma_{i+1}}\left[\left|\frac{1}{2}|Du|^2-\frac{|u|^{p+1}}{p+1}\right|+|Du|^2\right]dS(s'_{i+1}-s'_i)\to0$$

as $N \to 0$ because

$$\sum_{i=0}^{N-1} \int_{\Gamma_{i+1}} \left[\left| \frac{1}{2}|Du|^2 - \frac{|u|^{p+1}}{p+1} \right| + |Du|^2 \right] dS(s'_{i+1} - s'_i)$$

converges to an integral.

Now we sum up the right hand side of (15):

$$\begin{aligned}\sum_{i=0}^{N-1} J_i &= \sum_{i=0}^{N-1} \int_{\Omega_i} \left[\left(\frac{n-2}{2} - a \right) |Du|^2 + \left(a - \frac{n}{p+1} \right) |u|^{p+1} \right] dx \\ &= \int_{\Omega_r} \left[\left(\frac{n-2}{2} - a \right) |Du|^2 + \left(a - \frac{n}{p+1} \right) |u|^{p+1} \right] dx.\end{aligned}$$

We now change variables

$$x = \gamma(s) + x_2 e_2(s) + x_3 e_3(s) + \cdots + x_n e_n(s),$$

to obtain

$$\begin{aligned}&\int_{\Omega_r} \left[\left(\frac{n-2}{2} - a \right) |Du|^2 + \left(a - \frac{n}{p+1} \right) |u|^{p+1} \right] dx \\ &= \int_0^l \int_{\Gamma_s} \left[\left(\frac{n-2}{2} - a \right) |Du|^2 + \left(a - \frac{n}{p+1} \right) |u|^{p+1} \right] (1 + x_2 k_1(s)) dS\, ds,\end{aligned}$$

when $rk_0 < 1$. Finally we obtain

$$\begin{aligned}&\int_0^l \int_{\Gamma_s} \left[\left(\frac{n-2}{2} - a \right) |Du|^2 + \left(a - \frac{n}{p+1} \right) |u|^{p+1} \right] (1 + x_2 k_1(s)) dS\, ds \\ &\le \int_0^l \int_{\Gamma_s} \left[\left(\frac{1}{2}|Du|^2 - \frac{|u|^{p+1}}{p+1} \right) - \langle Du, \dot\gamma(s) \rangle^2 \right] dS ds.\end{aligned} \tag{23}$$

Now, we deduce from (23) that

$$\begin{aligned}&\int_0^l \int_{\Gamma_s} \left[\left(\frac{n-2}{2} - a \right) \phi - \frac{1}{2} \right] |Du|^2 + \left[\left(a - \frac{n}{p+1} \right) \phi + \frac{1}{p+1} \right] |u|^{p+1} dS\, ds \\ &- \int_0^l \int_{\Gamma_s} \langle Du, \dot\gamma(s) \rangle^2 dS ds \le 0,\end{aligned} \tag{24}$$

where $\phi := 1 + x_2 k_1(s)$. Now, $|\phi - 1| \le rk_0 \to 0$ as $r \to 0$. Therefore

$$\left(\frac{n-2}{2} - a \right) \phi - \frac{1}{2} \to \frac{n-3}{2} - a, \quad \text{and} \quad \left(a - \frac{n}{p+1} \right) \phi + \frac{1}{p+1} \to a - \frac{n-1}{p+1}$$

uniformly on $[0,\, l] \times \bar{B}_r(0)$ as $r \to 0$. Because $p > (n+1)/(n-3)$, it is possible to find some $a \in \mathbb{R}$ and $c > 0$ such that

$$\left(\frac{n-2}{2} - a \right) \phi - \frac{1}{2} \ge c, \qquad \left(a - \frac{n}{p+1} \right) \phi + \frac{1}{p+1} \ge c$$

on $[0,\, l] \times \bar{B}_r(0)$ as $r > 0$ sufficiently small. Thus (24) implies that $u = 0$ on Ω_r.

If γ is not a closed curve, the proof is similar. We need to extend the curve at the two end points $\gamma(0)$ and $\gamma(l)$ along the tangent directions as straight line segments so that the extended curve reaches the boundary of Ω_r at two points. Then the proof proceeds as in the case of closed curves. $\square$

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References

[BC] A. Bahri and J. M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 253-294.

[CZ] K.S.Chou and X.P.Zhu, *Some constancy results for nematic liquid crystals and harmonic maps*, Anal. Nonlin. H. Poncaré Inst. 12 (1995), 99-115.

[Da] E. N. Dancer, *A note on an equation with critical growth*, Bull. London Math. Soc. **20** (1988), 600-602.

[Di] W. Y. Ding, *Positive solutions of* $\Delta u + u^{(n+2)/(n-2)} = 0$, J. Partial Diff. Equations **2** (1989), 83-88.

[DZ] E. N. Dancer and K. Zhang, *Uniqueness of solutions for some elliptic equations and systems in nearly star-shaped domains*, To appear in, Nonlin. Anal. TMA.

[M] E. Mitidieri, *A Rellich-type identity and applications*, Comm. PDE **18** (1993), 125-151.

[P] S. I. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. **6** (1965), 1408-1411.

[PS] P. Pucci and J. Serrin, *A general variational identity*, Indiana Univ. Math. J. **35** (1986), 681-703.

[S] M. Spivak, *Differential Geometry*, vol. I-II, Publish or Perish, 1979.

[V] R. C. A. M. van der Vorst, *Variational identities and applications to differential systems*, Arch. Rational Mech. Anal. **116** (1992), 375-398.

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