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STABILIZATION OF SOLUTIONS TO HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION WITH LOCALIZED DAMPING

ELENI BISOGNIN, VANILDE BISOGNIN, OCTAVIO PAULO VERA VILLAGRÁN

Abstract. We study the stabilization of solutions to higher-order nonlinear Schrödinger equations in a bounded interval under the effect of a localized damping mechanism. We use multiplier techniques to obtain exponential decay in time of the solutions of the linear and nonlinear equations.

1. Introduction

In this work we consider the initial-value problem of the higher-order nonlinear Schrödinger equation with localized damping

$$iu_t + \alpha u_{xx} + i\beta u_{xxx} + |u|^2 u + ia(x)u = 0 \tag{1.1}$$

$$u(x,0) = u_0(x) \tag{1.2}$$

where $0 < x < L$ and $t > 0$, and with boundary conditions

$$u(0,t) = u(L,t) = 0 \quad \text{for all } t > 0 \tag{1.3}$$

$$u_x(L,t) = 0 \quad \text{for all } t > 0 \tag{1.4}$$

with $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, $u = u(x,t)$ a complex valued function, $a = a(x)$ a nonnegative everywhere function such that $a(x) \in C^\infty((0,L))$ and $a(x) \geq a_0 > 0$. Equation (1.1) is a particular case of the equation

$$\begin{gathered} iu_t + \omega u_{xx} + i\beta u_{xxx} + \gamma|u|^2 u + i\delta|u|^2 u_x + i\epsilon u^2 \overline{u}_x = 0 \quad x,t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{gathered} \tag{1.5}$$

where $\omega, \beta, \gamma, \delta$ are real numbers and $\beta \neq 0$. This equation was first proposed by Hasegawa and Kodama [9] as a model for the propagation of a signal in a optic fiber (see also [11]). The equation (1.5) can be reduced to other well known equations. For instance, setting $\omega = 1$, $\beta = \delta = \epsilon = 0$ in (1.5) we have the semilinear Schrödinger equation,

$$iu_t + u_{xx} + \gamma|u|^2 u = 0. \tag{1.6}$$

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If we let $\beta = \gamma = 0$ and $\omega = 1$ in (1.5) we obtain the nonlinear Schrödinger equation

$$iu_t + u_{xx} + i\delta|u|^2 u_x + i\epsilon u^2 \overline{u}_x = 0. \tag{1.7}$$

Letting $\alpha = \gamma = \epsilon = 0$ in (1.5) arises is the complex modified Korteweg-de Vries (KdV) equation

$$iu_t + i\beta u_{xxx} + i\delta|u|^2 u_x = 0. \tag{1.8}$$

The initial value problem for the equations (1.6), (1.7) and (1.8) has been extensively studied in the last few years, see for instance [3, 6, 10, 27] and references therein. In 1992, Laurey [14] considered the equation (1.5) and proved local well-posedness of the initial value problem associated for data in $H^s(\mathbb{R})$, $s > 3/4$, and global well-posedness in $H^s(\mathbb{R})$, $s \geq 1$. In 1997, Staffilani [30] established local well-posedness for data in $H^s(\mathbb{R})$, $s \geq 1/4$ in (1.5) improving Laurey's result. A similar result was given in [4, 5] with $w(t)$, $\beta(t)$ real functions. Recently, Sepúlveda and Vera [28] showed that C^∞ solutions $u(x,t)$ are obtained for all $t > 0$ if the initial data $u_0(x)$ decays faster than polynomially on $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and has a certain initial Sobolev regularity. In [2] Bisognin and Vera considered the equation (1.5) with $\delta = \epsilon = 0$ and proved the unique continuation property.

This paper concerns the exponential stabilization of the solution of (1.1) when the damping $a = a(x)$ is effective only on a subset of the interval $(0, L)$. This problem was extensively studied in the context of wave equations, see Dafermos [7], Haraux [8], Slemrod [29], Zuazua [36] and Nakao [17]. The same problem has been also studied for the KdV equation. Here we can mention the works of Komornik, Russell and Zhang [13]. Using a different damping mechanism they obtained the exponential decay with periodic boundary conditions. In [20], Menzala, Vasconcellos and Zuazua studied the nonlinear KdV equation inspired in the work of Rosier [23]. They studied the stabilization of solutions for the KdV equation in a bounded interval under the effect of a localized damping mechanism. Using compactness arguments, the smoothing effect of the KdV equation on the line and the unique continuation results, the authors deduced the exponential decay in time of the solutions of the linear equation and a local uniform stabilization result of the solutions of the nonlinear equation when the localized damping is active simultaneously only in a neighborhood of both extremes $x = 0$, $x = L$. The same result was obtained by the KdV coupled system by Menzala, Bisognin and Bisognin [21]. The main result of this paper says that the total energy $E(t)$ associated to (1.1) decays exponentially as $t \to +\infty$, for bounded sets of initial data. In order to prove the result we use multipliers together with compactness arguments and smoothing properties proved by M. Sepúveda and Vera [28] and the Unique Continuation Principle valid for this problem, see Bisognin and Vera [2].

This paper is organized as follows: In section two, we study the existence of global solution to the linear and nonlinear problem. In section three, we study the stabilization result of the problem. First we prove the exponential decay in the linear problem and of at end we prove the stabilization of the solution of the nonlinear problem. The notation that we use in this article is standard and can be found in Temam [32].

2. Stabilization of solutions of the linear problem

In this section we are interested in proving the global existence and uniqueness of the solution and the exponential decay of the solution of the linear problem associated to (1.1)-(1.4). We consider the problem

$$iu_t + \alpha u_{xx} + i\beta u_{xxx} + ia(x)u = 0 \tag{2.1}$$
$$u(x,0) = u_0(x), \quad \text{for all } x \in I \tag{2.2}$$
$$u(0,t) = u(L,t) = 0, \quad \text{for all } t > 0 \tag{2.3}$$
$$u_x(L,t) = 0, \quad \text{for all } t > 0 \tag{2.4}$$

where $I = (0,L)$, $a \in C^\infty(I)$, $a(x) \geq a_0 > 0$ is assumed to be nonnegative everywhere in an open non empty proper subset ω of I and we will prove the global existence of the problem (2.1)-(2.4).

We consider $a \equiv 0$ and the operator $A = -\beta\partial_x^3 + i\alpha\partial_x^2$ with domain

$$\mathcal{D}(A) = \{v \in H^3(I) : v(0) = v(L) = 0,\ v_x(L) = 0\} \subseteq L^2(I)$$

Lemma 2.1. *Let $a \equiv 0$ and $\beta > 0$. Then, the operator A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions in $L^2(I)$.*

Proof. It is easy to prove that A is closed. Let us to prove that A is dissipative. Integration by parts give us

$$(Av,v)_{L^2(I)} = \int_0^L (-\beta v_{xxx} + i\alpha v_{xx})\overline{v}\,dx = -\frac{\beta}{2}|v_x^2(0,t)|^2 - i\alpha\int_0^L |v_x|^2 dx.$$

Hence,

$$\mathrm{Re}(Av,v)_{L^2(I)} = -\frac{\beta}{2}|v_x^2(0,t)|^2 \leq 0,$$

where A is dissipative. On the other hand, the adjoint of the operator A is given by

$$H^* v = \beta v_{xxx} - i\alpha v_{xx}$$

with domain

$$\mathcal{D}(H^*) = \{v \in H^3(I) : v(0,t) = v(L,t) = 0,\ v_x(0,t) = 0\} \subseteq L^2(I).$$

A similar calculation shows that

$$(H^* v,v)_{L^2(I)} = \int_0^L (\beta v_{xxx} - i\alpha v_{xx})\overline{v}\,dx = -\frac{\beta}{2}|v_x^2(0,t)|^2 + i\alpha\int_0^L |v_x|^2\,dx.$$

Hence

$$\mathrm{Re}(Av,v)_{L^2(I)} = -\frac{\beta}{2}|v_x^2(0,t)|^2 \leq 0.$$

The conclusion of Lemma 2.1 follows from the Stone Theorem [19] of semigroup theory. $\square$

The above discussion proves the following result.

Theorem 2.2. *Let $u_0 \in \mathcal{D}(A)$, $a \equiv 0$ and $\beta > 0$. Then, there exists a unique function u such that $u \in C(0,+\infty : H^3(I)) \cap C^1(0,+\infty : L^2(I))$ which solves (2.1)-(2.4).*

The well-posedness of system (2.1)-(2.4), when $a \not\equiv 0$ can be handled in a similar way by considering the term $a(x)u$ as a linear perturbation of the case $a \equiv 0$.

Now, we will prove the exponential decay of the total energy $E(t)$ associated to (2.1)-(2.4) under suitable assumptions on the open subset ω of I. We denote by

$\{S(t)\}_{t\geq 0}$ the semigroup of contractions associated with A, and by $\mathcal{H}$ the Banach space $C([0,T]:L^2(I))\cap L^2(0,T:H^1(I))$ with the norm

$$\|v\|_{\mathcal{H}} = \sup_{[0,T]} \|v(\cdot,t)\|_{L^2(I)} + \Big(\int_0^T \|v(\cdot,t)\|^2_{H^1(I)}\Big)^{1/2} \tag{2.5}$$

Theorem 2.3. *Consider the solution of the problem* (2.1)-(2.4). *Then there exist* $c>0$ *and* $\mu>0$ *such that*

$$\|u(\cdot,t)\|^2_{L^2(I)} \leq c\|u_0\|^2_{L^2(I)} e^{-\mu t} \tag{2.6}$$

for all $t\geq 0$ *and* $u_0\in L^2(I)$.

For the proof of the above theorem we need the following result.

Lemma 2.4. *Let* $|\alpha|<3\beta$. *Then*

(1) *The map* $u_0\in L^2(I)\to S(t)u_0\in\mathcal{H}$ *is continuous.*
(2) *For* $u_0\in L^2(I)$, $\partial_x u(0,\cdot)$ *makes sense in* $L^2(I)$ *and*

$$\|u_x(0,\cdot)\|_{L^2(0,T)} \leq \frac{1}{\sqrt{\beta}}\,\|u_0\|_{L^2(I)} \tag{2.7}$$

$$\int_0^T |u_0|^2\,dx \leq \frac{1}{T}\int_0^T\int_0^L |u|^2\,dx\,dt + \beta\int_0^T |u_x(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)|u|^2\,dx. \tag{2.8}$$

Proof. (1) For $u_0\in L^2(I)$, let $u=S(t)u_0$ be the mild solution of (2.1)-(2.4). By Theorem 2.2, $u\in C(0,T:L^2(I))$ and

$$\|u\|_{C(0,T:L^2(I))} \leq \|u_0\|_{L^2(I)}. \tag{2.9}$$

To see that $u\in L^2(0,T:H^1(I))$ we first assume that $u_0\in\mathcal{D}(A)$. Let $\xi=\xi(x,t)\in C^\infty([0,L]\times[0,T])$. Multiplying the equation (2.1) by $\xi\overline{u}$ we have

$$i\xi\overline{u}u_t + \alpha\xi\overline{u}u_{xx} + i\beta\xi\overline{u}u_{xxx} + i\xi a(x)|u|^2 = 0. \tag{2.10}$$

Applying the conjugate, we have

$$-i\xi u\overline{u}_t + \alpha\xi u\overline{u}_{xx} - i\beta\xi u\overline{u}_{xxx} - i\xi a(x)|u|^2 = 0. \tag{2.11}$$

Subtracting (2.10) and (2.11) and integrating over $x\in(0,L)$, we have

$$\begin{aligned} &i\frac{d}{dt}\int_0^L \xi|u|^2\,dx - i\int_0^L \xi_t|u|^2\,dx + i\beta\int_0^L \xi\overline{u}u_{xxx}\,dx + i\beta\int_0^L \xi u\overline{u}_{xxx}\,dx \\ &+\alpha\int_0^L \xi\overline{u}u_{xx}\,dx - \alpha\int_0^L \xi u\overline{u}_{xx}\,dx + 2i\int_0^L \xi a(x)|u|^2\,dx = 0. \end{aligned} \tag{2.12}$$

Each term in the above equation is treated separately, integrating by parts and using the boundary conditions we obtain

$$\begin{aligned}
\int_0^L \xi\overline{u}u_{xxx}\,dx &= \int_0^L \xi_{xx}\overline{u}u_x\,dx + 2\int_0^L \xi_x|u_x|^2\,dx + \int_0^L \xi u_x\overline{u}_{xx}\,dx \\
&\quad + \xi(0,t)|u_x(0,t)|^2 \\
\int_0^L \xi u\overline{u}_{xxx}\,dx &= \int_0^L \xi_{xx}u\overline{u}_x\,dx + \int_0^L \xi_x|u_x|^2\,dx - \int_0^L \xi u_x\overline{u}_{xx}\,dx \\
\int_0^L \xi\overline{u}u_{xx}\,dx &= -\int_0^L \xi_x\overline{u}u_x\,dx - \int_0^L \xi|u_x|^2\,dx \\
\int_0^L \xi u\overline{u}_{xx}\,dx &= -\int_0^L \xi_x u\overline{u}_x\,dx - \int_0^L \xi|u_x|^2\,dx.
\end{aligned}$$

Replacing these expression in (2.12) and performing straightforward calculations,

$$\begin{aligned}
&i\frac{d}{dt}\int_0^L \xi|u|^2\,dx - i\int_0^L \xi_t|u|^2\,dx + i\beta\int_0^L \xi_{xx}(|u|^2)_x\,dx + 3i\beta\int_0^L \xi_x|u_x|^2\,dx \\
&+ i\beta\xi(0,t)|u_x(0,t)|^2 - 2i\alpha\operatorname{Im}\int_0^L \xi_x\overline{u}u_x\,dx + 2i\int_0^L \xi a(x)|u|^2\,dx = 0
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_0^L \xi|u|^2\,dx - \int_0^L \xi_t|u|^2\,dx + \beta\int_0^L \xi_{xx}(|u|^2)_x\,dx + 3\beta\int_0^L \xi_x|u_x|^2\,dx \\
&+ \beta\xi(0,t)|u_x(0,t)|^2 - 2\alpha\operatorname{Im}\int_0^L \xi_x\overline{u}u_x\,dx + 2\int_0^L \xi a(x)|u|^2\,dx = 0.
\end{aligned} \tag{2.13}$$

Let $\xi(x,t) \equiv x$, then in (2.13), we obtain

$$\frac{d}{dt}\int_0^L x|u|^2\,dx + 3\beta\int_0^L |u_x|^2\,dx - 2\alpha\operatorname{Im}\int_0^L \overline{u}u_x\,dx + 2\int_0^L xa(x)|u|^2\,dx = 0.$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\int_0^L x|u|^2\,dx + 3\beta\int_0^L |u_x|^2\,dx + 2\int_0^L xa(x)|u|^2\,dx \\
&= 2\alpha\operatorname{Im}\int_0^L \overline{u}u_x\,dx \le |\alpha|\int_0^L |u|^2\,dx + |\alpha|\int_0^L |u_x|^2\,dx
\end{aligned} \tag{2.14}$$

then

$$\frac{d}{dt}\int_0^L x|u|^2\,dx + (3\beta-|\alpha|)\int_0^L |u_x|^2\,dx + 2\int_0^L xa(x)|u|^2\,dx \le |\alpha|\int_0^L |u|^2\,dx$$

and

$$\frac{d}{dt}\int_0^L x|u|^2\,dx + (3\beta-|\alpha|)\int_0^L |u_x|^2\,dx + 2\int_0^L xa(x)|u|^2\,dx \le |\alpha|\|u\|^2_{L^2(0,L)}. \tag{2.15}$$

Integrating (2.15) over $t \in [0,T]$ we have

$$\begin{aligned}
&\int_0^L x|u|^2\,dx + (3\beta - |\alpha|)\int_0^T\int_0^L |u_x|^2\,dx\,dt + 2\int_0^T\int_0^L xa(x)|u|^2\,dx\,dt\\
&\le \int_0^L x|u_0|^2\,dx + |\alpha|\int_0^T \|u\|^2_{L^2(0,L)}\,dt\\
&\le L\|u_0\|^2_{L^2(0,L)} + T|\alpha|\|u_0\|^2_{L^2(0,L)}\\
&= [L + T|\alpha|]\|u_0\|^2_{L^2(0,L)}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^L x|u|^2\,dx + (3\beta - |\alpha|)\|u_x\|^2_{L^2(0,T:L^2(0,L))} + 2\int_0^T\int_0^L xa(x)|u|^2\,dx\,dt\\
&\le [L + T|\alpha|]\|u_0\|^2_{L^2(0,L)}.
\end{aligned}$$

Using that $a(x) \ge a_0 > 0$, we obtain

$$\int_0^L x|u|^2\,dx + (3\beta - |\alpha|)\|u_x\|^2_{L^2(0,T:L^2(0,L))} \le [L + T|\alpha|]\|u_0\|^2_{L^2(0,L)}. \tag{2.16}$$

In particular, using that $|\alpha| < 3\beta$,

$$(3\beta - |\alpha|)\|u_x\|^2_{L^2(0,T:L^2(0,L))} \le [L + T|\alpha|]\|u_0\|^2_{L^2(0,L)}.$$

and

$$\|u_x\|^2_{L^2(0,T:L^2(0,L))} \le \frac{1}{(3\beta - |\alpha|)}\,[L + T|\alpha|]\,\|u_0\|^2_{L^2(0,L)}. \tag{2.17}$$

By the density of $\mathcal{D}(A)$ in $L^2(I)$ the result extends to arbitrary $u_0 \in L^2(I)$.

We remark that: (a) The estimate (2.16) gives a smoothing effect. (b) In (2.14) using Young's estimate and assuming that $\beta > 0$ we have

$$2\alpha\,\mathrm{Im}\int_0^L \overline{u}u_x\,dx \le \frac{|\alpha|^2}{2\beta}\int_0^L |u|^2\,dx + 2\beta\int_0^L |u_x|^2\,dx.$$

Then, in we obtain (2.15),

$$\frac{d}{dt}\int_0^L x|u|^2\,dx + \beta\int_0^L |u_x|^2\,dx + 2\int_0^L xa(x)|u|^2\,dx \le \frac{|\alpha|^2}{2\beta}\|u\|^2_{L^2(0,L)}$$

and the assumption that $|\alpha| < 3\beta$ can be removed.

(2) We also assume $u_0 \in \mathcal{D}(A)$ and taking $\xi(x,t) = 1$ in (2.13), we have

$$\frac{d}{dt}\int_0^L |u|^2\,dx + \beta|u_x(0,t)|^2 + 2\int_0^L a(x)|u|^2\,dx = 0. \tag{2.18}$$

Hence, integrating (2.18) over $t \in [0,T]$ we have

$$\int_0^L |u|^2\,dx + \beta\int_0^T |u_x(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)|u|^2\,dx\,dt = \int_0^L |u_0|^2\,dx$$

Using $a(x) \ge a_0 > 0$, we obtain

$$\int_0^L |u|^2\,dx + \beta\int_0^T |u_x(0,t)|^2\,dt \le \int_0^L |u_0|^2\,dx$$

$$\beta\int_0^T |u_x(0,t)|^2\,dt \le \int_0^L |u_0|^2\,dx - \int_0^L |u|^2\,dx \le \int_0^L |u_0|^2\,dx;$$

therefore, (2.7) is proved. □

On the other hand, taking $\xi(x,t)=T-t$ in (2.13) we have

$$\begin{aligned} &\frac{d}{dt}\int_0^L (T-t)|u|^2\,dx+\int_0^L |u|^2\,dx\\ &+\beta(T-t)|u_x(0,t)|^2+2\int_0^L (T-t)a(x)|u|^2\,dx=0. \end{aligned} \tag{2.19}$$

Integrating (2.19) over $t\in[0,T]$ we have

$$\begin{aligned} &-T\int_0^L |u_0|^2\,dx+\int_0^T\int_0^L |u|^2\,dx\,dt\\ &+\beta\int_0^T (T-t)|u_x(0,t)|^2\,dt+2\int_0^T\int_0^L (T-t)a(x)|u|^2\,dx\,dt=0. \end{aligned}$$

Then

$$\begin{aligned} &T\int_0^L |u_0|^2\,dx\\ &=\int_0^T\int_0^L |u|^2\,dx\,dt+\beta\int_0^T (T-t)|u_x(0,t)|^2\,dt+2\int_0^T\int_0^L (T-t)a(x)|u|^2\,dx\,dt\\ &\le \int_0^T\int_0^L |u|^2\,dx\,dt+\beta\int_0^T T|u_x(0,t)|^2\,dt+2\int_0^T\int_0^L Ta(x)|u|^2\,dx\,dt \end{aligned}$$

and

$$\begin{aligned} \int_0^L |u_0|^2\,dx\le \frac{1}{T}\int_0^T\int_0^L |u|^2\,dx\,dt+\beta\int_0^T |u_x(0,t)|^2\,dt\\ +2\int_0^T\int_0^L a(x)|u|^2\,dx\,dt. \end{aligned} \tag{2.20}$$

Equation (2.20) holds trivially for any $u_0\in L^2(0,L)$.

Proof.][Proof of Theorem 2.3] To show the result,from (2.20), it suffices to prove

$$\frac{1}{T}\int_0^T\int_0^L |u|^2\,dx\,dt\le\beta\int_0^T |u_x(0,t)|^2\,dt+2\int_0^T\int_0^L a(x)|u|^2\,dx\,dt. \tag{2.21}$$

Let us argue by contradiction. Suppose that (2.21) is not valid. Then, there will exist a sequence of solutions $\{u_n\}$ of (2.1)-(2.4) such that

$$\lim_{n\to\infty}\frac{\|u_n\|^2_{L^2(0,T:L^2(I))}}{\beta\int_0^T |u_n'(0,t)|^2\,dt+2\int_0^T\int_0^L a(x)|u_n|^2\,dx\,dt}=+\infty.$$

Let

$$\lambda_n=\|u_n\|_{L^2(0,T:L^2(I))}\quad\text{and}\quad v_n(x,t)=\frac{u_n(x,t)}{\lambda_n}.$$

We have that v_n solves the (2.1)-(2.4) problem with initial data $v_n(x,0)=\frac{u_n(x,0)}{\lambda_n}$. Furthermore,

$$\|v_n\|_{L^2(0,T:L^2(I))}=1, \tag{2.22}$$

$$\beta\int_0^T |v_n'(0,t)|^2\,dt+2\int_0^T\int_0^L a(x)|v_n|^2\,dx\,dt\to 0,\quad\text{as}\quad n\to\infty. \tag{2.23}$$

In view of (2.20), it follows that $v_n(x,0)$ is bounded in $L^2(I)$. Thus

$$\|v_n(\cdot,t)\|_{L^2(I)} \leq c, \quad \text{for all } 0 \leq t \leq T.$$

According to (2.7)

$$\|v_n\|_{L^2(0,T:H^1(I))} \leq c(T)\|v_n(\cdot,t)\|_{L^2(I)} \leq \text{constant, for all } n \in \mathbb{N}. \tag{2.24}$$

Estimates (2.24) and (2.22) tell us that

$$i(v_n)_t = -\alpha(v_n)_{xx} - i\beta(v_n)_{xxx} - ia(x)v_n$$

is bounded in $L^2(0,T:H^{-2}(I))$. Since the embedding $H^1(I) \hookrightarrow L^2(I)$ is compact it follows that (v_n) is relatively compact in $L^2(0,T:L^2(I))$. By extracting a subsequence we may deduce that

$$\begin{gathered} v_n \rightharpoonup v \quad \text{weakly in } L^2(0,T:H^{-2}(I)),\\ v_n \to v \quad \text{strongly in } L^2(0,T:L^2(I)). \end{gathered}$$

Since

$$\|v_n\|_{L^2(0,T:L^2(I))} = 1 \tag{2.25}$$

it follows that

$$\|v\|_{L^2(0,T:L^2(I))} = 1. \tag{2.26}$$

By the weak lower semicontinuity, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \inf \big\{\beta \int_0^T |v_n'(0,t)|^2\,dt + 2\int_0^T \int_0^L a(x)|v_n|^2\,dx\,dt\big\}\\ &\geq \beta\int_0^T |v'(0,t)|^2\,dt + 2\int_0^T \int_0^L a(x)|v|^2\,dx\,dt \end{aligned}$$

which guarantees that $a(x)v \equiv 0$, and in particular, $v \equiv 0$ in $\omega \times (0,T)$. On the other hand, the limit v satisfies

$$iv_t + \alpha v_{xx} + i\beta v_{xxx} + ia(x)v = 0.$$

Using Holmgren's Uniqueness Theorem (see [22]) we deduce that $v \equiv 0$ in $I \times (0,T)$. This contradicts (2.22). Consequently, (2.21) has to be true. On the other hand, we have

$$\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(I)} + 2\int_0^L a(x)|u|^2\,dx \leq -\beta|u_x(0,t)|^2 \leq 0$$

and

$$\|u(\cdot,T)\|^2_{L^2(I)} = \|u_0\|^2_{L^2(I)} - \beta\int_0^T |u_x(0,t)|^2\,dt - 2\int_0^T \int_0^L a(x)|u|^2\,dx\,dt$$

which together with (2.20) give us the inequality

$$\begin{aligned} &(1+c)\|u(\cdot,T)\|^2_{L^2(I)}\\ &\leq (1+c)\big[\|u_0\|^2_{L^2(I)} - \beta\int_0^T |u_x(0,t)|^2\,dt - 2\int_0^T \int_0^L a(x)|u|^2\,dx\,dt\big]\\ &\leq c\|u_0\|^2_{L^2(I)} - \big[\beta\int_0^T |u_x(0,t)|^2\,dt + 2\int_0^T \int_0^L a(x)|u|^2\,dx\,dt\big]\\ &\leq c\|u_0\|^2_{L^2(I)}. \end{aligned}$$

Consequently,

$$\|u(\cdot,T)\|^2_{L^2(I)} \le \mu\|u_0\|^2_{L^2(I)} \quad \text{with } \mu = \frac{c}{1.+c} < 1$$

Therefore, by a semigroup property, the conclusion of the Theorem follows. □

3. Stabilization of the solution of the non-linear problem

In this section we prove the existence of a global solution (and uniqueness) and the exponential decay of the solution of the nonlinear problem (1.1)-(1.4). The proof of the result needs of the Unique Continuation Principle since we are dealing with a nonlinear equation.

Theorem 3.1 (local existence and uniqueness)**.** *Let $|\alpha| < 3\beta$ and $u_0 \in L^2(I)$. Then, there exist $T_0 > 0$ and a unique function $u \in L^\infty(0,T_0 : L^2(I)) \cap L^2(0,T_0 : H^1_0(I))$ that satisfies* (1.1)-(1.4)*.*

Proof. Let $T > 0$ and consider the set of functions $X(T) = \{u : u \in L^2(0,T : H^1_0(I))\}$ with the norm

$$\|u\|_{X(T)} = \Big(\int_0^T \|u(s)\|^2_{H^1(I)}\,ds\Big)^{1/2}.$$

We define the map $P : X(T) \to L^\infty(0,+\infty : L^2(I))$ given by

$$P(u)(t) = S(t)u_0 + \int_0^t S(t-\tau)g(u)(\tau)\,d\tau \tag{3.1}$$

where $g(u) = |u|^2u$. In order to prove local existence (and uniqueness) it is sufficient to prove that P maps $X(T)$ into itself continuously and it is a contraction for $T > 0$ sufficiently small. According to the results of section two it follows that the semigroup of contractions $\{S(t)\}_{t\ge 0}$ corresponding to the linear system satisfies the following properties:

$$\|S(t)u\|_{L^2} \le \|u_0\|_{L^2} \tag{3.2}$$

$$\|S(t)u_0\|_{L^2(0,T:H^1_0(I))} \le c(T)\|u_0\|_{L^2} \tag{3.3}$$

for all $T > 0$, where $c(T) = \frac{|\alpha|T+L}{3\beta-|\alpha|}$. It follows that $S(t)u_0 \in X(T)$. On the other hand, the function

$$J(t) = \int_0^t S(t-s)g(u)$$

is a solution to the problem

$$iJ_t + \alpha J_{xx} + i\beta J_{xxx} + ia(x)J = F,$$

where $F = -g(u)$. We can follow the same idea due to L. Rosier [23] (Proposition 4.1) to prove that $J \in X(T)$ and $F \to J$ which maps $L^2(0,T : L^2(I))$ to $X(T)$ is continuous. Furthermore, the map that associates to each u in $L^2(0,T : H^1_0(I))$ the element $g(u)$ in $L^1(0,T : L^2(I))$ is also continuous. Consequently, P maps $X(T)$ into $X(T)$ continuously. Now, let us prove that P is a contraction in a suitable ball of $X(T)$ provided that $T > 0$ is chosen sufficiently small. Let u and v be elements of $X(T)$, then

$$P(u) - P(v) = -\int_0^t S(t-\tau)[g(u) - g(v)]\,d\tau.$$

Direct calculation, (3.1), (3.3) and Holder's inequality yield

$$
\begin{aligned}
&\|P(u)(t) - P(v)(t)\|^2_{H^1(I)} \\
&= \Big\| \int_0^t S(t-\tau)[g(u) - g(v)]\, d\tau \Big\|^2_{H^1(I)} \\
&\le \Big[\int_0^t \|S(t-\tau)[g(u) - g(v)]\|_{H^1(I)} d\tau \Big]^2 \\
&\le \Big[\Big(\int_0^t d\tau \Big)^{1/2} \Big(\int_0^t \|S(t-\tau)[g(u) - g(v)]\|^2_{H^1(I)} d\tau \Big)^{1/2} \Big]^2 \\
&\le \Big(\int_0^t d\tau \Big) \Big(\int_0^t \|S(t-\tau)[g(u) - g(v)]\|^2_{H^1(I)} d\tau \Big) \\
&\le M^2 T \int_0^t \|g(u) - g(v)\|^2_{L^2(I)} d\tau \\
&\le T \Big(\frac{|\alpha| T + L}{3\beta - |\alpha|} \Big)^2 \int_0^T \|g(u) - g(v)\|^2_{L^2(I)}\, dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|P(u)(t) - P(v)(t)\|^2_{L^\infty(0,T:H^1(I))} \\
&\le T \left(\frac{|\alpha| T + L}{3\beta - |\alpha|} \right)^2 \int_0^T \|g(u) - g(v)\|^2_{L^2(I)}\, dt.
\end{aligned} \tag{3.4}
$$

On the other hand, using that $||u| - |v|| \le |u - v|$, we have

$$
\begin{aligned}
&\|g(u) - g(v)\|_{L^2(I)} \\
&= \||u|^2 u - |v|^2 v\|_{L^2(I)} \\
&= \||u|^2 (u - v) + (|u|^2 - |v|^2) v\|_{L^2(I)} \\
&= \||u|^2 (u - v) + (|u| + |v|)(|u| - |v|) v\|_{L^2(I)} \\
&\le \||u|^2 (u - v)\|_{L^2(I)} + \|(|u| + |v|)(|u| - |v|) v\|_{L^2(I)} \\
&\le \|u\|^2_{L^\infty(I)} \|u - v\|_{L^2(I)} + \big(\|u\|_{L^\infty(I)} + \|v\|_{L^\infty(I)} \big) \|v\|_{L^\infty(I)} \|u - v\|_{L^2(I)} \\
&\le \|u\|^2_{L^\infty(I)} \|u - v\|_{L^2(I)} + \Big(\|u\|_{L^\infty(I)} \|v\|_{L^\infty(I)} + \|v\|^2_{L^\infty(I)} \Big) \|u - v\|_{L^2(I)} \\
&\le \frac{3}{2} \Big(\|u\|^2_{L^\infty(I)} + \|v\|^2_{L^\infty(I)} \Big) \|u - v\|_{L^2(I)}.
\end{aligned}
$$

Thus

$$
\|g(u) - g(v)\|^2_{L^2(I)} \le \frac{3}{2} \Big(\|u\|^2_{L^\infty(I)} + \|v\|^2_{L^\infty(I)} \Big)^2 \|u - v\|^2_{L^2(I)}. \tag{3.5}
$$

Therefore, from (3.4), we have

$$
\begin{aligned}
&\|P(u)(t) - P(v)(t)\|^2_{L^\infty(0,T:H^1(I))} \\
&\le cT \Big(\frac{|\alpha| T + L}{3\beta - |\alpha|} \Big)^2 \int_0^t \big(\|u\|^2_{H^1(I)} + \|v\|^2_{H^1(I)} \big)^2 \|u - v\|^2_{L^2(I)} dt.
\end{aligned}
$$

Then

$$
\|P(u)(t) - P(v)(t)\|^2_{X(T)} \le cT \Big(\frac{|\alpha| T + L}{3\beta - |\alpha|} \Big)^2 \big(\|u\|^2_{X(T)} + \|v\|^2_{X(T)} \big)^2 \|u - v\|^2_{X(T)}
$$

and

$$\|P(u)(t) - P(v)(t)\|_{X(T)} \leq cT^{1/2}\big(\frac{|\alpha|T + L}{3\beta - |\alpha|}\big)\big(\|u\|^2_{X(T)} + \|v\|^2_{X(T)}\big)\|u - v\|_{X(T)}.$$

This shows that P is a contraction in the ball $\mathbb{B}_R = \{u \in X(T) : \|u\|_{X(T)} \leq R\}$ with

$$2cT^{1/2}\big(\frac{|\alpha|T + L}{3\beta - |\alpha|}\big)R^2 < 1. \tag{3.6}$$

Therefore, the proof will be complete if we show that for a suitable choice of R and T satisfying (3.6), the map P maps $\mathbb{B}_R$ into itself. Putting all the previous estimates together, we have

$$\|P(u)\|^2_{X(T)} = \int_0^T \|P(u)\|^2_{H^1(I)}\, dt.$$

From (3.5),

$$\|P(u)\|^2_{X(T)} \leq cT\Big(\frac{|\alpha|T + L}{3\beta - |\alpha|}\Big)^2\|u_0\|^6_{X(T)} \leq cT\Big(\frac{|\alpha|T + L}{3\beta - |\alpha|}\Big)^2 R^6 \tag{3.7}$$

for all $u \in \mathbb{B}_R$. Choosing $R = \|u_0\|_{L^2(I)}$ from (3.7) we deduce that

$$\|P(u)\|^2_{X(T)} \leq \big[cT^{1/2}\big(\frac{|\alpha|T + L}{3\beta - |\alpha|}\big)\|u_0\|^2_{L^2(I)}\big]\|u_0\|_{L^2(I)}.$$

Let us choose $T > 0$ sufficiently small and such that

$$cT^{1/2}\big(\frac{|\alpha|T + L}{3\beta - |\alpha|}\big)\|u_0\|^2_{L^2(I)} < 1. \tag{3.8}$$

Hence, P map $\mathbb{B}_R$ into itself. □

Theorem 3.2 (Global existence and uniqueness). *Let $|\alpha| < 3\beta$ and $u_0 \in L^2(I)$. Then, there exists a unique function $u \in L^\infty(0, T : L^2(I)) \cap L^2(0, T : H^1_0(I))$ that satisfies the problem* (1.1)-(1.4).

Proof. It follows from Theorem 3.1 that we can extend the solution u to the maximal interval of existence $0 \leq t < T_{\max}$. We need to prove that $T_{\max} = +\infty$. Let $T > 0$ such that $0 < T < T_{\max}$ and let us get bounds for the solution u in the interval $0 \leq t < T$. Due to Theorem 3.1 we know that the solution u belongs to $X(T)$ and satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)g(u(\tau))\, d\tau.$$

It follows that

$$-|u|^2u \in L^2(0, T : L^2(I)).$$

The global existence is an immediate consequence of the a priori estimate obtained by multiplying the equation in (1.1)-(1.4) by $\overline{u}$. In fact

$$i\overline{u}u_t + i\beta\overline{u}u_{xxx} + \alpha\overline{u}u_{xx} + |u|^4 + ia(x)|u|^2 = 0. \tag{3.9}$$

Applying conjugate in (3.9) we have

$$-iu\overline{u}_t - i\beta u\overline{u}_{xxx} + \alpha u\overline{u}_{xx} + |u|^4 - ia(x)|u|^2 = 0. \tag{3.10}$$

Subtracting (3.9) with (3.10), integrating over $x \in [0, L]$ and using boundary conditions we obtain

$$\frac{d}{dt}\int_0^L |u|^2\, dx + \beta|u_x(0, t)|^2 + \int_0^L a(x)|u|^2\, dx = 0. \tag{3.11}$$

Integrating over $t \in [0,T]$

$$\int_0^L |u|^2\,dx + \beta\int_0^T |u_x(0,t)|^2\,dt + \int_0^T\int_0^L a(x)|u|^2\,dx\,dt = \|u_0\|^2_{L^2(I)}. \tag{3.12}$$

Therefore, $u \in L^\infty(0,T : L^2(I))$ for any $0 < T < T_{\max}$.
Now, we multiply the equation in (1.1)-(1.4) by $x\overline{u}$

$$ix\overline{u}u_t + i\beta x\overline{u}u_{xxx} + \alpha x\overline{u}u_{xx} + x|u|^4 + ixa(x)|u|^2 = 0. \tag{3.13}$$

Applying conjugate in (3.9) we have

$$-ixu\overline{u}_t - i\beta xu\overline{u}_{xxx} + \alpha xu\overline{u}_{xx} + x|u|^4 - ixa(x)|u|^2 = 0. \tag{3.14}$$

Subtracting (3.13) with (3.14), integrating over $x \in [0,L]$ we obtain

$$\begin{gathered}
i\frac{d}{dt}\int_0^L x|u|^2\,dx + i\beta\int_0^L x\overline{u}u_{xxx}\,dx + i\beta\int_0^L xu_{xxx}\,dx \\
\alpha\int_0^L x\overline{u}u_{xx}\,dx - \alpha\int_0^L xu\overline{u}_{xx}\,dx + 2\int_0^L x|u|^4\,dx = 0.
\end{gathered}$$

Performing similar calculations as in section two, we obtain

$$\begin{aligned}
&i\frac{d}{dt}\int_0^L x|u|^2\,dx + 3i\beta\int_0^L |u_x|^2\,dx + i\beta|u_x(0,t)|^2 \\
&- 2i\alpha\,\mathrm{Im}\int_0^L \overline{u}u_x\,dx + 2i\int_0^L xa(x)|u|^2\,dx = 0
\end{aligned}$$

then

$$\begin{aligned}
&\frac{d}{dt}\int_0^L x|u|^2\,dx + 3\beta\int_0^L |u_x|^2\,dx + \beta|u_x(0,t)|^2 \\
&- 2\alpha\,\mathrm{Im}\int_0^L \overline{u}u_x\,dx + 2\int_0^L xa(x)|u|^2\,dx = 0.
\end{aligned} \tag{3.15}$$

Performing straightforward calculation we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^L x|u|^2\,dx + (3\beta - |\alpha|)\int_0^L |u_x|^2\,dx + \beta|u_x(0,t)|^2 + 2\int_0^L xa(x)|u|^2\,dx \\
&\le |\alpha|\int_0^L |u|^2\,dx.
\end{aligned} \tag{3.16}$$

Therefore, integrating over $t \in [0,T]$ we have

$$\begin{aligned}
\int_0^T\int_0^L |u_x|^2\,dx\,dt &\le \frac{1}{(3\beta - |\alpha|)}\Big[|\alpha|\int_0^T\int_0^L |u|^2\,dx\,dt + \int_0^L |u_0|^2\,dx\Big] \\
&\le \frac{1}{(3\beta - |\alpha|)}\|u_0\|^2_{L^2(I)}
\end{aligned} \tag{3.17}$$

and $u \in L^2(0,T : H_0^1(I))$, for any $0 < T < T_{\max}$. Estimates (3.11) and (3.17) allow us to conclude that $T_{\max} = +\infty$. Thus, the global existence follows. Uniqueness can be shown in the standard way using Gronwall's inequality. $\square$

Now, we have to prove the exponential decay of the solutions of the nonlinear problem (1.1)-(1.4). The proof of the result needs that the Unique Continuation Principle(UCP) holds because we are dealing with a nonlinear equation. The next

Theorem contains the result of (UCP) for the problem (1.1)-(1.4). The proof will be given later.

Theorem 3.3. *Assume that the set ω contains two sets of the form $(0,\delta)$ and $(L-\delta,L)$ for some $\delta>0$. Let $u\in L^\infty(0,T:L^2(I)))\cap L^2(0,T:H^1(I)))$ be the global solution of the problem*

$$iu_t+i\beta u_{xxx}+\alpha u_{xx}+\delta|u|^2u+ia(x)u=0 \quad \text{in } I\times(0,T) \tag{3.18}$$

$$u(0,t)=u(L,t),\quad t\in(0,T) \tag{3.19}$$

$$u_x(L,t)=0,\quad t\in(0,T) \tag{3.20}$$

$$u(x,t)\equiv 0 \quad \text{in } \omega\times(0,T) \tag{3.21}$$

with $\epsilon\geq 0$ and $T>0$, then necessarily $u\equiv 0$ in $I\times(0,T)$.

For the moment, let us assume that ω satisfies the (UCP). Then we have the following result.

Theorem 3.4. *Let $|\alpha|<3\beta$, $a=a(x)$ a non-negative function, $a\in C^\infty(I)$ such that $a(x)\geq a_0>0$ is assumed to be nonnegative everywhere in an open non empty proper subset ω. Let u be the global solution of the problem (1.1)-(1.4). Then, for any $L>0$ and $R>0$ there exist positive constants $c>0$ and $\mu>0$ such that*

$$E(t)\leq c\|u_0\|^2_{L^2(I)}e^{-\mu t}$$

for any $t\geq 0$ and any solution of (1.1)-(1.4) with $u_0\in L^2(I)$ such that $\|u_0\|_{L^2(I)}\leq R$.

Proof. We proceed as in he proof of Theorem 2.3. From (3.11) we have

$$\int_0^L|u|^2\,dx+\beta|u_x(0,t)|^2+2\int_0^T\int_0^L a(x)|u|^2\,dx\,dt=\|u_0\|^2_{L^2(I)}. \tag{3.22}$$

Next, we multiply the equation in (1.1)-(1.4) by $(T-t)\overline{u}$,

$$i(T-t)\overline{u}u_t+i\beta(T-t)\overline{u}u_{xxx}+\alpha(T-t)\overline{u}u_{xx}+(T-t)|u|^4+i(T-t)a(x)|u|^2=0. \tag{3.23}$$

Applying conjugate we have

$$-i(T-t)u\overline{u}_t-i\beta(T-t)u\overline{u}_{xxx}+\alpha(T-t)u\overline{u}_{xx}+(T-t)|u|^4-i(T-t)a(x)|u|^2=0. \tag{3.24}$$

Subtracting (3.23) with (3.24) and performing straightforward calculations we obtain

$$\frac{d}{dt}\int_0^L(T-t)|u|^2\,dx+\int_0^L|u|^2\,dx+\beta(T-t)|u_x(0,t)|^2+2\int_0^L(T-t)a(x)|u|^2\,dx=0. \tag{3.25}$$

Integrating over $t\in[0,T]$, we have

$$\begin{aligned}&-T\int_0^L|u_0|^2\,dx+\int_0^T\int_0^L|u|^2\,dx\,dt+\beta\int_0^T(T-t)|u_x(0,t)|^2\,dt\\&+2\int_0^T\int_0^L(T-t)a(x)|u|^2\,dx\,dt=0.\end{aligned}$$

and

$$\int_0^L |u_0|^2\,dx = \frac{1}{T}\int_0^T\int_0^L |u|^2\,dx\,dt + \frac{\beta}{T}\int_0^T (T-t)|u_x(0,t)|^2\,dt + \frac{2}{T}\int_0^T\int_0^L (T-t)a(x)|u|^2\,dx\,dt.$$

Consequently,

$$\int_0^L |u_0|^2\,dx \le \frac{1}{T}\int_0^T\int_0^L |u|^2\,dx\,dt + \beta\int_0^T |u_x(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)|u|^2\,dx\,dt. \tag{3.26}$$

To show the result it is sufficient to prove

$$\int_0^T\int_0^L |u|^2\,dx\,dt \le c\big\{\beta\int_0^T |u_x(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)|u|^2\,dx\,dt\big\} \tag{3.27}$$

for some positive constant c independent of the solution u.
Let us argue by contradiction. Suppose that (3.27) is not true. Then, there will exist a sequence of solutions u^n of (1.1)-(1.4) such that

$$\lim_{n\to\infty} \frac{\|u^n\|^2_{L^2(0,T:L^2(I)))}}{\beta\int_0^T |u^n_x(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)|u^n|^2\,dx\,dt} = +\infty.$$

Let $\lambda^n = \|u^n\|_{L^2(0,T:L^2(I))}$ and $v^n(x,t) = \frac{u^n}{\lambda_n}$. For each $n \in \mathbb{N}$ the function v^n satisfies

$$i(v^n)_t + \alpha(v^n)_{xx} + i\beta(v^n)_{xxx} + (\lambda^n)^2|v^n|^2v^n + ia(x)v^n = 0 \quad \text{in } I\times(0,T) \tag{3.28}$$

$$v^n_x(L,t) = 0, \quad \text{for all } t>0 \tag{3.29}$$

$$v^n(0,t) = v^n(L,t) = 0, \quad \text{for all } t>0 \tag{3.30}$$

$$v^n(x,0) = \frac{u^n(x,0)}{\lambda^n}, \quad \text{for all } x \in I. \tag{3.31}$$

We have

$$\|v^n\|_{L^2(0,T:L^2(I))} = 1, \tag{3.32}$$

$$\beta\int_0^T |u^n_x(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)|u^n|^2\,dx\,dt \to 0 \quad \text{as } n\to+\infty. \tag{3.33}$$

In view of (3.26) it follows that $v^n(x,0)$ is bounded in $L^2(I)$. Thus,

$$\|v^n(\cdot,t)\|_{L^2(I)} \le c \quad \text{for all} \quad 0\le t\le T. \tag{3.34}$$

According to (2.17)

$$\|v^n\|_{L^2(0,T:H^1(I))} \le c(T)\|v^n(\cdot,t)\|_{L^2(I)}, \quad \forall n\in\mathbb{N}. \tag{3.35}$$

On the other hand, $|v^n|^2v^n$ belongs to $L^2(0,T:L^1(I))$ and

$$\||v^n|^2v^n\|_{L^2(0,T:L^1(I))} \le \|v^n\|^2_{L^\infty(0,T:L^2(I))}\|v^n\|_{L^2(0,T:H^1(I))} \tag{3.36}$$

and by (3.35) we obtain a constant $c>0$ such that

$$\||v^n|^2v^n\|_{L^2(0,T:L^1(I))} \le c. \tag{3.37}$$

Since (λ^n) is a bounded sequence, because $\|u^n(\cdot,0)\|_{L^2(I)} \le R$, it follows by (3.18)-(3.21), (3.35) and (3.37) that

$$(v^n)_t = -\alpha(v^n)_{xx} - i\beta(v^n)_{xxx} - (\lambda^n)^2|v^n|^2v^n - ia(x)v^n$$

is bounded in $L^2(0,T:H^{-2}(I))$. Since the embedding $H^1(I)\hookrightarrow L^2(I)$ is compact it follows that (v^n) is relatively compact in $L^2(0,T:L^2(I))$. By extracting a subsequence we can deduce that

$$\begin{gathered} v^n \rightharpoonup v \quad \text{weakly in } L^2(0,T:H^{-2}(I)),\\ v^n \to v \quad \text{strongly in } L^2(0,T:L^2(I)). \end{gathered}$$

Since $\|v_n\|_{L^2(0,T:L^2(I))}=1$, then

$$\|v\|_{L^2(0,T:L^2(I))}=1. \tag{3.38}$$

By lower semicontinuity, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty}\inf\big\{\beta\int_0^T |v_x^n(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)|v^n|^2\,dx\,dt\big\}\\ &\geq \beta\int_0^T |v_x(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)|v|^2\,dx\,dt \end{aligned}$$

which guarantees that $av\equiv 0$, and in particular $v\equiv 0$ in $\omega\times(0,T)$. We now distinguish the following two situations:

(1) There exists a subsequence of (λ_n) also denoted by (λ_n) such that

$$\lambda_n\to 0 \quad \text{as } n\to\infty.$$

In this case, the limit v satisfies the linear problem

$$\begin{gathered} iv_t + i\beta v_{xxx} + \alpha v_{xx} + \delta|v|^2 v + ia(x)v = 0\\ v(0,t)=v(L,t) \quad \text{for all } t\in(0,T)\\ v_x(L,t)=0 \quad \text{for all } t\in(0,T)\\ v(x,t)=0 \quad \text{in } \omega\times(0,T) \end{gathered}$$

Then, by Holmgren's uniqueness Theorem (see [22]), $v\equiv 0$ in $I\times(0,T)$ and this contradicts (3.38).

(2) There exists a subsequence of (λ_n) also denoted by (λ_n) and $\lambda>0$ such that $\lambda_n\to\lambda$. In this case, the limit function v solves (3.29)-(3.31) and, by the (UCP) assumed to hold for the subset ω, we have that $v\equiv 0$ in $I\times(0,T)$, and again, in this case, we have a contradiction.

In the cases (1) and (2) we have a contradiction. Hence, (3.27) holds and the proof is complete. □

Proof of Theorem 3.3. From Theorem 3.2 we obtain that if $u_0\in L^2(I)$ then

$$u\in L^\infty(0,T:L^2(I))\cap L^2(0,T:H_0^1(I))$$

and $u_t\in L^2(0,T:H_0^{-2}(I))$. Consequently, we know that u is weakly continuous from $[0,T]$ into $L^2(I)$. According to the structure of ω, $u\equiv 0$ in $\{(0,\delta)\times(L-\delta,L)\}\times(0,T)$. Let us define the extended function

$$u(x,t)=\begin{cases} u(x,t) & \text{if } (x,t)\in(\delta,L-\delta)\times(0,T)\\ 0 & \text{if } (x,t)\in\{\mathbb{R}-(\delta,L-\delta)\}\times(0,T). \end{cases}$$

Then, u satisfies

$$iu_t + \alpha u_{xx} + i\beta u_{xxx} + \lambda|u|^2u + ia(x)u = 0 \quad \text{in } \mathbb{R}\times(0,T) \tag{3.39}$$

$$u(x,0)=\phi(x) \quad \text{in } \mathbb{R}. \tag{3.40}$$

and

$$\phi(x) = \begin{cases} u_0(x) & \text{if } x \in (\delta, L-\delta) \\ 0 & \text{if } x \in \{\mathbb{R} - (\delta, L-\delta)\}. \end{cases}$$

If we consider $v(x,t) = u(x+t,t)$, then v solves

$$iv_t + \alpha v_{xx} + i\beta v_{xxx} + \lambda |v|^2 v + ia(x)v = 0 \quad \text{in } \mathbb{R} \times (0,T) \tag{3.41}$$

$$u(x,0) = \phi(x) \quad \text{in } \mathbb{R}. \tag{3.42}$$

Since ϕ has compact support and belongs to $L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \phi^2(x) e^{2bx}\, dx < \infty, \quad \forall b > 0. \tag{3.43}$$

Thus, by regularizing properties (see [28]), $v \in C^\infty(I \times (0,T))$. Therefore, v is smooth as well, and applying the unique continuation result we have that $v \equiv 0$ and $u \equiv 0$, $x \in I$, $t \in (0,T)$. $\square$

Remark about the hypothesis $|\alpha| < 3\beta$. We consider the Gauge transformation

$$u(x,t) = e^{id_2 x + id_3 t} v(x - d_1 t, t) \equiv e^\theta v(\eta, \xi)$$

and $\theta = id_2 x + id_3 t$, $\eta = x - d_1 t$, $\xi = t$. Then

$$\begin{gathered} u_t = id_3 e^\theta v - d_1 e^\theta v_\eta + e^\theta v_\xi, \\ u_x = id_2 e^\theta v + e^\theta v_\eta, \\ u_{xx} = -d_2^2 e^\theta v + 2id_2 e^\theta v_\eta + e^\theta v_{\eta\eta}, \\ u_{xxx} = -id_2^3 e^\theta v - 3d_2^2 e^\theta v_\eta + 3id_2 e^\theta v_{\eta\eta} + e^\theta v_{\eta\eta\eta}. \end{gathered}$$

Replacing in (1.5), we have

$$\begin{aligned} &- d_3 e^\theta v - id_1 e^\theta v_\eta + ie^\theta v_\xi - \omega d_2^2 e^\theta v + 2i\omega d_2 e^\theta v_\eta + \omega e^\theta v_{\eta\eta} \\ &\beta d_3^3 e^\theta v - 3i\beta d_2^2 e^\theta v_\eta - 3\beta d_2 e^\theta v_{\eta\eta} + i\beta e^\theta v_{\eta\eta\eta} + \gamma |v|^2 e^\theta v \\ &- \delta d_2 |v|^2 e^\theta v + i\delta |v|^2 e^\theta v_\eta + \epsilon d_2 e^\theta v^2 \overline{v} + i\epsilon e^\theta v^2 v_\eta = 0 \end{aligned}$$

and

$$\begin{aligned} iv_\xi + (\omega - 3\beta d_2) v_{\eta\eta} + i\beta v_{\eta\eta\eta} + (2i\omega d_2 - 3i\beta d_2^2 - id_1 + i\delta |v|^2 + i\epsilon v^2) v_\eta \\ (\beta d_2^3 - \omega d_2^2 - d_3 + \gamma |v|^2 - \delta d_2 |v|^2) v + \epsilon d_2 v^2 \overline{v} = 0\,. \end{aligned}$$

Then

$$d_1 = \frac{\omega^2}{3\beta}, \quad d_2 = \frac{\omega}{3\beta}, \quad d_3 = \frac{-2\omega^3}{27\beta^2}.$$

This way in (1.5) we obtain

$$iv_\xi + i\beta v_{\eta\eta\eta} + i(\delta |v|^2 + \epsilon v^2) v_\eta + \big(\gamma - \frac{\omega\delta}{3\beta}\big)|v|^2 v + \frac{\epsilon\delta}{3\beta} v^2 \overline{v} = 0,$$

but $v^2 \overline{v} = vv\overline{v} = |v|^2 v$, then using the Gauge transformation we have the equivalent problem to (1.5)

$$\begin{gathered} iv_\xi + i\beta v_{\eta\eta\eta} + i\delta |v|^2 v_\eta + i\epsilon v^2 v_\eta + \big(\gamma + \frac{\epsilon\delta}{3\beta} - \frac{\omega\delta}{3\beta}\big)|v|^2 v = 0 \quad \eta, \xi \in \mathbb{R} \\ v(\eta, 0) = e^{-i\frac{\omega}{3\beta}\eta} u_0(\eta). \end{gathered} \tag{3.44}$$

Here, rescaling the equation, we take $\beta = 1$.

$$\begin{gathered} iv_t + iv_{xxx} + i\delta|v|^2v_x + i\epsilon v^2 v_x + \big(\gamma + \frac{\epsilon\delta}{3} - \frac{\omega\delta}{3}\big)|v|^2 v = 0 \quad x,t \in \mathbb{R} \\ v(x,0) = e^{-i\frac{\omega}{3}x}u_0(x). \end{gathered} \tag{3.45}$$

The above Gauge's transformation is a bicontinuous map from $L^p([0,T] : H^s(\mathbb{R})$ to itself, as far as $0 < T < \infty$. With this, the imposed assumption $|\omega| < 3\beta$ can be removed.

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Eleni Bisognin
Centro Universitário Franciscano-UNIFRA, Rua dos Andrade 1614, CEP: 97010-032, Santa Maria, R. S., Brasil
E-mail address: eleni@unifra.br

Vanilde Bisognin
Centro Universitário Franciscano-UNIFRA, Rua dos Andrade 1614, CEP: 97010-032, Santa Maria, R. S., Brasil
E-mail address: vanilde@unifra.br

Octavio Paulo Vera Villagrán
Departamento de Matemática, Universidad del Bío-Bío, Collao 1202, Casilla 5-C, Concepción, Chile
E-mail address: overa@ubiobio.cl octavipaulov@yahoo.com