Electronic Journal of Differential Equations, Vol. 2004(2004), No. 59, pp. 1–30.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

DAMPED SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH DEVIATING ARGUMENTS: SHARP RESULTS IN OSCILLATION PROPERTIES

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Abstract. This article presents a new approach for investigating the oscillation properties of second order linear differential equations with a damped term containing a deviating argument

$$x''(t) - [P(t)x(r(t))]' + Q(t)x(l(t)) = 0, \quad r(t) \le t.$$

To study this equation, a specially adapted version of Sturmian Comparison Method is developed and the following results are obtained:
(a) A comprehensive description of all critical (threshold) states with respect to its oscillation properties for a linear autonomous delay differential equation

$$y''(t) - py'(t-\tau) + qy(t-\sigma) = 0, \quad \tau > 0,\ \infty < \sigma < \infty.$$

(b) Two versions of Sturm-Like Comparison Theorems. Based on these Theorems, sharp conditions under which all solutions are oscillatory for specific realizations of $P(t), r(t)$ and $l(t)$ are obtained. These conditions are formulated as the unimprovable analogues of the classical Knezer Theorem which is well-known for ordinary differential equations ($P(t) = 0$, $l(t) = t$).
(c) Upper bounds for intervals, where any solution has at least one zero.

1. Introduction

It is well-known that many results for second order linear ordinary differential equations were obtained for the equation

$$y''(t) + a(t)y(t) = 0, \tag{1.1}$$

but not for the equation

$$x''(t) - (P(t)x(t))' + Q(t)x(t) = 0 \tag{1.2}$$

with damping term $(P(t)x(t))'$. The reason is the following: it is easy to transform (1.2) into (1.1) by the substitution

$$x(t) = y(t)\exp\Big\{\frac{1}{2}\int P(t)dt\Big\}, \tag{1.3}$$

where $a(t) := Q(t) - \frac{1}{4}P^2(t) - \frac{1}{2}P'(t)$. After this transformation the solutions of (1.2) and (1.1) have the same zeros.

2000 *Mathematics Subject Classification.* 34K11.
Key words and phrases. Linear differential equation with deviating arguments, second order, damping term, oscillation, Sturmian comparison method.
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Submitted March 23, 2004. Published April 19, 2004.

Consider now a second order linear delay differential equation without delay in the damping part

$$(lx)(t) := x''(t) - (P(t)x(t))' + Q(t)x(l(t)) = 0. \tag{1.4}$$

The substitution (1.3) transforms (1.4) into the following equation

$$y''(t) + a(t)y(t) + b(t)y(l(t)) = 0, \tag{1.5}$$

where

$$a(t) = -\frac{1}{4}P^2(t) - \frac{1}{2}P'(t),\ b(t) = Q(t)\exp\big\{-\frac{1}{2}\int_{l(t)}^t P(s)ds\big\}.$$

Hence every result on oscillation properties of (1.5) without a damping term implies the corresponding result for (1.4).

The situation is dramatically changed for an equation with delay in the damping term:

$$(lx)(t) := x''(t) - (P(t)x(r(t)))' + Q(t)x(l(t)) = 0. \tag{1.6}$$

There is no good substitution which transforms (1.6) into an equation without a damping term. Hence we have to investigate (1.6) itself.

It is important to note that the delay in the damping term significantly changes oscillating properties of this equation. Consider, for example, an autonomous equation

$$x''(t) - px'(t) + qx(t-\sigma) = 0, \quad -\infty < \sigma < \infty,\ t \geq t_0. \tag{1.7}$$

It is well known that for every $-\infty < p < \infty$ and $q > \frac{p^2}{4}$ all solutions of (1.7) are oscillatory.

On the other hand, an equation with a delay in the damping term

$$x''(t) - px'(t-\tau) + qx(t-\sigma) = 0, \quad \tau > 0,\ \sigma \leq \tau,\ p < 0 \tag{1.8}$$

has a nonoscillatory solution for any q.

Oscillation properties of damping equations were studied in [7, 8, 9, 10, 11] in which very general nonlinear differential equations with delay were considered. Unfortunately, in [7, 8, 9, 10, 11] there are no new results on (1.6) and even for (1.4). The purpose of the present paper is to obtain such results.

Our main tool is Sturmian Comparison Method (SCM) which was developed in [1] by one of the authors of this paper for various classes of differential and difference equations. For delay differential equations of the first and the second order this method was described in details in [1, Chap.4]. The further development of SCM for various kinds of differential equations one can find in the papers [2, 3, 4, 5].

We will explain here the main idea of this method. One of the most important results in the qualitative theory of ordinary differential equations is the classical

Theorem 1.1 (Sturmian Comparison Theorem)**.** *Suppose there exists a positive solution of the inequality*

$$\tilde{l}y := y''(t) + a(t)y(t) \geq 0, \quad t \in (t_1, t_2), \tag{1.9}$$

such that $y(t_1) = y(t_2) = 0$ *and* $b(t) \geq a(t)$, $t \in (t_1, t_2)$. *Then the inequality*

$$lx := x''(t) + b(t)x(t) \leq 0 \tag{1.10}$$

has no positive solutions on (t_1, t_2).

The above theorem implies the following result.

Theorem 1.2 (Sturmian Oscillation Comparison Theorem)**.** *Suppose $b(t) \geq a(t)$, $t \in (t_0, \infty)$, and all solutions of the equation $\tilde{l}y = 0$ are oscillatory. Then all solutions of the equation $lx = 0$ are also oscillatory.*

There are several investigations devoted to extensions of Theorem 1.2 to various classes of functional differential equations, including nonlinear differential equations with delays. We will not follow this direction, since SCM is concerned with the generalization just of Theorem 1.1 but not of Theorem 1.2 for various classes of functional differential equations.

It is worth to note that Theorem 1.1 is formulated for a finite interval (t_1, t_2), but not for a semiaxis. Thus it allows not only to obtain conditions of oscillation of all solutions but also to estimate the length of a sign-preservation interval.

The most important advantage of SCM is the fact, that for the application of this method it is sufficient to construct only one solution of (1.9) ("Testing inequality"), satisfying some required properties. Thus to the best of our knowledge this approach is the unique constructive method in the oscillation theory of functional differential equations.

We present here a version of SCM which is specially adapted to second order delay damped differential equation (1.6). However first we consider oscillation properties of (1.4) without delay in the damping term. To this end we do not need to develop a new theory.

2. Oscillation Properties of (1.4)

Consider (1.4) with monotone increasing delay argument $l(t) < t$ and denote by $k(t)$ the inverse function of $l(t)$. A classification of oscillation properties of (1.4) is based on the following statement which was published long time ago.

Theorem 2.1 ([1, Lemma 4.4 , Theorem 6.2.4])**.** *Let be $l(t_2) > t_1$, $\varphi(t) > 0$, $z(t) \geq 0$, $\varphi, z \in \mathbf{C}^2[l(t_1), k(t_2)]$,*

$$\int_{t_1}^{t_2} \varphi(s)ds = \pi, \quad 0 < \int_{l(t)}^{t} \varphi(s)ds < \frac{\pi}{2}, \quad t \in [t_1, k(t_2)]. \tag{2.1}$$

If

$$\begin{aligned} a(t) \geq \tilde{a}(t) := \varphi^2(t) + \frac{\varphi''(t)}{2\varphi(t)} - \frac{3}{4}\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2 + z'(t) - z^2(t) \\ - z(t)\frac{\varphi'(t)}{\varphi(t)} - 2\varphi(t)z(t)\cot\int_t^{k(t)} \varphi(s)ds \end{aligned} \tag{2.2}$$

and

$$b(t) \geq \tilde{b}(t) := 2l'(t)z[l(t)]\sqrt{\varphi(t)\varphi[l(t)]}\csc\int_{l(t)}^{t} \varphi(s)ds \exp\Big\{\int_{l(t)}^{t} z(s)ds\Big\}, \tag{2.3}$$

then (1.5) has no positive solution on $[l(t_1), t_2]$.

Corollary 2.2. *Suppose*

$$\varphi(t) > 0; \int^{\infty} \varphi(t)dt = +\infty, \quad \int_{l(t)}^{t} \varphi(s)ds < \frac{\pi}{2}; \quad z(t) \geq 0$$

and (2.2)-(2.3) hold on (t_0, ∞). Then all solutions of (1.5) are oscillatory and, moreover, every solution of (1.5) has at least one zero on any interval $(l(t_1), k(t_2))$ when $l(t_2) > t_1$ and $\int_{t_1}^{t_2} \varphi(t)dt = \pi$.

As was mentioned in Introduction, the substitution (1.3) in (1.4) does not change the oscillation properties of (1.4) and transforms it into (1.5), where

$$a(t) := -\frac{1}{4}P^2(t) - \frac{1}{2}P'(t), \quad b(t) := Q(t)\exp\big\{-\frac{1}{2}\int_{l(t)}^{t} P(s)ds\big\}.$$

Thus (2.1)–(2.3) represent easily verified conditions formulated in terms of (1.4) only.

One can illustrate the quality of these conditions by the following example.

Example 2.3. Let in (1.4) $P(t) := \frac{p}{t}$, $p = \text{const}$, $l(t) := \frac{t}{\mu}$, $\mu > 1$ and so $k(t) = \mu t$. Let in Theorem 2.1 $\varphi(t) := \frac{\nu}{t}$, $\nu > 0$, $z(t) := \frac{s}{t}$, $s = \text{const} > 0$. Then

$$t^2\tilde{a}(t) := \frac{1}{4} + \nu^2 - s^2 - 2\nu s\cot(\nu\ln\mu),$$
$$t^2\tilde{b}(t) := 2\nu s\mu^{s+\frac{1}{2}}\csc(\nu\ln\mu),$$
$$\exp\big\{-\frac{1}{2}\int_{l(t)}^{t} P(s)ds\big\} = \mu^{-\frac{p}{2}}; \quad t^2a(t) = -\frac{p^2}{4} + \frac{p}{2}.$$

Conditions (2.2)–(2.3) take a form

$$\begin{gathered} -\frac{p^2}{4} + \frac{p}{2} \geq \frac{1}{4} + \nu^2 - s^2 - 2\nu s\cot(\nu\ln\mu), \\ t^2Q(t)\mu^{-\frac{p}{2}} \geq 2\nu s\mu^{s+\frac{1}{2}}\csc(\nu\ln\mu). \end{gathered} \tag{2.4}$$

Denote by s_ν the positive root of a quadratic equation

$$s^2 + 2\nu s\cot(\nu\ln\mu) - \frac{1}{4}(p-1)^2 - \nu^2 = 0, \tag{2.5}$$

which exists for all $\mu > 1$ and $\nu > 0$ is sufficiently small. Then $(2.4)_1$ and $(2.4)_2$ turn into an equality and into the following inequality

$$t^2Q(t) \geq 2\nu s_\nu\frac{\mu^{s+\frac{1+p}{2}}}{\sin(\nu\ln\mu)}, \quad \forall t \tag{2.6}$$

respectively. Suppose

$$s_0 := \lim_{\nu\to 0} s_\nu = \frac{(p-1)^2\ln\mu}{2\sqrt{4+(p-1)^2\ln^2\mu}+4}.$$

Let $t_1 = T$ in Theorem 2.1 be sufficiently large. By (2.1) we have

$$\int_T^{t_2}\varphi(s)ds = \pi \Longleftrightarrow t_2 = T\exp\frac{\pi}{\nu}.$$

Then $(l(t_1), t_2) = (\mu T, T\exp\frac{\pi}{\nu})$, and we have the following statement.

Corollary 2.4. *Let*

$$\liminf_{t\to\infty} t^2Q(t) = C > \frac{2s_0\mu^{\frac{1+p}{2}+s_0}}{\ln\mu} := B(p). \tag{2.7}$$

Then all solutions of the equation

$$x''(t) - \Big[\frac{p}{t}x(t)\Big]' + Q(t)x\left(\frac{t}{\mu}\right) = 0 \tag{2.8}$$

are oscillatory. Moreover, there exists $\nu > 0$ such that every solution of (2.8) has at least one zero on any interval $(\frac{T}{\mu}, Te^{\frac{\pi}{\nu}})$ for sufficiently large T.

Remark. Condition (2.7) is the best possible for every p in the following sense. By direct calculations one can check, that (2.8) with $Q(t) := \frac{B(p)}{t^2}$ has a nonoscillatory solution $x(t) = t^{s_0 + \frac{1+p}{2}}$. Let us notice, that for $p = 0$ (i.e. for the equation without a damping term) this fact was mentioned in [1, p.177].

3. Critical states of the second order damped autonomous equation

Instead of (1.8) without loss of generality we will consider here the equation

$$x''(t) - px'(t-1) + qx(t-\sigma) = 0, \quad -\infty < \sigma < \infty,\ t \geq 0, \tag{3.1}$$

since one can transform (1.8) into (3.1) by rescaling.

Definition. We will say that (3.1) is **in a critical state (CS)** with respect to its oscillation properties if there exists an eventually positive solution $x(t) > 0, t \geq t_0$ of (3.1), while an equation

$$z''(t) - pz'(t-1) + (q+\epsilon)z(t-\sigma) = 0$$

has no such solution for every $\epsilon > 0$. In this case the pair of numbers $\{p; q\}$ is said to be **a critical pair**.

It is well known that for a wide class of autonomous linear differential equations with deviating arguments, in particular, for (3.1), the following statement holds:

> All solutions of the equation are oscillatory if and only if its characteristic quasipolynomial does not change its sign for every $\lambda \in (-\infty, \infty)$.

In particular, all solutions of (3.1) are oscillatory if and only if its characteristic quasipolynomial is eventually positive:

$$F_{p,q}(\lambda) := F(\lambda) = \lambda^2 - p\lambda e^{-\lambda} + qe^{-\lambda\sigma} > 0, \quad \lambda \in (-\infty, \infty).$$

On the other hand, (3.1) is in CS if and only if the following conditions hold

$$\begin{gathered} F(\lambda) \geq 0 \quad \forall \lambda \in (-\infty, \infty) \\ \exists \bar{\lambda} : F(\bar{\lambda}) = 0. \end{gathered} \tag{3.2}$$

Indeed, $(3.2)_2$ implies that (3.1) has a solution $x(t) = e^{\bar{\lambda} t}$, and $(3.2)_1$ provides that $F_{p,q+\epsilon}(\lambda) > 0$, for all λ.

Hence CS $\{p; q\}$ implies that $\bar{\lambda}$ is a solution of the system

$$\begin{gathered} F(\lambda) = 0 \\ F'(\lambda) = 0. \end{gathered} \tag{3.3}$$

Note that the existence of a solution of (3.3) for some pair $\{p, q\}$ does not imply that this pair is critical.

Rewrite (3.3) in the form

$$\begin{gathered} pe^{-\lambda}\lambda - qe^{-\lambda\sigma} - \lambda^2 = 0 \\ pe^{-\lambda}(1-\lambda) + qe^{-\lambda\sigma}\sigma - 2\lambda = 0. \end{gathered} \tag{3.4}$$

Then the pair $\{p,q\}$ (which may or may not be critical) can be presented in the parametric form

$$\begin{gathered} p=\frac{\lambda(2+\lambda\sigma)}{1+\lambda(\sigma-1)}e^{\lambda} \\ q=\frac{\lambda^2(1+\lambda)}{1+\lambda(\sigma-1)}e^{\lambda\sigma} \end{gathered} \tag{3.5}$$

where $-\infty<\lambda<\infty$. The pair $\{p,q\}$ belongs to the quadrant $\{p\ge 0,q\ge 0\}$ if

$$\begin{gathered} 0\le\lambda<\frac{1}{1-\sigma},\quad \text{when } -\infty<\sigma<1, \\ 0<\lambda<\infty,\quad \text{when } \sigma\ge 1. \end{gathered} \tag{3.6}$$

The pair $\{p,q\}$ represented by (3.5) cannot be in CS if

$$\lim_{\lambda\to-\infty}F(\lambda)=-\infty;\quad \lim_{\lambda\to\infty}F(\lambda)=\infty. \tag{3.7}$$

A pair $\{p<0,q>0\}$ cannot be in CS. For $-\infty<\sigma\le 1$ this fact follows from (3.7). For $\sigma>1$ this fact follows from the boundedness of $F(\lambda)$ from below.

Suppose (3.3) is solvable for two pairs $\{p,q_1\}$ and $\{p,q_2\}$, where $q_1<q_2$. Then the first pair is not a CS, but the second one may be CS.

Let us give without additional explanations the *full description* of CS for (3.1):
(1) $-\infty<\sigma<1$**:** A pair $\{p,q\}$ is a critical one if $\lambda\in[0,\frac{1}{1-\sigma})$ in (3.5). In this case

$$q=q(p)\approx\begin{cases}\frac{p^2}{4}, & 0<p\ll 1\\ \frac{1}{(1-\sigma)e}p+\frac{3-2\sigma}{(1-\sigma)^2}e^{\frac{\sigma}{1-\sigma}}+o(1), & p\to+\infty.\end{cases} \tag{3.8}$$

The last expression is the right asymptote for $q=q(p)$.
(2) $\sigma=1$**:** A pair $\{p,q\}$ is a critical one if $\lambda\in[0,\infty)$ in (3.5). In this case $(3.8)_1$ holds and also $q(p)\approx p\ln p$ as $p\to\infty$ (i.e. $\lambda\to\infty$).
(3) $1<\sigma<2$**:** A pair $\{p,q\}$ is a critical one if $\lambda\in(-\infty,-\frac{1}{\sigma-1})\bigcup[0,\infty)$ in (3.5). In this case $(3.8)_2$ holds for $p\to-\infty$ (i.e. $\lambda\to-\frac{1}{\sigma-1}-0$). This is the left asymptote, $(3.8)_1$ holds and, in addition,

$$q=q(p)\approx\begin{cases}|p|^\sigma|\ln|p||^{2-\sigma}, & p\to 0-0\\ p^\sigma(\ln p)^{2-\sigma}, & p\to+\infty \text{ (i.e. } \lambda\to+\infty).\end{cases} \tag{3.9}$$

(4) $\sigma=2$**:** A pair $\{p,q\}$ is a critical one if $\lambda\in(\infty,\infty)$ and $q(p)=\frac{p^2}{4}$.
(5) $\sigma>2$**:** A pair $\{p,q\}$ is a critical one if $\lambda\in(-\frac{1}{\sigma-1},\infty)$ in (3.5). In this case $q=q(p)$ has the same left asymptote as in the case (3), and

$$q(p)\approx\begin{cases}\frac{p^2}{4}, & p\to 0 \text{ (i.e.}\lambda\to 0)\\ p^\sigma(\ln p)^{2-\sigma}, & p\to+\infty.\end{cases}$$

4. The Sturm-like comparison theorems

Consider the inequality

$$(lx)(t):=x''(t)-[P(t)x(r(t))]'+Q(t)x(l(t))\le 0,\quad t\in(a,b), \tag{4.1}$$

where $r(t),l(t)$ are monotone increasing continuous differentiable functions,

$$\lim_{t\to\infty}r(t)=\lim_{t\to\infty}l(t)=\infty,\quad r(t)\le t.$$

(We do not assume that $l(t) \leq t$). Denote

$$e_{\text{ext}} := \{t\bar{\in}(a,b) : l(t) \in (a,b)\}, \quad e_{\text{int}} := \{t \in (a,b) : l(t)\bar{\in}(a,b)\}.$$

In particular, if $l(t) \leq t$, then $e_{\text{ext}} = (b, k(b))$, $e_{\text{int}} = (a, k(a))$. If $l(t) \geq t$, then $e_{\text{ext}} = (k(a), a)$, $e_{\text{int}} = (k(b), b)$. Here $k(t)$ and $q(t)$ are the inverse functions for $l(t)$ and $r(t)$, respectively. Consider also the inequality

$$(\tilde{l}y)(t) := y''(t) + q'(t)P[q(t)]y'(q(t)) + k'(t)\tilde{Q}[k(t)]y[k(t)] \geq 0, \quad t \in (a,b). \tag{4.2}$$

For arbitrary continuous on $(a, q(b)) \bigcup e_{\text{ext}}$ and possesing all needed derivatives function $y(t)$, we have the following equalities:

$$\begin{aligned}
\int_a^b x''(t)y(t)dt &= x'(b)y(b) - x'(a)y(a) - x(b)y'(b) + x(a)y'(b) + \int_a^b y''(t)x(t)dt,\\
\int_a^b Q(t)x(l(t))y(t)dt &= \int_{(a,b)\setminus e_{\text{int}}} [Q(t) - \tilde{Q}(t)]x(l(t))y(t)dt + \int_{e_{\text{int}}} Q(t)x(l(t))y(t)dt\\
&\quad - \int_{e_{\text{ext}}} \tilde{Q}(t)x(l(t))y(t)dt + \int_a^b k'(t)\tilde{Q}(k(t))y(k(t))x(t)dt,\\
-\int_a^b \left[P(t)x(r(t))\right]' y(t)dt &= -P(b)y(b)x(r(b)) + P(a)y(a)x(r(a))\\
&\quad + \int_a^{q(a)} P(t)x(r(t))y'(t)dt - \int_b^{q(b)} P(t)x(r(t))y'(t)dt\\
&\quad + \int_a^b q'(t)P(q(t))y'(q(t))x(t)dt.
\end{aligned}$$

These equalities imply the Main Identity-1:

$$\begin{aligned}
\int_a^b (lx)(t)y(t)dt &= \int_a^b (\tilde{l}y)(t)x(t)dt + \left[x'(b) - P(b)x(r(b))\right]y(b)\\
&\quad - \left[x'(a) - P(a)x(r(a))\right]y(a) + x(a)y'(a) - x(b)y'(b)\\
&\quad + \int_a^{q(a)} P(t)x(r(t))y'(t)dt - \int_b^{q(b)} P(t)x(r(t))y'(t)dt\\
&\quad + \int_{(a,b)\setminus e_{\text{int}}} [Q(t) - \tilde{Q}(t)]x(l(t))y(t)dt + \int_{e_{\text{int}}} Q(t)x(l(t))y(t)dt\\
&\quad - \int_a^b \tilde{Q}(t)x(l(t))y(t)dt.
\end{aligned} \tag{4.3}$$

This Identity is a basis of the first Sturm-like Comparison Theorem for Damped equation.

Theorem 4.1. *Let be $b > \max\{q(a); k(a)\}$ and assume that:*

1. $P(t) \geq 0$, $t \in (a, q(a)) \bigcup (b, q(b))$.
2. $Q(t) \geq 0$, $t \in e_{\text{int}}$.
3. $\tilde{Q}(t) \geq 0$, $t \in e_{\text{ext}}$.
4. $$Q(t) \geq \tilde{Q}(t), t \in (a,b) \setminus e_{\text{int}}. \tag{4.4}$$

(5) *Inequality* (4.2) *has a solution* $y(t)$ *such that*

$$y(a)=y(b)=0;\quad y(t)>0,\ t\in(a,b);\quad y(t)\le 0,\ t\in e_{\text{ext}}; \tag{4.5}$$

$$y'(t)\ge 0, t\in(a,q(a)); y'(t)\le 0, t\in(b,q(b)). \tag{4.6}$$

(6) *At least one of the inequalities* (4.4)-(4.6) *is strong on some subinterval.*

Then there is no positive solution of (4.1) *on* $(r(a),b)\bigcup e_{\text{ext}}$.

Proof. Suppose (4.1) has a positive solution on $(r(a),b)\bigcup e_{\text{ext}}$. Then the left hand-side of (4.3) is zero but all terms of the right hand-side are nonnegative and at least one is positive. We have a contradiction. □

Remark. For the case $P(t)\equiv 0$ (i.e. without the damping term) this Theorem was published in [1, Theorem 6.2.4, p.112].

Now, we state one more version of Sturm-like Comparison Theorem for (4.1), which is based on another Main Identity-2 which differs from (4.3). We will employ the preceding notation and the following:

$$\begin{gathered}\bar e_{\text{ext}}:=\{t\bar\in(a,b): l[q(t)]\in(a,b)\},\\ \bar e_{\text{int}}:=\{t\in(a,b): l[q(t)]\bar\in(a,b)\},\\ lq(t):=l[q(t)].\end{gathered} \tag{4.7}$$

The last notion will be applied to some other double superpositions.

In particular case $l(t)\le r(t)\le t$ we have

$$\bar e_{\text{int}}=(a,rk(a)),\quad \bar e_{\text{ext}}=(b,rk(b)). \tag{4.8}$$

If $l(t)\ge r(t)$, then

$$\bar e_{\text{int}}=(rk(b),b),\quad \bar e_{\text{ext}}=(rk(a),a). \tag{4.9}$$

An important particular case for us is $r(t):=t-1$, $l(t):=t-\sigma$, $q(t)=t+1$, $k(t)=t+\sigma$, $-\infty<\sigma<\infty$ and

$$\begin{gathered}\bar e_{\text{int}}=(a,a+\sigma-1),\quad \bar e_{\text{ext}}=(b,b+\sigma-1)\quad\text{for } \sigma\ge 1,\\ \bar e_{\text{int}}=(b+\sigma-1,b),\quad \bar e_{\text{ext}}=(a+\sigma-1,a)\quad\text{for } -\infty<\sigma\le 1.\end{gathered}$$

We introduce an operator $\tilde{\tilde{l}}y$ and the corresponding inequality

$$(\tilde{\tilde{l}}y)(t):=[r'(t)y'(r(t))]'+P(q(t))y'(t)+k'(t)\tilde Q(k(t))y(rk(t))\ge 0,\quad t\in(a,b). \tag{4.10}$$

Let $y(t)$ be an arbitrary smooth function, such that $y(a)=y(b)=0$. Then

$$\begin{aligned}&\int_{q(a)}^{q(b)}x''(t)y(r(t))dt\\ &=\int_a^b[r'(t)y'(r(t))]'x(t)dt+y'(a)\frac{x(q(a))}{q'(a)}-y'(b)\frac{x(q(b))}{q'(b)}\\ &-\int_{r(a)}^{a}q'(t)\left[\frac{y'(t)}{q'(t)}\right]'x(q(t))dt+\int_{r(b)}^{b}q'(t)\left[\frac{y'(t)}{q'(t)}\right]'x(q(t))dt.\end{aligned} \tag{4.11}$$

$$-\int_{q(a)}^{q(b)}[P(t)x(r(t))]'y(r(t))dt=\int_a^bP(q(t))y'(t)x(t)dt. \tag{4.12}$$

$$
\begin{aligned}
&\int_{q(a)}^{q(b)} Q(t)x(l(t))y(r(t))dt \\
&= \int_a^b k'(t)\tilde{Q}(k(t))y(rk(t))x(t)dt \\
&\quad + \int_{(a,b)\setminus \bar{e}_{\rm int}} q'(t)\left\{Q(q(t)) - \tilde{Q}(q(t))\right\} x(lq(t))y(t)dt \\
&\quad + \int_{\bar{e}_{\rm int}} q'(t)Q(q(t))x(lq(t))y(t)dt - \int_{\bar{e}_{\rm ext}} q'(t)\tilde{Q}(q(t))x(lq(t))y(t)dt.
\end{aligned}
\tag{4.13}
$$

Equalities (4.11)-(4.13) imply Main Identity-2:

$$
\begin{aligned}
&\int_{q(a)}^{q(b)} (lx)(t)y(r(t))dt \\
&= \int_a^b (\tilde{\tilde{l}}y)(t)x(t)dt + y'(a)\frac{x(q(a))}{q'(a)} \\
&\quad - y'(b)\frac{x(q(b))}{q'(b)} - \int_{r(a)}^{a} q'(t)\left[\frac{y'(t)}{q'(t)}\right]' x(q(t))dt + \int_{r(b)}^{b} q'(t)\left[\frac{y'(t)}{q'(t)}\right]' x(q(t))dt \\
&\quad + \int_{(a,b)\setminus \bar{e}_{\rm int}} q'(t)\left\{Q(q(t)) - \tilde{Q}(q(t))\right\} x(lq(t))y(t)dt \\
&\quad + \int_{\bar{e}_{\rm int}} q'(t)Q(q(t))x(lq(t))y(t)dt - \int_{\bar{e}_{\rm ext}} q'(t)\tilde{Q}(q(t))x(lq(t))y(t)dt.
\end{aligned}
\tag{4.14}
$$

Now we can formulate the second version of a Sturm-like Comparison Theorem for (4.1)

Theorem 4.2. *Let $b > \max\{q(a), k(a)\}$ and the following conditions hold:*

(1)
$$
Q(q(t)) \ge 0, t \in \bar{e}_{\rm int}, \quad \tilde{Q}(q(t)) \ge 0, \quad t \in \bar{e}_{\rm ext}. \tag{4.15}
$$

(2)
$$
Q(q(t)) \ge \tilde{Q}(q(t)), t \in (a,b) \setminus \bar{e}_{\rm int}. \tag{4.16}
$$

(3) *Inequality (4.10) has a solution $y(t)$ such that*
$$
y(a) = y(b) = 0; \quad y(t) > 0,\ t \in (a,b); \quad y(t) \le 0,\ t \in \bar{e}_{\rm ext}, \tag{4.17}
$$
$$
\Big[\frac{y'(t)}{q'(t)}\Big]' \le 0, t \in (r(a), a); \quad \Big[\frac{y'(t)}{q'(t)}\Big]' \ge 0, \quad t \in (r(b), b). \tag{4.18}
$$

(4) *At least one of the inequalities (4.15)-(4.18) is strong on one of the subintervals.*

Then (4.1) has no positive solution $x(t)$ on $(r(a), b) \bigcup \bar{e}_{\rm ext}$.

Remark. This Theorem, in contrast to Theorem 4.1 , does not impose any restriction on the sign of $P(t)$.

5. Testing Inequalities

In both Sturm-like Comparison Theorems there exists a requirement that an "associated inequality" has a solution satisfying several conditions. In this chapter we will construct the such inequalities ((4.2) and (4.10) in an explicit form. Later we will employ these inequalities as testing inequalities in applications of Sturm-like Comparison Theorems.

We will look for the solution of (4.2) in the form

$$y(t) = \exp\big\{\int_a^t z(s)ds\big\}\sin\int_a^t \varphi(s)ds, \quad t\in(a,b). \tag{5.1}$$

Here

$$\varphi(t)>0, \quad \int_a^b \varphi(s)ds=\pi, \quad \int_a^{p(a)}\varphi(s)ds<\frac{\pi}{2}, \quad \int_b^{p(b)}\varphi(s)ds<\frac{\pi}{2}, \tag{5.2}$$

where $p(t)=\max\{q(t),k(t)\}$. Then

$$\begin{aligned}
y'(t) &= \exp\big\{\int_a^t z(s)ds\big\}\big\{z(t)\sin\int_a^t\varphi(s)ds+\varphi(t)\cos\int_a^t\varphi(s)ds\big\},\\
y'(q(t)) &= \exp\big\{\int_a^t z(s)ds\big\}\exp\big\{\int_t^{q(t)} z(s)ds\big\}\\
&\quad\times\big\{\big[z(q(t))\cos\int_t^{q(t)}\varphi(s)ds-\varphi(q(t))\sin\int_t^{q(t)}\varphi(s)ds\big]\sin\int_a^t\varphi(s)ds\\
&\quad+\big[z(q(t))\sin\int_t^{q(t)}\varphi(s)ds+\varphi(q(t))\cos\int_t^{q(t)}\varphi(s)ds\big]\cos\int_a^t\varphi(s)ds\big\},\\
y''(t) &= \exp\big\{\int_a^t z(s)ds\big\}\big\{A(t)\sin\int_a^t\varphi(s)ds+B(t)\cos\int_a^t\varphi(s)ds\big\},
\end{aligned} \tag{5.3}$$

where

$$A(t):=z^2(t)+z'(t)-\varphi^2(t), \quad B(t):=\varphi'(t)+2\varphi'(t)z(t),$$

$$\begin{aligned}
y(k(t)) &= \exp\big\{\int_a^t z(s)ds\big\}\exp\big\{\int_t^{k(t)} z(s)ds\big\}\\
&\quad\times\big\{\cos\int_t^{k(t)}\varphi(s)ds\sin\int_a^t\varphi(s)ds+\sin\int_t^{k(t)}\varphi(s)ds\cos\int_a^t\varphi(s)ds\big\}.
\end{aligned}$$

Hence on the interval (a,b) we have

$$(\tilde{l}y)(t)\ge 0 \Longleftrightarrow S(t)\sin\int_a^t\varphi(s)ds+R(t)\cos\int_a^t\varphi(s)ds\ge 0$$

if and only if

$$
\begin{aligned}
S(t) :=\ & A(t) + q'(t)P(q(t)) \exp\big\{ \int_t^{q(t)} z(s)ds \big\} \\
& \times \big[z(q(t)) \cos \int_t^{q(t)} \varphi(s)ds - \varphi(q(t)) \sin \int_t^{q(t)} \varphi(s)ds \big] \\
& + k'(t)\tilde{Q}(k(t)) \exp\big\{ \int_t^{k(t)} z(s)ds \big\} \cos \int_t^{k(t)} \varphi(s)ds \geq 0, \\
R(t) :=\ & B(t) + q'(t)P(q(t)) \exp\Big\{ \int_t^{q(t)} z(s)ds \Big\} \\
& \times \big[z(q(t)) \sin \int_t^{q(t)} \varphi(s)ds + \varphi(q(t)) \cos \int_t^{q(t)} \varphi(s)ds \big] \\
& + k'(t)\tilde{Q}(k(t)) \exp\big\{ \int_t^{k(t)} z(s)ds \big\} \sin \int_t^{k(t)} \varphi(s)ds = 0.
\end{aligned}
\tag{5.4}
$$

If $\sin \int_t^{k(t)} \varphi(s)ds$ is a positive or a negative function, then (5.4) can be transformed into one of two equivalent systems on (a, b):

$$
\tilde{Q}(k(t)) \geq M(k(t)), \quad R(t) = 0 \tag{5.5}
$$

or

$$
Z(t) \geq 0, \quad R(t) = 0 \tag{5.6}
$$

where

$$
\begin{aligned}
Z(t) :=\ & A(t) \sin \int_t^{k(t)} \varphi(s)ds - B(t) \cos \int_t^{k(t)} \varphi(s)ds \\
& + q'(t)P(q(t)) \exp\big\{ \int_t^{q(t)} z(s)ds \big\} \\
& \times \big[z(q(t)) \sin \int_{q(t)}^{k(t)} \varphi(s)ds - \varphi(q(t)) \cos \int_{q(t)}^{k(t)} \varphi(s)ds \big],
\end{aligned}
$$

$$
\begin{aligned}
k'(t)M(k(t)) :=\ & -\exp\big\{ - \int_t^{k(t)} z(s)ds \big\} \Big\{ A(t) \cos \int_t^{k(t)} \varphi(s)ds \\
& + B(t) \sin \int_t^{k(t)} \varphi(s)ds + +q'(t)P(q(t)) \exp\big\{ \int_t^{q(t)} z(s)ds \big\} \\
& \times \big[z(q(t)) \cos \int_{q(t)}^{k(t)} \varphi(s)ds + \varphi(q(t)) \sin \int_{q(t)}^{k(t)} \varphi(s)ds \big] \Big\}.
\end{aligned}
$$

Thus if $\tilde{Q}(t)$ and $z(t)$ satisfy (5.4) (or (5.5), or (5.6), then (4.2) has a solution $y(t)$ of the form (5.1) for which (4.5)-(4.6) hold.

In case $l(t) \equiv t$, (5.4) changes its character because in this version $(5.4)_2$ does not contain $\tilde{Q}(t)$ and turns into the equation in $z(t)$ only:

$$
\begin{aligned}
R(t) :=\ & B(t) + q'(t)P[q(t)] \exp \int_t^{q(t)} z(s)ds \\
& \times \big[z(q(t)) \sin \int_t^{q(t)} \varphi(s)ds + \varphi(q(t)) \cos \int_t^{q(t)} \varphi(s)ds \big] = 0.
\end{aligned}
\tag{5.7}
$$

For (4.10), let us repeat all calculations done for (4.2). We will look for a solution of (4.10) of the same form (5.1), where $\varphi(t)$ and $z(t)$ satisfy the same conditions (5.2). After some calculations we have the following system on (a,b):

$$\begin{aligned}\tilde S(t) :=\ & r''(t)\big[z(r(t))\cos\int_{r(t)}^{t}\varphi(s)ds+\varphi(r(t))\sin\int_{r(t)}^{t}\varphi(s)ds\big]\\ &+[r'(t)]^2\big[A(r(t))\cos\int_{r(t)}^{t}\varphi(s)ds+B(r(t))\sin\int_{r(t)}^{t}\varphi(s)ds\big]\\ &+P(q(t))z(t)\exp\big\{\int_{r(t)}^{t}z(s)ds\big\}\\ &+k'(t)\tilde Q(k(t))\exp\big\{\int_{r(t)}^{rk(t)}z(s)ds\big\}\cos\int_{t}^{rk(t)}\varphi(s)ds\ge 0,\\ \tilde R(t) :=\ & r''(t)\big[-z(r(t))\sin\int_{r(t)}^{t}\varphi(s)ds+\varphi(r(t))\cos\int_{r(t)}^{t}\varphi(s)ds\big]\\ &+[r'(t)]^2\big[-A(r(t))\sin\int_{r(t)}^{t}\varphi(s)ds+B(r(t))\cos\int_{r(t)}^{t}\varphi(s)ds\big]\\ &+P(q(t))\varphi(t)\exp\big\{\int_{r(t)}^{t}z(s)ds\big\}\\ &+k'(t)\tilde Q(k(t))\exp\big\{\int_{r(t)}^{rk(t)}z(s)ds\big\}\sin\int_{t}^{rk(t)}\varphi(s)ds=0.\end{aligned}\tag{5.8}$$

Instead of (5.8) one of the following two equivalent systems can be applied

$$\tilde Q(t)\ge\tilde N(t),\quad \tilde R(t)=0,\tag{5.9}$$

or

$$\tilde Z(t)\ge 0,\quad \tilde R(t)=0,\tag{5.10}$$

where

$$\begin{aligned}&k'(t)\tilde N(k(t))\cos\int_{t}^{rk(t)}\varphi(s)ds\\ &:=\exp\big\{-\int_{r(t)}^{rk(t)}z(s)ds\big\}\Big\{-P(q(t))z(t)\exp\int_{r(t)}^{t}z(s)ds\\ &\quad -r''(t)\big[z(r(t))\cos\int_{r(t)}^{t}\varphi(s)ds+\varphi(r(t))\sin\int_{r(t)}^{t}\varphi(s)ds\big]\\ &\quad -[r'(t)]^2\big[A(r(t))\cos\int_{r(t)}^{t}\varphi(s)ds+B(r(t))\sin\int_{r(t)}^{t}\varphi(s)ds\big]\Big\},\end{aligned}$$

$$\begin{aligned}\tilde Z(t) :=\ & r''(t)\big[z(r(t))\sin\int_{r(t)}^{rk(t)}\varphi(s)ds-\varphi(r(t))\cos\int_{r(t)}^{rk(t)}\varphi(s)ds\big]\\ &+(r'(t))^2\big[A(r(t))\sin\int_{r(t)}^{rk(t)}\varphi(s)ds-B(r(t))\cos\int_{r(t)}^{rk(t)}\varphi(s)ds\big]\\ &+P(q(t))\big[z(t)\sin\int_{t}^{rk(t)}\varphi(s)ds-\varphi(t)\cos\int_{t}^{rk(t)}\varphi(s)ds\big]\exp\int_{r(t)}^{t}z(s)ds.\end{aligned}$$

6. Applications of Sturm-like Comparison Theorems to Investigation of some Typical Damped Equations

Consider the equation

$$x''(t) - px'(t-1) + Q(t)x(t-\sigma) = 0, \quad -\infty < \sigma < \infty,\ p > 0. \tag{6.1}$$

The equation

$$\frac{s(2+s\sigma)}{1+s(\sigma-1)}e^{s} = p \tag{6.2}$$

has (a unique) root $s > 0$. This root $s \in (0, \frac{1}{1-\sigma})$ for $-\infty < \sigma < 1$ and $s \in (0,\infty)$ for $\sigma \geq 1$. The number $q := \frac{s^2(s+1)}{1+s(\sigma-1)}e^{s\sigma}$ together with p forms a critical pair $\{p,q\}$ for the autonomous equation

$$z''(t) - pz(t-1) + qz(t-\sigma) = 0. \tag{6.3}$$

Theorem 6.1. *If*

$$\liminf_{t\to\infty} Q(t) = C > q, \tag{6.4}$$

then all solutions of (6.1) *are oscillatory and there exists* $\nu > 0$ *such that every solution has at least one zero on any interval* $(T-1, T+\frac{\pi}{\nu})$ *for sufficiently large* T.

Remark. The constant q in (6.4) is the best possible one. Indeed, (6.1) has a nonoscillatory solution $z(t) = e^{st}$ for $Q(t) \equiv q$.

Proof. The proof is based on Theorem 4.2. Let in (5.4) $\varphi(t) := \nu$; $z(t)$ =const, which will be chosen later. Then (5.4) has the form

$$\begin{gathered}\tilde{Q}(t)e^{z(\sigma-1)}\cos\nu(\sigma-1) \geq -e^{-z}\left[(z^2-\nu^2)\cos\nu + 2\nu z\sin\nu\right] - pz\\ \tilde{Q}(t)e^{z(\sigma-1)}\sin\nu(\sigma-1) = -e^{-z}\left[-(z^2-\nu^2)\sin\nu + 2\nu z\cos\nu\right] - p\nu\end{gathered} \tag{6.5}$$

We will choose z and $\tilde{Q}$ as a solution of a system, which is stronger than (6.5), with equality in $(6.5)_1$ instead of the inequality. Then this new system is equivalent to one of the following two systems:

$$\begin{gathered}F(\nu,z) := pe^{z} - (z^2-\nu^2)\frac{\sin\nu}{\nu} + 2z\cos\nu = 0\\ \tilde{Q} = -pz - e^{-z}\left[(z^2-\nu^2)\cos\nu + 2\nu z\sin\nu\right]\end{gathered}, \tag{6.6}$$

for $\sigma = 1$, and

$$\begin{gathered}\Phi(\nu,z) := (z^2-\nu^2)\frac{\sin\nu\sigma}{\nu} - 2z\cos\nu\sigma + pe^{z}\Big[z\frac{\sin\nu(\sigma-1)}{\nu} - cos\nu(\sigma-1)\Big] = 0\\ \tilde{Q} = \frac{e^{-\sigma z}}{\cos\nu(\sigma-1)}\left\{-(z^2-\nu^2)\cos\nu - 2z\nu\sin\nu - pe^{z}z\right\}\end{gathered} \tag{6.7}$$

for $\sigma \neq 1$.

Evidently, a possible solution $\{z,\tilde{Q}\}$ of (6.6) or (6.7) does not depend on t. Consider first (6.6):

$$F'_z(\nu,z) := pe^{z} - 2z\frac{\sin\nu}{\nu} + 2\cos\nu,$$

$$F(0,-s) = pe^{-s} - s^2 - 2s = \frac{s(2+s)e^{s}}{1}e^{-s} - s^2 - 2s = 0$$

$$F'_z(0,-s) = s(2+s) + 2s + 2 = s^2 + 4s + 2 \neq 0.$$

Then for sufficiently small ν there exists the implicit function $z = z(\nu)$ such that $z(0) = -s$ and $F(\nu, z(\nu)) = 0$. The number $z(\nu)$, which is constructed by this way, is taken as the function $z(t)$. From $(6.6)_2$ we obtain

$$\begin{aligned} \tilde{Q} = \tilde{Q}_\nu &= -pz_\nu - e^{-z_\nu}\left[(z_\nu^2 - \nu^2)\cos\nu + 2\nu z_\nu \sin\nu\right], \\ \lim_{\nu\to 0} \tilde{Q}_\nu &= s^2(2+s)e^s - e^s s^2 = s^2(s+1)e^s = q. \end{aligned} \tag{6.8}$$

Let us do the same calculations with (6.7):

$$\Phi(0,-s) = s^2\sigma + 2s + \frac{s^2\sigma + 2s}{1+s(\sigma-1)} e^s e^{-s}(-s(\sigma-1)-1) = 0,$$

$$\begin{aligned} &\Phi'_z(\nu, z) \\ &= 2z\frac{\sin\nu\sigma}{\nu} - 2\cos\nu\sigma + pe^z\big[z\frac{\sin\nu(\sigma-1)}{\nu} - \cos\nu(\sigma-1) + \frac{\sin\nu(\sigma-1)}{\nu}\big] \end{aligned}$$

$$\begin{aligned} \Phi'_z(0,-s) &= -2s\sigma - 2 + pe^{-s}[-s(\sigma-1) - 1 + \sigma - 1] \\ &= -2s\sigma - 2 + \frac{s^2\sigma + 2s}{1+s(\sigma-1)}[-s(\sigma-1) + \sigma - 2] \\ &= -\frac{\sigma^2(1+s)s^2 - \sigma s(s^2 - 2s - 2) - 2(s^2 - s - 1)}{1+s(\sigma-1)} \\ &:= -\frac{h(s,\sigma)}{1+s(\sigma-1)}, \end{aligned}$$

We will prove that $h(s,\sigma) > 0$ in two domains:

(1) Let $(\sigma, s) \in \{\sigma > 1; s > 0\}$. Then

$$h(s,\sigma) = s^3(\sigma-1)\sigma + s^2(\sigma^2 + 2\sigma - 2) + s(2\sigma+2) + 2 > s^2 + 4s + 2 > 0.$$

(2) Let $(\sigma, s) \in \{-\infty < \sigma < 1; 0 < s < \frac{1}{1-\sigma}\}$. Then $h(s,\sigma) > 0$ on the boundary of this domain. Indeed,

$$h\big(s, \frac{s-1}{s}\big) = s + 1 > 0; h(s,1) = s^2 + 4s + 2 > 0; h(0,\sigma) = 2 > 0.$$

The domain (σ, s) in case 2^o does not contains stationary points of $h(s,\sigma)$ since there are no solutions of the system

$$\begin{aligned} h'_s &= 0 \\ h'_\sigma &= 0 \end{aligned}$$

in this domain. Indeed,

$$h'_\sigma = 0 \Leftrightarrow \sigma = \frac{s^2 - 2s - 2}{2s(1+s)} \Leftrightarrow \frac{1}{1-\sigma} - s = -\frac{s^2(s+2)}{s^2 + 4s + 2} < 0;$$

i.e. the point (σ_0, s_0) belongs to outside of the domain for every s. Thus $\Phi'_z(0,-s) \neq 0$ for $\forall p > 0, \sigma \neq 1$ and we can choose $z = z_\nu$ as a solution of $(6.7)_1$. In addition, from $(6.7)_2$ we have

$$\lim_{\nu\to 0} \tilde{Q}_\nu = e^{\sigma s}\big\{-s^2 + s\frac{s(2+\sigma s)}{1+s(\sigma-1)} e^s e^{-s}\big\} = \frac{s^2(1+s)}{1+s(\sigma-1)} e^{\sigma s} = q. \tag{6.9}$$

Conditions (6.9) and (6.4) imply that for sufficiently small ν we have $\tilde{Q} < C$. Then by (6.4) we have $Q \geq \tilde{Q}$ which is condition 2 in Theorem 4.2. One needs to check only (4.18): For $t \in (a-1, a)$ we have

$$y''(t) \leq 0 \Longleftrightarrow A(t)\sin\nu(t-a) + B(t)\cos\nu(t-a) \leq 0$$
$$\Longleftrightarrow -A(t)\tan\nu(a-t) + B(t) \leq 0 \Longleftrightarrow -(z_\nu^2 - \nu^2)\frac{\tan\nu(a-t)}{\nu} + 2z_\nu \leq 0$$
$$\Longleftarrow s^2\frac{\tan\nu(a-t)}{\nu} + 2s > 0 \Longleftarrow s > 0.$$

From the condition $\int_a^b \varphi(t)dt = \pi$ we have $b = a + \frac{\pi}{\nu}$. Then for $t \in (b-1, b)$ we have

$$y''(t) \geq 0 \Longleftrightarrow A(t)\sin(\nu(t-b)+\pi) + B(t)\cos(\nu(t-b)+\pi) \geq 0$$
$$\Longleftrightarrow A(t)\sin\nu(t-b) + B(t)\cos\nu(t-b) \leq 0.$$

The end of the calculations is similar to the previous case. All conditions of Theorem 4.2 hold. Then Theorem 6.1 is proven. □

We will strengthen Theorem 6.1 by formulating the following result which will be called Knezer-like Theorem for (6.1).

Let be $p > 0$, $s > 0$, $q > 0$ denoted as in the beginning of section 6. Denote also the constant

$$K(p,\sigma) := \frac{\sigma^2 s^2(1+s) + \sigma s(-s^2+2s+2) - 2s^2 + 2s + 2}{8[1+s(\sigma-1)]}e^{s\sigma}. \tag{6.10}$$

Theorem 6.2. *If*

$$\liminf_{t\to\infty}\left\{[Q(t)-q]t^2\right\} = C > K(p,\sigma), \tag{6.11}$$

then all solutions of (6.1) are oscillatory and there exists $\nu > 0$ such that every solution has at least one zero on any interval $(T-1, T\exp\frac{\pi}{\nu})$ for sufficiently large T.

Proof. The proof is based on Theorem 4.2. Suppose first $\sigma \neq 1$ and set in (5.4),

$$\varphi(t) := \frac{\nu}{t}, \quad z(t) := -s + \frac{1}{2t} + \frac{\beta}{t^2},$$

where β will be chosen later. Substitute $k(t) = t+\sigma$, $q(t) = t+1$, $rk(t) = t+\sigma-1$ in (5.3)-(5.4). Write out several asymptotic, where we denote $f(t) \cong g(t)$ instead of $f(t) = g(t) + o(\frac{1}{t^2})$:

$$\frac{t}{\nu}\varphi(t+1) \cong 1 - \frac{1}{t} + \frac{1}{t^2}; \quad \cos\int_t^{t+\sigma}\varphi(s)ds \cong 1 - \frac{\nu^2\sigma^2}{2t^2},$$
$$\cos\int_{t+1}^{t+\sigma}\varphi(s)ds \cong 1 - \frac{\nu^2(\sigma-1)^2}{2t^2}; \quad \frac{t}{\nu\sigma}\sin\int_t^{t+\sigma}\varphi(s)ds \cong 1 - \frac{\sigma}{2t} + \frac{\sigma^2(2-\nu^2)}{6t^2};$$
$$\frac{t}{\nu(\sigma-1)}\sin\int_{t+1}^{t+\sigma}\varphi(s)ds \cong 1 - \frac{\sigma-1}{2t} + \frac{(2-\nu^2)(\sigma-1)^2 + 3\sigma + 3}{6t^2};$$
$$\frac{t}{\nu\sigma}\sin\int_{t-1}^{t+\sigma-1}\varphi(s)ds \cong 1 + \frac{2-\sigma}{2t} + \frac{2\sigma^2 - 6\sigma + 6 - \sigma^2\nu^2}{6t^2};$$
$$\frac{t}{\nu}\sin\int_{t-1}^{t}\varphi(s)ds \cong 1 + \frac{1}{2t} + \frac{2-\nu^2}{6t^2};$$

$$\frac{\nu\sigma}{t}\csc\int_t^{t+\sigma}\varphi(s)ds \cong 1+\frac{\sigma}{2t}+\frac{\sigma^2(2\nu^2-1)}{12t^2};$$
$$z(t+1)\cong -s+\frac{1}{2t}+\frac{2\beta-1}{2t^2};\quad \int_t^{t+\sigma} z(s)ds \cong -s\sigma+\frac{\sigma}{2t}+\frac{4\sigma\beta-\sigma^2}{4t^2},$$
$$\exp\int_t^{t+\sigma} z(s)ds \cong e^{-s\sigma}\Big[1+\frac{\sigma}{2t}+\frac{8\sigma\beta-\sigma^2}{8t^2}\Big];$$
$$\exp\Big\{-\int_t^{t+\sigma} z(s)ds\Big\} \cong e^{s\sigma}\Big[1-\frac{\sigma}{2t}+\frac{3\sigma^2-8\sigma\beta}{8t^2}\Big];$$
$$A(t):=z^2(t)+z'(t)-\varphi^2(t)\cong s^2-\frac{s}{t}-\frac{1+4s+4\nu^2+8\beta s}{4t^2};$$
$$\frac{t}{\nu}B(t)\cong -2s-\frac{2s}{t}+\frac{2\beta-2s}{t^2}.$$

Using these asymptotic results, rewrite $(5.4)_1$ as follows

$$\begin{aligned}
&\tilde Q(t+\sigma)\ge \tilde N(t+\sigma)\\
&:=\frac{\exp\{-\int_t^{t+\sigma-1} z(s)ds\}}{\cos\int_t^{t+\sigma-1}\varphi(s)ds}\Big\{-pz(t)-\exp\Big\{-\int_{t-1}^t z(s)ds\Big\}\\
&\quad\times\Big[A(t-1)\cos\int_{t-1}^t\varphi(s)ds+B(t-1)\sin\int_{t-1}^t\varphi(s)ds\Big]\Big\}=\dots\\
&=e^{s\sigma}\Big(1-\frac{\sigma-1}{2t}+\frac{3(\sigma-1)^2-8\beta(\sigma-1)+4(\sigma-1)^2\nu^2}{8t^2}\Big)\Big\{D_0+D_1\frac{1}{t}+D_2\frac{1}{t^2}\Big\},
\end{aligned}$$

where

$$D_0:=\frac{2s^2+\sigma s^3}{1+s(\sigma-1)}-s^2=\frac{s^2(1+s)}{1+s(\sigma-1)}=qe^{-s\sigma},$$
$$D_1=-\frac{2s+\sigma s^2}{2(1+s(\sigma-1))}+\frac{s^2}{2}+s=\frac{\sigma-1}{2}qe^{-s\sigma},$$
$$\begin{aligned}
D_2&=-\beta\frac{2s+\sigma s^2}{1+s(\sigma-1)}+\frac{s^2+8\beta s^2}{8}-\frac{s}{2}+\frac{1}{4}-\frac{2s-4\beta s+\nu^2(s^2+4s+2)}{2}\\
&=\beta\frac{(\sigma-1)s^2(1+s)}{1+s(\sigma-1)}+\frac{s^2+4s+2}{8}+\nu^2\frac{s^2+4s+2}{2}.
\end{aligned}$$

After these calculations we have

$$\tilde Q(t+\sigma)\ge \bar N(t+\sigma)\cong q+\frac{K(p,\sigma)+\nu^2L(\sigma,s)}{t^2}, \tag{6.12}$$

in which the explicit form of $L(\sigma,s)$ is unessential. Note also that (6.12) does not contain parameter β. Equality $(5.4)_2)$ turns into

$$\begin{aligned}
\tilde Q(t+\sigma)&=\frac{\exp\{-\int_t^{t+\sigma-1} z(s)ds\}}{\sin\int_t^{t+\sigma-1}\varphi(s)ds}\Big\{-p\frac{\nu}{t}+\exp\Big\{-\int_{t-1}^t z(s)ds\Big\}\\
&\quad\times\Big[A(t-1)\sin\int_{t-1}^t\varphi(s)ds--B(t-1)\cos\int_{t-1}^t\varphi(s)ds\Big]\Big\}\\
&\cong q+\frac{1}{t^2}\big\{L_1(s,\sigma)\nu^2-\beta R(s,\sigma)+\tilde K(p,\sigma)\big\},
\end{aligned}\tag{6.13}$$

where $R(s,\sigma):=\frac{1}{\sigma-1}[q(\sigma-1)^2+4e^{s\sigma}]$. The explicit form of L_1 and $\tilde K$ is unessential.

Evidently $R(s,\sigma)\neq 0,\ \forall s>0,\ \sigma\neq 1$, hence there exists β such that

$$K(p,\sigma)<L_1-\beta R+\tilde{K}=C_1<C \tag{6.14}$$

and choose $\nu>0$ small, such that

$$K(p,\sigma)+\nu^2 L(s,\sigma)<C_1<C, \tag{6.15}$$

which is possible due to (6.11). This inequality implies (2) in Theorem 4.2 for sufficiently large t. We need now to check (4.18) only. For $(4.18)_1)$ we have for $t\in(a-1,a)$:

$$\begin{gathered}
y''(t)\le 0 \Leftrightarrow s^2\sin\big(\nu\ln\frac{t}{a}\big)-\frac{2\nu s}{t}\cos\big(\nu\ln\frac{t}{a}\big)<0\Leftrightarrow\\
\Leftrightarrow -s^2\tan\big(\nu\ln\frac{a}{t}\big)-\frac{2\nu s}{t}<0\Leftrightarrow g(t):=s^2\tan\big(\nu\ln\frac{a}{t}\big)+\frac{2\nu s}{t}>0.
\end{gathered}$$

Since

$$g'(t)=-s^2\frac{\nu}{t\cos^2(\nu\ln\frac{a}{t})}-\frac{2\nu s}{t}<0,$$

it follows that $\inf_{t\in(a-1,a)}g(t)=g(a)=\frac{2\nu s}{a}>0$, for all $s>0$.

For $(4.18)_2$ due to $\int_a^b\varphi(s)ds=\pi$ we have for $t\in(b-1,b)$:

$$\begin{gathered}
y''(t)>0\Leftrightarrow s^2\sin\big(\pi+\nu\ln\frac{t}{a}\big)-\frac{2\nu s}{t}\cos\big(\pi+\nu\ln\frac{t}{a}\big)>0\Leftrightarrow\\
s^2\sin\big(\nu\ln\frac{t}{a}\big)-\frac{2\nu s}{t}\cos\big(\nu\ln\frac{t}{a}\big)<0,
\end{gathered}$$

etc. (see the above calculations). For the case $\sigma\neq 1$ Theorem 6.2 is proven.

Now suppose $\sigma=1$. In this case we have

$$k(t)=q(t)=t+1,\quad rk(t)=t,\quad q=s^2(s+1)e^{s\sigma},$$

where $s\ge 0$ is the unique root of the equation $s(2+s)e^s=p$. System (5.4) has now the form

$$\begin{aligned}
\tilde{Q}(t+2)\ge M(t):=&-pz(t+1)-\exp\big\{-\int_t^{t+1}z(s)ds\big\}\\
&\times\big[A(t)\cos\int_t^{t+1}\varphi(s)ds+B(t)\sin\int_t^{t+1}\varphi(s)ds\big]\\
\tilde{R}(t):=&-A(t)\sin\int_t^{t+1}\varphi(s)ds+B(t)\cos\int_t^{t+1}\varphi(s)ds\\
&+p\varphi(t+1)\exp\big\{\int_t^{t+1}z(s)ds\big\}=0.
\end{aligned}\tag{6.16}$$

Rewrite $(6.16)_2$ after the substitution $\varphi(t):=\frac{\nu}{t}$ and expressions $A(t)$ and $B(t)$ in the form

$$z'(t)+z^2(t)-\frac{\nu^2}{t^2}-\frac{p\nu}{t\sin[\nu\ln(1+\frac{1}{t})]}\exp\big\{\int_t^{t+1}z(s)ds\big\}-\frac{\nu^2[2z(t)-\frac{1}{t}]}{t^2\tan[\nu\ln(1+\frac{1}{t})]}=0. \tag{6.17}$$

This equality is an equation with respect to $z(t)$ only in (T,∞) for sufficiently large T.

All coefficients in (6.17) are analytic functions in a neighborhood of the infinity. Therefore there exists a solution $z(t)$ of (6.17) which has the following expansion in some neighborhood of the infinity :

$$z(t)=\sum_{n=0}^{\infty}\frac{a_n}{t^n},\quad t\in(T,\infty).$$

We will look for a few terms of this expansion:

$$z(t)=a_0+\frac{a_1}{t}+\frac{a_2}{t^2}+o(t^{-2}).$$

We have

$$\begin{aligned}
A(t)&:=z^2(t)+z'(t)-\varphi^2(t)=\big[a_0+\frac{a_1}{t}+\frac{a_2}{t^2}\big]^2-\frac{a_1}{t^2}-\frac{\nu^2}{t^2}+o(t^{-2})\\
&=a_0^2+\frac{a_0a_1}{t}+[a_1^2+2a_0a_2-a_1-\nu^2]\frac{1}{t^2}+o(t^{-2}),\\
B(t)&:=\varphi'(t)+2\varphi(t)z(t)=-\frac{\nu}{t^2}+\frac{2\nu}{t}\big[a_0+\frac{a_1}{t}+\frac{a_2}{t^2}+o(t^{-2})\big]\\
&=\frac{\nu}{t}\big[2a_0+\frac{2a_1-1}{t}+\frac{2a_2}{t^2}+o(t^{-2})\big].
\end{aligned}$$

Then

$$\begin{aligned}
\int_t^{t+1}z(s)ds&=\int_t^{t+1}\big[a_0+\frac{a_1}{s}+\frac{a_2}{s^2}+o(s^{-2})\big]ds\\
&=a_0+a_1\ln(1+\frac{1}{t})+\frac{a_2}{t(t+1)}+o(t^{-2})\\
&=a_0+a_1\big(\frac{1}{t}-\frac{1}{2t^2}\big)+\frac{a_2}{t^2}+o(t^{-2})\\
&=a_0+\frac{a_1}{t}-\frac{a_1-2a_2}{2t^2}+o(t^{-2}),
\end{aligned}$$

$$\begin{aligned}
\exp\big\{\int_t^{t+1}z(s)ds\big\}&=e^{a_0}\Big\{1+\big(\frac{a_1}{t}-\frac{a_1-2a_2}{2t^2}\big)+\frac{1}{2!}\big(\frac{a_1}{t}-\frac{a_1-2a_2}{2t^2}\big)^2+o(t^{-2})\Big\}\\
&=e^{a_0}\big\{1+\frac{a_1}{t}+\frac{a_1^2-a_1+2a_2}{2t^2}+o(t^{-2})\big\}.
\end{aligned}$$

Substituting these expressions and the previous asymptotic in (6.17) and taking into account that $p=s(2+s)e^s$, $q=s^2(1+s)e^s$, we obtain

$$\begin{aligned}
-\big[a_0^2+\frac{2a_0a_1}{t}+(a_1^2+2a_0a_1-a_1-\nu^2)\frac{1}{t^2}+o(t^{-2}\big]\frac{\nu}{t}\big[1-\frac{1}{2t}+\frac{2-\nu^2}{6t^2}+o(t^{-2}\big]\\
+\frac{\nu}{t}\big[2a_0+\frac{2a_1-1}{t}+\frac{2a_2}{t^2}+o(t^{-2}\big]\big[1-\frac{\nu^2}{2t^2}+o(t^{-2}\big]\\
+s(2+s)e^s\frac{\nu}{t}\big[1-\frac{1}{t}+\frac{1}{t^2}+o(t^{-2}\big]e^{a_0}\big[1+\frac{a_1}{t}+\frac{a_1^2-a_1+2a_2}{2t^2}+o(t^{-2}\big]=0.
\end{aligned}\tag{6.18}$$

Now we equate to zero all three coefficients of the left-hand side of the last equation. The first one has the form

$$-a_0^2+2a_0+s(2+s)e^{s+a_0}=0,\tag{6.19}$$

so $a_0 = -s$ is the unique root of this equation. Equating to zero the coefficient of $\frac{1}{t}$ we have:

$$\frac{s^2}{2} + 2sa_1 + 2a_1 - 1 + s(2+s)(a_1 - 1) = 0,$$

then

$$a_1 = \frac{1}{2}. \tag{6.20}$$

We will see further that the coefficient a_2 is not applied, so we will not look for this coefficient.

Thus, substituting $z(t) = -s + \frac{1}{2t} + \frac{a_2}{t^2} + o(t^{-2})$ into $(5.5)_1$) we have

$$\begin{aligned}
M(t) &\cong -2s(2+s)e^s\big[-s + \frac{1}{2t} + \frac{2a_2 - 1}{2t^2}\big] \\
&\quad - e^s\big[1 - \frac{1}{2t} - \frac{3 - 8a_2}{8t^2}\big]\Big\{\big[s^2 - \frac{s}{t} - \frac{1 + 8sa_2 + 4\nu^2}{4t^2}\big](1 - \frac{\nu^2}{2t^2}) \\
&\quad + \frac{\nu^2}{t^2}\big[-2s + \frac{a_2}{t^2}\big]\big[1 - \frac{1}{2t} + \frac{2-\nu^2}{6t^2}\big]\Big\} \\
&\cong s^2(1+s)e^s + \frac{1}{t^2}\big[\frac{e^s}{8}(s^2 + 4s + 2) + \nu^2 L_2(s)\big] \\
&\cong q + \frac{1}{t^2}\big[K(p,1) + \nu^2 L_2(s) + o(t^{-2})\big].
\end{aligned}$$

Note that the latter expression does not contain the coefficient a_2. We can choose ν so small that the inequality $K(p,1) + \nu L_2(s) < C$ holds together with (6.5), which was necessary to prove. For the case $\sigma = 1$ Theorem 6.2 is proven too. $\square$

Remark. Condition (6.11) is unimprovable. Indeed, the function $x_0(t) := \sqrt{t}e^{st}$ is a nonoscillatory solution of (6.1), where

$$Q(t+\sigma) := Q_0(t+\sigma) = \frac{px_0'(t+\sigma-1) - x_0''(t+\sigma)}{x_0(t)}. \tag{6.21}$$

We have $x_0'(t) = \sqrt{t}e^{st}(s + \frac{1}{2t})$,

$$\begin{aligned}
&x_0'(t+\sigma-1) \\
&= \sqrt{t}e^{st}e^{s(\sigma-1)}\Big\{\frac{1}{2t}\big[1 - \frac{\sigma-1}{2t} + o(t^{-1})\big] + s\big[1 - \frac{\sigma-1}{2t} - \frac{(\sigma-1)^2}{8t^2} + o(t^{-2}\big]\Big\},
\end{aligned}$$

$$x_0''(t) = \sqrt{t}e^{st}\big[s^2 + \frac{s}{t} - \frac{1}{4t^2} + o(t^{-2})\big],$$

$$\begin{aligned}
&x_0''(t+\sigma) \\
&= \sqrt{t}e^{st}e^{s\sigma}\big\{-\frac{1}{4t^2}[1 + o(1)] + \frac{s}{t}\big[1 - \frac{\sigma}{2t} + o(t^{-1})\big] + s^2\big[1 + \frac{\sigma}{2t} - \frac{\sigma^2}{8t^2} + o(t^{-2})\big]\big\}.
\end{aligned}$$

Substituting these expressions into (6.21) we obtain (after calculations)

$$Q_0(t) = q + \frac{K(p,\sigma)}{t^2} + o(t^{-2}).$$

Thus it is impossible to improve the constant $K(p,\sigma)$ in (6.11) for any $s > 0$ and any σ.

Consider an application of Theorem 4.1 and Theorem 4.2 to the following typical equation

$$lx := x''(t) - \big[\frac{p}{t-1}x(t-1)\big]' + Q(t)x(t-\sigma) = 0, \quad \sigma \neq 0. \tag{6.22}$$

Here $P(t) := \frac{p}{t-1}$, $l(t) = t-\sigma$, $r(t) = t-1$, $k(t) = t+\sigma$, and $q(t) = t+1$.
Case $p > 0$**:** In this case we will apply Theorem 4.1. Rewrite (5.4) as follows:

$$\begin{aligned}
\tilde{Q}(t+\sigma) &= -\frac{\exp\big\{-\int_t^{t+\sigma} z(s)ds\big\}}{\sin\int_t^{t+\sigma}\varphi(s)ds}\Big\{B(t) + \frac{p}{t}\exp\int_t^{t+1} z(s)ds \\
&\quad \times \Big[z(t+1)\sin\int_t^{t+1}\varphi(s)ds + \varphi(t+1)\cos\int_t^{t+1}\varphi(s)ds\Big]\Big\}, \\
\tilde{Q}(t+\sigma) &\geq M(t+\sigma) \\
&:= -\exp\big\{-\int_t^{t+\sigma} z(s)ds\big\}\Big\{A(t)\cos\int_t^{t+\sigma}\varphi(s)ds \\
&\quad + B(t)\sin\int_t^{t+\sigma}\varphi(s)ds + \frac{p}{t}\exp\int_t^{t+1} z(s)ds \\
&\quad \times \Big[z(t+1)\cos\int_{t+1}^{t+\sigma}\varphi(s)ds + \varphi(t+1)\sin\int_{t+1}^{t+\sigma}\varphi(s)ds\Big]\Big\}
\end{aligned} \tag{6.23}$$

Suppose $\varphi(t) := \frac{\nu}{t}$, $z(t) := \frac{\alpha}{t} + \frac{\beta}{t^2}$, where α and β will be chosen later. Let us first check (5) of Theorem 4.1. For $t \in [a, a+1]$ we have

$$\begin{aligned}
&y'(t) \geq 0 \Leftrightarrow z(t)\sin\big(\nu\ln\frac{t}{a}\big) + \frac{\nu}{t}\cos\big(\nu\ln\frac{t}{a}\big) \geq 0 \Leftrightarrow \\
&\Leftrightarrow \alpha\tan\big(\nu\ln\frac{t}{a}\big) + \nu \geq 0 \Leftrightarrow \alpha\ln\frac{t}{a}\,\frac{\tan\big(\nu\ln\frac{t}{a}\big)}{\nu\ln\frac{t}{a}} + 1 \geq 0.
\end{aligned} \tag{6.24}$$

Since $\lim_{\nu\to 0}\frac{\tan(\nu\ln\frac{t}{a})}{\nu\ln\frac{t}{a}} = 1$, Inequality (4.6) is fulfilled if for sufficiently small ν we have $\alpha\ln\frac{t}{a} + 1 > 0$, $t \in [a, a+1]$. The last one is true if

$$\alpha\ln\frac{a+1}{a} + 1 > 0 \Leftrightarrow \frac{\alpha}{a} + 1 > 0,$$

that is *for every* α, if a is sufficiently large. Since $\int_a^b \frac{\varphi(s)}{s}ds = \pi$ we will obtain similarly that $y'(t) \leq 0, \quad t \in [b, b+1]$.

Now, we return to $(6.23)_2$:

$$A(t) = z^2 + z' - \varphi^2 = \frac{\alpha^2 - \alpha - \nu^2}{t^2} + o(t^{-2});$$

$$B(t) = \frac{\nu(2\alpha-1)}{t^2} + \frac{2\nu\beta}{t^3} + o(t^{-3});$$

$$\begin{aligned}
&M(t+\sigma) = -\big\{\frac{\alpha^2-\alpha-\nu^2}{t^2} + \frac{p}{t}\big[\frac{\alpha}{t} + o(t^{-1})\big]\big\} \\
&\Rightarrow t^2 M(t+\sigma) = \alpha(1-p) - \alpha^2 + \nu^2 + o(1).
\end{aligned} \tag{6.25}$$

Let $\alpha = \alpha_0 = \frac{1-p}{2}$. Then

$$\sup_\alpha[\alpha(1-p) - \alpha^2] = \alpha_0(1-p) - \alpha_0^2 = \frac{(1-p)^2}{4}.$$

Hence
$$t^2M(t+\sigma) = \frac{(1-p)^2}{4} + \nu^2 + o(1).$$
From $(6.23)_1$ we have
$$\begin{aligned}
\tilde{Q}(t+\sigma) &\cong -\frac{1-\frac{\alpha\sigma}{t}}{\frac{\nu\sigma}{t}}\Big\{\frac{\nu(2\alpha-1)}{t^2} + \frac{2\nu\beta}{t^3} \\
&\quad + \frac{p}{t}\big(1+\frac{\alpha}{t}\big)\Big[\big(\frac{\alpha}{t+1} + \frac{\beta}{(t+1)^2}\big)\frac{\nu}{t} + \frac{\nu}{t}\big(1-\frac{\nu^2}{2t^2}\big)\Big]\Big\} \\
&\cong -\frac{t}{\sigma}\big\{\frac{2\alpha-1+p}{t^2} + \frac{2\beta+2p\alpha}{t^3} + o(t^{-3})\big\}.
\end{aligned}$$
Since $\alpha = \alpha_0 = (1-p)/2$,
$$t^2\tilde{Q}(t+\sigma) = -\frac{2}{\sigma}\beta - \frac{p(1-p)}{\sigma} + o(1). \tag{6.26}$$
If from the beginning we require in (6.11) that
$$\liminf_{t\to\infty} t^2Q(t) := C > \frac{(1+p)^2}{4}, \tag{6.27}$$
and we choose ν small such that
$$\frac{(1+p)^2}{4} + \nu^2 < C \tag{6.28}$$
and β such that
$$\frac{(1+p)^2}{4} + \nu^2 < -\frac{2}{\sigma}\beta - \frac{p(1-p)}{\sigma} < C,$$
then $M(t) < \tilde{Q}(t) < Q(t)$, and the choice of a function $\tilde{Q}(t)$ with required properties is completed.

Case $p < 0$: Since an application of Theorem 4.1 is impossible in this case, we employ Theorem 4.2. First consider (4.18) and rewrite $(4.18)_1$ for $t \in [a-1, a]$ as
$$\begin{aligned}
y''(t) \le 0 &\Leftrightarrow A(t)\sin(\nu\ln\frac{t}{a}) + B(t)\cos\big(\nu\ln\frac{t}{a}\big) \le 0 \\
&\Leftrightarrow -A(t)\sin(\nu\ln\frac{a}{t}) + B(t)\cos(\nu\ln\frac{a}{t}) \le 0 \\
&\Leftrightarrow -\frac{\alpha^2-\alpha-\nu^2}{t^2}\tan(\nu\ln\frac{a}{t}) + \frac{\nu(2\alpha-1)}{t^2} + \frac{2\nu\beta}{t^3} < 0 \\
&\Leftarrow (-\alpha^2+\alpha+\nu^2)\tan(\nu\ln\frac{a}{t}) + \nu(2\alpha-1) < 0.
\end{aligned}$$
Evidently, the condition $2\alpha - 1 < 0$ is necessary (put $t = a$). However it is also sufficient, since $\tan(\nu\ln\frac{a}{t}) \cong \nu/a$ and it becomes small for sufficiently large a. Thus, let $\alpha < 1/2$. Since $\int_a^b \frac{\nu}{s}ds = \nu\ln\frac{b}{a} = \pi$. Then $(4.18)_2$ can be rewritten for $t \in [b-1, b]$ in the form
$$\begin{aligned}
&A(t)\sin\int_a^t\varphi(s)ds + B(t)\cos\int_a^t\varphi(s)ds \ge 0 \\
&\Leftrightarrow A(t)\sin\big(\int_a^b\varphi(s)ds + \int_b^t\varphi(s)ds\big) + B(t)\cos\big(\int_a^b\varphi(s)ds + \int_b^t\varphi(s)ds\big) \ge 0 \\
&\Leftrightarrow -A(t)\sin\int_b^t\varphi(s)ds - B(t)\cos\int_b^t\varphi(s)ds \ge 0
\end{aligned}$$

which is proven above. Then for $\alpha < 1/2$ Condition (4.18) is fulfilled. Condition (2) of Theorem 4.2 holds if

$$t^2\tilde{M}(t+\sigma) = \alpha(1-p) - \alpha^2 + \nu^2 + o(1), \tag{6.29}$$

but here we can not assume $\alpha = \alpha_0 = \frac{1-p}{2}$ since $p < 0$, $\alpha_0 > 1/2$, and hence (6.19) is not satisfied.

Let $\alpha = \frac{1}{2}$ in (6.29), that is

$$t^2\tilde{M}(t+\sigma) = \frac{1-2p}{4} + \nu^2 + o(1). \tag{6.30}$$

Then (6.23) turns into

$$\begin{aligned}
&\tilde{Q}(t+\sigma)\exp\big\{\int_t^{t+\sigma-1} z(s)ds\big\}\sin\big(\nu\ln\frac{t+\sigma-1}{t}\big) + \frac{p}{t}\frac{\nu}{t}\\
&+\exp\big\{-\int_{t-1}^{t} z(s)ds\big\}\Big\{\big[-\frac{\alpha^2-\alpha-\nu^2}{t^2} + \frac{(2\alpha-1)\beta}{t^3}\big]\sin\big(\nu\ln\frac{t}{t-1}\big)\\
&+\big[\frac{\nu(2\alpha-1)}{t^2} + \frac{2\nu\beta}{t^3}\big]\cos(\nu\ln\frac{t}{t-1})\Big\} + o(t^{-3}) = 0,
\end{aligned}$$

or

$$\tilde{Q}(t+\sigma) = \frac{1}{\frac{\nu(\sigma-1)}{t}}\big\{\frac{\nu(2\alpha-1-p)}{t^2} + o(t^{-2})\big\}.$$

Since $\alpha = 1/2$,

$$t\tilde{Q}(t+\sigma) = -\frac{p}{\sigma-1} + o(t^{-1}). \tag{6.31}$$

We have $p < 0$, so we should require $\sigma > 1$ to satisfy Condition (1) of Theorem 4.2. Evidently, (6.30)–(6.31) imply $\tilde{Q}(t) > \tilde{M}(t)$, $t \gg t_0$, that is $(6.23)_2$ is satisfied.

Now we can formulate the final result for (6.22).

Theorem 6.3. *Suppose one of the two following conditions holds:*

(a) $p \geq 0$, $\sigma \neq 0$,

$$\liminf_{t\to+\infty}[t^2Q(t)] = C_1 > \frac{(1+p)^2}{4} \tag{6.32}$$

(b) $p < 0$, $\sigma > 1$,

$$\liminf_{t\to+\infty}[tQ(t)] = C_2 > \frac{|p|}{\sigma-1}. \tag{6.33}$$

Then all solutions of (6.22) *are oscillatory and there exists* $\nu > 0$ *such that all solutions have at least one zero on any interval* $(T-1, T\exp\frac{\pi}{\nu})$ *for sufficiently large* T.

Remarks: (1) Equation (6.22) for case $p > 0$, $\sigma \neq 0$ has a nonoscillatory solution $x_0(t) = t^{\frac{p+1}{2}}$ and

$$Q(t) = Q_0(t) := \frac{\big[\frac{p}{t-1}x_0(t-1)\big]' - x_0''(t)}{x_0(t-\sigma)} == \frac{1}{t^2}\frac{p(\gamma-1)(1-\frac{1}{t})^{\gamma-2} - \gamma(\gamma-1)}{(1-\frac{\sigma}{t})^{\gamma}}.$$

It is easy to see that

$$\lim_{t\to\infty} t^2Q_0(t) = \frac{(1-p)^2}{4}.$$

Thus (6.32) which is apparently exact in order (t^{-2}), but is not exact in constant. Condition (6.33), apparently, is not exact, both in order and in constant.

(2) The condition $\sigma > 1$ in the case $p < 0$ is not accidental. We remember that an autonomous (3.1) with $p < 0$, $\tau = 1$, $\sigma \le 1$ cannot be oscillatory for any q, but for $\sigma > 1$ there exist such numbers q.

We will investigate here oscillation properties of another equation, which contains unbounded delay in damped term. Consider the equation

$$x''(t) - \big[\frac{p}{t}x\big(\frac{t}{\mu}\big)\big]' + Q(t)x(t) = 0, \quad p > 0,\ \mu > 1. \tag{6.34}$$

Let be $\varphi(t) := \nu/t$, $\nu > 0$. Then

$$A(t) := z^2(t) + z'(t) - \frac{\nu^2}{t^2}; \quad B(t) := \frac{\nu}{t}\big[2z(t) - \frac{1}{t}\big].$$

Substituting these equalities in (5.4) we have

$$\begin{aligned} S(t) &:= z^2(t) - z'(t) - \frac{\nu^2}{t^2} + \frac{p}{t}\exp\big\{\int_t^{\mu t} z(s)ds\big\} \\ &\quad \times \big[z(\mu t)\cos(\nu\ln\mu) - \frac{\nu}{\mu t}\sin(\nu\ln\mu)\big] + \tilde{Q}(t) \ge 0 \\ R(t) &:= \frac{\nu}{t}\big[2z(t) - \frac{1}{t}\big] + \frac{p}{t}\exp\big\{\int_t^{\mu t} z(s)ds\big\} \\ &\quad \times \big[z(\mu t)\sin(\nu\ln\mu) + \frac{\nu}{\mu t}\cos(\nu\ln\mu)\big] = 0 \end{aligned}$$

or

$$\begin{aligned} \tilde{Q}(t) \ge M(t) &:= -z'(t) - z^2(t) + \frac{\nu^2}{t^2} - \frac{p}{t}\exp\big\{\int_t^{\mu t} z(s)ds\big\} \\ &\quad \times \big[z(\mu t)\cos(\nu\ln\mu) - \frac{\nu}{\mu t}\sin(\nu\ln\mu)\big] \\ N[z](t) &:= 2tz(t) - 1 + p\exp\big\{\int_t^{\mu t} z(s)ds\big\} \\ &\quad \times \big[\mu t z(\mu t)\frac{1}{\mu\nu}\sin(\nu\ln\mu) + \frac{1}{\mu}\cos(\nu\ln\mu)\big] = 0. \end{aligned} \tag{6.35}$$

Equation $(6.35)_2$ has an analytic solution $z(t)$ in a neighborhood of infinity, such that $z(\infty) = 0$.

Let us find the first term of an expansion $z(t) = \frac{\alpha}{t} + \dots$:

$$\begin{aligned} &2\alpha - 1 + p\mu^{\alpha}\big[\alpha\frac{1}{\mu\nu}\sin(\nu\ln\mu) + \frac{1}{\mu}\cos(\nu\ln\mu)\big] = 0 \\ \Leftrightarrow F(\alpha;\nu) &:= 2\alpha - 1 + p\mu^{\alpha-1}\big[\alpha\ln\mu\frac{\sin(\nu\ln\mu)}{\nu\ln\mu} + \cos(\nu\ln\mu)\big] = 0. \end{aligned} \tag{6.36}$$

Consider also the limit equation of (6.36), where $\nu \to 0$:

$$F(\alpha;0) := 2\alpha - 1 + p\mu^{\alpha-1}(\alpha\ln\mu + 1) = 0. \tag{6.37}$$

One can show that (6.37) has the unique root α_0 (we omit calculations) and, moreover, $\alpha_0 \in [0, \frac{1}{2})$ for $0 < p \le \mu$ and $\alpha_0 \in (-\frac{1}{\ln\mu}, 0)$ for $p > \mu$.

On the other hand

$$F'_{\alpha}(\alpha_0, 0) = 2p\mu^{\alpha_0-1}\ln\mu(2 + \ln\mu) > 0.$$

Then (6.36) has a solution α_ν for $0 < \nu < \nu_0$. That is $z(t) = \frac{\alpha_\nu}{t} + o(t^{-1})$ is a solution of $(6.35)_2$. Thus

$$M(t)t^2 = \alpha_\nu - \alpha_\nu^2 + \nu^2 - p\mu^{\alpha_\nu - 1}[\alpha_\nu \cos(\nu \ln \mu) - \nu \sin(\nu \ln \mu)] + o(t^{-2}).$$

Since

$$\alpha_\nu - \alpha_\nu^2 + \nu^2 - p\mu^{\alpha_\nu - 1}[\alpha_\nu \cos(\nu \ln \mu) - \nu \sin(\nu \ln \mu)] = \alpha_0^2 \left[1 + p\mu^{\alpha_0 - 1} \ln \mu\right] + o(\nu^2),$$

the equality

$$\liminf_{t\to\infty} t^2 Q(t) = C > \alpha_0^2 \left[1 + p\mu^{\alpha_0 - 1} \ln \mu\right] := K(p;\mu) \tag{6.38}$$

implies $Q(t) > \tilde{Q}(t)$ for $t > T$. Then conditions (2)–(4) of Theorem 4.1 hold. Since $p > 0$, Condition (1) also holds.

We check now (4.6) of (5). For $t \in (a, \mu a)$ we have

$$\begin{gathered}
y'(t) \ge 0 \Leftrightarrow \frac{\alpha_0}{t} \sin\left(\nu \ln \frac{t}{a}\right) + \frac{\nu}{t} \cos\left(\nu \ln \frac{t}{a}\right) \ge 0 \\
\Leftrightarrow \alpha_0 \tan\left(\nu \ln \frac{t}{a}\right) + \nu \ge 0 \Leftrightarrow \alpha_0 \ln \mu \frac{\tan\left(\nu \ln \frac{t}{a}\right)}{\nu \ln \frac{t}{a}} + 1 \ge 0.
\end{gathered} \tag{6.39}$$

Since $\lim_{\nu\to 0} \frac{\tan\left(\nu \ln \frac{t}{a}\right)}{\nu \ln \frac{t}{a}} = 1$ uniformly on $t \in (a, \mu a)$ then an inequality $\alpha_0 \ln \mu + 1 > 0$ implies (6.39) for sufficiently small $\nu > 0$. Since

$$\int_a^b \varphi(s)ds = \pi \Leftrightarrow b = ae^{\frac{\pi}{\nu}}$$

it follows that $y'(t) \le 0$, $t \in (b, \mu b)$, which is the second part of (4.6). Thus we have the following statement which is the exact analogue of the classical Knezerian Theorem:

Theorem 6.4. *Assume* (6.38). *Then all solutions of* (6.34) *are oscillatory, and, moreover, there exists* $\nu > 0$ *such that any solution has at least one zero on any interval* $\left(\frac{T}{\mu}, Te^{\frac{\pi}{\nu}}\right)$ *for sufficiently large* T.

Remark. The constant $K(p,\mu)$ is an unimprovable in (6.38). Indeed, by direct calculations one can check that $x_0(t) = t^{1-\alpha_0}$ is a non-oscillatory solution of (6.34) with $Q_0(t) = Kt^{-2}$.

We will investigate in detail a particular case of (6.34) with $p = \mu$, i.e. the following equation

$$x''(t) - \left[\frac{\mu}{t} x\left(\frac{t}{\mu}\right)\right]' + Q(t)x(t) = 0, \quad \mu > 1. \tag{6.40}$$

If $p = \mu$ (and only in this case) then (6.37) has the root $\alpha_0 = 0$. And so, in(6.38) $K(\mu,\mu) = 0$. It means that there is no information about the main term of Knezerian Minorant for $Q(t)$ in (6.40).

Let $\varphi(t) := \frac{\nu}{t \ln t}$ in (5.4). Hence

$$\begin{gathered}
\varphi'(t) = -\frac{\nu(1 + \ln t)}{t^2 \ln^2 t}; \quad \int_t^{\mu t} \varphi(s)ds = \frac{\nu \ln \mu}{\ln t}\left[1 - \frac{\ln \mu}{2 \ln t} + o\left(\frac{1}{\ln t}\right)\right], \\
A(t) := z^2(t) + z'(t) - \frac{\nu^2}{t^2 \ln^2 t}; \quad B(t) := \frac{\nu}{t \ln t}\left[2z(t) - \frac{1}{t} - \frac{1}{t \ln t}\right].
\end{gathered}$$

Substituting all these equalities in (5.4), with $p=\mu$, we obtain:

$$\begin{gathered}
z^2(t)+z'(t)-\frac{\nu^2}{t^2\ln^2 t}+\frac{\mu}{t}\exp\big\{\int_t^{\mu t} z(s)ds\big\} \\
\times\big[z(\mu t)\cos\int_t^{\mu t}\varphi(s)ds-\frac{\nu}{\mu t\ln(\mu t)}\sin\int_t^{\mu t}\varphi(s)ds\big]+\tilde{Q}(t)\ge 0\,, \\
\frac{\nu}{t\ln t}\big[2z(t)-\frac{1}{t}-\frac{1}{t\ln t}\big]\frac{\mu}{t}\exp\int_t^{\mu t} z(s)ds \\
\times\big[z(\mu t)\sin\int_t^{\mu t}\varphi(s)ds+\frac{\nu}{\mu t\ln(\mu t)}\cos\int_t^{\mu t}\varphi(s)ds\big]=0.
\end{gathered}\tag{6.41}$$

We will look for the main part of the solution $z(t)$ of $(6.41)_2$ in a neighborhood of the infinity, that is a solution $z(t)$ of the form

$$tz(t)=\sum_{n=1}^{\infty}\frac{a_n}{(\ln t)^n}.$$

Let us find the number β in $z(t)=\frac{\beta}{t\ln t}+o(\frac{1}{t\ln t})$. Since

$$\exp\big\{\int_t^{\mu t} z(s)ds\big\}=1+\frac{\beta\ln\mu}{\ln t}+o\big(\frac{1}{\ln t}\big),$$

Equation $(6.41)_2$ has the form

$$\begin{gathered}
\frac{\nu}{t\ln t}\big(\frac{2\beta-1}{t\ln t}-\frac{1}{t}\big)+\frac{\mu}{t}\big(1+\frac{\beta\ln\mu}{\ln t}\big) \\
\times\Big[\frac{\beta}{\mu t\ln(\mu t)}\sin\big(\frac{\nu\ln\mu}{\ln t}\big)+\frac{\nu}{\mu t\ln(\mu t)}\cos\big(\frac{\nu\ln\mu}{\ln t}\big)+o\big(\frac{1}{\ln t}\big)\Big]=0.
\end{gathered}\tag{6.42}$$

Applying the equality

$$\frac{1}{\ln(\mu t)}=\frac{1}{\ln t}\big(1-\frac{\ln\mu}{\ln t}+o\big(\frac{1}{\ln t}\big)\big),$$

and multiplying all terms of $(6.41)_2$ by $t^2\ln t$, write its main terms:

$$\frac{2\beta-1}{\ln t}-1+\big(1-\frac{\ln\mu}{t}\big)\big(1+\frac{\beta\ln\mu}{\ln t}\big)\big(1+\frac{\beta\ln\mu}{\ln t}\big)+o\big(\frac{1}{\ln t}\big)=0$$

or

$$\frac{2\beta-1}{\ln t}-1+1+\frac{(2\beta-1)\ln\mu}{\ln t}+o\big(\frac{1}{\ln t}\big)=0\Rightarrow\beta=\frac{1}{2}.\tag{6.43}$$

Thus, $z(t)=\frac{1}{2t\ln t}+o(\frac{1}{t\ln t})$ and we can write $(6.41)_1$ as

$$\begin{gathered}
\tilde{Q}(t)+\frac{1}{4t^2\ln^2 t}-\frac{1+\ln t}{2t^2\ln^2 t}-\frac{\nu^2}{2t^2\ln^2 t}+\frac{\mu}{t}\big[1+\frac{\ln\mu}{2\ln t}+o\big(\frac{1}{\ln t}\big)\big] \\
\big\{\frac{1}{2\mu t\ln(\mu t)}\cos\frac{\nu\ln\mu}{\ln t}-\frac{\nu}{\mu t\ln(\mu t)}\frac{\nu\ln\mu}{\ln t}+o\big(\frac{1}{\ln t}\big)\big\}\ge 0.
\end{gathered}$$

After some calculations we obtain

$$t^2\ln^2 t\tilde{Q}(t)\ge\frac{1+\ln\mu}{4}+o(1).$$

Since

$$\int_a^b\frac{\nu}{s\ln s}ds=\pi\Leftrightarrow\nu\ln\Big[\frac{\ln b}{\ln a}\Big]=\pi\Leftrightarrow b=ae^{\exp\frac{\pi}{\nu}},$$

we can formulate the following result.

Theorem 6.5. *Suppose in* (6.40)

$$\liminf_{t\to\infty}[t^2\ln^2 t\ Q(t)] = C > \frac{1+\ln\mu}{4}. \tag{6.44}$$

Then all its solutions are oscillatory. Moreover, there exists $\nu>0$ such that every solution has at least one zero in any interval $(\frac{T}{\mu}, T^{\exp\frac{\pi}{\nu}})$ for sufficiently large T.

This result is the best possible one in the following sense. The function $x_0(t) := t\sqrt{\ln t}$ is a nonoscillatory solution of (6.40) with

$$Q(t) = Q_0(t) := \frac{1}{x_0(t)}\big\{\big[\frac{\mu}{t}x_0\big(\frac{t}{\mu}\big)\big]' - x_0''(t)\big\}.$$

On the other hand, we can show that

$$\lim_{t\to\infty}\big\{t^2\ln^2 t\ Q_0(t)\big\} = \frac{1+\ln\mu}{4}. \tag{6.45}$$

Remark. This phenomenon is well-known for ordinary differential (1.2). Indeed, substituting (1.3), where $P(t) := \frac{1}{t}$, we transform (1.2) into (1.1) with $a(t) = Q(t) + \frac{1}{4t^2}$. Then by generalized Kneser Theorem [12, Theorem 7.1, Exercise 1.2, Ch.11] the following statement holds

If

$$\liminf_{t\to\infty}\big\{t^2\ln^2 t\ Q(t)\big\} = C > \frac{1}{4}, \tag{6.46}$$

then all solutions of the equation

$$x''(t) + \big[\frac{x(t)}{t}\big]' + Q(t)x(t) = 0 \tag{6.47}$$

are oscillatory.

For differential equations with deviating arguments such phenomenon is described for the first time here. Note, by the way, that (6.44) turns into (6.46) when $\mu = 1$.

Now, we apply Theorem 4.1 and Theorem 4.2 to the investigation of oscillation properties of (1.6) with *strong* deviating arguments $r(t)$ or $l(t)$. We apply this term when not only $t - r(t)$ is unbounded but $\lim_{t\to\infty}\frac{r(t)}{t} = 0$ holds as well. For example, the delay

$$t - r(t) = t - \frac{t}{\mu} = \frac{\mu-1}{\mu}t, \quad \mu > 1$$

which was considered in the previous paragraph, is unbounded, but not strong in our sense. A typical example of the strong deviating argument is $r(t) = t^{1/\mu}$, $\mu > 1$. Investigation of such equations usually meets difficulties. Below we will apply our method to a typical equation with a strong delay. Consider the equation

$$x''(t) - \big[\frac{p}{t^\beta}x(t^{\frac{1}{\mu}})\big]' + Q(t)x(t^{\frac{1}{\mu}}) = 0, \quad \mu > 1;\ p > 0,\ \beta \ge \frac{1}{\mu}, \tag{6.48}$$

and let us apply Theorem 4.1. Choose $z(t)$ and $\tilde{Q}(t)$ on (T,∞) for T sufficiently large such that (5.6) is satisfied. We will seek $z(t)$ in the form

$$z(t) = \frac{s}{t\ln t} + \frac{\gamma}{t\ln^2 t}, \tag{6.49}$$

where parameters s and γ will be chosen later, and $\varphi(t)$ is define as $\varphi(t) = \nu/(t\ln t)$. Then

$$\begin{gathered}
A(t) = \frac{1}{t^2\ln t}\big[-s + \frac{s^2 - s - \nu^2 - \gamma}{\ln t} + o\big(\frac{1}{\ln t}\big)\big],\\
B(t) = \frac{\nu}{t^2\ln t}\big[-1 + \frac{2s-1}{\ln t} + o\big(\frac{1}{\ln t}\big)\big],\\
r(t) = l(t) = t^{1/\mu}, \quad q(t) = k(t) = t^{\mu}, \quad \int_t^{t^\mu}\varphi(s)ds = \nu\ln\mu,\\
\exp\int_t^{t^\mu} z(s)ds = \mu^s\big[1 + \frac{\gamma(\mu-1)}{\mu\ln t} + o\big(\frac{1}{\ln t}\big)\big].
\end{gathered}$$

Substituting all these expressions in (5.6) we obtain

$$\begin{gathered}
Z(t) \ge 0 \Leftrightarrow \big[-s + \frac{s^2 - s - \nu^2 - \gamma}{\ln t} + o\big(\frac{1}{\ln t}\big)\big]\sin(\nu\ln\mu)\\
-\big[\nu - 1 + \frac{2s-1}{\ln t} + o\big(\frac{1}{\ln t}\big)\big]\nu\cos(\nu\ln\mu) - \frac{p\nu\mu^s}{t^{\beta\mu-1}}\big[1 + \frac{\gamma(\mu-1)}{\mu\ln t} + o\big(\frac{1}{\ln t}\big)\big] \ge 0.
\end{gathered} \tag{6.50}$$

Consider two cases: $\beta = \frac{1}{\mu}$ and $\beta > 1/\mu$.

Case $\beta = \frac{1}{\mu}$: Equation (6.50) turns into

$$\begin{gathered}
\big[-s\sin(\nu\ln\mu) + \nu\cos(\nu\ln\mu) - \nu p\mu^s\big]\\
+\frac{\nu}{\ln t}\big[s^2 - s - \gamma - \nu^2)\frac{\sin(\nu\ln\mu)}{\nu} - p\mu^{s-1}(\mu-1)\gamma\big] + o\big(\frac{1}{\ln t}\big) \ge 0.
\end{gathered} \tag{6.51}$$

Put $s = s_\nu$ for sufficiently small $\nu > 0$, where s_ν is the root of the equation

$$F(\nu, s) := \frac{\sin(\nu\ln\mu)}{\nu\ln\mu}s\ln\mu - \cos(\nu\ln\mu) + p\nu^s = 0. \tag{6.52}$$

By the Implicit Function Theorem such a root exists since there exists the unique root $s = s_0 < \frac{1}{\ln\mu}$ of the equation

$$F(0, s) := s\ln\mu - 1 + p\nu^s = 0, \tag{6.53}$$

and $F_s'(0, s_0) = \ln\mu(1 + p\mu^{s_0}) \ne 0$. The coefficient γ in the second bracket in (6.50) is negative, hence by choosing $\gamma < \gamma_0$ one can carry out (6.50). The strict value of γ_0 is not important. Thus assume

$$z(t) := \frac{s_\nu}{t\ln t} + \frac{\gamma}{t\ln^2 t} \tag{6.54}$$

and $\nu > 0$ is sufficiently small. Substituting (6.54) in $(5.6)_2)$, we obtain

$$\begin{aligned}
\mu t^{\mu-1}\tilde{Q}(t^\mu) = [\sin(\nu\ln\mu)]^{-1}\Big\{&\exp\big\{-\int_t^{t^\mu} z(s)ds\big\}B(t)\\
&+ \mu t^{\mu-1}\frac{p}{t}\big[z(t^\mu)\sin(\nu\ln\mu) + \frac{\nu}{t^\mu\mu\ln t}\cos(\nu\ln\mu)\big]\Big\},
\end{aligned}$$

hence

$$\mu t^{\mu+1}\ln t\tilde{Q}(t^\mu) = D(p) + \epsilon(\nu, t),$$

where $\lim_{\nu\to 0, t\to\infty}\epsilon(\nu, t) = 0$ and

$$D(p) := \frac{1}{\ln\mu}\big\{\mu^{-s_0} - p(s_0\ln\mu + 1)\big\} = \frac{s_0^2\ln\mu}{\mu^{s_0}}. \tag{6.55}$$

Proceeding to a new time by the substitution of t instead of $t^{\frac{1}{\mu}}$ we obtain

$$t^{\frac{\mu+1}{\mu}} \ln t\tilde{Q}(t) = D(p) + \epsilon(\nu, t^{\frac{1}{\mu}}). \tag{6.56}$$

By the way, note that

$$\lim_{p\to 0} D(p) = \frac{1}{e\ln\mu}. \tag{6.57}$$

Hence we have the following statement.

Theorem 6.6. *Let be* $\beta = \frac{1}{\mu}$, $p > 0$, $\mu > 1$ *in* (6.48) *and*

$$\liminf_{t\to\infty} \big\{t^{\frac{\mu+1}{\mu}} \ln t\tilde{Q}(t)\big\} = C > D(p). \tag{6.58}$$

Then all solutions of (6.48) *are oscillatory and there exists* $\nu > 0$ *such that every solution has at least one zero in any interval* $(T^{1/\mu}, T^{\exp(\pi/\nu)})$ *for sufficiently large* T.

Proof. It follows from Theorem 4.1 and the fact, that a pair $\{\tilde{Q}(t); z(t)\}$ constructed above satisfies (5.6). The interval $(T^{\frac{1}{\mu}}, T^{\exp(\pi/\nu)})$ coincides with the interval (a, b) in Theorem 4.1. Indeed

$$\int_T^b \varphi(t)dt = \pi \Leftrightarrow \nu\int_T^b \frac{dt}{t\ln t} = \pi \Rightarrow b = T^{\exp(\pi/\nu)}.$$

$\square$

Remark. Condition (6.58) is unimprovable in the same sense: there exists (6.48) such that

$$\lim_{t\to\infty} t^{\frac{\mu+1}{\mu}} \ln t\tilde{Q}(t) = D(p), \tag{6.59}$$

but, nevertheless, this equation has a nonoscillatory solution.

Indeed, let $x_0(t) := t(\ln t)^{-s_0}$, where s_0 is defined in (6.53). Then $x_0(t)$ is a nonoscillatory solution of (6.48), where

$$Q(t) = Q_0(t) := \frac{1}{x_0(t^{\frac{1}{\mu}})}\big\{\big[\frac{p}{t^{\frac{1}{\mu}}} x_0(t^{\frac{1}{\mu}})\big]' - x_0''(t^{\frac{1}{\mu}})\big\}.$$

After some calculations we get

$$Q(t) = \big\{t^{\frac{\mu+1}{\mu}} \ln t\big\}^{-1}\big\{D(p) - \frac{s_0(s_0+1)}{\mu^{s_0}\ln t}\big\},$$

hence (6.58) holds.

Case $\beta > 1/\mu$**:** One can write (6.50) as

$$[-s\sin(\nu\ln\mu) + \nu\cos(\nu\ln\mu)] + \frac{1}{\ln t}(s^2 - s - \gamma - \nu^2)\sin(\nu\ln\mu) \geq 0. \tag{6.60}$$

After calculations as in the previous case with $s_\nu = (\nu\cos(\nu\ln\mu))/(\sin(\nu\ln\mu)$, we obtain

$$t^{\frac{\mu+1}{\mu}} \ln t\tilde{Q}(t) = \frac{1}{e\ln\mu} + \epsilon(\nu, t^{\frac{1}{\mu}}). \tag{6.61}$$

Thus we have the statement.

Theorem 6.7. *Let* $\beta > 1/\mu$, $p > 0$ *in* (6.48). *Then the condition*

$$\liminf_{t\to\infty} \big\{t^{\frac{\mu+1}{\mu}} \ln tQ(t)\big\} = C > \frac{1}{e\ln\mu}$$

implies the assertion of Theorem 6.6.

Now we show that it is possible to extend previous results to nonlinear damped equations. Here we do not intend to cover the widest class of such equations, but to present the sketch of the main idea only. Consider the nonlinear equation

$$z''(t) - [P(t)z(r(t))]' + F(t, z(l(t))) = 0, \quad t \in (a, b), \tag{6.62}$$

where $F(t, u)$ satisfies the following "one-sided" estimation:

$$vF(t, v) \geq \tilde{Q}(t)v^2, \ \forall t \in (a, b)\backslash e_{\text{int}}, \quad vF(t, v) \geq 0, \ t \in e_{\text{int}}, \ \forall v \in (-\infty, \infty). \tag{6.63}$$

Theorem 6.8. *Suppose for $P(t), r(t), l(t), \tilde{Q}(t)$ Conditions (1), (3), (5), (6) of Theorem 4.1 and* (6.63) *hold. Then* (6.62) *has no solution which is positive on $(r(a), b) \bigcup e_{\text{ext}}$.*

Proof. Suppose by contradiction that (6.62) has a solution $z_0(t) > 0$, for $t \in (r(a), b) \cup e_{\text{ext}}$. Then the linear equation

$$(lx)(t) := x''(t) - [P(t)x(r(t))]' + \frac{F(t, z_0(l(t))}{z_0(l(t))} x(l(t)) = 0, \ t \in (a, b) \tag{6.64}$$

also has a positive solution $x(t) := z_0(t)$ on $t \in (r(a), b) \cup e_{\text{ext}}$. Equation (6.64) has the form of (4.1) with $Q(t) := \frac{F(t, x_0(l(t))}{x_0(l(t))}$. Equalities (6.63) and (6.64) imply that conditions (2) and (4) of Theorem 4.1 hold. Then the statement of Theorem 6.8 is a consequence of Theorem 4.1. □

Corollary 6.9. *Let*

$$\varphi(t) > 0, \quad \int^{\infty} \varphi(t)dt = \infty$$

and let $\{\tilde{Q}(t); z(t)\}$ be a solution of (5.5). *Suppose* (6.63) *holds on* (T, ∞) *for sufficiently large T and $\tilde{Q}(t) \geq 0$ on (T, ∞). Then all solutions of* (6.62) *are oscillatory. If ,in addition,* (5.2) *holds, then every solution of* (6.62) *has at least one zero on any interval $(r(a), b) \bigcup e_{\text{ext}}$.*

Acknowledgement. The first author was partially supported by Israel Ministry of Absorbtion.

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